Generalized Riemann-Liouville k-Fractional Integrals Associated With Ostrowski Type Inequalities and Error Bounds of Hadamard Inequalities

YOUNG CHEL KWUN[1], **GHULAM FARID**[ID][2], **WAQAS NAZEER**[3], **SALEEM ULLAH**[4], **AND SHIN MIN KANG**[5,6]

[1] Department of Mathematics, Dong-A University, Busan 49315, South Korea
[2] Department of Mathematics, COMSATS University Islamabad, Attock Campus, Attock 43600, Pakistan
[3] Division of Science and Technology, University of Education, Lahore 54000, Pakistan
[4] Department of Mathematics, Air University, Islamabad 44000, Pakistan
[5] Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, South Korea
[6] Center for General Education, China Medical University, Taichung 40402, Taiwan

Corresponding authors: Ghulam Farid (faridphdsms@hotmail.com) and Shin Min Kang (smkang@gnu.ac.kr)

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ABSTRACT Ostrowski inequality provides the estimation of a function to its integral mean. It is useful in error estimations of quadrature rules in numerical analysis. The objective of this paper is to define a more general form of Riemann–Liouville k-fractional integrals with respect to an increasing function, which are used to obtain fractional integral inequalities of Ostrowski type. A simple and straightforward approach is followed to establish these inequalities. The applications of established results are also briefly discussed and succeeded to get bounds of some fractional Hadamard inequalities.

INDEX TERMS Fractional inequalities, Hadamard inequality, Ostrowski inequality, Riemann-Liouville fractional integrals, Generalized fractional integrals.

I. INTRODUCTION AND PRELIMINARY RESULTS

Ostrowski [18] inequality is very important and useful in the subjects of mathematical analysis, numerical analysis and other fields of mathematics and engineering. It was introduced by Ostrowski in 1938. It is stated in the following theorem.

Theorem 1: Suppose that g be a differentiable function on J^o, the interior of an interval J in $\mathbb{R}$. For $a, b \in J^o$, $a < b$, let $|g'(t)| \leq M$ for all $t \in [a, b]$. Then for $x \in [a, b]$ the following inequality holds true

$$\left| g(x) - \frac{1}{b-a} \int_a^b g(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)M. \tag{1}$$

As an application point of view it provides the estimations of the Hadamard inequality. It also establishes the error bounds, of relations in special means and of several numerical quadrature rules of integration like rectangular, trapezoidal, Simpson and other in very general form [1], [2]. Many authors have been working continuously on inequality (1) and have produced very interesting results

(see, [2]–[5], [8], [10], [13] and references there in). We are motivated to study this inequality for the Riemann-Liouville k-fractional integrals in a general form with respect to an increasing function. As a result we get several fractional integral inequalities, such inequalities are useful in the theory of fractional differential equations. Also we apply these fractional inequalities to obtain bounds of the Hadamard inequalities for Riemann-Liouville fractional integrals given in [11], [12], [19], and [20].

The Hadamard inequality is a fascinating interpretation of convex functions in the coordinate plane, and it is stated as follows:

Theorem 2: If g is a convex function on an interval J of real numbers, then the inequality

$$g\left(\frac{a+b}{2} \right) \leq \int_a^b g(t)dt \leq \frac{g(a) + g(b)}{2}$$

for $a, b \in J$, $a < b$ holds.

The Ostrowski inequality is associated with the Hadamard inequality in the sense that this provides its estimations. The main goal of this paper is to develop fractional versions

of the Ostrowski type inequalities and bounds of associated fractional Hadamard type inequalities. It is worthful to give a brief description of already developed work in this regard.

In [5] a different and straightforward proof of the Ostrowski inequality is given by utilizing its conditions in a sound way. By following this new and keen method in [4] some Riemann-Liouville fractional inequalities of Ostrowski type have been studied. These inequalities generalize the Ostrowski inequality in fractional calculus point of view. Also this method of studying Ostrowski inequality do not need to define kernels and establish identities for the sake of its proof. Fractional Ostrowski type inequalities are useful in studying bounds of the fractional Hadamard type inequalities. In [19] a fractional version of the Hadamard inequality using Riemann-Liouville fractional integrals is proved which leads to a number of its versions for convex and related functions in fractional calculus. In the next theorem we state this version of the fractional Hadamard inequality for Riemann-Liouville fractional integrals defined in Definition 1.

Theorem 3: Suppose that g be a positive convex function defined on the interval $[a, b]$, $0 \leq a < b$, $g \in L_1[a, b]$. Then the following inequalities hold for Riemann-Liouville fractional integrals

$$g\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left(\mathcal{I}_{a+}^\alpha g(b) + \mathcal{I}_{b-}^\alpha g(a)\right) \leq \frac{g(a)+g(b)}{2}$$

with $\alpha > 0$.

In [12] it is generalized via Riemann-Liouville k-fractional integrals. In the next theorem we state k-fractional version of the Hadamard inequality for Riemann-Liouville k-fractional integrals defined in Definition 2.

Theorem 4: Suppose that g be a positive convex function defined on the interval $[a, b]$, $0 \leq a < b$, $g \in L_1[a, b]$. Then the following inequalities hold for Riemann-Liouville k-fractional integrals

$$g\left(\frac{a+b}{2}\right) \leq \frac{\Gamma_k(\alpha+k)}{2(b-a)^{\frac{\alpha}{k}}}\left(\mathcal{I}_{a+}^{\alpha,k} g(b) + \mathcal{I}_{b-}^{\alpha,k} g(a)\right) \leq \frac{g(a)+g(b)}{2}$$

with $\alpha, k > 0$.

