Received October 11, 2018, accepted October 18, 2018, date of publication October 24, 2018,
date of current version November 30, 2018.

Digital Object Identifier 10.1109/ACCESS.2018.2877921

Complex Multi-Fuzzy Soft Set: Its Entropy and Similarity Measure

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This work was supported by Universiti Kebangsaan Malaysia under Grant GUP-2017-105.

ABSTRACT We develop a novel mathematical tool known as complex multi-fuzzy soft set (CMFSS) which has the ability to handle uncertainties, imprecision, and vagueness of information that are inherent in the data by considering the amplitude and phase terms of the complex numbers simultaneously. This CMFSS constitutes of a hybrid structure of multi-fuzzy set and soft set which are defined in a complex setting. The structure is flexible as it allows for a greater range of values for the membership function by extending them to the unit circle in a complex plane through modification of the multi-fuzzy soft set by the inclusion of an additional term called the phase term in order to take into account of the periodic nature of the data. Accordingly, the novelty of this paper lies in the complex multi-membership functions which consider more range of values while handling the uncertainty of the periodic data. In this paper, the concept of complex multi-fuzzy soft set is introduced. We then define its basic operations of complement, union and intersection and study some related properties, with supporting proofs. Subsequently, by means of level soft sets, we present an algorithm to solve a CMFSSs decision making problem, to illustrate the effectiveness and practicality of the proposed concept. Finally, we introduce axiomatic definitions of entropy and similarity measure for CMFSSs, and some formulas have also been put forward to calculate them. Numerical examples are given to demonstrate that the proposed entropy measure for CMFSSs is an important concept for measuring uncertainty in the information/data. Furthermore, some theorems are proposed showing how the entropy of CMFSS can be found from the similarity measure of CMFSS.

INDEX TERMS Complex multi-fuzzy set, decision making, fuzzy set, multi-fuzzy set, soft set, uncertain information.

I. INTRODUCTION

Many problems in the areas such as economics, management, engineering, environmental science and social science involve data which have the properties of fuzzy, imprecise and uncertain. The classical methods are not always successful in solving these problems since the properties may be quite complex. In recent times, a number of theories have been proposed for handling the uncertainties and vagueness in an effective way. Some of these theories are fuzzy set [1], intuitionistic fuzzy sets [2], rough sets [3] and multi-fuzzy sets [4], [5]. The concept of the multi-fuzzy sets theory was proposed by Sebastian and Ramakrishnan which is a more general fuzzy set using ordinary fuzzy sets as building blocks, and its membership function is an ordered sequence of ordinary fuzzy membership functions. The notion of multi-fuzzy sets provides a new method to represent some problems which are difficult to explain in other extensions of fuzzy set theory, such as color of pixels. Moreover, these above mentioned theories have their own difficulties due to the inadequacy of the parameterization tool. Aiming to overcome these difficulties, Molodtsov [6] suggested the concept of soft set which involves dealing with uncertainties. Soft set is free from the inadequacy of the parameterization tools, therefore, it can be applied into practice easily. At the same time, there had been some practical applications in soft set theory that are used in decision making [7]–[9]. Maji *et al.* [10] extended the soft set to the notion of fuzzy soft set, by combining the theories of soft set and fuzzy set. Due to the capability of dealing with uncertain and fuzzy parameters, fuzzy soft set theory was quite versatile and extensively applied in the decision making problems. Maji and Roy [11] described a new approach of object recognition from an uncertain multi-observer data in order to handle decision making based on fuzzy soft sets. Feng *et al.* [12] applied level soft sets to describe an adjustable decision making approach for fuzzy soft set. The works in [13] and [14] described the essentials of fuzzy soft set as a basis for decision-making. Yang *et al.* [15] introduced the concept of multi-fuzzy soft set by combining the multi-fuzzy set and soft set models, and applied it to decision making, while Dey and Pal [16] generalized the notion of multi-fuzzy soft set. Zhanga and Shu [17] extended the idea of multi-fuzzy soft set and introduced the notion of possibility multi-fuzzy soft set and applied it to a decision making problem.

Entropy and similarity measures are measures that are important in the study of fuzzy set and its hybrid structures as these two components play an important role in measuring uncertain information which is available in the data. The degree of fuzziness in fuzzy set and other extended higher order fuzzy sets was first mentioned by Zadeh [18]. Then, De Luca and Termini [19] suggested a certain set of axioms for fuzzy entropy. On the other hand, similarity measure which is an important tool for determining the degree of similarity between two objects has received much more attention than entropy, as shown by an extensive number of literatures on the subject. Pappis and his collaborators have issued a series of papers [20], [21] which took an axiomatic view of similarity measures. The entropy and similarity measures for many other sets such as interval-valued fuzzy set [22], fuzzy soft set [23] and intuitionistic fuzzy soft set [24] have been widely applied in solving problems related to decision making, image processing and pattern recognition.

The fuzzy set model can be combined with other mathematical models to extend the range of the membership function from the real field to the complex field. For example, Ramot *et al.* [25] extended this range from the interval [0,1] to the unit circle in the complex plane and called it complex fuzzy set, an area which had been progressing rapidly, and is now known as complex fuzzy logic [26]. The complex fuzzy sets were used to represent information which involve uncertainty and periodicity. Since its inception, a lot of extensions of complex fuzzy set models have been developed, such as complex intuitionistic fuzzy set [27], complex neutrosophic set [28], complex fuzzy soft set [29], complex fuzzy soft multisets [30] and complex intuitionistic fuzzy soft set [31]. These models have been used to represent the uncertainty and periodicity aspects of an object together, in a single set.

Recently, Al-Qudah and Hassan [32] developed a hybrid model of complex fuzzy sets and multi-fuzzy sets, called the complex multi-fuzzy set. This model is useful for handling problems with the properties of multidimensional characterization. To make this model more functional for the purpose of attaining an improved new decision making results, we will develop it into complex multi-fuzzy soft set (CMFSS) in order to incorporate the advantages of soft set and apply them to the complex multi-fuzzy set models. Our proposed model will have the ability to handle uncertainties, imprecision and vagueness of two-dimensional multi-fuzzy information by capturing the amplitude terms and phase terms of the complex numbers simultaneously.

The main contributions of our research are as follows. Firstly, we introduce the concept of CMFSS, which combines the advantages of both the complex multi-fuzzy set and soft set. Secondly, we define some concepts related to the notion of CMFSS as well as some basic operations namely the complement, union, intersection, AND and OR. The basic properties and relevant laws pertaining to this concept such as the De Morgan's laws are also verified. Thirdly, in terms of the application, the CMFSS will be used together with a generalized algorithm to determine the degree and the total time of the influence of the economic factors on the sectors that promotes the Malaysian economy and then deduced results that help in making decision to determine the most important factor from these factors. Lastly, we present the axiomatic definition of entropy and similarity measures of CMFSSs and study the basic relations between them. Moreover, numerical example is given to demonstrate that the proposed entropy measure for CMFSSs is an important concept for measuring uncertain information.

The paper is organized in the following way. Fundamentals of multi- fuzzy set theory, soft set theory, multi-fuzzy soft sets and complex multi-fuzzy sets are presented in Section 2. In Section 3, the concept of complex multi-fuzzy soft set with its operation rules are introduced. In Section 4, the basic set theoretic operations of complex multi-fuzzy soft set such as complement, union and intersection along with some propositions are presented. In Section 5, we discuss an application of this concept in the area of economics. In Section 6, we introduce the axiomatic definition of entropy for CMFSS, along with an illustrative example. In Section 7, the similarity measure between CMFSSs and the relations between the entropy and similarity measures are studied. Finally, conclusions are presented in Section 8.

II. PRELIMINARIES

In this section, we summarize some of the important concepts pertaining to multi- fuzzy sets, soft sets, multi-fuzzy soft sets and complex multi-fuzzy sets that are relevant to this paper. These concepts are stated below.

A. SOFT SETS AND FUZZY SOFT SETS

Molodtsov [6] defined soft set in the following way. Let U be an initial universe, E be a set of parameters under consideration and $\mathcal{A} \subseteq E$. Let $P(U)$ denote the power set of U.

Definition 1 (See [6]): A pair $(\tilde{F}, \mathcal{A})$ is called a soft set over U, where $\tilde{F}$ is a mapping given by $\tilde{F} : \mathcal{A} \to P(U)$.

Maji *et al.* [10] applied the concept of fuzzy sets to soft set theory to introduce the hybrid structure called fuzzy soft sets, which can be seen as a fuzzy generalization of crisp soft set. They defined the concept of fuzzy soft sets as follows.

Definition 2 (See [10]): Let U be an initial universal and E be a set of parameters. Let $P(U)$ denote the power set of all fuzzy subsets of U and $\mathcal{A} \subseteq E$. A pair $(\hat{F}, \mathcal{A})$ is called a fuzzy soft set over U, where $\hat{F}$ is a mapping given by $\hat{F} : \mathcal{A} \to P(U)$.

Liu *et al.* [23] presented the axioms that entropy and similarity measures of fuzzy soft set must satisfy. Similarity measure and entropy are important tools for dealing with uncertain data.

Definition 3 (See [23]): A real function $S : FS(U, E) \times FS(U, E) \to [0, 1]$ is called a similarity measure for fuzzy soft set, if it satisfies the following properties.

($S1$) $S((U_A, \phi_A) = 0$ for any $A \in E$, and $S((F, A), (F, A)) = 1$ for any $(F, A) \in FS(U, E)$.

($S2$) $S((F, A), (G, B)) = S((G, B), (F, A))$ for any $(F, A), (G, B) \in FS(U, E)$.

($S3$) For any $(F, A), (G, B), (H, C) \in FS(U, E)$, if $(F, A) \subseteq (G, B) \subseteq (H, C)$, then $S((H, C), (F, A)) \leq min(S((H, C), (G, B)), S((G, B), (F, A)))$.

Definition 4 (See [23]): A real function $E : FS(U, E) \to [0, +\infty)$ is called an entropy on $FS(U, E)$, if E has the following properties.

($E1$) $E(F, A) = 0$ if (F, A) is a soft set.

($E2$) $E(F, A) = 1$ if $F(e) = [0.5]$ for any $e \in A$, where $[0.5]$ is the fuzzy set with the membership function $[0.5](x) = 0.5$ for each $x \in U$.

($E3$) Let (F, A) be crisper set than (G, B); that is, for any $e \in A$ and $x \in U$, $F(e)(x) \leq G(e)(x)$ if $G(e)(x) \leq 0.5$ and $F(e)(x) \geq G(e)(x)$ if $G(e)(x) \geq 0.5$. Than $E(F, A) \leq E(G, B)$.

($E4$) $E(F, A) = E(F^c, A)$, where (F^c, A) is the complement of fuzzy soft set (F, A) given by $F^c(e) = (F(e))^c$ for each $e \in A$.

B. MULTI-FUZZY SETS AND MULTI-FUZZY SOFT SETS

The notion of fuzzy sets [1] was generalized to multi-fuzzy sets by Sebastian and Ramakrishnan [4] in 2011. Sebastian and Ramakrishnan introduced multi-fuzzy sets theory as a mathematical tool to deal with life problems that have multidimensional characterization properties. The definition of soft sets is given as follows:

Definition 5 (See [4]): Let k be a positive integer and U be a non-empty set. A multi-fuzzy set $\mathcal{A}$ in U is a set of ordered sequences $\mathcal{A} = \{\langle x, \mu_1(x), \ldots, \mu_k(x)\rangle : x \in U\}$, where $\mu_i : U \longrightarrow L_i = [0, 1]$, $i = 1, 2, \ldots, k$.

The function $\mu_{\mathcal{A}}(x) = (\mu_1(x), \ldots, \mu_k(x))$ is called the multi-membership function of multi-fuzzy sets A, k is called a dimension of $\mathcal{A}$. The set of all multi-fuzzy sets of dimension k in U is denoted by $M^kFS(U)$.

Combining multi-fuzzy sets and soft sets, Yang *et al.* [15] proposed the following hybrid model called multi-fuzzy soft set, which can be seen as an extension of both multi-fuzzy set and crisp soft set. Some of the basic concepts pertaining to multi-fuzzy sets are as follows.

Definition 6 (See [15]): Let U be an initial universal and E be a set of parameters. A pair $(F, \mathcal{A})$ is called a multi-fuzzy soft set of dimension k over U, where F is a mapping given by $F : \mathcal{A} \to M^kFS(U)$. A multi-fuzzy soft set is a mapping from parameters to $M^kFS(U)$. It is a parameterized family of multi-fuzzy subsets of U. For $e \in \mathcal{A}$, $F(e)$ may be considered as the set of e-approximate elements of the multi-fuzzy soft set $(F, \mathcal{A})$.

Definition 7 (See [15]): Let $\mathcal{A}, \mathcal{B} \subseteq E$. Let $(F, \mathcal{A})$ and $(G, \mathcal{B})$ be two multi-fuzzy soft sets of dimension k over U, $(F, \mathcal{A})$ is said to be a multi-fuzzy soft subset of $(G, \mathcal{B})$ if

1) $\mathcal{A} \subseteq \mathcal{B}$ *and*
2) $\forall e \in \mathcal{B}, F(e) \sqsubseteq G(e)$.

Here, we write $(F, \mathcal{A}) \sqsubseteq (G, \mathcal{B})$.

C. COMPLEX MULTI FUZZY SETS

The novelty of the complex multi fuzzy sets introduced by Al-Qudah and Hassan [32] lies in the ability of complex multi-membership functions to allow for more range of values of the membership functions while handling uncertainty in the data that is periodic in nature. Some of the basic concepts pertaining to complex multi- fuzzy sets are as follows.

