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Construction of Period *qp* PGISs With Degrees Equal to or Larger Than Four

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ABSTRACT The *degree* of a perfect Gaussian integer sequence (PGIS) is defined as the number of distinct nonzero Gaussian integers within one period of the sequence. This paper focuses on constructing PGISs with degrees equal to or larger than four and period of $N = qp$, where q and p are distinct primes. The study begins with the partitioning of a ring $\mathbb{Z}_N$ into four subsets, after which degree-4 PGISs can be constructed from either the time or frequency domain. In these two approaches, nonlinear constraint equations are derived to govern the coefficients for the associative sequences to be perfect. By transforming nonlinear constraint equations into a system of linear equations, the construction of degree-4 PGISs becomes straightforward. To construct PGISs with degrees larger than four, further partitioning of $\mathbb{Z}_N$ should be carried out; here, two cases, the even period $N = 2p$ and the odd period $N = qp$, are treated separately. We can adopt the Legendre sequences of the prime period p to construct PGISs of period $2p$ with degrees larger than four. For the case of period qp, we introduce the Jacobi symbols to partition $\mathbb{Z}_N$ into seven subsets and construct PGISs with more diverse degrees.

INDEX TERMS Gaussian integers, Jacobi symbol, Legendre sequence, PACF, PGIS.

I. INTRODUCTION

Sequences with an ideal periodic autocorrelation function (PACF) [1]–[9] are widely used in modern communication systems for such applications as channel estimation [1]–[4], synchronization [3], [5], peak-to-average power ratio (PAPR) reduction [6], [7], modulation [8] and CDMA systems [9]. A sequence is regarded as perfect if it has an ideal PACF. In practical systems, binary or quadri-phase sequences are preferred due to their simple implementation [10]–[16]. However, perfect binary sequences of length $N > 4$ and perfect quadri-phase sequences of length $N > 16$ have yet to be found [5].

A Gaussian integer sequence (GIS) is a sequence with elements that are complex numbers $a+bi$, where $i = \sqrt{-1}$, and a, b are integers. As the implementation of GISs is simpler than that of other perfect sequences (PSs) with real or complex coefficients, the construction of perfect Gaussian integer sequences (PGISs) has become an important research topic [17]–[27]. A general form of even-period PGIS presented in [17]; here, the PGIS is constructed by linearly combining a set of base sequences. Yang *et al.* [18] constructed PGISs of an odd prime period p by using cyclotomic classes with respect to the multiplicative group of $\mathbf{GF}(p)$. Ma *et al.* [19] later presented PGISs with a period of $p(p+2)$ based on Whiteman's generalized cyclotomy of order two over $\mathbb{Z}_{p(p+2)}$, where p and $(p+2)$ are twin primes. Chang *et al.* [20] initialized the *degree* concept of a sequence and constructed two, three and four degrees PGISs of composite period $N = mp$. Lee *et al.* [21], [22] focused on constructing degree-2 PGISs of various periods using two-tuple-balanced sequences and cyclic difference sets. Applying Zero-padding, convolution and the Legendre symbols to generate PGISs with diverse degrees were developed by Pei and Chang [23]. A systematic method for constructing sparse PGISs in which most of the elements are zero appeared in [24]. Lee and Hong [25] and Lee and Chen [26] constructed the families of PGISs with high energy efficiency; and they used a short PGIS together with the polynomial or trace computation over an extension field to construct a family of the long PGISs [25], [26]. Recently, new PGISs

TABLE 1. Summary of Some Known PGISs.

Reference	Length	Degree	Construction method
[17]	$N = 2m$, odd m $N = 2m$, even m	3,4,6 3,4,5,6	based on base sequences set (time-domain approach)
[18]	$N = 2f + 1$ (prime length) $N = 4f + 1$, odd f (prime length)	3 5	based on cyclotomic classes (time-domain approach)
[19]	$N = p(p + 2)$, p and $p + 2$ are twin primes	2	based on Whiteman's generalized cyclotomy of order two
[20]	$p = 2f + 1$ (p is prime) $N = mp$ (arbitrary composite length)	2,3 3,4	both time and frequency domain approaches up-sampling technique is applied
[21]	$N = 2^m - 1$ (N is prime)	2	adopt from various binary sequences
[22]	diverse length	2	based on cyclic difference sets
[23]	arbitrary length	diverse degrees	zero-padding, convolution, Legendre sequences, etc. algorithms
[24]	$N = 4m$, m is positive integer	diverse degrees	based on base sequences set
[25], [26]	diverse length	2	polynomial or trace computation over an extension field
[27]	$N = p^k$, $k \geq 2$, p is prime	$\leq k + 1$	ring partition, base sequences set
Proposed PGISs	$N = 2p$ (p is prime) $N = pq$ (p and q are primes)	$4, 5$ ≥ 4	ring partition, decomposition and transformation, Legendre and Jacobi sequences, base sequences set

of period p^k with degrees equal to or less than $k + 1$ was proposed [27].

This paper focuses on constructing PGISs with degrees equal to or larger than four and period of $N = qp$, where q and p are distinct primes. The available degrees and distinct *sequence patterns* of a set of PGISs determine the carrier-to-interference ratio of a PGIS-CDMA scheme, when this set of PGISs are applied for channelization in a CDMA system [9]. In this paper, *sequence pattern* refers to the distribution of nonzero elements within one period of the sequence; in this case, more sequence patterns and different degrees contribute to the larger size of a sequence family which has the advantage or demonstrates more potential to applications. However, the construction of PGISs with degrees larger than three is especially challenging because such construction requires derivation of integer solutions from nonlinear constraint equations and matching of criteria to achieve perfect associative sequences where the number of nonlinear constraint equations is one less than the degree of the sequence [27], [28].

To solve nonlinear constraint equations, we provide three methods to decompose and transform three nonlinear equations with eight variables into a system of four linear equations with four variables. This approach renders the construction of degree-4 PGISs simpler and more straightforward. In addition, the construction of PGISs in this paper is based on the partitioning of a ring $\mathbb{Z}_N$. Thus, the proposed methods are significantly different from other studies of composite period PGISs addressed in [17], [19], and [20]. Examining above mentioned different PGISs, there are various methods that can be applied for constructing PGIS, where the variation of these approaches might depend on mathematical structure or tools, techniques or algorithms. TABLE 1 summarizes the PGISs obtained in this paper and other previously presented in the literature in terms of length, degree, and construction method.

This work is the first in the literature to introduce Jacobi symbols to process the partitioning of $\mathbb{Z}_N$, where $N = qp$ is odd. We present the construction of PGISs from both the time and frequency domains, and the results demonstrate that two different approaches derive the same perfect sequences. These two approaches are designed to satisfy the time domain ideal periodic autocorrelation function (PACF) requirement and the frequency domain flat magnitude spectrum criterion, respectively.

The rest of the paper is organized as follows. The definition and set partition of $\mathbb{Z}_N$ are addressed in Section II. Sections III and IV present the construction of degree-4 PGISs from the time and frequency domains, respectively. The study of period $N = 2p$ PGISs with degrees larger than four is presented in Section V. Section VI presents the construction of PGISs of the odd period $N = qp$ with more diverse degrees, and conclusions are drawn in Section VII.

II. PRELIMINARIES

Let $N = pq$, where p and q are distinct prime numbers. In addition, $\mathbf{s} = \{s[n]\}_{n=0}^{N-1}$ denotes a sequence of period N, where $s[n]$ is the nth component of $\mathbf{s}$. Let $\mathbf{R_s} = \{R_s[\tau]\}_{\tau=0}^{N-1}$ be the PACF of $\mathbf{s}$ expressed as,

$$R_s[\tau] = \sum_{n=0}^{N-1} s[n]s^*[(n-\tau)_N], \tag{1}$$

where the superscript * denotes the complex conjugate operation, and $(\cdot)_N$ is the modulo N operation. Define $\mathbf{s}_{-1} = \{s[(-n)_N]\}_{n=0}^{N-1}$. As can be easily shown, $\mathbf{R_s} = \mathbf{s} \otimes \mathbf{s}^*_{-1}$, where $\otimes$ denotes the circular convolution operation. Let $\mathbf{S} = \{S[n]\}_{n=0}^{N-1}$ denote the discrete Fourier transform (DFT) of $\mathbf{s}$. The DFT of $\mathbf{R_s}$ is then given by $\mathbf{S} \circ \mathbf{S}^* = \{|S[n]|^2\}_{n=0}^{N-1}$, where $\circ$ and $|\cdot|$ denote the component-wise product operation and the Euclidean norm, respectively.

The sequence **s** is said to be perfect if and only if it has an ideal PACF, i.e., $\mathbf{R_s} = E \cdot \delta_N$, where $E = \sum_{n=0}^{N-1} |s[n]|^2$ is the energy of sequence **s**, and δ_N is a delta sequence of period N. The DFT pair-relationship between $\mathbf{R_s} = E \cdot \delta_N$ and $\mathbf{S} \circ \mathbf{S}^*$ indicates that a sequence **s** is perfect if and only if the spectrum magnitude of **s** is flat, i.e., $|S[n]| = \sqrt{E}$, $0 \le n \le N-1$.