Another fractional version of the Hadamard inequality is stated in next theorem [20].

Theorem 5: Suppose that g be a positive convex function defined on the interval $[a, b]$, $0 \leq a < b$, $g \in L_1[a, b]$. Then the following inequalities hold for Riemann-Liouville fractional integrals

$$g\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left(\mathcal{I}_{(\frac{a+b}{2})+}^\alpha g(b) + \mathcal{I}_{(\frac{a+b}{2})-}^\alpha g(a)\right)$$
$$\leq \frac{g(a)+g(b)}{2}$$

with $\alpha > 0$.

In [12] the following k-fractional version of the above fractional Hadamard inequality is proved.

Theorem 6: Suppose that g be a positive convex function defined on the interval $[a, b]$, $0 \leq a < b$, $g \in L_1[a, b]$.

Then the following inequalities hold for Riemann-Liouville k-fractional integrals

$$g\left(\frac{a+b}{2}\right) \leq \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[\mathcal{I}_{(\frac{a+b}{2})+}^{\alpha,k} g(b) + \mathcal{I}_{(\frac{a+b}{2})-}^{\alpha,k} g(a)\right]$$
$$\leq \frac{g(a)+g(b)}{2}$$

with $\alpha, k > 0$.

For more general results in fractional calculus we suggest the readers [6], [7], [9], [14], [15], [21] and their cited references.

The results of this paper are used for some estimations of these Hadamard type fractional integral inequalities stated in above theorems. The method we have adopted also provide the technique to prove such results independently.

In the following we give the definitions of Riemann-Liouville fractional integrals, Riemann-Liouville k-fractional integrals and general form of Riemann-Liouville fractional integrals with respect to an increasing function.

Definition 1: Let $g \in L[a, b]$. Then the left-sided and right-sided Riemann Liouville fractional integrals of order $\alpha > 0$ with $a \geq 0$ are defined as:

$$\mathcal{I}_{a+}^\alpha g(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}g(t)dt, \quad x > a$$

and

$$\mathcal{I}_{b-}^\alpha g(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}g(t)dt, \quad x < b$$

where $\Gamma(.)$ is the Gamma function.

A generalization of Riemann-Liouville fractional integrals are the left-sided and right-side Riemann-Liouville k-fractional integrals defined as follows [17].

Definition 2: Let $g \in L_1[a, b]$. Then the k-fractional integrals of order $\alpha, k > 0$ with $a \geq 0$ are defined as:

$$\mathcal{I}_{a+}^{\alpha,k} g(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1}g(t)dt, \quad x > a \quad (2)$$

and

$$\mathcal{I}_{b-}^{\alpha,k} g(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^b (t-x)^{\frac{\alpha}{k}-1}g(t)dt, \quad x < b, \quad (3)$$

where $\Gamma_k(.)$ is the k-Gamma function.

A generalization of the Riemann-Liouville fractional integrals with respect to an increasing function is given as follows [16].

Definition 3: Let $f_1 : [a, b] \rightarrow \mathbb{R}$ be an integrable function. Also let f_2 be an increasing and positive function on $(a, b]$, having a continuous derivative f_2' on (a, b). The left-sided and right-sided fractional integrals of a function f_1 with respect to another function f_2 on $[a, b]$ of order $\alpha > 0$ are defined as:

$$\mathcal{I}_{f_2,a^+}^\alpha f_1(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (f_2(x)-f_2(t))^{\alpha-1}f_2'(t)f_1(t)dt, \quad x > a$$

and

$$\mathcal{I}_{f_2,b^-}^\alpha f_1(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (f_2(t)-f_2(x))^{\alpha-1}f_2'(t)f_1(t)dt, \quad x < b.$$

We organize the paper as follows:

In Section II we define a more general form of Riemann-Liouville k-fractional integrals with respect to an increasing function and use them to obtain Ostrowski-type inequalities. Utilizing a simple inequality via an increasing function and assumptions of Ostrowski inequality several fractional integral inequalities are obtained. These results provide the Ostrowski type inequalities for Riemann-Liouville fractional integrals which are published in [4]. In Section III fractional versions of inequalities of Section II are presented. As an application point of view in Section IV some of the results are applied to find estimations of the Hadamard type fractional inequalities for Riemann-Liouville fractional and k-fractional integrals.

II. K-FRACTIONAL INTEGRAL INEQUALITIES IN A GENERAL FORM

We define a more general form of Riemann-Liouville k-fractional integrals with respect to an increasing function as follows:

Definition 4: Let $f_1 : [a, b] \rightarrow \mathbb{R}$ be an integrable function. Also let f_2 be an increasing and positive function on $(a, b]$, having a continuous derivative f_2' on (a, b). The left-sided and right-sided fractional integrals of a function f_1 with respect to another function f_2 on $[a, b]$ of order $\alpha, k > 0$ are defined as

$$\mathcal{I}_{f_2, a^+}^{\alpha, k} f_1(x) = \frac{1}{k \Gamma_k(\alpha)}$$
$$\times \int_a^x (f_2(x) - f_2(t))^{\frac{\alpha}{k} - 1} f_2'(t) f_1(t) dt, \quad x > a$$

and

$$\mathcal{I}_{f_2, b^-}^{\alpha, k} f_1(x) = \frac{1}{k \Gamma_k(\alpha)}$$
$$\times \int_x^b (f_2(t) - f_2(x))^{\frac{\alpha}{k} - 1} f_2'(t) f_1(t) dt, \quad x < b,$$

where $\Gamma_k(.)$ is the k-Gamma function.

Remark 1: In the above Definition 4.

(i) If we take $k = 1$, then we get the Definition 3 of Riemann-Liouville fractional integrals with respect to an increasing function.