Definition 8 (See [32]): Let k be a positive integer and U be a non-empty set. A complex multi-fuzzy set (CMFS) $\mathcal{A}$, defined on a universe of discourse U, is characterised by a multi-membership function $\mu_{\mathcal{A}}(x) = (\mu^j_{\mathcal{A}}(x))_{j \in k}$, that assigns to any element $x \in U$ a complex-valued grade of multi- membership functions in $\mathcal{A}$. $\mu_{\mathcal{A}}(x)$ may all lie within the unit circle in the complex plane, and are thus of the form $\mu_{\mathcal{A}}(x) = (r^j_{\mathcal{A}}(x).e^{i\omega^j_{\mathcal{A}}(x)})_{j\in k}$, $(i = \sqrt{-1})$, $(r^j_{\mathcal{A}}(x))_{j\in k}$ are real-valued functions and $(r^j_{\mathcal{A}}(x))_{j\in k} \in [0, 1]$. The CMFS $\mathcal{A}$ may be represented as the set of ordered sequence

$$\begin{aligned}\mathcal{A} &= \{(x \left(\mu^j{}_{\mathcal{A}}(x) = a_j\right)_{j\in k}) : x \in U\} \\ &= \{x, ((r^j_{\mathcal{A}}(x).e^{i\omega^j_{\mathcal{A}}(x)})_{j\in k}) : x \in U\}.\end{aligned}$$

where $\mu^j{}_{\mathcal{A}} : U \to L_j = \{a_j : a_j \in C, |a_j| \leq 1\}$ for $j = 1, 2, \ldots, k$.

The function $(\mu_{\mathcal{A}}(x) = r^j_{\mathcal{A}}(x).e^{i\omega^j_{\mathcal{A}}(x)})_{j\in k}$ is called the complex multi-membership function of complex multi-fuzzy set $\mathcal{A}$, k is called the dimension of $\mathcal{A}$. The set of all complex multi-fuzzy sets of dimension k in U is denoted by $CM^kFS(U)$.

We now present the theoretic operations of complex multi-fuzzy sets.

Definition 9 (See [32]): Let $\mathcal{A} = \{x, ((r^j_{\mathcal{A}}(x).e^{i\omega^j_{\mathcal{A}}(x)})_{j\in k}) : x \in U\}$ and $\mathcal{B} = \{x, ((r^j_{\mathcal{B}}(x).e^{i\omega^j_{\mathcal{B}}(x)})_{j\in k}) : x \in U\}$ be two complex multi-fuzzy sets of dimension k in X. We define the following relations and operations. 1.

1) $\mathcal{A} \subset \mathcal{B}$ *if and only if* $r^j_{\mathcal{A}}(x) \leq r^j_{\mathcal{B}}(x)$ *and* $\omega^j_{\mathcal{A}}(x) \leq \omega^j_{\mathcal{B}}(x)$, *for all* $x \in U$ *and* $j = 1, 2, \ldots, k$.
2) $\mathcal{A} = \mathcal{B}$ *if and only if* $r^j_{\mathcal{A}}(x) = r^j_{\mathcal{B}}(x)$ *and* $\omega^j_{\mathcal{A}}(x) = \omega^j_{\mathcal{B}}(x)$, *for all* $x \in U$ *and* $j = 1, 2, \ldots, k$.
3) $\mathcal{A} \cup \mathcal{B} = \{\langle x, r^j_{\mathcal{A}\cup\mathcal{B}}(x).e^{i\omega^j_{\mathcal{A}\cup\mathcal{B}}(x)}\rangle : x \in\} = \{\langle x, \max(r^j_{\mathcal{A}}(x), r^j_{\mathcal{B}}(x)).e^{i\max[\omega^j_{\mathcal{A}}(x), \omega^j_{\mathcal{B}}(x)]}\rangle : x \in U\}$, *for all* $j = 1, 2, \ldots, k$.

4) $\mathcal{A} \cap \mathcal{B} = \{\langle x, r^j_{\mathcal{A}\cap\mathcal{B}}(x).e^{i\omega^j_{\mathcal{A}\cap\mathcal{B}}(x)}\rangle : x \in\} = \{\langle x, \min(r^j_{\mathcal{A}}(x), r^j_{\mathcal{B}}(x)).e^{i\min[\omega^j_{\mathcal{A}}(x),\omega^j_{\mathcal{B}}(x)]}\rangle : x \in U\}$, *for all* $j = 1, 2, \ldots, k$.
5) $\mathcal{A}^c = \{x, [r^j_{\mathcal{A}^c}(x).e^{i\omega^j_{\mathcal{A}^c}(x)}]_{j\in k} : x \in X\} = \{x, ([1 - r^j_{\mathcal{A}}(x)].e^{i[2\pi-\omega^j_{\mathcal{A}}(x)]})_{j\in k} : x \in U\}$, *for all* $j = 1, 2, \ldots, k$.

III. COMPLEX MULTI-FUZZY SOFT SET

Theory of fuzzy sets [1] is the popular generalization of classical set theory, whose membership grades are within the real valued interval [0, 1]. Fuzzy set is used successfully in modeling uncertainty in many fields of everyday life. However, there are many problems like complete colour characterization of colour images, taste recognition of food items and decision making problems with multi aspects which cannot be characterized by a single membership function of Zadeh's fuzzy sets. In order to overcome these problems, Sebastian and Ramakrishnan [4] proposed the concept of multi-fuzzy sets theory as a mathematical tool to deal life problems that have multidimensional characterization properties. In fact, the multi-fuzzy set has an obstacle to retrieve full information with correct meaning. Al-Qudah and Hassan [32] gave a solution for this obstacle by generalizing the range of membership function of multi-fuzzy set from [0, 1] to the unit circle in the complex plane by adding an additional term to represent phase. Complex multi-fuzzy sets [32] theory consists of multi-membership functions such that each membership function is composed of amplitude term and phase term that handle the uncertainty and periodicity, simultaneously. However, the complex multi-fuzzy set has one major drawback, which is the lack of an adequate parameterization tool to facilitate the representation of the parameters in a comprehensive manner. Our proposed complex multi-fuzzy soft set has the added advantage of complex multi-fuzzy set. This model incorporates the advantages of complex multi-fuzzy sets and the adequate parameterization tool.

Now, we begin proposing the definition of complex multi-fuzzy soft set, and give an illustrative example of it.

Let U be a universe, $\mathcal{E}$ be a set of parameters and $\mathcal{A} \subseteq \mathcal{E}$. Let $CM^k(U)$ be a set of all complex multi-fuzzy subsets of dimension k in U.

Definition 10: A pair $(\mathcal{F}, \mathcal{A})$ is called a complex multi-fuzzy soft set of dimension k over U, where $\mathcal{F}$ is a mapping given by

$$\mathcal{F} : \mathcal{A} \to CM^k(U).$$

A complex multi-fuzzy soft set of dimension k ($\mathcal{CM}^k\mathcal{FSS}$) is a mapping from parameters to $CM^k(U)$. It is a parameterized family of complex multi-fuzzy subsets of U, and it can be written as:

$$(\mathcal{F}, \mathcal{A}) = \{\langle e, \mathcal{F}(e)\rangle : e \in \mathcal{A}, \mathcal{F}(e) \in CM^k(U)\},$$

where

$$\mathcal{F}(e) = \{\langle x, \mu^j_{\mathcal{F}(e)}(x) = r^j_{\mathcal{F}(e)}(x).e^{i\omega^j_{\mathcal{F}(e)}(x)}\rangle : e \in \mathcal{A}, x \in U, j = 1, 2, \ldots, k\}.$$

where $(\mu^j_{\mathcal{F}(e)}(x))_{j\in K}$ is a complex-valued grade of multi-membership function $\forall x \in U$. By definition, the values of $(\mu^j_{\mathcal{F}(e)}(x))_{j\in K}$ may all lie within the unit circle in the complex plane, and are thus of the form $[\mu^j_{\mathcal{F}(e)}(x) = r^j_{\mathcal{F}(e)}(x).e^{i\omega^j_{\mathcal{F}(e)}(x)}]_{j\in K}$, where $(i = \sqrt{-1})$, each of the amplitude terms $(r^j_{\mathcal{F}(e)}(x))_{j\in K}$ and the phase terms $(\omega^j_{\mathcal{F}(e)}(x))_{j\in K}$ are both real-valued, and $(r^j_{\mathcal{F}(e)}(x))_{j\in K} \in [0, 1]$. The set of all $\mathcal{CM}^k\mathcal{FSS}$s in U is denoted by $\mathcal{CM}^k\mathcal{FSS}(U)$.

To illustrate this notion, let us consider the following example.

Example 11: Assume that a person wishes to buy a travel ticket from one of the travel agencies. Suppose that $U = \{x_1, x_2, x_3\}$ is the universe consisting of three travel agencies. It is well known that a year has four seasons and the prices of the travel ticket are different for each season. Suppose the parameter set $\mathcal{A} = \{e_1, e_2, e_3\}$, where e_1, e_2 and e_3 stand for economic class, business class and first class, simultaneously. The price of the ticket in each of these classes has three levels: high, medium and low. We define a complex multi-fuzzy soft set of dimension three as follows:

$$\begin{aligned}
\mathcal{F}(e_1) = \{&x_1/(0.9e^{i2\pi(2/4)}, 0.2e^{i2\pi(4/4)}, 0.5e^{i2\pi(1/4)}),\\
&x_2/(0.2e^{i2\pi(1/4)}, 0.7e^{i2\pi(3/4)}, 0.7e^{i2\pi(2/4)}),\\
&x_3/(0.7e^{i2\pi(3/4)}, 0.5e^{i2\pi(4/4)}, 0.1e^{i2\pi(1/4)})\},\\
\mathcal{F}(e_2) = \{&x_1/(0.6e^{i2\pi(3/4)}, 0.5e^{i2\pi(1/4)}, 0.1e^{i2\pi(4/4)}),\\
&x_2/(0.2e^{i2\pi(1/4)}, 0.5e^{i2\pi(2/4)}, 0.3e^{i2\pi(4/4)}),\\
&x_3/(0.6e^{i2\pi(2/4)}, 0.5e^{i2\pi(3/4)}, 0.3e^{i2\pi(4/4)})\},\\
\mathcal{F}(e_3) = \{&x_1/(0.7e^{i2\pi(3/4)}, 0.3e^{i2\pi(2/4)}, 0.3e^{i2\pi(1/4)}),\\
&x_2/(0.6e^{i2\pi(2/4)}, 0.6e^{i2\pi(2/4)}, 1e^{i2\pi(4/4)}),\\
&x_3/(0.6e^{i2\pi(2/4)}, 1e^{i2\pi(3/4)}, 0.6e^{i2\pi(4/4)})\}.
\end{aligned}$$

In the context of this example, the amplitude terms represent the degrees of belongingness to the set of prices items and the phase terms represent the degrees of belongingness to the phase of seasons.

For example, let us consider the approximation $\mathcal{F}(e_1)$. In the complex multi-fuzzy value $x_1/(0.9e^{i2\pi(2/4)}, 0.2e^{i2\pi(4/4)}, 0.5e^{i2\pi(1/4)})$, the first membership value $(0.9e^{i2\pi(2/4)})$ indicates that the price of the ticket is very high in the summer, since the amplitude term 0.9 is very close to one and the phase term (2/4) represents the second season of the year (the summer season). While the second membership value $0.2e^{i2\pi(4/4)}$ indicates that the price of the ticket is very low in the winter, since the amplitude term 0.2 is very close to zero and the phase term (4/4) represents the fourth season of the year (the winter season). All the other expressions in the rest of the example can be interpreted in a similar manner.

Now, we present the concept of the subset and equality operations on two CMFSSs in the following definition.

Definition 12: Let $\mathcal{A}, \mathcal{B} \in \mathcal{E}$. Let $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ be two $\mathcal{CM}^k\mathcal{FSS}s$ over U. Now, $(\mathcal{F}, \mathcal{A})$ is said to be a complex multi-fuzzy soft subset of $(\mathcal{G}, \mathcal{B})$ if,

1) $\mathcal{A} \subseteq \mathcal{B}$ *and*
2) $\forall e \in \mathcal{A}, \mathcal{F}(e) \sqsubseteq \mathcal{G}(e)$.

In this case, we write $(\mathcal{F}, \mathcal{A}) \sqsubseteq (\mathcal{G}, \mathcal{B})$.

Definition 13: Let $\mathcal{A}, \mathcal{B} \in \mathcal{E}$. Let $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ be two $\mathcal{CM}^k\mathcal{FSS}s$ over U. $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ are said to be a complex multi-fuzzy soft equal if $(\mathcal{F}, \mathcal{A})$ is a complex multi-fuzzy soft subset of $(\mathcal{G}, B)$ and $(\mathcal{G}, \mathcal{B})$ is a complex multi-fuzzy soft subset of $(\mathcal{F}, \mathcal{A})$.

In the following, we put forward the definition of a null CMFSS and the definition of an absolute CMFSS.

Definition 14: A $\mathcal{CM}^k\mathcal{FSS}s$ $(\mathcal{F}, \mathcal{A})$ over U is said to be a null complex multi-fuzzy soft set, denoted by $(\mathcal{F}, \mathcal{A})_{\tilde{\phi}_k}$, if $\mathcal{F}(e) = 0_k$, for all $e \in \mathcal{A}$ (i.e., $r^j_{\mathcal{F}(e)}(x) = 0$ and $\omega^j_{\mathcal{F}(e)}(x) = 0\pi$, for all $e \in \mathcal{A}, x \in U, j = 1, 2, \ldots, k$).