Let $\mathbb{Z}_N$ denote the ring $\{0, 1, \ldots, N-1\}$ with integer multiplication modulo N and integer addition modulo N. Here we define $\mathbb{Z}_N^\times = \mathbb{Z}_N \backslash \{0\}$. Three subsets of $\mathbb{Z}_N^\times$ are defined as follows:

$$S_p = \{np | n = 1, 2, \ldots, q-1\},$$
$$S_q = \{kq | k = 1, 2, \ldots, p-1\},$$

and

$$S_1 = \{n | gcd(n, N) = 1, n \in \mathbb{Z}_N^\times\}.$$

We have $\mathbb{Z}_N^\times = S_p \bigcup^* S_q \bigcup^* S_1$, where $\bigcup^*$ denotes disjoint union. We define three sets as follows:

$$S_a = \{(m-k)(\text{mod } N) | (m, k) \in S_p \times S_q\}, \tag{2}$$
$$S_b = \{(m+k)(\text{mod } N) | (m, k) \in S_p \times S_q\}, \tag{3}$$
$$S_c = \{(k-m)(\text{mod } N) | (m, k) \in S_p \times S_q\}. \tag{4}$$

Lemma 1: $S_a = S_b = S_c = S_1$.

Proof: Given that the cardinalities of S_p and S_q are $|S_p| = (q-1)$ and $|S_q| = (p-1)$, respectively, this implies that the cardinality of S_1 is given by $|S_1| = (pq-1) - (p-1) - (q-1) = (p-1)(q-1)$. Based on the definitions of S_a, S_b, and S_c presented in (2) $\sim$ (4), respectively, where the number of elements in set $\{(m, k) \in S_p \times S_q\}$ is $(p-1)(q-1)$, we derive $|S_a| = |S_b| = |S_c| = (p-1)(q-1) = |S_1|$. In addition, because p and q are distinct primes, all $(p-1)(q-1)$ elements of set $S_a = \{(m-k)(\text{mod } N) | (m, k) \in S_p \times S_q\}$ are distinct, and these elements do not belong to S_p and S_q. This derives that $\mathbb{Z}_N^\times = S_p \bigcup^* S_q \bigcup^* S_a$, which is a partition of $\mathbb{Z}_N^\times$. These suggest that $S_a = S_1$ is true. Similarly, $\mathbb{Z}_N^\times = S_p \bigcup^* S_q \bigcup^* S_b = S_p \bigcup^* S_q \bigcup^* S_c$. ■

Next, we define three base sequences $\mathbf{s}_i = \{s_i[n]\}_{n=0}^{N-1}$, $i = 1, q, p$, of periodic N using S_1, S_q and S_p, respectively, as follows:

$$s_i[n] = \begin{cases} 1, & n \in S_i, \\ 0, & \text{otherwise}. \end{cases} \tag{5}$$

Theorem 1: $\mathbf{s}_1 = \mathbf{s}_p \otimes \mathbf{s}_q$.

Proof: Based on the result of *Lemma* 1, where $S_1 = S_b = \{(m+k)(\text{mod } N) | (m, k) \in S_p \times S_q\}$, we can demonstrate that $\mathbf{s}_1 = \mathbf{s}_p \otimes \mathbf{s}_q$ is true. ■

III. TIME DOMAIN CONSTRUCTION OF DEGREE-4 PGIS

Sequence $\mathbf{s} = \{s[n]\}_{n=0}^{N-1}$ of periodic N is defined as

$$s[n] = \begin{cases} a_3, & n = 0, \\ a_0, & n \in S_1, \\ a_1, & n \in S_q, \\ a_2, & n \in S_p. \end{cases} \tag{6}$$

Theorem 2: The N elements of PACF $\mathbf{R_s} = \{R_s[\tau]\}_{\tau=0}^{N-1}$ of sequence **s**, defined in (6), have at most 4 distinct values, R_k, $k = 0, 1, 2, 3$, which are given by

$$R_s[\tau] = \begin{cases} R_3, & \tau = 0, \\ R_0, & \tau \in S_1, \\ R_1, & \tau \in S_q, \\ R_2, & \tau \in S_p, \end{cases} \tag{7}$$

where

$$\begin{cases} R_3 = |a_3|^2 + (p-1)(q-1)|a_0|^2 + (p-1)|a_1|^2 \\ \quad +(q-1)|a_2|^2, \\ R_0 = (p-2)(q-2)|a_0|^2 + (q-2)(a_0a_2^* + a_2a_0^*) \\ \quad +(p-2)(a_0a_1^* + a_1a_0^*) + (a_2a_1^* + a_1a_2^*) \\ \quad +(a_0a_3^* + a_3a_0^*), \\ R_1 = (p-2)(q-1)|a_0|^2 + (p-2)|a_1|^2 \\ \quad +(q-1)(a_0a_2^* + a_2a_0^*) + (a_3a_1^* + a_1a_3^*), \\ R_2 = (p-1)(q-2)|a_0|^2 + (q-2)|a_2|^2 \\ \quad +(p-1)(a_0a_1^* + a_1a_0^*) + (a_3a_2^* + a_2a_3^*). \end{cases} \tag{8}$$

Proof: The PACF of **s** is given by (1), where

$$\begin{aligned} R_s[\tau] &= \sum_{n=0}^{N-1} s[n]s^*[(n-\tau)_N] \\ &= s[0]s^*[(-\tau)_N] + \sum_{n\in S_1} s[n]s^*[(n-\tau)_N] \\ &\quad + \sum_{n\in S_q} s[n]s^*[(n-\tau)_N] + \sum_{n\in S_p} s[n]s^*[(n-\tau)_N] \\ &= a_3s^*[(-\tau)_N] + a_0 \sum_{n\in S_1} s^*[(n-\tau)_N] \\ &\quad + a_1 \sum_{n\in S_q} s^*[(n-\tau)_N] + a_2 \sum_{n\in S_p} s^*[(n-\tau)_N]. \end{aligned} \tag{9}$$

For a fixed τ, the value of $R_s[\tau]$ is calculated over the entire domain of parameter n through (9), where the bottom equation of (9) consists of four parts, and the numbers of n belonging to these four parts are 1, $(p-1)(q-1)$, $(p-1)$, and $(q-1)$, respectively. The details of derivation are presented below.

1) When $\tau = 0$, we derive $R_s[0] = |a_3|^2 + (p-1)(q-1)|a_0|^2 + (p-1)|a_1|^2 + (q-1)|a_2|^2 = R_3$.

2) When $\tau \in S_1$, first, $a_3s^*[(-\tau)_N] = a_3a_0^*$, which results from $n = 0$. Then, for the second part of (9), both τ and n belong to the same set S_1. For a fixed $\tau \in S_1$, among the $(p-1)(q-1)$ numbers of n in $\sum_{n\in S_1} s^*[(n-\tau)_N]$, there is exactly one $n(=\tau)$ that makes $s^*[(n-\tau)_N] = s^*[0]$. By *Lemma 1*, the number of items of $(n-\tau)_N \in S_q$ is $(p-2)$, that of $(n-\tau)_N \in S_p$ is $(q-2)$, and the number of the other $(n-\tau)_N \in S_1$ is $(p-2)(q-2)$. Thus, we have $a_0 \sum_{n\in S_1} s^*[(n-\tau)_N] = a_0a_3^* + (p-2)(q-2)|a_0|^2 + (q-2)a_0a_2^* + (p-2)a_0a_1^*$.

Third, for the third part of (9), the number of n in $\sum_{n\in S_q} s^*[(n-\tau)_N]$ is $(p-1)$. Given $\tau = mq + kp \in S_1$, there exists a single $n = mq \in S_q$ that makes $(n-\tau)_N = (-kp)_N \in S_p$, and the remaining $(p-2)$ n make $(n-\tau)_N \in S_1$. This infers that $a_1 \sum_{n\in S_q} s^*[(n-\tau)_N] = (p-2)a_1a_0^* + a_1a_2^*$.

Fourth, the number of n in $\sum_{n\in S_p} s^*[(n-\tau)_N]$ is $(q-1)$. For $\tau = mq + kp$, there exists a single $n = kp \in S_p$ that makes $(n-\tau)_N = (-mq)_N \in S_q$, the number the remaining items of $(n-\tau)_N \in S_1$ is $(q-2)$. This means that $a_2 \sum_{n\in S_p} s^*[(n-\tau)_N] = (q-2)a_2a_0^* + a_2a_1^*$.

The summation of the results of these four parts results in $R_0 = (p-2)(q-2)|a_0|^2 + (q-2)(a_0a_2^* + a_2a_0^*) + (p-2)(a_0a_1^* + a_1a_0^*) + (a_2a_1^* + a_1a_2^*) + (a_0a_3^* + a_3a_0^*)$.