(ii) If we take $f_2(x) = x$, then we get the Definition 2 of Riemann-Liouville k-fractional integrals.

(iii) If we take $f_2(x) = x$ and $k = 1$, then we get the Definition 1 of Riemann-Liouville fractional integrals.

Next we give Ostrowski-type inequality due to Riemann-Liouville k-fractional integrals with respect to an increasing function.

Theorem 7: Let $f_1 : J \longrightarrow \mathbb{R}$ where J is an interval in $\mathbb{R}$, be a function differentiable in J^o, the interior of J and $a, b \in J^o$, $a < b$. Also let $f_2 : [a, b] \longrightarrow \mathbb{R}$ be a differentiable and strictly increasing function with $f_2' \in L[a, b]$ and $|f_1'(t)| \leq M$ for all $t \in [a, b]$. Then for $\alpha, \beta \geq 0$ and $k > 0$, the following inequality for general form of

Riemann-Liouville k-fractional integrals holds

$$\left| f_1(x)((f_2(b) - f_2(x))^{\frac{\beta}{k}} + (f_2(x) - f_2(a))^{\frac{\alpha}{k}}) \right.$$
$$\left. - (\Gamma_k(\beta + k) \mathcal{I}_{f_2, b^-}^{\beta, k} f_1(x) + \Gamma_k(\alpha + k) \mathcal{I}_{f_2, a^+}^{\alpha, k} f_1(x)) \right|$$
$$\leq M \left(x \left((f_2(x) - f_2(a))^{\frac{\alpha}{k}} - (f_2(b) - f_2(x))^{\frac{\beta}{k}} \right) \right.$$
$$\left. + \Gamma_k(\beta + k) \mathcal{I}_{f_2, b^-}^{\beta, k} I(x) - \Gamma_k(\alpha + k) \mathcal{I}_{f_2, a^+}^{\alpha, k} I(x)) \right), \quad (4)$$

where I is the identity function.

Proof: Let $x \in [a, b]$ and $t \in [a, x]$. Since the function f_2 is strictly increasing therefore for $\alpha \geq 0$ and $k > 0$, the following inequality holds true

$$(f_2(x) - f_2(t))^{\frac{\alpha}{k}} \leq (f_2(x) - f_2(a))^{\frac{\alpha}{k}}. \quad (5)$$

From (5) and the boundedness condition on f_1', the following inequalities are the simple consequences

$$\int_a^x (M - f_1'(t))(f_2(x) - f_2(t))^{\frac{\alpha}{k}} dt$$
$$\leq (f_2(x) - f_2(a))^{\frac{\alpha}{k}} \int_a^x (M - f_1'(t)) dt \quad (6)$$
$$\int_a^x (M + f_1'(t))(f_2(x) - f_2(t))^{\frac{\alpha}{k}} dt$$
$$\leq (f_2(x) - f_2(a))^{\frac{\alpha}{k}} \int_a^x (M + f_1'(t)) dt. \quad (7)$$

From (6) and (7) after integrating and simple calculation and by using Definition 4, we get the following resulting inequalities

$$(f_2(x) - f_2(a))^{\frac{\alpha}{k}} f_1(x) - \Gamma_k(\alpha + k) \mathcal{I}_{f_2, a^+}^{\alpha, k} f_1(x)$$
$$\leq M \left(x(f_2(x) - f_2(a))^{\frac{\alpha}{k}} - \Gamma_k(\alpha + k) \mathcal{I}_{f_2, a^+}^{\alpha, k} I(x) \right) \quad (8)$$

and

$$\Gamma_k(\alpha + k) \mathcal{I}_{f_2, a^+}^{\alpha, k} f_1(x) - (f_2(x) - f_2(a))^{\frac{\alpha}{k}} f_1(x)$$
$$\leq M \left(x(f_2(x) - f_2(a))^{\frac{\alpha}{k}} - \Gamma_k(\alpha + k) \mathcal{I}_{f_2, a^+}^{\alpha, k} I(x) \right). \quad (9)$$

Therefore from (8) and (9) we have the following modulus inequality

$$\left| (f_2(x) - f_2(a))^{\frac{\alpha}{k}} f_1(x) - \Gamma_k(\alpha + k) \mathcal{I}_{f_2, a^+}^{\alpha, k} f_1(x) \right|$$
$$\leq M \left(x(f_2(x) - f_2(a))^{\frac{\alpha}{k}} - \Gamma_k(\alpha + k) \mathcal{I}_{f_2, a^+}^{\alpha, k} I(x) \right). \quad (10)$$

Now on the other hand let $x \in [a, b]$, $t \in [x, b]$ and $\beta \geq 0$ and $k > 0$. Then the following inequality holds true

$$(f_2(t) - f_2(x))^{\frac{\beta}{k}} \leq (f_2(b) - f_2(x))^{\frac{\beta}{k}}. \quad (11)$$

From (11) and the boundedness of f_1', the following inequalities are their simple consequences

$$\int_x^b (M - f_1'(t))(f_2(t) - f_2(x))^{\frac{\beta}{k}} dt$$
$$\leq (f_2(b) - f_2(x))^{\frac{\beta}{k}} \int_a^x (M - f_1'(t)) dt \quad (12)$$

$$\int_x^b (M + f_1'(t))(f_2(t) - f_2(x))^{\frac{\beta}{k}} dt$$

$$\leq (f_2(b) - f_2(x))^{\frac{\beta}{k}} \int_a^x (M + f_1'(t)) dt. \quad (13)$$

Following the same way as we have adopted for (5), (6) and (7) one can get from (11), (12) and (13) the following modulus inequality

$$\left| (f_2(b) - f_2(x))^{\frac{\beta}{k}} f_1(x) - \Gamma_k(\beta + k)\mathcal{I}_{f_2, b-}^{\beta, k} f_1(x) \right|$$

$$\leq M \left(\Gamma_k(\beta + k)\mathcal{I}_{f_2, b-}^{\beta, k} I(x) - x(f_2(b) - f_2(x))^{\frac{\beta}{k}} \right). \quad (14)$$

Modulus inequalities (10) and (14) constitute the inequality (4).