Definition 15: A $\mathcal{CM}^k\mathcal{FSS}s$ $(\mathcal{F}, \mathcal{A})$ over U is said to be absolute complex multi-fuzzy soft set, denoted by $(\mathcal{F}, \mathcal{A})_{\tilde{\text{U}}_k}$, if $\mathcal{F}(e) = 1_k$, for all $e \in \mathcal{A}$ (i.e., $r^j_{\mathcal{F}(e)}(x) = 1$ and $\omega^j_{\mathcal{F}(e)}(x) = 2\pi$, for all $e \in \mathcal{A}, x \in U, j = 1, 2, \ldots, k$).

IV. BASIC OPERATIONS ON COMPLEX MULTI-FUZZY SOFT SETS

In this section, we introduce some basic theoretic operations on CMFSSs such as the complement, union and intersection. We also give some properties on CMFSSs, which are associative, De Morgan's law and other pertaining laws.

A. COMPLEMENT OF COMPLEX MULTI-FUZZY SOFT SETS

We define the complement operation for CMFSS, give an illustrative example and a proof of a proposed proposition.

Definition 16: Let $(\mathcal{F}, \mathcal{A})$ be a $\mathcal{CM}^k\mathcal{FSS}s$ over U. The complement of $(\mathcal{F}, \mathcal{A})$ is denoted by $(\mathcal{F}, \mathcal{A})^c$ and is defined by $(\mathcal{F}, \mathcal{A})^c = (\mathcal{F}^c, \neg\mathcal{A})$, where $\mathcal{F}^c : \neg\mathcal{A} \to CM^k(U)$ is a mapping given by

$$\mathcal{F}^c(e) = \{\langle x, \mu^j_{\mathcal{F}^c(e)}(x) = r^j_{\mathcal{F}^c(e)}(x).e^{i\omega^j_{\mathcal{F}^c(e)}(x)}\rangle : e \in \neg\mathcal{A}, x \in U, j = 1, 2, \ldots, k\}.$$

where the complement of the amplitude term is $r^j_{\mathcal{F}^c(e)}(x) = 1 - r^j_{\mathcal{F}(e)}(x)$ and the complement of the phase term is $\omega^j_{\mathcal{F}^c(e)}(x) = 2\pi - i\omega^j_{\mathcal{F}(e)}(x)$.

Example 17: Consider the approximation given in Example 11, where

$$\begin{aligned}\mathcal{F}(e_1) = \{&x_1/(0.9e^{i2\pi(2/4)}, 0.2e^{i2\pi(4/4)}, 0.5e^{i2\pi(1/4)}),\\ &x_2/(0.2e^{i2\pi(1/4)}, 0.7e^{i2\pi(3/4)}, 0.7e^{i2\pi(2/4)})\\ &x_3/(0.7e^{i2\pi(3/4)}, 0.5e^{i2\pi(4/4)}, 0.1e^{i2\pi(1/4)})\}.\end{aligned}$$

By using the complex multi-fuzzy complement, we obtain the complement of the approximation given by

$$\begin{aligned}\mathcal{F}^c(e_1) = \{&x_1/(0.1e^{i2\pi(2/4)}, 0.8e^{i2\pi(0/4)}, 0.5e^{i2\pi(3/4)}),\\ &x_2/(0.8e^{i2\pi(3/4)}, 0.3e^{i2\pi(1/4)}, 0.3e^{i2\pi(2/4)}),\\ &x_3/(0.3e^{i2\pi(3/4)}, 0.5e^{i2\pi(0)}, 0.9e^{i2\pi(3/4)})\}.\end{aligned}$$

Proposition 18: If $(\mathcal{F}, \mathcal{A})$ is a $\mathcal{CM}^k\mathcal{FSS}s$ over U, then

1) $((\mathcal{F}, \mathcal{A})^c)^c = (\mathcal{F}, \mathcal{A})$,
2) $((\mathcal{F}, \mathcal{A})_{\tilde{\phi}_k})^c = (\mathcal{F}, \mathcal{A})_{\tilde{\text{U}}_k}$, *where* $(\tilde{F}, \mathcal{A})_{\tilde{\phi}_k}$ *and* $(\mathcal{F}, \mathcal{A})_{\tilde{\text{U}}_k}$ *are the null and the absolute complex multi-fuzzy soft sets, respectively.*
3) $((\mathcal{F}, \mathcal{A})_{\tilde{\text{U}}_k})^c = (\mathcal{F}, \mathcal{A})_{\tilde{\phi}_k}$.

Proof: We will provide the proof of assertion 1 since the proof of assertions 2 and 3 are straightforward from Definitions 14, 15 and 16. Suppose that $(\mathcal{F}, \mathcal{A})$ is a complex multi-fuzzy soft set of dimension k over U. The complement $(\mathcal{F}, \mathcal{A})$, denoted by $(\mathcal{F}, \mathcal{A})^c = (\mathcal{F}^c, \neg\mathcal{A})$ is defined as:

$$\begin{aligned}(\mathcal{F}, \mathcal{A})^c &= \{\langle e, r^j_{\mathcal{F}^c(e)}(x).e^{i\omega^j_{\mathcal{F}^c(e)}(x)}\rangle : e \in \neg\mathcal{A},\\ &\qquad x \in U, j = 1, 2, \ldots, k\},\\ &= \{\langle e, [1 - r^j_{\mathcal{F}(e)}(x)].e^{i[2\pi - \omega^j_{\mathcal{F}(e)}(x)]}\rangle :\\ &\qquad e \in \neg\mathcal{A}, x \in U, j = 1, 2, \ldots, k\},\end{aligned}$$

Now let $(\mathcal{F}, \mathcal{A})^c = (\mathcal{G}, B) = (\mathcal{F}^c, \neg\mathcal{A})$. Then we obtain the following:

$$\begin{aligned}(\mathcal{G}, \mathcal{B})^c &= \{\langle e, [1 - r^j_{\mathcal{F}^c(e)}(x)].e^{i[2\pi - \omega^j_{\mathcal{F}^c(e)}(x)]}\rangle :\\ &\qquad e \in \neg(\neg\mathcal{A}), x \in U, j = 1, 2, \ldots, k\},\\ &= \{\langle e, [1 - (1 - r^j_{\mathcal{F}(e)}(x))].e^{i[2\pi - (2\pi - \omega^j_{\mathcal{F}(e)}(x))]}\rangle :\\ &\qquad e \in \neg(\neg\mathcal{A}), x \in U, j = 1, 2, \ldots, k\},\\ &= \{\langle e, r^j_{\mathcal{F}(e)}(x).e^{i\omega^j_{\mathcal{F}(e)}(x)}\rangle : e \in \mathcal{A}, x \in U,\\ &\qquad j = 1, 2, \ldots, k\},\\ &= (\mathcal{F}, \mathcal{A}).\end{aligned}$$

B. UNION AND INTERSECTION OF COMPLEX MULTI-FUZZY SOFT SETS

In this part, we will now introduce the definitions of union and intersection operations of two CMFSSs.

Definition 19: The union of two $\mathcal{CM}^k\mathcal{FSS}s$ $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ over U, denoted by $(\mathcal{F}, \mathcal{A})\tilde{\cup}(\mathcal{G}, \mathcal{B})$, is a complex multi-fuzzy soft set $(\mathcal{H}, C)$, where $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$, $\forall e \in \mathcal{C}$ and $x \in U$,

$$\mathcal{H}(e) = \begin{cases}\mathcal{F}(e) = [r^j_{\mathcal{F}(e)}(x)e^{i\omega^j_{\mathcal{F}(e)}(x)}]_{j\in k} & \text{if } e \in \mathcal{A} - \mathcal{B},\\ \mathcal{G}(e) = [r^j_{\mathcal{G}(e)}(x)e^{i\omega^j_{\mathcal{G}(e)}(x)}]_{j\in k} & \text{if } e \in \mathcal{B} - \mathcal{A},\\ \mathcal{F}(e) \sqcup \mathcal{G}(e) = [(r^j_{\mathcal{F}(e)}(x) \vee r^j_{\mathcal{G}(e)}(x)) & \\ \qquad \cdot\, e^{i[\omega^j_{\mathcal{F}(e)}(x) \cup \omega^j_{\mathcal{G}(e)}(x)]}]_{j\in k} & \text{if } e \in \mathcal{A} \cap \mathcal{B}.\end{cases}$$

We write $(\mathcal{H}, \mathcal{C}) = (\mathcal{H}, \mathcal{A})\tilde{\cup}(\mathcal{H}, \mathcal{B})$, where $\vee$ denotes the max operator, whereas the phase term $(e^{\omega^j_{\mathcal{F}(e) \cup\, \mathcal{G}(e)}(x)})_{j\in k}$ of the functions lie in the interval $[0, 2\pi]$ and can be calculated using any one of the following operators:

1) Sum: $\omega^j_{\mathcal{F}(e)\cup\,\mathcal{G}(e)}(x) = \omega^j_{\mathcal{F}(e)}(x) + \omega^j_{\mathcal{G}(e)}(x)$, for all $j = 1, 2, \ldots, k$.
2) Max: $\omega^j_{\mathcal{F}(e)\cup\,\mathcal{G}(e)}(x) = \max(\omega^j_{\mathcal{F}(e)}(x), \omega^j_{\mathcal{G}(e)}(x))$, for all $j = 1, 2, \ldots, k$.
3) Min: $\omega^j_{\mathcal{F}(e)\cup\,\mathcal{G}(e)}(x) = \min(\omega^j_{\mathcal{F}(e)}(x), \omega^j_{\mathcal{G}(e)}(x))$, for all $j = 1, 2, \ldots, k$.
4) "Winner Takes All":
$$\omega^j_{\mathcal{F}(e)\cup\,\mathcal{G}(e)}(x) = \begin{cases} \omega^j_{\mathcal{F}(e)}(x) & r^j_{\mathcal{F}(e)} > r^j_{\mathcal{G}(e)} \\ \omega^j_{\mathcal{G}(e)}(x) & r^j_{\mathcal{F}(e)} < r^j_{\mathcal{G}(e)} \end{cases}$$
for all $j = 1, 2, \ldots, k$.

Definition 20: The intersection of two $\mathcal{CM}^k\mathcal{FSS}$s $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ over U, denoted by $(\mathcal{F}, \mathcal{A})\tilde{\cap}(\mathcal{G}, \mathcal{B})$, is a complex multi-fuzzy soft set $(\mathcal{H}, \mathcal{C})$, where $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$, $\forall e \in \mathcal{C}$ and $x \in U$,
$\mathcal{H}(e) =$
$$\begin{cases} \mathcal{F}(e) = [r^j_{\mathcal{F}(e)}(x)e^{i\omega^j_{\mathcal{F}(e)}(x)}]_{j\in k} & \text{if } e \in \mathcal{A} - \mathcal{B}, \\ \mathcal{G}(e) = [r^j_{\mathcal{G}(e)}(x)e^{i\omega^j_{\mathcal{G}(e)}(x)}]_{j\in k} & \text{if } e \in \mathcal{B} - \mathcal{A}, \\ \mathcal{F}(e) \sqcap \mathcal{G}(e) = [(r^j_{\mathcal{F}(e)}(x) \wedge r^j_{\mathcal{G}(e)}(x)) & \\ \quad \cdot\, e^{i[\omega^j_{\mathcal{F}(e)}(x)\cap\omega^j_{\mathcal{G}(e)}(x)]}]_{j\in k} & \text{if } e \in \mathcal{A} \cap \mathcal{B}. \end{cases}$$

We write $(\mathcal{H}, \mathcal{C}) = (\mathcal{H}, \mathcal{A})\tilde{\cap}(\mathcal{H}, \mathcal{B})$, where $\wedge$ denotes the max operator, whereas the phase term $(e^{\omega^j_{\mathcal{F}(e)\cap\,\mathcal{G}(e)}(x)})_{j\in k}$ of the function lie in the interval $[0, 2\pi]$ and can be calculated using any one of the operators given in Deffinition 19.

We will now give some theorems on the union, intersection and complement of CMFSSs. These theorems illustrate the relationship between the set theoretic operations that have been mentioned above.

Theorem 21: Let $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ be two $\mathcal{CM}^k\mathcal{FSS}$s over U. Then the following properties hold true.
1) $(\mathcal{F}, \mathcal{A})\tilde{\cup}(\mathcal{F}, \mathcal{A})_{\tilde{\phi}_k} = (\mathcal{F}, \mathcal{A})$.
2) $(\mathcal{F}, \mathcal{A})\tilde{\cap}(\mathcal{F}, \mathcal{A})_{\tilde{\phi}_k} = (\mathcal{F}, \mathcal{A})_{\tilde{\phi}_k}$.
3) $(\mathcal{F}, \mathcal{A})\tilde{\cup}(\mathcal{F}, \mathcal{A})_{\tilde{U}_k} = (\mathcal{F}, \mathcal{A})_{\tilde{U}_k}$,
4) $(\mathcal{F}, \mathcal{A})\tilde{\cap}(\mathcal{F}, \mathcal{A})_{\tilde{U}_k} = (\mathcal{F}, \mathcal{A})$,

Proof: The proofs are straightforward by using Definitions 19 and 20.