3) When $\tau \in S_q$, first, $a_3s^*[(-\tau)_N] = a_3a_1^*$. Then, to evaluate $\sum_{n\in S_q} s^*[(n-\tau)_N]$, we assume $\tau = mq \in S_q$. Among $p-1$ numbers of n belonging to S_q, there is exactly one n among $n = kq, k = 1, \ldots, (p-1)$, which contributes $s^*[(n-\tau)_N] = s^*[0]$, and the remaining $(p-2)$ n result in $(n-\tau)_N \in S_q$. This means that $a_1 \sum_{n\in S_q} s^*[(n-\tau)_N] = a_1a_3^* + (p-2)|a_1|^2$.

Next, in the part of $\sum_{n\in S_p} s^*[(n-\tau)_N]$, given that n and τ do not overlap, we derive $(n-\tau)_N \in S_1$ by *Lemma 1*. We have $a_2 \sum_{n\in S_p} s^*[(n-\tau)_N] = (q-1)a_2a_0^*$.

Finally, to the part of $\sum_{n\in S_1} s^*[(n-\tau)_N]$, given that $\tau = mq$, there are $(q-1)$ numbers of $n = mq+kp, k = 1, \ldots, (q-1)$, in $\sum_{n\in S_1} s^*[(n-\tau)_N]$, which makes $(n-\tau)_N = kp \in S_p$. Thus, the number of $(n-\tau)_N \in S_1$ is $(p-1)(q-1)-(q-1) = (p-2)(q-1)$. This implies that $a_0 \sum_{n\in S_1} s^*[(n-\tau)_N] = (p-2)(q-1)|a_0|^2 + (q-1)a_0a_2^*$.

We conclude from the above four results that $R_1 = (p-2)(q-1)|a_0|^2 + (p-2)|a_1|^2 + (q-1)(a_0a_2^* + a_2a_0^*) + (a_3a_1^* + a_1a_3^*)$ is true.

4) When $\tau \in S_p$, to show that $R_2 = (p-1)(q-2)|a_0|^2 + (q-2)|a_2|^2 + (p-1)(a_0a_1^* + a_1a_0^*) + (a_3a_2^* + a_2a_3^*)$ is true is similar to that of deriving R_1. The detailed derivation of R_2 is no longer included here for brevity. ■

Corollary 1: Sequence $\mathbf{s} = \{s[n]\}_{n=0}^{N-1}$, defined in (6), is a degree-4 PGIS of period N if and only if it matches the following criteria,

$$R_0 = R_1 = R_2 = 0. \tag{10}$$

Proof: Based on the definition of a PGIS, $\mathbf{R_s} = E\cdot\delta_N \Leftrightarrow R_0 = R_1 = R_2 = 0$. ■

Let $a_n = x_n + y_ni, n = 0, 1, 2, 3$, where x_n and y_n are integers. When these four $a_n = x_n + y_ni$ coefficients are inserted to (8), we derive a system of three nonlinear equations to govern the coefficients of sequence $\mathbf{s} = \{s[n]\}_{n=0}^{N-1}$ to be a degree-4 PGIS. These constraint equations are expressed below

$$\begin{cases} (p-2)(q-2)(x_0^2+y_0^2) + 2(q-2)(x_0x_2+y_0y_2) \\ \quad +2(p-2)(x_0x_1+y_0y_1) \\ \quad +2(x_1x_2+y_1y_2+x_0x_3+y_0y_3) = 0, \\ (p-2)(q-1)(x_0^2+y_0^2) + (p-2)(x_1^2+y_1^2) \\ \quad +2(q-1)(x_0x_2+y_0y_2) + 2(x_1x_3+y_1y_3) = 0, \\ (q-2)(x_2^2+y_2^2) + (p-1)(q-2)(x_0^2+y_0^2) \\ \quad +2(p-1)(x_0x_1+y_0y_1) + 2(x_2x_3+y_2y_3) = 0. \end{cases} \tag{11}$$

There are eight unknown parameters and the number of equations is only three in (11); this system of three nonlinear equations might exist numerous integer solutions. To derive the solutions, first, the bottom equation of (11) can be replaced by subtracting from the top equation of (11), after which it becomes

$$\begin{aligned}(x_0-x_2)((q-2)(x_0-x_2)+2(x_1-x_3)) \\ +(y_0-y_2)((q-2)(y_0-y_2)+2(y_1-y_3)) = 0.\end{aligned} \tag{12}$$

Second, the nonlinear equation (12) can be decomposed into two parts, which results in a linear system of two equations. We provide three different decomposition methods, which are respectively presented below

$$\begin{cases} x_2+y_2 = x_0+y_0, \\ (2-q)x_2+(q-2)y_2-2x_3+2y_3 \\ = (q-2)(y_0-x_0)-2x_1+2y_1. \end{cases} \tag{13}$$

$$\begin{cases} -x_2+(q-2)y_2+2y_3 = -x_0+(q-2)y_0+2y_1, \\ (q-2)x_2+y_2+2x_3 = (q-2)x_0+y_0+2x_1. \end{cases} \tag{14}$$

$$\begin{cases} (2-q)x_2-2x_3 = (2-q)x_0-2x_1, \\ (2-q)y_2-2y_3 = (2-q)y_0-2y_1. \end{cases} \tag{15}$$

Third, we can combine the first two equations of (11) with the above three sets of linear equations to form three different linear systems of four equations with x_2, y_2, x_3 and y_3 as the variables, respectively. These three systems can be respectively expressed using the matrix notation $\mathbf{A}_i\mathbf{x} = \mathbf{b}_i$, $i = 1, 2, 3$. In these equations, $\mathbf{A}_i$ is the coefficient matrix of size 4×4 and $\mathbf{b}_i$ is a data column vector, in which the elements of two terms consist of integers and constants belonging to set $\{x_0, y_0, x_1, y_1\}$. In addition, the elements of column vector $\mathbf{x} = [x_2\ \ y_2\ \ x_3\ \ y_3]^T$ are considered as the variables. We have

$$\mathbf{A}_1 = \begin{bmatrix} 2(q-2)x_0+2x_1 & 2(q-2)y_0+2y_1 & 2x_0 & 2y_0 \\ 2(q-1)x_0 & 2(q-1)y_0 & 2x_1 & 2y_1 \\ 1 & 1 & 0 & 0 \\ 2-q & q-2 & -2 & 2 \end{bmatrix}, \tag{16}$$

$$\mathbf{A}_2 = \begin{bmatrix} 2(q-2)x_0+2x_1 & 2(q-2)y_0+2y_1 & 2x_0 & 2y_0 \\ 2(q-1)x_0 & 2(q-1)y_0 & 2x_1 & 2y_1 \\ -1 & q-2 & 0 & 2 \\ q-2 & 1 & 2 & 0 \end{bmatrix}, \tag{17}$$

$$\mathbf{A}_3 = \begin{bmatrix} 2(q-2)x_0+2x_1 & 2(q-2)y_0+2y_1 & 2x_0 & 2y_0 \\ 2(q-1)x_0 & 2(q-1)y_0 & 2x_1 & 2y_1 \\ 2-q & 0 & -2 & 0 \\ 0 & 2-q & 0 & -2 \end{bmatrix},$$

$$\mathbf{b}_1 = [\triangle_1 \ \triangle_2 \ x_0+y_0 \ \ (q-2)(y_0-x_0)+2(y_1-x_1)]^T,$$

$$\mathbf{b}_2 = [\triangle_1 \ \triangle_2 \ (q-2)y_0+2y_1-x_0 \ \ (q-2)x_0+y_0+2x_1]^T,$$

$$\mathbf{b}_3 = [\triangle_1 \ \triangle_2 \ (2-q)x_0-2x_1 \ (2-q)y_0-2y_1]^T, \tag{18}$$

where $\triangle_1 = (2-p)(q-2)(x_0^2+y_0^2) - 2(p-2)(x_0x_1+y_0y_1)$ and $\triangle_2 = (2-p)(q-1)(x_0^2+y_0^2) - 2(p-2)(x_1^2+y_1^2)$.

Given that the elements of three coefficient matrices $\mathbf{A}_i$ are integers, the elements of matrix inverse $\mathbf{A}_i^{-1}$ can still be integers or at most rational numbers. By choosing constants x_0, y_0, x_1, and y_1 such that all $|\mathbf{A}_i| \neq 0$, we can always adjust these four constants and derive the integer solutions of four variables (x_2, y_2, x_3, y_3) from equations $\mathbf{x}_i = \mathbf{A}_i^{-1}\mathbf{b}_i$, $i = 1, 2, 3$. These eight parameters $x_n, y_n, n = 0, 1, 2, 3$, meet the system of three nonlinear equations (11).