Corollary 1: Under the same assumptions of Theorem 7 for $\alpha = \beta$ in (4) the following fractional integral inequality holds true

$$\left| f_1(x)((f_2(b) - f_2(x))^{\frac{\alpha}{k}} + (f_2(x) - f_2(a))^{\frac{\alpha}{k}}) \right.$$

$$\left. - \Gamma_k(\alpha + k)(\mathcal{I}_{f_2, b-}^{\alpha, k} f_1(x) + \mathcal{I}_{f_2, a+}^{\alpha, k} f_1(x)) \right|$$

$$\leq M \left(x \left((f_2(x) - f_2(a))^{\frac{\alpha}{k}} - (f_2(b) - f_2(x))^{\frac{\alpha}{k}} \right) \right.$$

$$\left. + \Gamma_k(\alpha + k) \left(\mathcal{I}_{f_2, b-}^{\alpha, k} I(x) - \mathcal{I}_{f_2, a+}^{\alpha, k} I(x) \right) \right). \quad (15)$$

Corollary 2: If we take $f_2(x) = x$ in (4), then we get the following fractional integral inequality for Riemann-Liouville *k*-fractional integrals

$$\left| f_1(x) \left((b - x)^{\frac{\beta}{k}} + (x - a)^{\frac{\alpha}{k}} \right) - \left(\Gamma_k(\beta + k)\mathcal{I}_{b-}^{\beta, k} f_1(x) \right. \right.$$

$$\left. \left. + \Gamma_k(\alpha + k)\mathcal{I}_{a+}^{\alpha, k} f_1(x) \right) \right|$$

$$\leq M \left(\frac{\beta}{\beta + k}(b - x)^{\frac{\beta}{k}+1} + \frac{\alpha}{\alpha + k}(x - a)^{\frac{\alpha}{k}+1} \right).$$

Remark 2: (i) If we take $f_2(x) = x$ and $k = 1$ in (4), then we get fractional integral inequality for Riemann-Liouville fractional integrals [4, Th. 1.2].
(ii) If we take $\alpha = \beta = k = 1$ and $f_2(x) = x$ in (4), then we get Ostrowski inequality (1).
In the following we give a more general form of fractional Ostrowski type inequality due to Riemann-Liouville *k*-fractional integrals with respect to an increasing function.

Theorem 8: Let $f_1 : J \longrightarrow \mathbb{R}$ where J is an interval in $\mathbb{R}$, be a mapping differentiable in J^o, the interior of J and $a, b \in J^o$, $a < b$. Also let $f_2 : [a, b] \longrightarrow \mathbb{R}$ be differentiable and strictly increasing function with $f_2' \in L[a, b]$. If $m < f_1'(t) \leq M$ for all $t \in [a, b]$, then for $\alpha, \beta \geq 0$ and $k > 0$, the following inequalities for general form of Riemann-Liouville *k*-fractional integrals hold

$$((f_2(x) + f_2(a))^{\frac{\alpha}{k}} - (f_2(b) - f_2(x))^{\frac{\beta}{k}})f_1(x)$$

$$- (\Gamma_k(\alpha + k)\mathcal{I}_{f_2, a+}^{\alpha, k} f_1(x) + \Gamma_k(\beta + k)\mathcal{I}_{f_2, b-}^{\beta, k} f_1(x))$$

$$\leq M \left(x(f_2(x) - f_2(a))^{\frac{\alpha}{k}} - \Gamma_k(\alpha + k)\mathcal{I}_{f_2, a+}^{\alpha, k} I(x) \right)$$

$$- m \left(\Gamma_k(\beta + k)\mathcal{I}_{f_2, b-}^{\beta, k} I(x) - x(f_2(b) - f_2(x))^{\frac{\beta}{k}} \right) \quad (16)$$

and

$$(\Gamma_k(\alpha + k)\mathcal{I}_{f_2, a+}^{\alpha, k} f_1(x) + \Gamma_k(\beta + k)\mathcal{I}_{f_2, b-}^{\beta, k} f_1(x))$$

$$- ((f_2(b) - f_2(x))^{\frac{\beta}{k}} + (f_2(x) - f_2(a))^{\frac{\alpha}{k}})f_1(x)$$

$$\leq M \left(\Gamma_k(\beta + k)\mathcal{I}_{f_2, b-}^{\beta, k} I(x) - x(f_2(b) - f_2(x))^{\frac{\beta}{k}} \right)$$

$$- m \left(x(f_2(x) - f_2(a))^{\frac{\alpha}{k}} - \Gamma_k(\alpha + k)\mathcal{I}_{f_2, a+}^{\alpha, k} I(x) \right), \quad (17)$$