Theorem 22: Let $(\mathcal{F}, \mathcal{A})$, $(\mathcal{G}, \mathcal{B})$ and $(\mathcal{Q}, \mathcal{D})$ be three $\mathcal{CM}^k\mathcal{FSS}$s over U of dimension k. Then the following associative laws hold true.
1) $(\mathcal{F}, \mathcal{A})\tilde{\cup}((\mathcal{G}, \mathcal{B})\tilde{\cup}(\mathcal{Q}, \mathcal{D})) = ((\mathcal{F}, \mathcal{A})\tilde{\cup}((\mathcal{G}, \mathcal{B})) \cup ((\mathcal{Q}, \mathcal{D})$,
2) $(\mathcal{F}, \mathcal{A})\tilde{\cap}((\mathcal{G}, \mathcal{B})\tilde{\cap}(\mathcal{Q}, \mathcal{D})) = ((\mathcal{F}, \mathcal{A})\tilde{\cap}((\mathcal{G}, \mathcal{B})) \cap ((\mathcal{Q}, \mathcal{D})$,

Proof: 1. Assume that $((\mathcal{G}, \mathcal{B})\tilde{\cup}(\mathcal{Q}, \mathcal{D})) = (\mathcal{M}, \mathcal{N})$ where $\mathcal{N} = \mathcal{B} \cup \mathcal{D}$. By Definition 4.2.1 we have $((\mathcal{G}, \mathcal{B})\tilde{\cup}(\mathcal{Q}, \mathcal{D}))$ to be a $\mathcal{CM}^k\mathcal{FSES}$ $(\mathcal{M}, \mathcal{N})$, where $\mathcal{N} = \mathcal{B} \cup \mathcal{D}$ and $\forall e \in \mathcal{N}$, such that
$$\begin{aligned}\mathcal{M}(e) &= \mathcal{G}(e) \sqcup \mathcal{Q}(e) \\ &= \left[(r^j_{\mathcal{G}(e)}(x) \vee r^j_{\mathcal{Q}(e)}(x)) \cdot e^{i[\omega^j_{\mathcal{G}(e)}(x)\vee\,\omega^j_{\mathcal{Q}(e)}(x)]}\right]_{j\in k},\end{aligned}$$

Suppose that $(\mathcal{Y}, \mathcal{R}) = ((\mathcal{F}, \mathcal{A})\tilde{\cup}(\mathcal{M}, \mathcal{N}))$ where $\mathcal{Y} = \mathcal{A} \cup \mathcal{N}$. By Definition 19, then we have $((\mathcal{F}, \mathcal{A})\tilde{\cup}(\mathcal{M}, \mathcal{N}))$ to be a $\mathcal{CM}^k\mathcal{FSES}$ $(\mathcal{Y}, \mathcal{R})$, where $\mathcal{R} = \mathcal{A} \cup \mathcal{N}$ and $\forall e \in \mathcal{R}$, such that
$$\mathcal{Y}(e) = \begin{cases} \mathcal{F}(e) = [r^j_{\mathcal{F}(e)}(x)e^{i\omega^j_{\mathcal{F}(e)}(x)}]_{j\in k} & \text{if } e\in\mathcal{A}-\mathcal{P}, \\ \mathcal{M}(e) = [r^j_{\mathcal{M}(e)}(x)e^{i\omega^j_{\mathcal{M}(e)}(x)}]_{j\in k} & \text{if } e\in\mathcal{P}-\mathcal{A}, \\ \mathcal{F}(e)\sqcup\mathcal{M}(e) = [(r^j_{\mathcal{F}(e)}(x) \vee r^j_{\mathcal{M}(e)}(x)) & \\ \quad \cdot\, e^{i[\omega^j_{\mathcal{F}(e)}(x)\vee\,\omega^j_{\mathcal{M}(e)}(x)]}]_{j\in k} & \text{if } e\in\mathcal{A}\cap\mathcal{P}. \end{cases}$$

Now let $(\mathcal{F}, \mathcal{A})\tilde{\cup}((\mathcal{G}, \mathcal{B})\tilde{\cap}(\mathcal{Q}, \mathcal{D})) = ((\mathcal{F}, \mathcal{A})\tilde{\cup}(\mathcal{M}, \mathcal{N}))$. We consider the case when $e \in \mathcal{A} \cap \mathcal{N}$ as the other cases are trivial. Hence,
$$\begin{aligned}&(\mathcal{F}, \mathcal{A})\tilde{\cup}((\mathcal{G}, \mathcal{B})\tilde{\cap}(\mathcal{Q}, \mathcal{D})) \\ &\quad= (\mathcal{F}, \mathcal{A})\tilde{\cup}(\mathcal{M}, \mathcal{N}) \\ &\quad= \mathcal{F}(e) \sqcup \mathcal{M}(e) \\ &\quad= \mathcal{F}(e) \sqcup (\mathcal{G}(e) \sqcup \mathcal{Q}(e)) \\ &\quad= \Big[\big(r^j_{\mathcal{F}(e)}(x) \vee r^j_{\mathcal{G}(e)\sqcup\mathcal{Q}(e)}(x)\big) \\ &\qquad\quad \cdot\, e^{i\left(\omega^j_{\mathcal{F}(e)}(x)\vee\,\omega^j_{\mathcal{G}(e)\sqcup\mathcal{Q}(e)}(x)\right)}\Big]_{j\in k} \\ &\quad= \Big[\big(r^j_{\mathcal{F}(e)}(x) \vee [r^j_{\mathcal{G}(e)}(x) \vee r^j_{\mathcal{Q}(e)}(x)]\big) \\ &\qquad\quad \cdot\, e^{i\left(\omega^j_{\mathcal{F}(e)}(x)\vee[\omega^j_{\mathcal{G}(e)}(x)\vee\,\omega^j_{\mathcal{Q}(e)}(x)]\right)}\Big]_{j\in k} \\ &\quad= \Big[\big([r^j_{\mathcal{F}(e)}(x) \vee r^j_{\mathcal{G}(e)}(x)] \vee r^j_{\mathcal{Q}(e)}(x)\big) \\ &\qquad\quad \cdot\, e^{i\left([\omega^j_{\mathcal{F}(e)}(x)\vee\,\omega^j_{\mathcal{G}(e)}(x)]\vee\omega^j_{\mathcal{F}(e)}(x)\right)}\Big]_{j\in k} \\ &\quad= \Big[\big(r^j_{\mathcal{F}(e)\sqcup\mathcal{G}(e)}(x) \vee r^j_{\mathcal{Q}(e)}(x)\big) \\ &\qquad\quad \cdot\, e^{i\left(\omega^j_{\mathcal{F}(e)\sqcup\mathcal{G}(e)}(x),\omega^j_{\mathcal{F}(e)\sqcup\mathcal{Q}(e)}(x)\right)}\Big]_{j\in k} \\ &\quad= (\mathcal{F}(e) \sqcup \mathcal{G}(e)) \sqcup \mathcal{Q}(e) \\ &\quad= ((\mathcal{F}, \mathcal{A})\tilde{\cup}(\mathcal{G}, \mathcal{B}))\tilde{\cup}(\mathcal{Q}, \mathcal{D})\end{aligned}$$

Therefore, we have
$$(\mathcal{F}, \mathcal{A})\tilde{\cup}((\mathcal{G}, \mathcal{B})\tilde{\cap}(\mathcal{Q}, \mathcal{D})) = ((\mathcal{F}, \mathcal{A})\tilde{\cup}(\mathcal{G}, \mathcal{B}))\tilde{\cup}(\mathcal{Q}, \mathcal{D}).$$

2. The proof is similar to that in part (1) and therefore is omitted.

Theorem 23: Let $(\mathcal{F}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ be two $\mathcal{CM}^k\mathcal{FSS}$s over U. Then the following De Morgan's law holds true.
1) $((\mathcal{F}, \mathcal{A})\tilde{\cup}(\mathcal{G}, \mathcal{B}))^c = (\mathcal{F}, \mathcal{A})^c\tilde{\cap}(\mathcal{G}, \mathcal{B})^c$,
2) $((\mathcal{F}, \mathcal{A})\tilde{\cap}(\mathcal{G}, \mathcal{B}))^c = (\mathcal{F}, \mathcal{A})^c\tilde{\cup}(\mathcal{G}, \mathcal{B})^c$.

Proof: Suppose that $(\mathcal{F}, \mathcal{A})\tilde{\cup}(\mathcal{G}, \mathcal{B}) = (\mathcal{H}, \mathcal{C})$, where $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ and $\forall e \in \mathcal{C}$,
$$\mathcal{H}(e) = \begin{cases} \mathcal{F}(e) = [r^j_{\mathcal{F}(e)}(x)e^{i\omega^j_{\mathcal{F}(e)}(x)}]_{j\in k} & \text{if } e \in \mathcal{A} - \mathcal{B}, \\ \mathcal{G}(e) = [r^j_{\mathcal{G}(e)}(x)e^{i\omega^j_{\mathcal{G}(e)}(x)}]_{j\in k} & \text{if } e \in \mathcal{B} - \mathcal{A}, \\ \mathcal{F}(e) \sqcup \mathcal{G}(e) = [(r^j_{\mathcal{F}(e)}(x) \vee r^j_{\mathcal{G}(e)}(x)) & \\ \quad \cdot\, e^{i[\omega^j_{\mathcal{F}(e)}(x)\cup\omega^j_{\mathcal{G}(e)}(x)]}]_{j\in k} & \text{if } e \in \mathcal{A} \cap \mathcal{B}. \end{cases}$$

Since $(\mathcal{F},\mathcal{A})\tilde{\cup}(\mathcal{G},\mathcal{B}) = (\mathcal{H},\mathcal{C})$, then we have $((\mathcal{F},\mathcal{A})\tilde{\cup}(\mathcal{G},\mathcal{B}))^c = (\mathcal{H},\mathcal{C})^c = (\mathcal{H}^c,\mathcal{C})$. Hence $\forall e \in C$,

$$\mathcal{H}^c(e) = \begin{cases} \mathcal{F}^c(e) = [r^j_{\mathcal{F}^c(e)}(x)e^{i\omega^j_{\mathcal{F}^c(e)}(x)}]_{j\in k} & \text{if } e\in\mathcal{A}-\mathcal{B}, \\ \mathcal{G}^c(e) = [r^j_{\mathcal{G}^c(e)}(x)e^{i\omega^j_{\mathcal{G}^c(e)}(x)}]_{j\in k} & \text{if } e\in\mathcal{B}-\mathcal{A}, \\ \mathcal{F}^c(e)\sqcap\mathcal{G}^c(e) = [(r^j_{\mathcal{F}^c(e)}(x)\wedge r^j_{\mathcal{G}^c(e)}(x)) & \\ \qquad\cdot e^{i[\omega^j_{\mathcal{F}^c(e)}(x)\cap\omega^j_{\mathcal{G}^c(e)}(x)]}]_{j\in k} & \text{if } e\in\mathcal{A}\cap\mathcal{B}. \end{cases}$$

Since $(\mathcal{F},\mathcal{A})^c = (\mathcal{F}^c,\mathcal{A})$ and $(\mathcal{G},\mathcal{B})^c = (\mathcal{G}^c,\mathcal{B})$ then we have $(\mathcal{F},\mathcal{A})^c\tilde{\cap}(\mathcal{G},\mathcal{B})^c = (\mathcal{F}^c,\mathcal{A})\tilde{\cap}(\mathcal{G}^c,\mathcal{B})$. Suppose that $(\mathcal{F}^c,\mathcal{A})\tilde{\cap}(\mathcal{G}^c,\mathcal{B}) = (\mathcal{T},\mathcal{J})$, where $\mathcal{J} = \mathcal{A}\cup\mathcal{B}$. Hence $\forall e \in \mathcal{J}$,

$$\mathcal{T}(e) = \begin{cases} \mathcal{F}^c(e) = [r^j_{\mathcal{F}^c(e)}(x)e^{i\omega^j_{\mathcal{F}^c(e)}(x)}]_{j\in k} & \text{if } e\in\mathcal{A}-\mathcal{B}, \\ \mathcal{G}^c(e) = [r^j_{\mathcal{G}^c(e)}(x)e^{i\omega^j_{\mathcal{G}^c(e)}(x)}]_{j\in k} & \text{if } e\in\mathcal{B}-\mathcal{A}, \\ \mathcal{F}^c(e)\sqcap\mathcal{G}^c(e) = [(r^j_{\mathcal{F}^c(e)}(x)\wedge r^j_{\mathcal{G}^c(e)}(x)) & \\ \qquad\cdot e^{i[\omega^j_{\mathcal{F}^c(e)}(x)\cap\omega^j_{\mathcal{G}^c(e)}(x)]}]_{j\in k} & \text{if } e\in\mathcal{A}\cap\mathcal{B}. \end{cases}$$

Therefore, $(\mathcal{H}^c,\mathcal{C})$ and $(\mathcal{T},\mathcal{J})$ are the same operators, for all $e \in \mathcal{C}(\mathcal{J})$, which implies that $((\mathcal{F},\mathcal{A})\tilde{\cup}(\mathcal{G},\mathcal{B}))^c = (\mathcal{F},\mathcal{A})^c\tilde{\cap}(\mathcal{G},\mathcal{B})^c$ and this completes the proof.

(2) The proof is similar to that in part (1) and therefore is omitted.

V. AN APPLICATION OF COMPLEX MULTI-FUZZY SOFT SET IN DECISION MAKING

Like most of the decision making problems, complex multi-fuzzy soft set based decision making involves the evaluation of all the objects which are decision alternatives. In general, there actually does not exist a unique criterion for the evaluation of decision alternatives. Thus sometime it is not very efficient to select the optimal object from the considered alternatives. In this section, we will further modify the algorithm of [12] and [15] to solve a decision making problem which is based on the concept of the CMFSS.