Example 1: In the case of $p = 3$, $q = 7$, given that $a_0 = -6+6i$, $a_1 = 12+6i$, we derive $a_2 = 12-12i$ and $a_3 = 3+87i$ from $\mathbf{A}_1\mathbf{x} = \mathbf{b}_1$ of (16), where a degree-4 PGIS of period $N = 21$ is given by

$$(a_3, a_0, a_0, a_2, a_0, a_0, a_2, a_1, a_0, a_2, a_0, a_0, a_2, a_0, a_1, a_2, a_0, a_0, a_2, a_0, a_0).$$

In the case of $p = 3$, $q = 5$, given that $a_0 = 10-10i$, $a_1 = -20-10i$, we derive $a_2 = -2+20i$ and $a_3 = 13-49i$ from $\mathbf{A}_2\mathbf{x} = \mathbf{b}_2$ of (17), where a degree-4 PGIS of period $N = 15$ is given by

$$(a_3, a_0, a_0, a_2, a_0, a_1, a_2, a_0, a_0, a_2, a_1, a_0, a_2, a_0, a_0).$$

In the case of $p = 2$, $q = 5$, given that $a_0 = -25+25i$, $a_1 = 50+25i$, we derive $a_2 = 5-21i$, $a_3 = 5+94i$ from $\mathbf{A}_3\mathbf{x} = \mathbf{b}_3$ of (18), where a degree-4 PGIS of period $N = 10$ is given by

$$(a_3, a_0, a_2, a_0, a_2, a_1, a_2, a_0, a_2, a_0).$$

IV. FREQUENCY DOMAIN CONSTRUCTION OF DEGREE-4 PGIS

Let sequence $\mathbf{s}$ be constructed by linearly combining three base sequences defined in (5) and δ_N, that is

$$\mathbf{s} = a_3\delta_N + a_0\mathbf{s}_1 + a_1\mathbf{s}_q + a_2\mathbf{s}_p, \tag{19}$$

where a_k, $k = 0, 1, 2, 3$, are four distinct nonzero Gaussian integers. We can easily show that the DFTs of δ_N, $\mathbf{s}_q$, and $\mathbf{s}_p$, denoted as $\mathbf{S}_\delta$, $\mathbf{X}_q$, and $\mathbf{X}_p$, are respectively given by

$$\mathbf{S}_\delta = [\underbrace{1 \ 1 \ \cdots \ 1}_{N}],$$

$$\mathbf{X}_q = [\underbrace{(p-1) \ -1 \cdots -1}_{p} \ \underbrace{(p-1) \ -1 \cdots -1}_{p} \cdots \underbrace{(p-1) \ -1 \cdots -1}_{p}],$$

$$\mathbf{X}_p = [\underbrace{(q-1) \ -1 \cdots \ -1}_{q} \ \underbrace{(q-1) \ -1 \cdots -1}_{q} \cdots \underbrace{(q-1) \ -1 \cdots \ -1}_{q}].$$

By *Theorem* 1, the DFT of $\mathbf{s}_1$, denoted as $\mathbf{X}_1$, is given by

$$\mathbf{X}_1 = \mathbf{X}_p \circ \mathbf{X}_q.$$

Theorem 3: Sequence $\mathbf{s} = a_3\delta_N + a_0\mathbf{s}_1 + a_1\mathbf{s}_q + a_2\mathbf{s}_p$, defined in (19), is a degree-4 PGIS if and only if it fulfills the following constraint equations

$$\begin{aligned}|a_0(p-1)(q-1) + a_2(q-1) + a_1(p-1) + a_3| \\ = |a_0 - a_2 - a_1 + a_3| = |(q-1)(a_2-a_0) + a_3 - a_1| \\ = |(p-1)(a_1-a_0) + a_3 - a_2|. \end{aligned} \tag{20}$$

Proof: Based on the patterns of $\mathbf{S}_\delta$, $\mathbf{X}_q$, $\mathbf{X}_p$ and $\mathbf{X}_1$, the DFT of $\mathbf{s}$, denoted as $\mathbf{S} = a_3\mathbf{S}_\delta + a_0\mathbf{X}_1 + a_1\mathbf{X}_q + a_2\mathbf{X}_p$, is a four-valued vector, and the values of its elements belong to set $\{a_0(p-1)(q-1)+a_2(q-1)+a_1(p-1)+a_3, a_0-a_2-a_1+a_3, (q-1)(a_2-a_0)+a_3-a_1, (p-1)(a_1-a_0)+a_3-a_2\}$. The flat magnitude spectrum requirement for a sequence $\mathbf{s}$ to be a PGIS, which is the absolute value of all elements of $\mathbf{S}$ should be the same, indicates that the constraint equations should be fulfilled. ■

Let $a_n = x_n + y_n i$, $n = 0, 1, 2, 3$, where x_n and y_n are integers. By substituting these four a_n into (20), we derive a system of three nonlinear equations, shown in equation (21), as shown at the top of the next page, for constructing a degree-4 PGIS. We can easily show that two systems of three nonlinear equations, represented by (21) and (11), respectively, are equivalent.

The construction of a PGIS from either the time or frequency domain approach is based on the same partitioning of ring $\mathbb{Z}_N = \{0\} \bigcup \mathbb{Z}_N^\times (= S_1 \bigcup^* S_p \bigcup^* S_q)$; thus, it makes sense that these two different approaches should construct the same perfect sequence. To construct PGIS with degrees larger than four, subsets S_1, S_p, and S_q should be further partitioned into smaller subsets. To address this topic, the case of $N = 2p$ and odd $N = qp$ are addressed separately in the Sections V and VI, respectively. This decision is based on the idea that there exists primitive root (mod $2p$) but no primitive root (mod qp) for odd prime q [29].

V. HIGHER DEGREE PGIS CONSTRUCTION OF PERIOD N=2P

When $N = 2p$, where p is an odd prime, $\mathbb{Z}_{2p}^\times$ can be partitioned into $S_1 \bigcup^* S_2 \bigcup^* S_p$, where $S_p = \{p\}$,

$$\begin{cases} (p-2)(q-2)(x_0^2+y_0^2)+2(q-2)(x_0x_2+y_0y_2)+2(p-2)(x_0x_1+y_0y_1) \\ \quad +2(x_1x_2+y_1y_2+x_0x_3+y_0y_3)=0, \\ (p-2)(x_0^2+y_0^2+x_1^2+y_1^2)+2(x_0x_2+y_0y_2+x_1x_3+y_1y_3) \\ \quad +2(2-p)(x_1x_0+y_1y_0)-2(x_0x_3+y_0y_3+x_1x_2+y_1y_2)=0, \\ (q-2)(x_0^2+y_0^2+x_2^2+y_2^2)+2(x_3x_2+y_3y_2+x_1x_0+y_1y_0) \\ \quad +2(2-q)(x_2x_0+y_2y_0)-2(x_0x_3+y_0y_3+x_1x_2+y_1y_2)=0. \end{cases} \tag{21}$$

$S_1 = \{2k-1 | k = 1, 2, \ldots, p, k \neq \frac{p+1}{2}\}$ and $S_2 = \{2k | k = 1, 2, \ldots, p-1\}$. As there exist primitive root α (mod $2p$) of order $\phi(2p) = 2p(1-\frac{1}{2})(1-\frac{1}{p}) = (p-1)$ where $\alpha^{\phi(2p)} \equiv 1 (\mathrm{mod}\ 2p)$, in which $\phi(2p)$ is *Euler function*; here, the subset S_1 is a cyclic group with cardinality given by $|S_1| = (p-1)$. In addition, because $(p-1)$ is even, there exists a subgroup $S_{11} = \{\alpha^{2n}\}_{n=0}^{\frac{(p-3)}{2}}$ of S_1 with cardinality $|S_{11}| = \frac{(p-1)}{2}$, $S_1 = S_{11} \bigcup^* S_{12} (= \{\alpha^{2n+1}\}_{n=0}^{\frac{(p-3)}{2}})$. The partition of $S_2 = S_{21} \bigcup^* S_{22}$ can be formed according to $S_{21} = \{n+p | n \in S_{11}\}$ and $S_{22} = \{n+p | n \in S_{12}\}$, respectively.

Theorem 4 [29]: If p is an odd prime, then there are $\frac{1}{2}(p-1)$ quadratic residues and an equal number of quadratic nonresidues (mod p).

Lemma 2: For odd prime p, p is a quadratic residue (mod $2p$).

Proof: The proof is omitted for brevity. ■

Lemma 3: For an odd prime p, k is a quadratic residue (mod $2p$) $\Leftrightarrow$ $k+p$ is quadratic residue (mod $2p$); k is a quadratic nonresidue (mod $2p$) $\Leftrightarrow$ $k+p$ is a quadratic nonresidue (mod $2p$).