where I is an identity function.
Proof: Proof is followed from the proof of Theorem 7 so we left it for the reader.
Remark 3: (i) If we take $f_2(x) = x$ and $k = 1$ in (16) and (17), then we get the fractional integral inequalities for Riemann-Liouville fractional integrals [4, Th. 1.3].
(ii) If we take $m = -M$ in Theorem 8, then with some rearrangements we get Theorem 7.
Another related fractional integral inequality of Ostrowski type is established in the next result.
Theorem 9: Let $f_1 : J \longrightarrow \mathbb{R}$ where J is an interval in $\mathbb{R}$, be a mapping differentiable in J^o, the interior of J and $a, b \in J^o$, $a < b$. Also let $f_2 : [a, b] \longrightarrow \mathbb{R}$ be differentiable and strictly increasing function with $f_2' \in L[a, b]$. If $|f_1'(t)| \leq M$ for all $t \in [a, b]$, then for $\alpha, \beta \geq 0$ and $k > 0$, the following inequality for Riemann-Liouville *k*-fractional integrals in general form holds

$$\left| ((f_2(b) - f_2(x))^{\frac{\beta}{k}} f_1(b) + (f_2(x) - f_2(a)^{\frac{\alpha}{k}} f_1(a)) \right.$$

$$\left. - (\Gamma_k(\beta + k)\mathcal{I}_{f_2, x+}^{\beta, k} f_1(b) + \Gamma_k(\alpha + k)\mathcal{I}_{f_2, x-}^{\alpha, k} f_1(a)) \right|$$

$$\leq M \left(b(f_2(b) - f_2(x))^{\frac{\beta}{k}} - a(f_2(x) - f_2(a))^{\frac{\alpha}{k}} \right.$$

$$\left. + \Gamma_k(\alpha + k)\mathcal{I}_{f_2, x-}^{\alpha, k} a - \Gamma_k(\beta + k)\mathcal{I}_{f_2, x+}^{\beta, k} b \right). \quad (18)$$

Proof: Let $x \in [a, b]$ and $t \in [a, x]$. Since the function f_2 is strictly increasing therefore for $\alpha \geq 0$ and $k > 0$, the following inequality holds true

$$(f_2(t) - f_2(a))^{\frac{\alpha}{k}} \leq (f_2(x) - f_2(a))^{\frac{\alpha}{k}}. \quad (19)$$

From (19) and given condition on f_1', the following inequalities are trivial to exist

$$\int_a^x (M - f_1'(t))(f_2(t) - f_2(a))^{\frac{\alpha}{k}} dt$$

$$\leq (f_2(x) - f_2(a))^{\frac{\alpha}{k}} \int_a^x (M - f_1'(t)) dt \quad (20)$$

$$\int_a^x (M + f_1'(t))(f_2(t) - f_2(a))^{\frac{\alpha}{k}} dt$$

$$\leq (f_2(x) - f_2(a))^{\frac{\alpha}{k}} \int_a^x (M + f_1'(t)) dt. \quad (21)$$

From (20) and (21) after integrating and simple calculation and by using Definition 4, we get

$$\Gamma_k(\alpha + k)\mathcal{I}_{f_2, x-}^{\alpha, k} f_1(a) - (f_2(x) - f_2(a))^{\frac{\alpha}{k}} f_1(a)$$

$$\leq M \left(\Gamma_k(\alpha + k)\mathcal{I}_{f_2, x-}^{\alpha, k} a - a(f_2(x) - f_2(a))^{\frac{\alpha}{k}} \right) \quad (22)$$

and

$$(f_2(x) - f_2(a))^{\frac{\alpha}{k}} f_1(a) - \Gamma_k(\alpha + k) \mathcal{I}^{\alpha,k}_{f_2,x^-} f_1(a)$$
$$\leq M \left(\Gamma_k(\alpha + k) \mathcal{I}^{\alpha,k}_{f_2,x^-} a - a(f_2(x) - f_2(a))^{\frac{\alpha}{k}} \right). \quad (23)$$

From (22) and (23) we have the following modulus inequality

$$\left| (f_2(x) - f_2(a))^{\frac{\alpha}{k}} f_1(a) - \Gamma_k(\alpha + k) \mathcal{I}^{\alpha,k}_{f_2,x^-} f_1(a) \right|$$
$$\leq M \left(\Gamma_k(\alpha + k) \mathcal{I}^{\alpha,k}_{f_2,x^-} a - a(f_2(x) - f_2(a))^{\frac{\alpha}{k}} \right). \quad (24)$$

Now on the other hand let $x \in [a, b]$, $t \in [x, b]$ and $\beta \geq 0$ and $k > 0$, the following inequality holds true

$$(f_2(b) - f_2(t))^{\frac{\beta}{k}} \leq (f_2(b) - f_2(x))^{\frac{\beta}{k}}. \quad (25)$$

From (25) and given condition on f'_1 following inequalities can be obtained

$$\int_x^b (M - f'_1(t))(f_2(b) - f_2(t))^{\frac{\beta}{k}} dt$$
$$\leq (f_2(b) - f_2(x))^{\frac{\beta}{k}} \int_a^x (M - f'_1(t)) dt \quad (26)$$

and

$$\int_x^b (M + f'_1(t))(f_2(b) - f_2(t))^{\frac{\beta}{k}} dt$$
$$\leq (f_2(b) - f_2(x))^{\frac{\beta}{k}} \int_a^x (M + f'_1(t)) dt. \quad (27)$$

Following the same way as we have adopted for (19), (20) and (21) one can get from (25), (26) and (27) the following inequality

$$\left| (f_2(b) - f_2(x))^{\frac{\beta}{k}} f_1(b) - \Gamma_k(\beta + k) \mathcal{I}^{\beta,k}_{f_2,x^+} f_1(b) \right|$$
$$\leq M \left(b(f_2(b) - f_2(x))^{\frac{\beta}{k}} - \Gamma_k(\beta + k) \mathcal{I}^{\beta,k}_{f_2,x^+} b \right). \quad (28)$$

Modulus inequalities (24) and (28) constitute the inequality (18).