Let $U = \{x_1, x_2, \ldots, x_n\}$ and $(\mathcal{F},\mathcal{A})$ be a $\mathcal{CM}^k\mathcal{FSS}$ over U. For each $e \in \mathcal{A}$, $\mathcal{F}(e) = \{\frac{(r^j_{\mathcal{F}(e)}(x_1).e^{i\omega^j_{\mathcal{F}(x_1)}(x_1)})_{j\in k}}{x_1}, \ldots, \frac{(r^j_{\mathcal{F}(e)}(x_n).e^{i\omega^j_{\mathcal{F}(e)}(x_n)})_{j\in k}}{x_n}\}$. $\mathcal{F}(e)$ can be expressed in matrix form as follows

$$\mathcal{F}(e) = \begin{pmatrix} r^1_{\mathcal{F}(e)}(x_1).e^{i\omega^1_{\mathcal{F}(e)}(x_1)} & \ldots & r^k_{\mathcal{F}(e)}(x_1).e^{i\omega^k_{\mathcal{F}(e)}(x_1)} \\ r^1_{\mathcal{F}(e)}(x_2).e^{i\omega^1_{\mathcal{F}(e)}(x_2)} & \ldots & r^k_{\mathcal{F}(e)}(x_2).e^{i\omega^k_{\mathcal{F}(e)}(x_2)} \\ \vdots & \vdots & \vdots \\ r^1_{\mathcal{F}(e)}(x_n).e^{i\omega^1_{\mathcal{F}(e)}(x_n)} & \ldots & r^k_{\mathcal{F}(e)}(x_n).e^{i\omega^k_{\mathcal{F}(e)}(x_n)} \end{pmatrix}$$

Then we convert the above $\mathcal{CM}^k\mathcal{FSS}$ $(\mathcal{F},\mathcal{A})$ into multi-fuzzy soft set $(F,\mathcal{A})$, where

$$F(e) = \begin{pmatrix} \mu^1_{F(e)}(x_1) & \ldots & \mu^k_{F(e)}(x_1) \\ \mu^1_{F(e)}(x_2) & \ldots & \mu^k_{F(e)}(x_2) \\ \vdots & \vdots & \vdots \\ \mu^1_{F(e)}(x_n) & \ldots & \mu^k_{F(e)}(x_n) \end{pmatrix}$$

Suppose $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_k)^T$,$(\sum_{j=1}^k \varpi_j = 1)$ is the relative weight of parameter e. We define an induced fuzzy set $\mu_{F(e)}$ with respect to e in U as follows.

$$\mu_F(e) = \begin{pmatrix} \mu^1_{F(e)}(x_1) & \ldots & \mu^k_{F(e)}(x_1) \\ \mu^1_{F(e)}(x_2) & \ldots & \mu^k_{F(e)}(x_2) \\ \vdots & \vdots & \vdots \\ \mu^1_{F(e)}(x_n) & \ldots & \mu^k_{F(e)}(x_n) \end{pmatrix} \begin{pmatrix} \varpi_1 \\ \vdots \\ \varpi_k \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n \varpi_j\mu^j_{F(e)}(x_1) \\ \sum_{j=1}^n \varpi_j\mu^j_{F(e)}(x_2) \\ \vdots \\ \sum_{j=1}^n \varpi_j\mu^j_{F(e)}(x_n) \end{pmatrix}$$

Thus, if $\varpi(e)$ is given, we can change a multi-fuzzy soft set $F(e)$ into an induced fuzzy soft set $\mu_F(e)$. Hence, by using this method, we can change a multi-fuzzy soft set to an induced fuzzy soft set. Once the induced fuzzy soft set of a multi-fuzzy soft set has been arrived at, it may be necessary to choose the best alternative from the alternatives based on Feng's algorithm [12]. Henceforth, we present an algorithm to select an optimal decision.

Algorithm

Step 1. Input the $\mathcal{CM}^k\mathcal{FSS}$ $(\mathcal{F},\mathcal{A})$.

Step 2. Convert the $\mathcal{CM}^k\mathcal{FSS}$ $(\mathcal{F},\mathcal{A})$ to the multi-fuzzy soft set $(F,\mathcal{A})$ by obtaining the weighted aggregation values of $\mu^j_{F(e)}(x)$,$\forall e \in \mathcal{A}$,$\forall x \in U$ and $j = 1, 2, \ldots, k$ as in the following equation.

$$\mu^j_{F(e)}(x) = \nu_1 r^j_{\mathcal{F}(e)}(x) + \nu_2(1/2\pi)\omega^j_{\mathcal{F}(e)}(x)$$

where $r^j_{\mathcal{F}(e)}(x)$ and $\omega^j_{\mathcal{F}(e)}(x)$ (for $j = 1, 2, \ldots, k$) are the amplitude and phase terms in the $\mathcal{CM}^k\mathcal{FSS}$ $(\mathcal{F},\mathcal{A})$, respectively. $\mu^j_{F(e)}(x)$ is the multi-membership function in the multi-fuzzy soft set $(F,\mathcal{A})$ and ν_1, ν_2 are the weights for the amplitude terms (degrees of influence) and the phase terms (times of influence), respectively, where ν_1 and $\nu_2 \in [0, 1]$ and $\nu_1 + \nu_2 = 1$.

Step 3. Change the multi-fuzzy soft set $(F,\mathcal{A})$ into the normalized multi-fuzzy soft set, that is, if there exists some $x \in U$ and $e \in \mathcal{A}$ such that $\sum_{j=1}^k \mu^j_{F(e)}(x) = l > 1$, then we change $(\mu^1_{F(e)}(x), \mu^2_{F(e)}(x), \ldots, \mu^k_{F(e)}(x))$ as $\frac{1}{l}(\mu^1_{F(e)}(x), \mu^2_{F(e)}(x), \ldots, \mu^k_{F(e)}(x))$.

Step 4. Input the relative weight $\varpi(e_j)$ of every parameter $e_j \in E$. Compute the induced fuzzy soft set $\Delta_F = (\hat{F}, \mathcal{A})$.

Step 5. Input a threshold fuzzy sets $\lambda \longrightarrow [0, 1]$ (or give two threshold values $t \in [0, 1]$; or choose the mid-level decision rule; or choose the top-level decision rule) for decision making.

Step 6. Compute the level soft set $L(\Delta_F; \lambda)$ of Δ_F with respect to the threshold fuzzy set λ (or the t-level soft set $L(\Delta_F; t)$; or the mid-level soft set $L(\Delta_F; \text{mid})$; or the top-level soft set $L(\Delta_F; \max)$).

Step 7. Present the level soft set $L(\Delta_F; \lambda)$ (or $L(\Delta_F; t)$); or $L(\Delta_F;$ mid); or $L(\Delta_F; \max)$ in tabular forms and compute the choice values c_j of $x_j, \forall j$.

Step 8. The optimal decision is to select x_ℓ such that $c_\ell = \max_j c_j$

Step 9. If ℓ has more than one value, then any one of x_ℓ may be an optimal decision.

We illustrate the proposed algorithm by the following example.

Example 24: The sectors that are influencing positively the financial situation in Malaysia are tourism and industrial sectors. Both sectors are affected by some economic factors. For example, suppose that we are interested in understanding the most important economic factors (indicators) that affect those sectors. In this case, we will consider four factors represented in the following universal set $U = \{x_1, x_2, x_3, x_4\}$, where x_1 = the role of promotion and advertising, x_2 = goods and services tax (GST), x_3 = the plunge in commodity and oil prices and x_4 = the exchange rate variability. The problem is to arrange these four factors in descending order from the most significant to the least one. Suppose $\mathcal{A} = \{e_1, e_2\}$ is a set of parameters that represents the major sectors of the Malaysian economy, where e_1 stands for "industry sector"t, which includes the following industries; automotive industry, pharmaceutical industry, food industry and oil refining industry; e_2 stands for "tourism sector", which includes four types of tourism: religious tourism, medical tourism, leisure tourism and cultural tourism. Now the team of economic analysts is requested to make a decision about the four factors, through determining the degree and the overall time of the impact of these factors on the both sectors, in order to construct a complex multi-fuzzy soft set of dimension four as follows:

$$
\begin{aligned}
(\mathcal{F}, \mathcal{A}) &= \Big\{ \mathcal{F}(e_1) \\
&= \Big\{ \frac{(0.9e^{i2\pi(\frac{3}{12})}, 0.2e^{i2\pi(\frac{8}{12})}, 0.8e^{i2\pi(\frac{2}{12})}, 0.5e^{i2\pi(\frac{5}{12})})}{x_1}, \\
&\quad \frac{(0.8e^{i2\pi(\frac{2}{12})}, 0.4e^{i2\pi(\frac{6}{12})}, 0.3e^{i2\pi(\frac{1}{12})}, 0.5e^{i2\pi(\frac{5}{12})})}{x_2}, \\
&\quad \frac{(0.8e^{i2\pi(\frac{4}{12})}, 0.4e^{i2\pi(\frac{3}{12})}, 0.5e^{i2\pi(\frac{1}{12})}, 0.8e^{i2\pi(\frac{4}{12})})}{x_3}, \\
&\quad \frac{(0.1e^{i2\pi(\frac{6}{12})}, 0.8e^{i2\pi(\frac{3}{12})}, 0.2e^{i2\pi(\frac{10}{12})}, 0.5e^{i2\pi(\frac{9}{12})})}{x_4} \Big\}, \\
\mathcal{F}(e_2) &= \Big\{ \frac{(0.9e^{i2\pi(\frac{5}{12})}, 0.6e^{i2\pi(\frac{4}{12})}, 0.2e^{i2\pi(\frac{11}{12})}, 0.2e^{i2\pi(\frac{1}{12})})}{x_1}, \\
&\quad \frac{(0.6e^{i2\pi(\frac{6}{12})}, 0.5e^{i2\pi(\frac{3}{12})}, 0.3e^{i2\pi(\frac{12}{12})}, 0.8e^{i2\pi(\frac{8}{12})})}{x_2}, \\
&\quad \frac{(0.9e^{i2\pi(\frac{2}{12})}, 0.4e^{i2\pi(\frac{3}{12})}, 0.1e^{i2\pi(\frac{2}{12})}, 0.7e^{i2\pi(\frac{8}{12})})}{x_3}, \\
&\quad \frac{(0.4e^{i2\pi(\frac{7}{12})}, 0.3e^{i2\pi(\frac{5}{12})}, 0.2e^{i2\pi(\frac{1}{12})}, 0.5e^{i2\pi(\frac{5}{12})})}{x_4} \Big\} \Big\}.
\end{aligned}
$$

In our example, the amplitude terms of the membership values represent the degree of influence of the mentioned factors on those sectors, whereas the phase term represents the phase of this influence or the period of this influence. Both of the amplitude and phase terms lie in [0, 1]. An amplitude term with value close to 0 (1) implies that the above mentioned factors has a very little (strong) influence on those sectors and a phase term with value close to 0 (1) implies that the above mentioned factors take a very short (long) time to influence.

In the following discussion, we will describe an example of possible scenarios that might occur in this context, instead of providing the complete set for $(\mathcal{F}, \mathcal{A})$. For example, let us consider the approximation $\mathcal{F}(e_1) = \Big\{ \frac{(0.9e^{i2\pi(\frac{3}{12})}, 0.2e^{i2\pi(\frac{8}{12})}, 0.8e^{i2\pi(\frac{2}{12})}, 0.5e^{i2\pi(\frac{5}{12})})}{x_1}, \ldots \Big\}$, the term $\frac{(0.9e^{i2\pi(\frac{3}{12})}, 0.2e^{i2\pi(\frac{8}{12})}, 0.8e^{i2\pi(\frac{2}{12})}, 0.5e^{i2\pi(\frac{5}{12})})}{x_1}$ under the parameter (e_1) for the first economic factor (x_1) represents the influence of the role of advertising and promotion on the industry sector, where the first membership value $0.9e^{i2\pi(\frac{3}{12})}$ indicates that there is a high influence of the role of advertising and promotion on automotive industry, since the amplitude term 0.9 is very close to one and this influence needs 3 months, which is considered as a short time. While the second membership value $0.2e^{i2\pi(\frac{8}{12})}$ indicates that there is a low influence of the role of advertising and promotion on pharmaceutical industry with degree 0.2. The time required for this effect is about 11 months, which is very long time. All the other expressions in the rest of the example can be interpreted in a similar manner.

Now, convert the complex multi-fuzzy soft set $(\mathcal{F}, \mathcal{A})$ to multi-fuzzy soft set $(F, \mathcal{A})$. To implement this step, we assume that the weight for the amplitude term is $\nu_1 = 0.4$ and the weight for the phase term is $\nu_2 = 0.6$ to obtain the weighted aggregation values of $\mu^j_{F(e)}(x), \forall e \in \mathcal{A}, \forall x \in U$ and $j = 1, 2, 3, 4$. We calculate $\mu^j_{F(e)}(x)$, when $e = e_1$ and $x = x_1$ as shown below.

$$
\begin{aligned}
\mu^1_{F(e_1)}(x_1) &= \nu_1 r^1_{\mathcal{F}(e_1)}(x_1) + \nu_2(1/2\pi)\omega^1_{\mathcal{F}(e_1)}(x_1) \\
&= (0.4)(0.9) + (0.6)(1/2\pi)(2\pi)(3/12) \\
&= 0.51 \\
\mu^2_{F(e_1)}(x_1) &= \nu_1 r^2_{\mathcal{F}(e_1)}(x_1) + \nu_2(1/2\pi)\omega^2_{\mathcal{F}(e_1)}(x_1) \\
&= (0.4)(0.2) + (0.6)(1/2\pi)(2\pi)(8/12) \\
&= 0.48
\end{aligned}
$$

TABLE 1. Values of $(F, \mathcal{A})$.