Proof: The proof is omitted for brevity. ■

Lemma 4: All $\frac{(p-1)}{2}$ elements of cyclic group $S_{11} = \{\alpha^{2n}\}_{n=0}^{\frac{(p-3)}{2}}$ are quadratic residues (mod $2p$), and all elements of $S_{12} = \{\alpha^{2n+1}\}_{n=0}^{\frac{(p-3)}{2}}$ are quadratic nonresidues (mod $2p$).

Proof: Given that cyclic group $S_{11} = \{\alpha^{2n}\}_{n=0}^{\frac{(p-3)}{2}}$ consists of elements that are either "$1''(=\alpha^0)$ or primitive root with an even exponent α^{2n}, to each $x = \alpha^{2n} \in S_{11}$, there exists $a = (\alpha^{2n})^2 \in S_{11}$ by closure property, where α is a primitive root. This implies that the congruence $x^2 \equiv a (\mathrm{mod}\ 2p)$ has a solution, indicating that all $\frac{(p-1)}{2}$ elements of cyclic group S_{11} are quadratic residues (mod $2p$). For $a = \alpha^{2n+1} \in S_{12}$, given that $S_{12} = \{\alpha^{2n+1}\}_{n=0}^{\frac{(p-3)}{2}}$ consists of elements that are primitive root α with odd exponents, there is no solution to congruence $x^2 \equiv a (\mathrm{mod}\ 2p)$ and proves that all elements of S_{12} are quadratic nonresidues (mod $2p$). ■

Lemma 5: There are p quadratic residues and $(p-1)$ quadratic nonresidues (mod $2p$).

Proof: The proof is omitted for brevity. ■

We define three base sequences $\mathbf{c}_i = \{c_i[n]\}_{n=0}^{N-1}$, $i = 1, 2, 3$, of period $N = 2p$ as follows:

$$c_1[n] = \begin{cases} 1, & n \in S_{11}, \\ -1, & n \in S_{12}, \\ 0, & \text{otherwise}. \end{cases} \tag{22}$$

$$c_2[n] = \begin{cases} 1, & n \in S_{21}, \\ -1, & n \in S_{22}, \\ 0, & \text{otherwise}. \end{cases} \tag{23}$$

$$c_3[n] = \begin{cases} 1, & n \in S_p, \\ 0, & \text{otherwise}. \end{cases} \tag{24}$$

The cyclotomic classes of order 2 with respect to $\mathbf{GF}(p)$ is defined as $D_m^{(2)} = \left\{\beta^{m+2n} | 0 \leq n < \frac{p-1}{2}\right\}$, $0 \leq m < 2$ where β is a primitive root (mod p). We also define two base sequences $\{s_{D0}[n]\}$ and $\{s_{D1}[n]\}$, with the associative DFTs denoted by $\{S_{D0}[n]\}$ and $\{S_{D1}[n]\}$, of period p based on $D_0^{(2)}$ and $D_1^{(2)}$, respectively.

When $p \equiv 3 (\mathrm{mod}\ 4)$, the DFTs of $\{s_{D0}[n]\}$ and $\{s_{D1}[n]\}$ are respectively given by [20]

$$S_{D0}[n] = \begin{cases} \dfrac{p-1}{2}, & n = 0, \\ -\left(1+i\sqrt{p}\right)/2, & n \in D_0^{(2)}, \\ -\left(1-i\sqrt{p}\right)/2, & n \in D_1^{(2)}. \end{cases} \tag{25}$$

$$S_{D1}[n] = \begin{cases} \dfrac{p-1}{2}, & n = 0, \\ -\left(1-i\sqrt{p}\right)/2, & n \in D_0^{(2)}, \\ -\left(1+i\sqrt{p}\right)/2, & n \in D_1^{(2)}. \end{cases} \tag{26}$$

When $p \equiv 1 (\mathrm{mod}\ 4)$, the DFTs of $\{s_{D0}[n]\}$ and $\{s_{D1}[n]\}$ are given by [20]

$$S_{D0}[n] = \begin{cases} \dfrac{p-1}{2}, & n = 0, \\ -\left(1-\sqrt{p}\right)/2, & n \in D_0^{(2)}, \\ -\left(1+\sqrt{p}\right)/2, & n \in D_1^{(2)}. \end{cases} \tag{27}$$

$$S_{D1}[n] = \begin{cases} \dfrac{p-1}{2}, & n = 0, \\ -\left(1+\sqrt{p}\right)/2, & n \in D_0^{(2)}, \\ -\left(1-\sqrt{p}\right)/2, & n \in D_1^{(2)}. \end{cases} \tag{28}$$

With $c_2[n]$ defined in (23), we can express $\mathbf{c}_2 = \mathbf{c}_{21} - \mathbf{c}_{22}$. In this expression, $\mathbf{c}_{21} = \{c_{21}[n]\}_{n=0}^{2p-1}$, where $c_{21}[n] = 1$

when $n \in S_{21}$, and $c_{21}[n] = 0$ otherwise; and $\mathbf{c}_{22} = \{c_{22}[n]\}_{n=0}^{2p-1}$, where $c_{22}[n] = 1$ when $n \in S_{22}$, and $c_{22}[n] = 0$ otherwise. Given that S_2 consists of even numbers of $\mathbb{Z}_{2p}^{+}$, it is easy to show that $\mathbf{c}_{21}$ and $\mathbf{c}_{22}$ are the up-sampled of sequences $\{s_{D1}[n]\}$ and $\{s_{D0}[n]\}$ by a factor of two, respectively. Let $\mathbf{C}_{21} = \{C_{21}[n]\}_{n=0}^{2p-1}$ and $\mathbf{C}_{22} = \{C_{22}[n]\}_{n=0}^{2p-1}$ denote the DFTs of $\mathbf{c}_{21}$ and $\mathbf{c}_{22}$, respectively. Thus, we have $\{C_{21}[n]\}_{n=0}^{p-1} = \{S_{D1}[n]\}_{n=0}^{p-1}$ and $\{C_{22}[n]\}_{n=0}^{p-1} = \{S_{D0}[n]\}_{n=0}^{p-1}$, and $C_{21}[n+p] = C_{21}[n]$ and $C_{22}[n+p] = C_{22}[n]$ are true.

The results of (25) $\sim$ (28) can be used to derive the DFTs of $\mathbf{c}_2$, denoted by $\mathbf{C}_2 = \{C_2[n]\}_{n=0}^{2p-1}$, where the first p elements of $\{C_2[n]\}_{n=0}^{p-1}$ are expressed below.

When $p \equiv 3 (\mathrm{mod}\ 4)$,

$$C_2[n] = \begin{cases} 0, & n = 0, \\ i\sqrt{p}, & n \in D_0^{(2)}, \\ -i\sqrt{p}, & n \in D_1^{(2)}. \end{cases} \quad (29)$$

When $p \equiv 1 (\mathrm{mod}\ 4)$,

$$C_2[n] = \begin{cases} 0, & n = 0, \\ -\sqrt{p}, & n \in D_0^{(2)}, \\ +\sqrt{p}, & n \in D_1^{(2)}. \end{cases} \quad (30)$$

The other p elements of $\{C_2[n]\}_{p}^{2p-1}$ can be obtained from $C_2[n+p] = C_2[n]$.

By the definition of $c_1[n]$ shown in (22), we can express $\mathbf{c}_1 = \mathbf{c}_{11} - \mathbf{c}_{12}$, where nonzero entries of $\mathbf{c}_{11}$ and $\mathbf{c}_{12}$ are defined with respect to elements of set S_{11} and S_{12}, respectively.

Lemma 6: $\mathbf{c}_{11} = \mathbf{c}_{21} \otimes \mathbf{c}_3$, $\mathbf{c}_{12} = \mathbf{c}_{22} \otimes \mathbf{c}_3$.