Corollary 3: Under the same assumptions of Theorem 9 for $\alpha = \beta$ in (18) we get the following fractional integral inequality

$$\left| ((f_2(b) - f_2(x))^{\frac{\alpha}{k}} f_1(b) + (f_2(x) - f_2(a))^{\frac{\alpha}{k}} f_1(a)) \right.$$
$$\left. - \Gamma_k(\alpha + k) (\mathcal{I}^{\alpha,k}_{f_2,x^+} f_1(b) + \mathcal{I}^{\alpha,k}_{f_2,x^-} f_1(a)) \right|$$
$$\leq M \left(b(f_2(b) - f_2(x))^{\frac{\alpha}{k}} - a(f_2(x) - f_2(a))^{\frac{\alpha}{k}} \right.$$
$$\left. + \Gamma_k(\alpha + k)(\mathcal{I}^{\alpha,k}_{f_2,x^-} a - \mathcal{I}^{\alpha,k}_{f_2,x^+} b) \right).$$

Corollary 4: If we take $f_2(x) = x$ in (18), then we get the following fractional integral inequality for Riemann-Liouville *k*-fractional integrals

$$\left| \left((b - x)^{\frac{\beta}{k}} f_1(b) + (x - a)^{\frac{\alpha}{k}} f_1(a) \right) \right.$$
$$\left. - \left(\Gamma_k(\beta + k) \mathcal{I}^{\beta,k}_{x^+} f_1(b) + \Gamma_k(\alpha + k) \mathcal{I}^{\alpha,k}_{x^-} f_1(a) \right) \right|$$
$$\leq M \left(\frac{\beta}{\beta + k}(b - x)^{\frac{\beta}{k}+1} + \frac{\alpha}{\alpha + k}(x - a)^{\frac{\alpha}{k}+1} \right).$$

Remark 4: (i) If we take $f_2(x) = x$ and $k = 1$, then we get fractional integral inequality for Riemann-Liouville fractional integral [4, Th. 1.4].

(ii) A more general form of Theorem 9 like Theorem 8 holds which we leave for the reader.

III. FRACTIONAL INTEGRAL INEQUALITIES IN GENERAL FORM

In this section we present the particular results of previous section. These results can also be proved independently following the method used in previous section. Theorem 7 takes the particular form as follows:

Theorem 10: Suppose that the assumptions of the Theorem 7 hold true. Then we have

$$\left| ((f_2(b) - f_2(x))^\beta + (f_2(x) - f_2(a))^\alpha) f_1(x) \right.$$
$$\left. - (\Gamma(\beta + 1) \mathcal{I}^\beta_{f_2,b^-} f_1(x) + \Gamma(\alpha + 1) \mathcal{I}^\alpha_{f_2,a^+} f_1(x)) \right|$$
$$\leq M \left(x \left((f_2(x) - f_2(a))^\alpha - (f_2(b) - f_2(x))^\beta \right) \right.$$
$$\left. + \Gamma(\beta + 1) \mathcal{I}^\beta_{f_2,b^-} id_{[x,b]}(x) - \Gamma(\alpha + 1) \mathcal{I}^{\alpha,g}_{a^+} id_{[a,x]}(x) \right);$$
$$x \in [a, b], \quad (29)$$

where $id_{[.,.]}$ denotes identity function on $[., .]$.

Theorem 8 is modified into the following form in particular.

Theorem 11: Suppose that the assumptions of Theorem 8 hold true. Then we have

$$((f_2(x) - f_2(a))^\alpha + (f_2(b) - f_2(x))^\beta) f_1(x)$$
$$- (\Gamma(\alpha + 1) \mathcal{I}^\alpha_{f_2,a^+} f_1(x) + \Gamma(\beta + 1) \mathcal{I}^\beta_{f_2,b^-} f_1(x))$$
$$\leq M \left((f_2(x) - f_2(a))^\alpha x - \Gamma(\alpha + 1) \mathcal{I}^\alpha_{f_2,a^+} id_{[a,x]}(x) \right)$$
$$- m \left(\Gamma(\beta + 1) \mathcal{I}^\beta_{f_2,b^-} id_{[x,b]}(x) - x(f_2(b) - f_2(x))^\beta \right);$$
$$x \in [a, b]$$

and

$$(\Gamma(\alpha + 1) \mathcal{I}^\alpha_{f_2,a^+} f_1(x) + \Gamma(\beta + 1) \mathcal{I}^\beta_{f_2,b^-} f_1(x))$$
$$- ((f_2(x) - f_2(a))^\alpha + (f_2(b) - f_2(x))^\beta) f_1(x)$$
$$\leq M \left(\Gamma(\beta + 1) \mathcal{I}^\beta_{f_2,b^-} id_{[x,b]}(x) - x(f_2(b) - f_2(x))^\beta \right)$$
$$- m \left((f_2(x) - f_2(a))^\alpha x - \Gamma(\alpha + 1) \mathcal{I}^\alpha_{f_2,a^+} id_{[a,x]}(x) \right);$$
$$x \in [a, b],$$

where $id_{[.,.]}$ denotes identity function on $[., .]$.

Theorem 9 in particular takes the following form.