U	e_1	e_2
x_1	$(0.51, 0.48, 0.42, 0.45)$	$(0.61, 0.44, 0.63, 0.13)$
x_2	$(0.42, 0.46, 0.17, 0.45)$	$(0.54, 0.35, 0.72, 0.72)$
x_3	$(0.52, 0.31, 0.25, 0.57)$	$(0.46, 0.31, 0.14, 0.68)$
x_4	$(0.34, 0.47, 0.28, 0.65)$	$(0.51, 0.37, 0.13, 0.45)$

TABLE 2. Tabular representation of the normalized multi-fuzzy soft set.

U	e_1	e_2
x_1	$(0.274, 0.26, 0.23, 0.241)$	$(0.335, 0.243, 0.348, 0.072)$
x_2	$(0.28, 0.307, 0.113, 0.3)$	$(0.233, 0.15, 0.31, 0.31)$
x_3	$(0.325, 0.194, 0.156, 0.356)$	$(0.29, 0.195, 0.09, 0.428)$
x_4	$(0.167, 0.230, 0.284, 0.319)$	$(0.35, 0.253, 0.089, 0.308)$

TABLE 3. Tabular representation of the induced fuzzy soft set $\Delta_F = (\hat{F}, \mathcal{A})$.

U	e_1	e_2
	$\varpi(e_1) = (0.3, 0.2, 0.3, 0.2)$	$\varpi(e_2) = (0.4, 0.3, 0.2, 0.1)$
x_1	0.251	0.284
x_2	0.24	0.231
x_3	0.254	0.235
x_4	0.245	0.245

$$
\begin{aligned}
\mu^3_{F(e_1)}(x_1) &= \nu_1 r^3_{\mathcal{F}(e_1)}(x_1) + \nu_2(1/2\pi)\omega^3_{\mathcal{F}(e_1)}(x_1)\\
&= (0.4)(0.8) + (0.6)(1/2\pi)(2\pi)(2/12)\\
&= 0.42\\
\mu^4_{F(e_1)}(x_1) &= \nu_1 r^{44}_{\mathcal{F}(e_1)}(x_1) + \nu_2(1/2\pi)\omega^4_{\mathcal{F}(e_1)}(x_1)\\
&= (0.4)(0.5) + (0.6)(1/2\pi)(2\pi)(5/12)\\
&= 0.45
\end{aligned}
$$

Then, for $e = e_1$ and $x = x_1$, the multi-fuzzy soft value

$$
(\mu^1_{F(e_1)}(x_1), \mu^2_{F(e_1)}(x_1), \mu^3_{F(e_1)}(x_1), \mu^4_{F(e_1)}(x_1)) = (0.51, 0.48, 0.42, 0.45).
$$

In the same way, we calculate the other multi-fuzzy soft values, $\forall e \in \mathcal{A}$ and $\forall x \in U$ and the results are displayed in Table 1.

Then we convert $(F, \mathcal{A})$ into the normalized multi-fuzzy soft set with its tabular representation as in Table 2.

Suppose that experts in the economic field would like to determine the most important factor from these factors that affect the sectors. Assume that the weights for the parameters in A are set as follows: for the parameter "industry sector", $\varpi(e_1) = (0.3, 0.2, 0.3, 0.2)$, for the parameter "tourism sector"ht, $\varpi(e_2) = (0.4, 0.3, 0.2, 0.1)$. Thus we have an induced fuzzy soft set $\Delta_F = (\hat{F}, \mathcal{A})$ with its tabular representation as in Table 3.

As an adjustable approach, one can use different rules (or the thresholds) in decision making problem. For example, if we deal with this problem by mid-level decision rule, to obtain the mid-threshold values of $\Delta_F = (\hat{F}, \mathcal{A})$, $\forall e \in A$ and $j = 1, 2, 3, 4$. We calculate $\mathrm{mid}_{\Delta_{F(e)}}$, when $e = e_1$ as shown below.

TABLE 4. Tabular representation of the mid-level soft set $L(\Delta_F;\ \mathrm{mid})$ with choice values.

U	e_1	e_2	Choice value
x_1	1	1	2
x_2	0	0	0
x_3	1	0	1
x_4	0	0	0

$$
\begin{aligned}
&\mathrm{mid}_{\Delta_{F(e_1)}}\\
&= \frac{\sum_{j=1}^{n} (\hat{F}, \mathcal{A})_{(e_1, x_n)}}{n}\\
&= \frac{(\hat{F}, \mathcal{A})_{(e_1,x_1)} + (\hat{F}, \mathcal{A})_{(e_1,x_2)} + (\hat{F}, \mathcal{A})_{(e_1,x_3)} + (\hat{F}, \mathcal{A})_{(e_1,x_4)}}{4}\\
&= \frac{0.251 + 0.24 + 0.254 + 245}{4} = 0.248
\end{aligned}
$$

Similarly, we can calculate $\mathrm{mid}_{\Delta_{F(e_2)}}$. The mid-threshold of $\Delta_F = (\hat{F}, \mathcal{A})$ is a fuzzy set

$$
\mathrm{mid}_{\Delta_F} = \{(e_1, 0.248), (e_2, 0.249)\}.
$$

For $n = 1, 2$ and $m = 1, 2, 3$, if $(\hat{F}, \mathcal{A})_{(e_n, x_m)} \geq \mathrm{mid}_{\Delta_F(e_n)}$, then x_m gets a "1"t, otherwise, x_m gets a "0". We can present the mid-level soft set $L(\Delta_F;\ \mathrm{mid})$ of Δ_F with choice values with tabular representation as in Table 4.

Clearly, the maximum choice value is 2 from Table 4 and so the optimal decision is to select x_1. Therefore, the expert should select "the role of promotion and advertising" as the most effective factor based on the specified weights for different parameters, followed by "plunge in commodity and oil prices", then "goods and services" and "exchange rate variability".

To show the advantages of our proposed method using CMFSS as compared to that of multi-fuzzy soft set as proposed by Yang *et al.* [15], which is a generalization of fuzzy soft set [10], let us consider Example 24 above. The multi-fuzzy soft set can describe this problem as follows:

$$
(\mathcal{F}, \mathcal{A}) = \left\{\mathcal{F}(e_1) = \left\{\frac{(0.9,\ 0.2,\ 0.8,\ 0.5)}{x_1}\right\}, \ldots\right.
$$

Note that the CMFSS is a generalization of the concept of multi-fuzzy soft set by adding the phase term to the definition of multi-fuzzy soft set. Thus as shown in the decision making problem above, the CMFSS can the ability to handle uncertainties, imprecise and vagueness information that is simultaneously captured by the amplitude terms and phase terms of the complex numbers, whereas multi-fuzzy soft set cannot handle problems that utilizes the time factor, as its structure lacks the phase term, it is clear that it is not possible to apply multi-fuzzy soft set to describe the effect of the economic factors (indicators) on the sectors at a certain period of time, since it is unable to represent variables in two dimensions. In other words multi-fuzzy soft set cannot represent both the degree and phase of the influence simultaneously. However, the structure of complex multi-fuzzy soft set provides the ability to describe these two variables simultaneously.

Thus, our proposed model has ability to provide succinct, elegant and comprehensive representation of two-dimensional multi-fuzzy information.

VI. ENTROPY ON COMPLEX MULTI-FUZZY SOFT SETS

Entropy is one of the fundamental properties of fuzzy sets as its answers the most important question when dealing with fuzzy sets: How fuzzy is a fuzzy set? The concept of entropy answers this fundamental question as it provides a tool to measure the degree of fuzziness of a fuzzy set. In this section, we introduce the concepts of entropy of CMFSSs. Furthermore, we give some theorems and examples.

Definition 25: A real valued function $\hat{E}:\mathcal{CM}^k\mathcal{FSS}(U)\to[0,1]$ *is called an entropy on* $\mathcal{CM}^k\mathcal{FSS}$*, if* $\hat{E}$ *satisfies the following axiomatic requirements:*

($\hat{E}1$) $\hat{E}(\mathcal{F},\mathcal{E})=0\iff r^j_{\mathcal{F}(e)}(x)=1$ *and* $\omega^j_{\mathcal{F}(e)}(x)=2\pi,\ \forall e\in\mathcal{E},x\in U,j=1,2,\ldots,k.$

($\hat{E}2$) $\hat{E}(\mathcal{F},\mathcal{E})=1\iff r^j_{\mathcal{F}(e)}(x)=0.5$ *and* $\omega^j_{\mathcal{F}(e)}(x)=\pi,\ \forall e\in\mathcal{E},x\in U,j=1,2,\ldots,k.$

($\hat{E}3$) $\hat{E}(\mathcal{F},\mathcal{E})=\hat{E}((\mathcal{F},\mathcal{E})^c).$

($\hat{E}4$) *if* $(\mathcal{F},\mathcal{E})\subseteq(\mathcal{G},\mathcal{E})$*, i.e.,* $r^j_{\mathcal{F}(e)}(x)\le r^j_{\mathcal{G}(e)}(x)$ *and* $\omega^j_{\mathcal{F}(e)}(x)\le\omega^j_{\mathcal{G}(e)}(x),\ \forall e\in\mathcal{E},x\in U,j=1,2,\ldots,k.$*, then* $\hat{E}(\mathcal{F},\mathcal{E})\ge\hat{E}(\mathcal{G},\mathcal{E})$.

Theorem 26: Let $U=\{x_1,x_2,\ldots,x_p\}$ *be the universal set of elements and* $E=\{e_1,e_2,\ldots,e_m\}$ *be the universal set of parameters. Hence* $(\mathcal{F},\mathcal{E})=\{\mathcal{F}(e_l)=r^j_{\mathcal{F}(e_l)}(x).e^{i\omega^j_{\mathcal{F}(e_l)}(x)}\mid l=1,2,\ldots,m\}$ *is a family of* $\mathcal{CM}^k\mathcal{FSS}$*. Define* $\hat{E}(\mathcal{F},\mathcal{E})$ *as follows:*

$$\hat{E}(\mathcal{F},\mathcal{E})=\frac{1}{2m}\sum_{l=1}^{m}[\hat{E}^r_l(\mathcal{F},\mathcal{E})+\frac{\hat{E}^\omega_l(\mathcal{F},\mathcal{E})}{2\pi}],$$

where

$$\hat{E}^r_l(\mathcal{F},\mathcal{E})=\frac{1}{nk}\sum_{p=1}^{n}\sum_{j=1}^{k}[1-|r^j_{\mathcal{F}(e_l)}(x_p)-r^j_{\mathcal{F}^c(e_l)}(x_p)|]$$

and

$$\hat{E}^\omega_l(\mathcal{F},\mathcal{E})=\frac{1}{nk}\sum_{p=1}^{n}\sum_{j=1}^{k}[2\pi-|\omega^j_{\mathcal{F}(e_l)}(x_p)-\omega^j_{\mathcal{F}^c(e_l)}(x_p)|],$$

then $\hat{E}(\mathcal{F},\mathcal{E})$ *is an entropy of* $\mathcal{CM}^k\mathcal{FSS}$.

Proof: We show that the $\hat{E}(\mathcal{F},\mathcal{E})$ satisfies the all conditions given in Definition 25.

($\hat{E}1$) $\hat{E}(\mathcal{F},\mathcal{E})=0$

$$\iff\sum_{l=1}^{m}[\hat{E}^r_l(\mathcal{F},\mathcal{E})+\frac{\hat{E}^\omega_l(\mathcal{F},\mathcal{E})}{2\pi}]=0$$

$$\iff\hat{E}^r_l(\mathcal{F},\mathcal{E})=0\text{ and }\hat{E}^\omega_l(\mathcal{F},\mathcal{E})=0$$

$$\iff\forall e_l\in\mathcal{E},x_p\in U,j=1,2,\ldots,k,$$

$$\sum_{p=1}^{n}\sum_{j=1}^{k}([1-|r^j_{\mathcal{F}(e_l)}(x_p)-r^j_{\mathcal{F}^c(e_l)}(x_p)|])=0$$

$$\text{and }\sum_{p=1}^{n}\sum_{j=1}^{k}([2\pi-|\omega^j_{\mathcal{F}(e_l)}(x_p)-\omega^j_{\mathcal{F}^c(e_l)}(x_p)|])=0$$

$$\iff\forall e_l\in\mathcal{E},x\in U,j=1,2,\ldots,k,$$

$$|r^j_{\mathcal{F}(e_l)}(x_p)-r^j_{\mathcal{F}^c(e_l)}(x_p)|=1$$

$$\text{and }|\omega^j_{\mathcal{F}(e_l)}(x_p)-\omega^j_{\mathcal{F}^c(e_l)}(x_p)|=2\pi.$$

$$\iff\forall e_l\in\mathcal{E},x\in U,j=1,2,\ldots,k,$$

$$r^j_{\mathcal{F}(e_l)}(x_p)=1\text{ and }\omega^j_{\mathcal{F}(e_l)}(x_p)=2\pi.$$