Proof: From *Lemma 3*, when k is a quadratic residue (mod $2p$), then $k + p$ is also a quadratic residue (mod $2p$). Since nonzero entries of $\mathbf{c}_{11}$ are defined with respect to elements of set S_{11}, where S_{11} consists of quadratic residue (mod $2p$), this derives $\mathbf{c}_{11} = \mathbf{c}_{21} \otimes \mathbf{c}_3$ when S_{21} also consists of quadratic residues (mod $2p$). Based on the result of $\mathbf{c}_{11} = \mathbf{c}_{21} \otimes \mathbf{c}_3$, $\mathbf{c}_{12}$ should be equal to $\mathbf{c}_{22} \otimes \mathbf{c}_3$. ■

It is easy to show that the DFT of $\mathbf{c}_3$ is given by

$$\mathbf{C}_3 = [1 \ \ -1 \ \ 1 \ \ -1 \ \cdots \ 1 \ \ -1].$$

By *Lemma 6*, we have

$$\mathbf{C}_1 = \mathbf{C}_2 \circ \mathbf{C}_3,$$

Note that the value of $\{c_2[n]\}$ is assigned as "1″ when $n \in S_{21}$, where S_{21} consists of quadratic residues (mod $2p$), and the value of $\{c_2[n]\}$ is "-1″ when $n \in S_{22}$ where S_{22} consists of quadratic nonresidues (mod $2p$). This implies that sequence $\mathbf{c}_2$, defined in (23), can be obtained from upsampling a *Legendre Sequence* of period p by a factor of two, where the Legendre symbol $\left(\frac{n}{p}\right)$ is defined as in [23] and [30]

$$\left(\frac{n}{p}\right) = \begin{cases} 1, & n \text{ is quadratic residue (mod } p), \\ -1, & n \text{ is quadratic nonresidue (mod } p), \\ 0, & n \equiv 0 (\mathrm{mod}\ p). \end{cases} \quad (31)$$

The DFT of the Legendre Sequence is *Gauss sum* $\{G[k]\}_{k=0}^{p-1}$ [23], which is defined as

$$G[k] = \sum_{n=0}^{p-1} \left(\frac{n}{p}\right) e^{-i2\pi nk/p} = \begin{cases} \left(\frac{k}{p}\right)\sqrt{p}, & p \equiv 1 (\mathrm{mod}\ 4), \\ -\left(\frac{k}{p}\right) i\sqrt{p}, & p \equiv 3 (\mathrm{mod}\ 4). \end{cases} \quad (32)$$

The results shown in (32) are the same as those shown in (29) and (30). Aside from $C_1[0] = C_1[p] = C_2[0] = C_2[p] = G[0] = 0$, the remaining elements of $\mathbf{C}_i$ are magnitude flat. Applying these $\mathbf{C}_i$ for PGIS construction can help avoid solving complex constrain equations to obtain the coefficients of sequences, where two equivalent systems of three nonlinear equations are shown in (11) and (21), respectively. In the case of prime period p, Pei constructed a sequence $\{f[n]\}$ defined as [23]

$$f[n] = \begin{cases} a, & n = 0, \\ \left(\frac{n}{p}\right)\sqrt{p}c + bi, & n \neq 0. \end{cases} \quad (33)$$

The DFT of $\{f[n]\}$ is a degree-3 PGIS, given that integers b and c as well as Gaussian integer a meet the requirement stated by $|a|^2 = b^2 + pc^2$. Here, we can adopt Pei's algorithm to construct the PGIS of period $2p$, described in *Theorem 5*.

Theorem 5: Let d and r be two Gaussian integers, and g and h be simple integers. Given that $|d|^2 = |r|^2 = h^2 + pg^2$, the DFT of sequence

$$\mathbf{s} = d\delta_{2p} + \eta\mathbf{c}_{11} - \eta^*\mathbf{c}_{12} + \eta^*\mathbf{c}_{21} - \eta\mathbf{c}_{22} + r\mathbf{c}_3$$

is a PGIS of period $N = 2p$, where $\eta = g\sqrt{p} - hi$.

Proof: Given that $\{|s[n]|\}$ is magnitude flat, the DFT of $\mathbf{s}$ is a perfect sequence. For $\mathbf{s}$ to be PGIS, all coefficients of the DFT of $\mathbf{s}$ should be Gaussian integers. For both $d \cdot \delta_{2p}$ and $r\mathbf{c}_3$ two terms, the coefficients of their DFTs belong to $\{d, r, -r\}$.

Next, we should prove that both the coefficients of DFTs of $\eta\mathbf{c}_{11} - \eta^*\mathbf{c}_{12}$ and $\eta^*\mathbf{c}_{21} - \eta\mathbf{c}_{22}$ are Gaussian integers. Let $\{X_2[n]\}_{n=0}^{2p-1}$ be the DFT of $\eta^*\mathbf{c}_{21} - \eta\mathbf{c}_{22}$.

When $p \equiv 3 (\mathrm{mod}\ 4)$, based on the results shown in (25) $\sim$ (28), the parts of $\{X_2[n]\}_{n=0}^{p-1}$ are given by

$$X_2[n] = \begin{cases} (p-1)hi, & n = 0, \\ -(h+pg)i, & n \in D_0^{(2)}, \\ (pg-h)i, & n \in D_1^{(2)}. \end{cases} \quad (34)$$

When $p \equiv 1 (\mathrm{mod}\ 4)$, it becomes

$$X_2[n] = \begin{cases} (p-1)hi, & n = 0, \\ pg - hi, & n \in D_0^{(2)}, \\ -pg - hi, & n \in D_1^{(2)}. \end{cases} \quad (35)$$

The elements of $X_2[n]$ in both (34) and (35) are Gaussian integers. In addition, we have $X_2[p+n] = X_2[n]$, which implies that the coefficients of the DFT of $\eta^*\mathbf{c}_{21} - \eta\mathbf{c}_{22}$

are Gaussian integers. Due to the fact that $\mathbf{C}_1 = \mathbf{C}_2 \circ \mathbf{C}_3$, the derivation process to prove that the coefficients of the DFT of $\eta\mathbf{c}_{11} - \eta^*\mathbf{c}_{12}$ are Gaussian integers is similar. Thus, the explanation is omitted here for brevity. ■

Corollary 2: When $p \equiv 1 \pmod 4$, the DFT of $\mathbf{s} = d\delta_{2p} + \eta\mathbf{c}_{11} - \eta^*\mathbf{c}_{12} + \eta^*\mathbf{c}_{21} - \eta\mathbf{c}_{22} + r\mathbf{c}_3$ is given by

$$S[n] = \begin{cases} d + r, & n \in \{0\} \bigcup^* S_2, \\ d - r + 2(pg - hi), & n \in S_{11}, \\ d - r - 2(pg + hi), & n \in S_{12}, \\ d - r + 2(p-1)hi, & n \in S_p. \end{cases} \tag{36}$$

Sequence $\mathbf{S} = \{S[n]\}$ of (36) is a degree-4 PGIS.

Proof: Taking DFT upon $\mathbf{s}$ derives the results. ■

Corollary 3: When $p \equiv 1 \pmod 4$, the DFT of $\mathbf{s} = d\delta_{2p} + \eta\mathbf{c}_{11} - \eta^*\mathbf{c}_{12} - \eta\mathbf{c}_{21} + \eta^*\mathbf{c}_{22} + r\mathbf{c}_3$ is given by

$$S[n] = \begin{cases} d + r, & n = 0, \\ d - r - 2hi, & n \in S_1, \\ d + r - 2pg, & n \in S_{21}, \\ d + r + 2pg, & n \in S_{22}, \\ d - r + 2(p-1)hi, & n \in S_p. \end{cases} \tag{37}$$

Sequence $\mathbf{S} = \{S[n]\}$ of (37) is a degree-5 PGIS.

Proof: Taking DFT upon $\mathbf{s}$ derives the results. ■

Example 2: When $d = 2+5i$, $r = 5-2i$, and $g\sqrt{p}+hi = 2\sqrt{5} + 3i$, two PGISs of period $N = 10$ associated with (36) and (37), are respectively given by

$$(7+3i, 17+i, 7+3i, -23+i, 7+3i, \\ -3+31i, 7+3i, -23+i, 7+3i, 17+i), \tag{38}$$

$$(7+3i, -3+i, 27+3i, -3+i, -13+3i, \\ -3+31i, -13+3i, -3+i, 27+3i, -3+i). \tag{39}$$

The degrees of PGISs in (38) and (39) are four and five, respectively.

Corollary 4: When $p \equiv 3 \pmod 4$, the DFT of $\mathbf{s} = d\delta_{2p} + \eta\mathbf{c}_{11} - \eta^*\mathbf{c}_{12} + \eta^*\mathbf{c}_{21} - \eta\mathbf{c}_{22} + r\mathbf{c}_3$ is given by

$$S[n] = \begin{cases} d + r, & n = 0, \\ d - r - 2hi, & n \in S_1, \\ d + r - 2pgi, & n \in S_{21}, \\ d + r + 2pgi, & n \in S_{22}, \\ d - r + 2(p-1)hi, & n \in S_p. \end{cases} \tag{40}$$

Sequence $\mathbf{S} = \{S[n]\}$ of (40) is a degree-5 PGIS.

Proof: Taking DFT upon $\mathbf{s}$ derives the results. ■

Corollary 5: When $p \equiv 3 \pmod 4$, the DFT of $\mathbf{s} = d\delta_{2p} + \eta\mathbf{c}_{11} - \eta^*\mathbf{c}_{12} - \eta\mathbf{c}_{21} + \eta^*\mathbf{c}_{22} + r\mathbf{c}_3$ is given by

$$S[n] = \begin{cases} d + r, & n \in \{0\} \bigcup^* S_2, \\ d - r + 2(pg - h)i, & n \in S_{11}, \\ d - r - 2(pg + h)i, & n \in S_{12}, \\ d - r + 2(p-1)hi, & n \in S_p. \end{cases} \tag{41}$$

Sequence $\mathbf{S} = \{S[n]\}$ of (41) is a degree-4 PGIS.