Theorem 12: Suppose that the assumptions of Theorem 9 hold true. Then we have

$$\left| ((f_2(b) - f_2(x))^\beta + (f_2(x) - f_2(a))^\alpha) f_1(x) \right.$$
$$\left. - (\Gamma(\beta + 1) \mathcal{I}^\beta_{f_2,x^+} f_1(b) + \Gamma(\alpha + 1) \mathcal{I}^\alpha_{f_2,x^-} f_1(a)) \right|$$
$$\leq M \left((f_2(b) - f_2(x))^\beta b - (f_2(x) - f_2(a))^\alpha a \right.$$
$$\left. + \Gamma(\alpha + 1) \mathcal{I}^\alpha_{f_2,x^-} a - \Gamma(\beta + 1) \mathcal{I}^\beta_{f_2,x^+} b \right); \quad x \in [a, b].$$
$$(30)$$

All these results of this section also provide the fractional integral inequalities for Riemann-Liouville fractional integrals when the function f_2 behaves as an identity function.

IV. APPLICATIONS

In this section we give applications of the results proved in Section II. First we apply Theorem 7 and get the following result.

Theorem 13: Under the assumptions of Theorem 7, we have

$$\left| f_1(a)(f_2(b) - f_2(a))^{\frac{\beta}{k}} + f_1(b)(f_2(b) - f_2(a))^{\frac{\alpha}{k}} \right.$$
$$\left. - \left(\Gamma_k(\beta + k)\mathcal{I}^{\beta,k}_{f_2,b^-}f_1(a) + \Gamma_k(\alpha + k)\mathcal{I}^{\alpha,k}_{f_2,a^+}f_1(b) \right) \right|$$
$$\leq M \left((b - a) \left((f_2(b) - f_2(a))^{\frac{\beta}{k}} + (f_2(b) - f_2(a))^{\frac{\alpha}{k}} \right) \right.$$
$$\left. + \Gamma_k(\beta + k)\mathcal{I}^{\beta,k}_{f_2,b^-}a - \Gamma_k(\alpha + k)\mathcal{I}^{\alpha k}_{f_2,a^+}b \right). \quad (31)$$

Proof: By taking $x = a$ and $x = b$ in (16), then adding resulting inequalities we get (31).

Corollary 5: Under the assumptions of Theorem 7 for $\alpha = \beta$ in (31), we have

$$\left| (f_1(a) + f_1(b))(f_2(b) - f_2(a))^{\frac{\alpha}{k}} \right.$$
$$\left. - \Gamma_k(\alpha + k) \left(\mathcal{I}^{\alpha,k}_{f_2,b^-}f_1(a) + \mathcal{I}^{\alpha,k}_{f_2,a^+}f_1(b) \right) \right|$$
$$\leq M \left(2(b - a)(f_2(b) - f_2(a))^{\frac{\alpha}{k}} \right.$$
$$\left. + \Gamma_k(\alpha + k)(\mathcal{I}^{\beta,k}_{f_2,b^-}a - \mathcal{I}^{\alpha k}_{f_2,a^+}b) \right). \quad (32)$$

In the following an estimation of the Hadamard inequality for Riemann-Liouville *k*-fractional integrals [11, Th. 2.1] is established.

Corollary 6: Under the assumptions of Theorem 7 for $f_2(x) = x$ in (32) an estimation of the Hadamard inequality for Riemann-Liouville *k*-fractional integrals is obtained as follows

$$\left| \frac{f_1(a) + f_1(b)}{2} - \frac{\Gamma_k(\alpha + k)}{2(b - a)^{\frac{\alpha}{k}}} \left(\mathcal{I}^{\alpha,k}_{b^-}f_1(a) + \mathcal{I}^{\alpha,k}_{a^+}f_1(b) \right) \right|$$
$$\leq \frac{M\alpha(b - a)}{\alpha + k}.$$

In the following an estimation of the Hadamard inequality for Riemann-Liouville fractional integral [19, Th. 2] is established.

Corollary 7: Under the assumptions of Theorem 7 for $f_2(x) = x$ and $k = 1$ in (32) an estimation of the Hadamard inequality for Riemann-Liouville fractional integral is obtained as follows

$$\left| \frac{f_1(a) + f_1(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^{\alpha}} \left(\mathcal{I}^{\alpha}_{b^-}f_1(a) + \mathcal{I}^{\alpha}_{a^+}f_1(b) \right) \right|$$
$$\leq \frac{M\alpha(b - a)}{\alpha + 1}.$$

Next we apply Theorem 9 to study some more *k*-fractional and fractional integral inequalities and the estimations of fractional Hadamard integral inequalities.

Theorem 14: Under the assumptions of Theorem 9, we have

$$\left| f_1(b) \left(f_2(b) - g\left(\frac{a+b}{2} \right) \right)^{\frac{\beta}{k}} + f_1(a) \left(g\left(\frac{a+b}{2} \right) - f_2(a) \right)^{\frac{\alpha}{k}} \right.$$
$$\left. - \left(\Gamma_k(\beta + k)\mathcal{I}^{\beta,k}_{f_2,\frac{a+b}{2}^+}f_1(b) + \Gamma_k(\alpha + k)\mathcal{I}^{\alpha,k}_{f_2,\frac{a+b}{2}^-}f_1(a) \right) \right|$$
$$\leq M \left(b\left(f_2(b) - g\left(\frac{a+b}{2} \right) \right)^{\frac{\beta}{k}} - a\left(g\left(\frac{a+b}{2} \right) - f_2(a) \right)^{\frac{\alpha}{k}} \right.$$
$$\left. + \Gamma_k(\alpha + k)\mathcal{I}^{\alpha,k}_{f_2,\frac{a+b}{2}^-}a - \Gamma_k(\beta + k)\mathcal{I}^{\beta,k}_{f_2,\frac{a+b}{2}^+}b \right). \quad (33)$$

Proof: By taking $x = \frac{a+b}{2}$ in (30) resulting inequality (33) can be obtained.