($\hat{E}2$) For $(\mathcal{F},\mathcal{E})\in\mathcal{CM}^k\mathcal{FSS}(U)$, we have $\hat{E}(\mathcal{F},\mathcal{E})=1$

$$\iff\sum_{l=1}^{m}[\hat{E}^r_l(\mathcal{F},\mathcal{E})+\frac{\hat{E}^\omega_l(\mathcal{F},\mathcal{E})}{2\pi}]=2m$$

$$\iff\hat{E}^r_l(\mathcal{F},\mathcal{E})=1\text{ and }\hat{E}^\omega_l(\mathcal{F},\mathcal{E})=2\pi$$

$$\iff\forall e_l\in\mathcal{E},x_p\in U,j=1,2,\ldots,k,$$

$$\frac{1}{nk}\sum_{p=1}^{n}\sum_{j=1}^{k}([1-|r^j_{\mathcal{F}^c(e_l)}(x_p)-r^j_{\mathcal{F}(e_l)}(x_p)|])=1$$

$$\text{and }\frac{1}{nk}\sum_{p=1}^{n}\sum_{j=1}^{k}([2\pi-|\omega^j_{\mathcal{F}^c(e_l)}(x_p)-\omega^j_{\mathcal{F}(e_l)}(x_p)|])=2\pi$$

$$\iff\forall e_l\in\mathcal{E},x_p\in U,j=1,2,\ldots,k,$$

$$\sum_{p=1}^{n}\sum_{j=1}^{k}([1-|r^j_{\mathcal{F}^c(e_l)}(x_p)-r^j_{\mathcal{F}(e_l)}(x_p)|])=nk$$

and

$$\sum_{p=1}^{n}\sum_{j=1}^{k}([2\pi-|\omega^j_{\mathcal{F}^c(e_l)}(x_p)-\omega^j_{\mathcal{F}(e_l)}(x_p)|])=2\pi(nk)$$

$$\iff\forall e_l\in\mathcal{E},x\in U,j=1,2,\ldots,k,$$

$$[1-|r^j_{\mathcal{F}^c(e_l)}(x_p)-r^j_{\mathcal{F}(e_l)}(x_p)|]=1$$

$$\text{and }[2\pi-|\omega^j_{\mathcal{F}^c(e_l)}(x_p)-\omega^j_{\mathcal{F}(e_l)}(x_p)|]=2\pi$$

$$\iff\forall e_l\in\mathcal{E},x\in U,j=1,2,\ldots,k,$$

$$|r^j_{\mathcal{F}^c(e_l)}(x_p)-r^j_{\mathcal{F}(e_l)}(x_p)|=0$$

$$\text{and }|\omega^j_{\mathcal{F}^c(e_l)}(x_p)-\omega^j_{\mathcal{F}(e_l)}(x_p)|=0$$

$$\iff\forall e_l\in\mathcal{E},x\in U,j=1,2,\ldots,k,$$

$$r^j_{\mathcal{F}(e_l)}(x_p)=\frac{1}{2}\text{ and }\omega^j_{\mathcal{F}(e_l)}(x_p)=\pi.$$

($\hat{E}3$) For $(\mathcal{F},\mathcal{E})\in\mathcal{CM}^k\mathcal{FSS}(U)$, we have

$$\hat{E}^r_l(\mathcal{F},\mathcal{E})=\frac{1}{nk}\sum_{p=1}^{n}\sum_{j=1}^{k}([1-|r^j_{\mathcal{F}(e_l)}(x_p)-r^j_{\mathcal{F}^c(e_l)}(x_p)|])$$

$$=\frac{1}{nk}\sum_{p=1}^{n}\sum_{j=1}^{k}([1-|r^j_{\mathcal{F}^c(e_l)}(x_p)-r^j_{\mathcal{F}(e_l)}(x_p)|])$$

$$=\hat{E}^r_l((\mathcal{F},\mathcal{E})^c)$$

Similarly, we can prove $\hat{E}^\omega_l(\mathcal{F},\mathcal{E})=\hat{E}^\omega_l((\mathcal{F},\mathcal{E})^c)$ it is clear that $\hat{E}(\mathcal{F},\mathcal{E})=\hat{E}((\mathcal{F},\mathcal{E})^c)$.

($\hat{E}4$) Let $(\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})\in\mathcal{CM}^k\mathcal{FSS}(U)$. If $(\mathcal{F},\mathcal{E})\subseteq(\mathcal{G},\mathcal{E})$

$$
\begin{aligned}
&\Longrightarrow \forall e_l\in\mathcal{E},\quad x\in U,\ j=1,2,\ldots,k,\\
&\quad r^j_{\mathcal{F}(e_l)}(x_p)\le r^j_{\mathcal{G}(e_l)}(x_p) \text{ and } \omega^j_{\mathcal{F}(e_l)}(x_p)\le\omega^j_{\mathcal{G}(e_l)}(x_p),\\
&\Longrightarrow \forall e_l\in\mathcal{E},x\in U,j=1,2,\ldots,k,\\
&\quad |r^j_{\mathcal{F}(e_l)}(x_p)-r^j_{\mathcal{F}^c(e_l)}(x_p)|\le|r^j_{\mathcal{G}(e_l)}(x_p)-r^j_{\mathcal{G}^c(e_l)}(x_p)|\\
&\quad \text{and } |\omega^j_{\mathcal{F}(e_l)}(x_p)-\omega^j_{\mathcal{F}^c(e_l)}(x_p)|\le|\omega^j_{\mathcal{G}(e_l)}(x_p)-\omega^j_{\mathcal{G}^c(e_l)}(x_p)|\\
&\Longrightarrow \forall e_l\in\mathcal{E},x\in U,j=1,2,\ldots,k,\\
&\quad 1-|r^j_{\mathcal{F}(e_l)}(x_p)-r^j_{\mathcal{F}^c(e_l)}(x_p)|\ge 1\\
&\quad -|r^j_{\mathcal{G}(e_l)}(x_p)-r^j_{\mathcal{G}^c(e_l)}(x_p)|\\
&\quad \text{and } 2\pi-|\omega^j_{\mathcal{F}(e_l)}(x_p)-\omega^j_{\mathcal{F}^c(e_l)}(x_p)|\ge 2\pi\\
&\quad -|\omega^j_{\mathcal{G}(e_l)}(x_p)-\omega^j_{\mathcal{G}^c(e_l)}(x_p)|\\
&\Longrightarrow \frac{1}{nk}\sum_{p=1}^{n}\sum_{j=1}^{k}([1-|r^j_{\mathcal{F}(e_l)}(x_p)-r^j_{\mathcal{F}^c(e_l)}(x_p)|])\\
&\quad \ge\frac{1}{nk}\sum_{p=1}^{n}\sum_{j=1}^{k}([1-|r^j_{\mathcal{G}(e_l)}(x_p)-r^j_{\mathcal{G}^c(e_l)}(x_p)|])\\
&\quad \text{and } \frac{1}{nk}\sum_{p=1}^{n}\sum_{j=1}^{k}([2\pi-|\omega^j_{\mathcal{F}(e_l)}(x_p)-\omega^j_{\mathcal{F}^c(e_l)}(x_p)|])\\
&\quad \ge\frac{2}{nk}\sum_{p=1}^{n}\sum_{j=1}^{k}([2\pi-|\omega^j_{\mathcal{G}(e_l)}(x_p)-\omega^j_{\mathcal{G}^c(e_l)}(x_p)|])\\
&\Longrightarrow \hat{E}^r_l(\mathcal{F},\mathcal{E})\ge\hat{E}^r_l(\mathcal{G},\mathcal{E}) \text{ and } \hat{E}^\omega_l(\mathcal{F},\mathcal{E})\ge\hat{E}^\omega_l(\mathcal{G},\mathcal{E})\\
&\Longrightarrow \hat{E}^r_l(\mathcal{F},\mathcal{E})+\hat{E}^\omega_l(\mathcal{F},\mathcal{E})\ge\hat{E}^r_l(\mathcal{G},\mathcal{E})+\hat{E}^\omega_l(\mathcal{G},\mathcal{E})\\
&\Longrightarrow \frac{1}{2m}\sum_{l=1}^{m}[\hat{E}^r_l(\mathcal{F},\mathcal{A})+\frac{\hat{E}^\omega_l(\mathcal{F},\mathcal{E})}{2\pi}]\\
&\quad \ge\frac{1}{2m}\sum_{l=1}^{m}[\hat{E}^r_l(\mathcal{G},\mathcal{E})+\frac{\hat{E}^\omega_l(\mathcal{G},\mathcal{E})}{2\pi}]\\
&\Longrightarrow \hat{E}(\mathcal{F},\mathcal{E})\ge\hat{E}(\mathcal{G},\mathcal{E})
\end{aligned}
$$

This completes the proof of Theorem 26.

A. APPLICATION USING ENTROPY MEASURE

Prediction of a bushfire danger rating in advance can reduce costs of damage and save people's lives. Fire-fighting management currently uses a three-level fire danger rating system that is based on the fire danger index. The fire danger index is determined using the data (observations and/or predictions) of three primary meteorological indicators (factors), that is, "maximum temperature", "wind speed" and "relative humidity". It is well-known that the three indicators change seasonally. Hence, the same data for an indicator mean different things at different times; for instance, a 20° temperature might mean a cool day in summer, a warm day in winter or a fair day in spring. This phenomenon indicates that data are of semantic uncertainty and periodicity.

Assume that there is a set of two experts $\{P_1,P_2\}$ to evaluate the three levels of fire danger index levels. Let $U=\{x_1,x_2,x_3\}$ be the set of levels, $\mathcal{E}=\{e_1,e_2,e_3\}$ is a set of parameters that represents the meteorological indicators where e_1 stands for "maximum temperature" which includes three levels: cool, warm and fair, e_2 stands for "wind speed" includes three speeds: light, moderate and strong, and e_3 stands for "relative humidity" which can be categorized as high, medium and low.

The following $\mathcal{CM}^k\mathcal{FSS}(\mathcal{F},\mathcal{E})$ describes the evaluation of expert P_1.

$$
\begin{aligned}
(\mathcal{F},\mathcal{E}) &=\Big\{\mathcal{F}(e_1)=\Big\{\frac{(0.8e^{i2\pi(6/12)},0.9e^{i2\pi(6/12)},0.7e^{i2\pi(7/12)})}{x_1},\\
&\quad\frac{(0.5e^{i2\pi(4/12)},0.4e^{i2\pi(3/12)},0.3e^{i2\pi(1/12)})}{x_2},\\
&\quad\frac{(0.1e^{i2\pi(10/12)},0.2e^{i2\pi(11/12)},0.2e^{i2\pi(10/12)})}{x_3}\Big\},\\
\mathcal{F}(e_2)&=\Big\{\frac{(0.3e^{i2\pi(5/12)},0.2e^{i2\pi(4/12)},0.2e^{i2\pi(6/12)})}{x_1},\\
&\quad\frac{(0.9e^{i2\pi(2/12)},0.8e^{i2\pi(2/12)},0.7e^{i2\pi(1/12)})}{x_2},\\
&\quad\frac{(0.4e^{i2\pi(7/12)},0.3e^{i2\pi(5/12)},0.4e^{i2\pi(5/12)})}{x_3}\Big\},\\
\mathcal{F}(e_3)&=\Big\{\frac{(0.2e^{i2\pi(4/12)},0.3e^{i2\pi(3/12)},0.3e^{i2\pi(4/12)})}{x_1},\\
&\quad\frac{(0.6^{i2\pi(5/12)},0.7e^{i2\pi(8/12)},0.4e^{i2\pi(9/12)})}{x_2},\\
&\quad\frac{(0.2e^{i2\pi(2/12)},0.3e^{i2\pi(6/12)},0.4e^{i2\pi(7/12)})}{x_3}\Big\}\Big\}.
\end{aligned}
$$

The following $\mathcal{CM}^k\mathcal{FSS}(\mathcal{G},\mathcal{E})$ describes the evaluation of expert P_2.

$$
\begin{aligned}
(\mathcal{G},\mathcal{E})&=\Big\{\mathcal{G}(e_1)\\
&=\Big\{\frac{(0.3e^{i2\pi(7/12)},0.5e^{i2\pi(6/12)},0.4e^{i2\pi(7/12)})}{x_1},\\
&\quad\frac{(0.1e^{i2\pi(2/12)},0.2e^{i2\pi(3/12)},0.1e^{i2\pi(1/12)})}{x_2},\\
&\quad\frac{(0.8e^{i2\pi(8/12)},0.7e^{i2\pi(9/12)},0.2e^{i2\pi(10/12)})}{x_3}\Big\},\\
\mathcal{G}(e_2)&=\{\frac{(0.9e^{i2\pi(9/12)},0.6e^{i2\pi(10/12)},0.2e^{i2\pi(11/12)})}{x_1},\\
&\quad\frac{(0.9e^{i2\pi(2/12)},0.4e^{i2\pi(3/12)},0.1e^{i2\pi(1/12)})}{x_2},\\
&\quad\frac{(0.4e^{i2\pi(7/12)},0.3e^{i2\pi(5/12)},0.2e^{i2\pi(6/12)})}{x_3}\},\\
\mathcal{G}(e_3)&=\{\frac{(0.6e^{i2\pi(6/12)},0.5e^{i2\pi(3/12)},0.3e^{i2\pi(12/12)})}{x_1},\\
&\quad\frac{(0.1^{i3\pi(5/12)},0.3e^{i2\pi(4/12)},0.4e^{i2\pi(3/12)})}{x_2},\\
&\quad\frac{(0.8e^{i2\pi(5/12)},0.9e^{i2\pi(7/12)},0.7e^{i2\pi(8/12)})}{x_3}\}\Big\}.
\end{aligned}
$$

Then we have

$$\begin{aligned}\hat{E}_1^r(\mathcal{F},\mathcal{E}) &= \frac{1}{nk}\sum_{p=1}^{n}\sum_{j=1}^{k}[1-|r^j_{\mathcal{F}(e_1)}(x_p)-r^j_{\mathcal{F}^c(e_1)}(x_p)|] \\ &= \frac{1}{3\times 3}[(0.4+0.2+0.6)+(1+0.8+0.6) \\ &\quad +(0.2+0.4+0.4)] \\ &= 0.551\end{aligned}$$

$$\begin{aligned}\hat{E}_1^\omega(\mathcal{F},\mathcal{E}) &= \frac{1}{nk}\sum_{p=1}^{n}\sum_{j=1}^{k}[2\pi-|\omega^j_{\mathcal{F}(e_1)}(x_p)-\omega^j_{\mathcal{F}^c(e_1)}(x_p)|] \\ &= \frac{1}{3\times 3}[(2\pi+2\pi+2\pi\frac{10}{12}) \\ &\quad +(2\pi\frac{8}{12}+2\pi\frac{6}{12}+2\pi\frac{2}{12}) \\ &\quad +(2\pi\frac{4}{12}+2\pi\frac{2}{12}+2\pi\frac{4}{12})] \\ &= 1.11\pi\end{aligned}$$