Proof: Taking DFT upon $\mathbf{s}$ derives the results. ■

We make conclusion that the DFT of $\mathbf{s} = d\delta_{2p} + \eta\mathbf{c}_{11} - \eta^*\mathbf{c}_{12} + \eta^*\mathbf{c}_{21} - \eta\mathbf{c}_{22} + r\mathbf{c}_3$ constructs a degree-5 PGIS of period $N = 2p$ only when $p \equiv 1 \pmod 4$, which is represented in (37). When $p \equiv 3 \pmod 4$, we apply $\mathbf{s} = d\delta_{2p} + \eta\mathbf{c}_{11} - \eta^*\mathbf{c}_{12} + \eta^*\mathbf{c}_{21} - \eta\mathbf{c}_{22} + r\mathbf{c}_3$, and taking the DFT operation upon $\mathbf{s}$ can obtain a degree-5 PGIS of period $N = 2p$, which is given in (40).

VI. HIGHER DEGREE PGIS CONSTRUCTION OF PERIOD N = QP

In the case of $N = qp$, where both p and q are odd primes, because there is no primitive root, $S_1 = \{n | gcd(n, N) = 1, n \in \mathbb{Z}_N^\times\}$ is a multiplicative group but not a cyclic one. Moreover, there exists more than one subgroup of S_1 with cardinality $\frac{(q-1)(p-1)}{2}$, e.g., $S_{1a} = \{1, 2, 4, 8\}$, $S_{1b} = \{1, 4, 7, 13\}$ and $S_{1c} = \{1, 4, 11, 14\}$ are subgroups of $S_1 = \{1, 2, 4, 7, 8, 11, 13, 14\}$ with respect to $\mathbb{Z}_{15}^+$. Among the three different partitioning of S_1 based on S_{1a}, S_{1b} and S_{1a}, respectively, only the partition $S_1 = S_{1a} \bigcup^* S_{12}$, where $S_{12} = \{7, 11, 13, 14\}$, can construct base sequences, defined in (22), (44) and (45), which possesses the desired magnitude flat property over nonzero elements of their DFTs. The reason is that the elements of set S_{1a} are either quadratic residues (mod 15) or the value of associative *Jacobi symbol* is "1", analyzed in the sequel. This relies on introducing Jacobi symbol as a tool to find the proper partitioning of S_1 and to construct PGISs with higher degrees. The Jacobi symbol, defined as follows, generalizes the Legendre symbol with respect to quadratic congruence of odd composite modulo.

Definition 1 [29]: Let $m = \prod_{n=1}^r p_n$ where the p_n are odd primes. Let $\left(\frac{a}{p_n}\right)$ be Legendre symbol for each n such that $1 \le n \le r$, where $gcd(a, m) = 1$. Then $\left(\frac{a}{m}\right) = \prod_{n=1}^r \left(\frac{a}{p_n}\right)$ is called a *Jacobi symbol*.

Theorem 6: There are $\frac{(p-1)(q-1)}{4} + \frac{(p-1)}{2} + \frac{(q-1)}{2}$ quadratic residues (mod qp).

Proof: 1) The congruence $x^2 \equiv k \pmod{qp}$ is equivalent to the system of simultaneous congruences

$$\begin{cases} x^2 \equiv & k \pmod q, \\ x^2 \equiv & k \pmod p. \end{cases} \tag{42}$$

By the Chinese Remainder Theorem [29], the system of simultaneous linear congruences

$$\begin{cases} x \equiv & a_1 \pmod q, \\ x \equiv & a_2 \pmod p. \end{cases} \tag{43}$$

has a unique solution (mod qp) for each set of parameters a_1 and a_2, that is, $x = px_1a_1 + qx_2a_2$ where $px_1 \equiv 1 \pmod q$ and $qx_2 \equiv 1 \pmod p$.

2) There exist $\frac{(q-1)}{2}$ and $\frac{(p-1)}{2}$ quadratic residues (mod q) and (mod p), respectively. Let S_{qr} and S_{pr} be the respective sets of these two quadratic residues. The number of linear combination $x = px_1a_1 + qx_2a_2$ for each set of parameters a_1 and a_2 over $\{(a_1, a_2) | (a_1, a_2) \in S_{qr} \times S_{pr}\}$ is $\frac{(q-1)}{2} \cdot \frac{(p-1)}{2}$.

These are the parts of quadratic residues (mod qp) that belong to S_1.

3) The number of linear combination $x = px_1a_1 + qx_2a_2$ over $\{(a_1, a2)|(a_1, a_2) \in S_{qr} \times \{0\}\}$ is $\frac{(q-1)}{2}$. These are the parts of quadratic residues (mod qp) that belong to S_q.

4) Finally, the number of linear combination $x = px_1a_1 + qx_2a_2$ over $\{(a_1, a2)|(a_1, a_2) \in \{0\} \times S_{pr}\}$ is $\frac{(p-1)}{2}$. These are the parts of quadratic residues (mod qp) that belong to S_p.

Above all, there are $\frac{(p-1)(q-1)}{4} + \frac{(p-1)}{2} + \frac{(q-1)}{2}$ quadratic residues (mod qp). ■

Lemma 7: There are $\frac{(q-1)(p-1)}{2}$ Jacobi symbols $\left(\frac{a}{qp}\right) = 1$ and equal number of such Jacobi symbols $\left(\frac{a}{qp}\right) = -1$.

Proof: The proof is omitted for brevity. ■

S_1 can be partitioned into the disjoint union of $S_{11} = \{a| \left(\frac{a}{qp}\right) = 1, a \in S_1\}$ and $S_{12} = \{a| \left(\frac{a}{qp}\right) = -1, a \in S_1\}$ two subsets. S_q can be partitioned into $S_q = S_{q1} \bigcup^* S_{q2}$, where S_{q1} and S_{q2} consist of quadratic residues and quadratic nonresidues (mod qp), respectively, belonging to S_q. Similarly, $S_p = S_{p1} \bigcup^* S_{p2}$, where S_{p1} and S_{p2} consist of quadratic residues and quadratic nonresidues (mod qp), respectively, belonging to S_p. With these partitions of S_1, S_q and S_p, the base sequence $\mathbf{c}_1$, which is defines in (23), is still applied. However, the period of this base sequence becomes $N = qp$. Here, $\mathbf{c}_2$ and $\mathbf{c}_3$ are redefined as

$$c_2[n] = \begin{cases} 1, & n \in S_{q1}, \\ -1, & n \in S_{q2}, \\ 0, & \text{otherwise}. \end{cases} \tag{44}$$

$$c_3[n] = \begin{cases} 1, & n \in S_{p1}, \\ -1, & n \in S_{p2}, \\ 0, & \text{otherwise}. \end{cases} \tag{45}$$

Similar to the $N = 2p$ case, the base sequence $\mathbf{c}_2$, can be obtained from upsampling a *Legendre Sequence* [23], [30] of prime period p by a factor of q, when the period of base sequence $\mathbf{c}_2$ becomes $N = qp$. To the base sequence $\mathbf{c}_3$, defined in (45), it can be obtained from upsampling a *Legendre Sequence* of period q by a factor of p.

Lemma 8: $\mathbf{c}_{11} = \mathbf{c}_{21} \otimes \mathbf{c}_{31} + \mathbf{c}_{22} \otimes \mathbf{c}_{32}$, $\mathbf{c}_{12} = \mathbf{c}_{21} \otimes \mathbf{c}_{32} + \mathbf{c}_{22} \otimes \mathbf{c}_{31}$, and $\mathbf{c}_1 = \mathbf{c}_{11} - \mathbf{c}_{12}$.

Proof: 1) To prove $\mathbf{c}_{11} = \mathbf{c}_{21} \otimes \mathbf{c}_{31} + \mathbf{c}_{22} \otimes \mathbf{c}_{32}$, the number of nonzero elements of sequence $\mathbf{c}_{11}$ is $\frac{(p-1)(q-1)}{2}$, where the associative entries of these elements are assigned with respect to the elements of set S_1, in which the value of Jacobi symbols is $\left(\frac{n}{qp}\right) = 1$. When both elements of S_{p1} and S_{q1} are quadratic residues (mod qp), the $\frac{(p-1)(q-1)}{4}$ nonzero elements of $\mathbf{c}_{21} \otimes \mathbf{c}_{31}$ contribute half parts of $\frac{(p-1)(q-1)}{2}$ nonzero elements of sequence $\mathbf{c}_{11}$ where the associative Jacobi symbols are with $\left(\frac{n}{qp}\right) = \left(\frac{n}{q}\right)\left(\frac{n}{p}\right) = 1 \cdot 1 = 1$. Meanwhile, when both S_{p2} and S_{q2} consist of quadratic nonresidues (mod qp), the $\frac{(p-1)(q-1)}{4}$ nonzero elements of $\mathbf{c}_{22} \otimes \mathbf{c}_{32}$ contribute the parts of other half nonzero elements of sequence $\mathbf{c}_{11}$ where the associative Jacobi symbols are with $\left(\frac{n}{qp}\right) = \left(\frac{n}{q}\right)\left(\frac{n}{p}\right) = (-1) \cdot (-1) = 1$.