Corollary 8: Under the assumptions of Theorem 9 for $\alpha = \beta$ in (33), we have

$$\left| f_1(b) \left(f_2(b) - g\left(\frac{a+b}{2} \right) \right)^{\frac{\alpha}{k}} + f_1(a) \left(g\left(\frac{a+b}{2} \right) - f_2(a) \right)^{\frac{\alpha}{k}} \right.$$
$$\left. - \Gamma_k(\alpha + k) \left(\mathcal{I}^{\alpha,k}_{f_2,\frac{a+b}{2}^+}f_1(b) + \mathcal{I}^{\alpha,k}_{f_2,\frac{a+b}{2}^-}f_1(a) \right) \right|$$
$$\leq M \left(b\left(f_2(b) - g\left(\frac{a+b}{2} \right) \right)^{\frac{\alpha}{k}} - a\left(g\left(\frac{a+b}{2} \right) - f_2(a) \right)^{\frac{\alpha}{k}} \right.$$
$$\left. + \Gamma_k(\alpha + k)\mathcal{I}^{\alpha,k}_{f_2,\frac{a+b}{2}^-}a - \Gamma_k(\alpha + k)\mathcal{I}^{\alpha,k}_{f_2,\frac{a+b}{2}^+}b \right). \quad (34)$$

In the following an estimation of the Hadamard inequality for Riemann-Liouville *k*-fractional integrals [12, Th. 2.1] is established.

Corollary 9: Under the assumptions of Theorem 9 for $f_2(x) = x$ in (34) an estimation of the Hadamard inequality for Riemann-Liouville *k*-fractional integrals is obtained as follows

$$\left| \frac{f_1(a) + f_1(b)}{2} - \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha + k)}{(b - a)^{\frac{\alpha}{k}}} \right.$$
$$\left. \times \left(\mathcal{I}^{\alpha,k}_{\frac{a+b}{2}^-}f_1(b) + \mathcal{I}^{\alpha,k}_{\frac{a+b}{2}^+}f_1(a) \right) \right| \leq \frac{M\alpha(b - a)}{2(\alpha + k)}.$$

In the following an estimation of the Hadamard inequality for Riemann-Liouville fractional integrals [20, Th. 4] is established.

Corollary 10: Under the assumptions of Theorem 9 for $f_2(x) = x$ and $k = 1$ in (34) an estimation of the Hadamard inequality for Riemann-Liouville fractional integral is obtained as follows

$$\left| \frac{f_1(a) + f_1(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha + 1)}{(b - a)^{\alpha}} \left(\mathcal{I}^{\alpha}_{\frac{a+b}{2}^-}f_1(b) + \mathcal{I}^{\alpha}_{\frac{a+b}{2}^+}f_1(a) \right) \right|$$
$$\leq \frac{M\alpha(b - a)}{2(\alpha + 1)}.$$

Note that all the applications of this section also hold for Section III. Remarks of this section are also applicable for the Section III.

CONCLUDING REMARKS

This paper provides a new and elegant technique to prove fractional integral inequalities of Ostrowski type. Here Riemann-Liouville *k*-fractional integrals in a general form are utilized. Applying Theorem 7 and Theorem 9 some interesting results have been obtained which are connected to already published work. Several similar results can be obtained as an application of Theorem 7, Theorem 8 and Theorem 9. Some of these results are actually very useful to establish the error bounds of the Hadamard inequalities in fractional calculus. Adopting the method developed in this paper several fractional integrals can be used to establish new fractional integral inequalities.

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YOUNG CHEL KWUN received the Ph.D. degree in mathematics from Dong-A University, Busan, South Korea. He is currently a Mathematician from South Korea. He is currently a Professor with Dong-A University. He has published over 100 research articles in different international journals. His research interests are nonlinear analysis, decision theory, and system theory and control.

GHULAM FARID received the Ph.D. degree in mathematics from Abdus Salam School of Mathematical Sciences, GC University, Lahore Pakistan in 2012. He is currently an Assistant Professor with the Department of Mathematics, COMSATS University Islamabad, Attock Campus, Attock. He also received two research grants from COMSATS University Islamabad and the one from Higher Education Commission of Pakistan. He received the research productivity award each year from COMSATS University Islamabad.

WAQAS NAZEER received the Ph.D. degree in mathematics from the Abdus Salam School of Mathematical Sciences, Government College University, Lahore, Pakistan. He is currently an Assistant Professor with the University of Education, Lahore. He is also a Mathematician from Pakistan. During his studies, he was funded by the Higher Education Commission of Pakistan. He has published over 100 research articles in different international journals. His research interests are analysis and graph theory. He received the outstanding performance award for the Ph.D. degree.

SALEEM ULLAH received the Ph.D. degree in mathematics from COMSATS University Islamabad, in 2013, under the supervision of Prof. Dr. M. A. Noor, with a focus on numerical optimization and variational inequalities. During the Ph.D. degree, he introduced some new self-adaptive methods for solving the general variational inequalities and related optimization problems. It was shown that several unrelated problems that arise in pure and applied sciences, can be studied in the framework of general variational inequalities. It is known that the general variational inequalities are equivalent to the fixed point problem and the Wiener–Hopf equations. This equivalent formulation is used to study the existence of solution and to develop various iterative methods for solving the general variational inequalities. He is currently an Assistant Professor with the Department of Mathematics, Air University, Islamabad, where he is also teaching M.S. degree courses and actively involved in research.

SHIN MIN KANG received the Ph.D. degree in mathematics from Dong-A University, Busan, South Korea. He is currently as Professor with Gyeongsang National University, South Korea. He is also a Mathematician from South Korea. He has published over 200 research articles in different international journals. His research interests are fixed point theory, nonlinear analysis, and variational inequality.

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