2) On the other hand, the entries of nonzero elements of $\mathbf{c}_{12}$ are assigned based on the elements of set S_1 where the associative value of Jacobi symbols is $\left(\frac{n}{qp}\right) = -1$. This means that the $\frac{(p-1)(q-1)}{4}$ nonzero elements of $\mathbf{c}_{21} \otimes \mathbf{c}_{32}$ contribute half parts of $\frac{(p-1)(q-1)}{2}$ nonzero elements of sequence $\mathbf{c}_{12}$ where the associative Jacobi symbols are with $\left(\frac{n}{qp}\right) = \left(\frac{n}{q}\right)\left(\frac{n}{p}\right) = (-1)\cdot 1 = -1$. The remaining half of the nonzero elements of $\mathbf{c}_{12}$ can be obtained from $\mathbf{c}_{22} \otimes \mathbf{c}_{31}$ using a similar procedure as that described above.

3) Finally, by the definition of $c_1[n]$ shown in (22), we can express $\mathbf{c}_1 = \mathbf{c}_{11} - \mathbf{c}_{12}$. ■

Let $\{F[k]\}_{k=0}^{q-1}$ be the DFT of Legendre Sequence of period q, which is also the Gauss sum defined as

$$F[k] = \begin{cases} \left(\frac{k}{q}\right)\sqrt{q}, & q \equiv 1 (\text{mod } 4), \\ -\left(\frac{k}{q}\right) i\sqrt{q}, & q \equiv 3 (\text{mod } 4). \end{cases} \tag{46}$$

The DFTs of $\mathbf{c}_i$, denoted by $\mathbf{C}_i = \{C_i[n]\}_{n=0}^{qp-1}$, $i = 2, 3$, where $\{C_2[n]\}_{n=0}^{p-1} = \{G[n]\}_{n=0}^{p-1}$ and $C_2[kp+n] = C_2[n]$, $k = 1, \ldots, q-1$, $\{C_3[n]\}_{n=0}^{p-1} = \{F[n]\}_{n=0}^{q-1}$ and $C_3[kp+n] = C_3[n]$, $k = 1, \ldots, p-1$, are expressed as:

$$\mathbf{C}_2 = [\underbrace{G[0]\ G[1] \cdots G[p-1]}_{1^{st} \text{ pattern}} \cdots \underbrace{G[0]\ G[1] \cdots G[p-1]}_{q^{th} \text{ pattern}}],$$

$$\mathbf{C}_3 = [\underbrace{F[0]\ F[1] \cdots F[q-1]}_{1^{st} \text{ pattern}} \cdots \underbrace{F[0]\ F[1] \cdots F[q-1]}_{p^{th} \text{ pattern}}].$$

We can use $\mathbf{C}_2$ and $\mathbf{C}_3$ to derive $\mathbf{C}_1$ described in *Lemma 9*.

Lemma 9: $\mathbf{c}_1 = \mathbf{c}_2 \otimes \mathbf{c}_3$, and the associative DFT is $\mathbf{C}_1 = \mathbf{C}_2 \circ \mathbf{C}_3$.

Proof: *Lemma 9* can be proven by the results of *Lemma 8*. ■

Aside from $C_2[n] = G[0] = 0$, $n = kp$, $k = 0, 1, \ldots, (q-1)$, $C_3[n] = F[0] = 0$, $n = lq$, $l = 0, 1, \ldots, (p-1)$, and $C_1[n] = 0$, $n \in \{kp|k = 0, 1, \ldots, (q-1)\} \cup \{lq|l = 0, 1, \ldots, (p-1)\}$, the nonzero elements of three $\mathbf{C}_i$ have the magnitude flat property, which is expressed as $|C_2[n]| = |G[n]| = \sqrt{p}$, $|C_3[n]| = |F[n]| = \sqrt{q}$ and $|C_1[n]| = |G[n]F[n]| = \sqrt{qp}$. We can adopt these desired properties to construct the PGIS of period $N = qp$ according to *Theorem 7*, the algorithm of which is similar to that of the $N = 2p$ case addressed in Section V.

Theorem 7: Let a be Gaussian integer, and b, k, d, f, g, and h be simple integers. Given that $|a|^2 = b^2 + (qp)k^2 = d^2 + pf^2 = g^2 + qh^2$, the DFT of sequence

$$\mathbf{s} = a\delta_N + \eta\mathbf{c}_{11} - \eta^*\mathbf{c}_{12} + \upsilon\mathbf{c}_{21} - \upsilon^*\mathbf{c}_{22} + \xi\mathbf{c}_{31} - \xi^*\mathbf{c}_{32}$$

is a PGIS of period $N = qp$, where $\eta = bi + k\sqrt{qp}$, $\upsilon = di + f\sqrt{p}$, and $\xi = gi + h\sqrt{q}$.

Proof: Based on *Lemma 8* and 9, the proof of *Theorem 7* is similar to that of *Theorem 5*, where the coefficients of the

DFTs of $a \cdot \delta_N$, $\eta \mathbf{c}_{11} - \eta^* \mathbf{c}_{12}$, $\upsilon \mathbf{c}_{21} - \upsilon^* \mathbf{c}_{22}$, and $\xi \mathbf{c}_{31} - \xi^* \mathbf{c}_{32}$ four vectors are all Gaussian integers. The detailed explanations are omitted here for brevity. ∎

Example 3: Let $a = 6 + i$, $bi + k\sqrt{qp} = 4i + \sqrt{21}$, $di + f\sqrt{p} = 3i + 2\sqrt{7}$, and $gi + h\sqrt{q} = 5i + 2\sqrt{3}$, which $|a|^2 = b^2 + (qp)k^2 = d^2 + pf^2 = g^2 + qh^2 = 37$ is held, a degree-9 PGIS of period 21 is given by

$$\mathbf{s} = (a_0, a_1, a_2, a_3, a_1, a_4, a_3, a_5, a_2, a_6, a_7, a_2, a_3, a_7, a_8, a_6, a_1, a_4, a_6, a_7, a_4), \quad (47)$$

where $a_0 = 6 + 77i$, $a_1 = 27 - 23i$, $a_2 = -15 - 11i$, $a_3 = 6 + 14i$, $a_4 = 27 + 17i$, $a_5 = 6 - 16i$, $a_6 = 6 - 14i$, $a_7 = -15 + 5i$, and $a_8 = 6 - 4i$.

VII. CONCLUSIONS

This paper addresses the construction of PGISs of period $N = qp$, where q and p are distinct primes. Based on the partitioning of a ring $\mathbb{Z}_N$ into four subsets, four base sequences can be defined to construct degree-4 PGISs from either the time or frequency domain. Both approaches are challenged by the solution of two equivalent systems of three nonlinear constraint equations with eight variables. To address this problem, we present three methods that can be used to decompose the three nonlinear constraint equations and form three systems of four linear equations with four variables. Here each individual system derives a unique set of integer solutions. The proposed scheme is significant because it simplifies the construction of degree-4 PGISs. To construct PGISs with degrees larger than four, the study of period $N = 2p$ PGISs can be considered by up-sampling the Legendre sequences of period p by a factor of two, followed by adjusting associative coefficients to meet the flat-spectrum property. When $N = qp$ is odd, we introduce Jacobi symbols as a tool to partition ring $\mathbb{Z}_N$ further into seven subsets and apply an algorithm similar to that in the $N = 2p$ case to construct PGISs with more diverse degrees. Finally, the properties and many theorems related to the construction of PGISs and partitioning of ring $\mathbb{Z}_N$ are derived in this paper.

NOTATION AND SYMBOLS

$\mathbb{Z}_N^\times$	$\{1, 2, \ldots, N-1\}$
$\mathbb{Z}_N$	$= \{0\} \cup \mathbb{Z}_N^\times$
$\mathbf{s}$	$= \{s[n]\}_{n=0}^{N-1}$ PGIS
$\mathbf{S}$	DFT of $\mathbf{s}$
$\mathbf{R_s}$	PACF of $\mathbf{s}$
$\mathbf{c}_i$	$= \{c_i[n]\}_{n=0}^{N-1}$ base sequence
$\mathbf{C}_i$	DFT of $\mathbf{c}_i$
$a_k + b_k i$	a Gaussian integer, $i = \sqrt{-1}$
S_i	a subset of $\mathbb{Z}_N^\times$
$\mathbf{A}_i$	coefficient matrix
$G[k]$	Gauss sum defined in (32)
$F[k]$	Gauss sum defined in (46)
$\left(\frac{n}{p}\right)$	Legendre or Jacobi symbol
$\bigcup^*$	disjoint union
$\otimes$	circular convolution

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