Similarly,

$$\begin{aligned}&\hat{E}_2^r(\mathcal{F},\mathcal{E}) = 0.555, \quad \hat{E}_2^\omega(\mathcal{F},\mathcal{E}) = 1.29\pi,\\ &\hat{E}_3^r(\mathcal{F},\mathcal{E}) = 0.644, \quad \hat{E}_3^\omega(\mathcal{F},\mathcal{E}) = 1.33\pi,\\ &\hat{E}_1^r(\mathcal{G},\mathcal{E}) = 0.511, \quad \hat{E}_1^\omega(\mathcal{G},\mathcal{E}) = 1.148\pi,\\ &\hat{E}_2^r(\mathcal{G},\mathcal{E}) = 0.488, \quad \hat{E}_2^\omega(\mathcal{G},\mathcal{E}) = 1.037\pi,\\ &\hat{E}_3^r(\mathcal{G},\mathcal{E}) = 0.577, \quad \hat{E}_3^\omega(\mathcal{G},\mathcal{E}) = 1.296\pi,\end{aligned}$$

Hence the entropy of the $\mathcal{CM}^k\mathcal{FSS}s$ $(\mathcal{F},\mathcal{E})$ and $(\mathcal{F},\mathcal{E})$ are as given below:

$$\hat{E}(\mathcal{F},\mathcal{E}) = \frac{1}{2\times 3}\sum_{l=1}^{3}[\hat{E}_l^r(\mathcal{F},\mathcal{E})+\frac{\hat{E}_l^\omega(\mathcal{F},\mathcal{E})}{2\pi}] = 0.569$$

$$\hat{E}(\mathcal{G},\mathcal{E}) = \frac{1}{2\times 3}\sum_{l=1}^{3}[\hat{E}_l^r(\mathcal{G},\mathcal{E})+\frac{\hat{E}_l^\omega(\mathcal{G},\mathcal{E})}{2\pi}] = 0.553$$

Entropy is an important notion for measuring uncertain information. The less uncertainty information has the larger possibility to select the optimal. From the computation we have $\hat{E}(\mathcal{G},\mathcal{E})) \leq \hat{E}(\mathcal{F},\mathcal{E})$. Therefore, the expert P_2 has larger possibility to make the decision on fire fighting management than expert P_1. For expert P_2, the value $\hat{E}_3^r(\mathcal{G},\mathcal{E}) = 0.577$ has the largest entropy value between the fire danger index levels. This points out that the parameters related with the fire fighting namely, maximum temperature, wind speed and relative humidity have to be given proper attention for fire-fighting management system.

VII. SIMILARITY MEASURE BETWEEN COMPLEX MULTI-FUZZY SOFT SETS

Similarity measures quantify the extent to which different patterns, images, or sets are alike. Such measures are used extensively in the application of fuzzy soft sets. Based on the axioms for the entropy of fuzzy soft sets [23], we give a definition of a similarity measure for CMFSS as follows.

Definition 27: A real valued function $\hat{S}:\mathcal{CM}^k\mathcal{FSS}(U)\times\mathcal{CM}^k\mathcal{FSS}(U)\to[0,1]$ is called a similarity measure between two $\mathcal{CM}^k\mathcal{FSS}s$ $(\mathcal{F},\mathcal{E})$ and $(\mathcal{G},\mathcal{E})$, if $\hat{S}$ satisfies the following axiomatic requirements:

($\hat{S}1$) $\hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) = \hat{S}((\mathcal{G},\mathcal{E}),(\mathcal{F},\mathcal{E}))$,

($\hat{S}2$) $\hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) = 1 \Longleftrightarrow (\mathcal{F},\mathcal{E}) = (\mathcal{G},\mathcal{E})$,

($\hat{S}3$) $\hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) = 0 \Longleftrightarrow \forall e\in\mathcal{E}, x\in U, j = 1,2,\ldots,K$, the following conditions are satisfied:
$r^j_{\mathcal{F}(e)}(x) = 1,\ r^j_{\mathcal{G}(e)}(x) = 0$ õr
$r^j_{\mathcal{F}(e)}(x) = 0,\ r^j_{\mathcal{G}(e)}(x) = 1$
and
$\omega^j_{\mathcal{F}(e)}(x) = 2\pi,\ \omega^j_{\mathcal{G}(e)}(x) = 0$ õr
$\omega^j_{\mathcal{F}(e)}(x) = 0,\ \omega^j_{\mathcal{G}(e)}(x) = 2\pi$,

($\hat{S}4$) $\forall(\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})$ and $(\mathcal{Q},\mathcal{E}) \in \mathcal{CM}^k\mathcal{FSS}(U)$,
if $(\mathcal{F},\mathcal{E})\subseteq(\mathcal{G},\mathcal{E})\subseteq(\mathcal{Q},\mathcal{E})$,
then $\hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{Q},\mathcal{E}))\leq\hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))$
and $\hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{Q},\mathcal{E}))\leq\hat{S}((\mathcal{G},\mathcal{E}),(\mathcal{Q},\mathcal{E}))$.

Now, we introduce the formula to calculate the similarity between two $\mathcal{CM}^k\mathcal{FSS}s$ as follows:

Theorem 28: Let $U = \{x_1,x_2,\ldots,x_p\}$ be the universal set of elements and $\mathcal{E} = \{e_1,e_2,\ldots,e_m\}$ be the universal set of parameters. $(\mathcal{F},\mathcal{E}) = \{\mathcal{F}(e_l) = r^j_{\mathcal{F}(e_l)}(x).e^{i\omega^j_{\mathcal{F}(e_l)}(x)}|l = 1,2,\ldots,m\}$ and $(\mathcal{G},\mathcal{E}) = \{\mathcal{G}(e_l) = r^j_{\mathcal{G}(e_l)}(x).e^{i\omega^j_{\mathcal{G}(e_l)}(x)}|l = 1,2,\ldots,m\}$ are two families of $\mathcal{CM}^k\mathcal{FSS}s$. Define $\hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))$ as follows:

$$\begin{aligned}&\hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))\\ &\quad = \frac{1}{2m}\sum_{l=1}^{m}[\hat{S}_l^r((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))+\frac{\hat{S}_l^\omega((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))}{2\pi}],\end{aligned}$$

where

$$\begin{aligned}&\hat{S}_l^r((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))\\ &\quad = 1-\frac{1}{n}\sum_{p=1}^{n}max\{(|r^j_{\mathcal{F}(e_l)}(x_p)-r^j_{\mathcal{G}(e_l)}(x_p)|)_{j\in k}\},\end{aligned}$$

and

$$\begin{aligned}&\hat{S}_l^\omega((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))\\ &\quad = 2\pi-\frac{1}{n}\sum_{p=1}^{n}max\{(|\omega^j_{\mathcal{F}(e_l)}(x_p)-\omega^j_{\mathcal{G}(e_l)}(x_p)|)_{j\in k}\},\end{aligned}$$

then $\hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))$ is a similarity measure between two $\mathcal{CM}^k\mathcal{FSS}s$ $(\mathcal{F},\mathcal{E})$ and $(\mathcal{G},\mathcal{E})$.

Proof: It is sufficient to show that $\hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))$ satisfies the requirements listed in Definition 27.

($\hat{S}1$) For

$$\begin{aligned}&\hat{S}_l^r((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))\\ &\quad = 1-\frac{1}{n}\sum_{p=1}^{n}max\{(|r^j_{\mathcal{F}(e_l)}(x_p)-r^j_{\mathcal{G}(e_l)}(x_p)|)_{j\in k}\},\\ &\quad = 1-\frac{1}{n}\sum_{p=1}^{n}max\{(|r^j_{\mathcal{G}(e_l)}(x_p)-r^j_{\mathcal{F}(e_l)}(x_p)|)_{j\in k}\}\\ &\quad = \hat{S}_l^r((\mathcal{G},\mathcal{E}),(\mathcal{F},\mathcal{E}))\end{aligned}$$

and

$$
\begin{aligned}
&\hat{S}_l^{\omega}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))\\
&\quad = 2\pi - \frac{1}{n}\sum_{p=1}^{n} max\{(|\omega^j_{\mathcal{F}(e_l)}(x_p) - \omega^j_{\mathcal{G}(e_l)}(x_p)|)_{j\in k}\},\\
&\quad = 2\pi - \frac{1}{n}\sum_{p=1}^{n} max\{(|\omega^j_{\mathcal{G}(e_l)}(x_p) - \omega^j_{\mathcal{F}(e_l)}(x_p)|)_{j\in k}\}\\
&\quad = \hat{S}_l^{\omega}((\mathcal{G},\mathcal{E}),(\mathcal{F},\mathcal{E}))
\end{aligned}
$$

So we have

$$
\begin{aligned}
&\hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))\\
&\quad = \frac{1}{2m}\sum_{l=1}^{m}[\hat{S}_l^r((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) + \frac{\hat{S}_l^{\omega}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))}{2\pi}]\\
&\quad = \frac{1}{2m}\sum_{l=1}^{m}[\hat{S}_l^r((\mathcal{G},\mathcal{E}),(\mathcal{F},\mathcal{E})) + \frac{\hat{S}_l^{\omega}((\mathcal{G},\mathcal{E}),(\mathcal{F},\mathcal{E}))}{2\pi}]\\
&\quad = \hat{S}((\mathcal{G},\mathcal{E}),(\mathcal{F},\mathcal{E})).
\end{aligned}
$$

$(\hat{S}2)$ $\hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) = 1$

$$
\begin{aligned}
&\iff \frac{1}{2m}\sum_{l=1}^{m}[\hat{S}_l^r((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) + \frac{\hat{S}_l^{\omega}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))}{2\pi}] = 1\\
&\iff \hat{S}_l^r((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) = 1 \text{ and } \hat{S}_l^{\omega}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))) = 2\pi\\
&\iff 1 - \frac{1}{n}\sum_{p=1}^{n} max\{(|r^j_{\mathcal{F}(e_l)}(x_p) - r^j_{\mathcal{G}(e_l)}(x_p)|)_{j\in k}\} = 1\\
&\qquad \text{and}\\
&\qquad 2\pi - \frac{1}{n}\sum_{p=1}^{n} max\{(|\omega^j_{\mathcal{F}(e_l)}(x_p) - \omega^j_{\mathcal{G}(e_l)}(x_p)|)_{j\in k}\} = 2\pi,\\
&\qquad \forall e_l \in \mathcal{A}, x \in U, j = 1,2,\ldots,k,\\
&\iff \frac{1}{n}\sum_{p=1}^{n} max\{(|r^j_{\mathcal{F}(e_l)}(x_p) - r^j_{\mathcal{G}(e_l)}(x_p)|)_{j\in k}\} = 0\\
&\qquad \text{and}\\
&\qquad \frac{1}{n}\sum_{p=1}^{n} max\{(|\omega^j_{\mathcal{F}(e_l)}(x_p) - \omega^j_{\mathcal{G}(e_l)}(x_p)|)_{j\in k}\} = 0,\\
&\qquad \forall e \in \mathcal{E}, x \in U, j = 1,2,\ldots,k,\\
&\iff max\{(|r^j_{\mathcal{F}(e_l)}(x_p) - r^j_{\mathcal{G}(e_l)}(x_p)|)_{j\in k}\} = 0\\
&\qquad \text{and}\\
&\qquad max\{(|\omega^j_{\mathcal{F}(e_l)}(x_p) - \omega^j_{\mathcal{G}(e_l)}(x_p)|)_{j\in k}\} = 0,\\
&\qquad \forall e \in \mathcal{A}, x \in U, j = 1,2,\ldots,k,\\
&\iff r^j_{\mathcal{F}(e_l)}(x_p) = r^j_{\mathcal{G}(e_l)}(x_p)\\
&\qquad \text{and}\\
&\qquad \omega^j_{\mathcal{F}(e_l)}(x_p) = \omega^j_{\mathcal{G}(e_l)}(x_p),\\
&\qquad \forall e_l \in \mathcal{E}, x \in U, j = 1,2,\ldots,k,\\
&\iff (\mathcal{F},\mathcal{E}) = (\mathcal{G},\mathcal{E}).
\end{aligned}
$$

$(\hat{S}3)$ $\hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) = 0$

$$
\begin{aligned}
&\iff \frac{1}{2m}\sum_{l=1}^{m}[\hat{S}_l^r((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) + \frac{\hat{S}_l^{\omega}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))}{2\pi}] = 0\\
&\iff \hat{S}_l^r((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) = 0 \text{ and } \hat{S}_l^{\omega}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))) = 0\\
&\iff 1 - \frac{1}{n}\sum_{p=1}^{n} max\{(|r^j_{\mathcal{F}(e_l)}(x_p) - r^j_{\mathcal{G}(e_l)}(x_p)|)_{j\in k}\} = 0\\
&\qquad \text{and}\\
&\qquad 2\pi - \frac{1}{n}\sum_{p=1}^{n} max\{(|\omega^j_{\mathcal{F}(e_l)}(x_p) - \omega^j_{\mathcal{G}(e_l)}(x_p)|)_{j\in k}\} = 0,\\
&\qquad \forall e_l \in \mathcal{E}, x \in 7U, j = 1,2,\ldots,k,\\
&\iff \frac{1}{n}\sum_{p=1}^{n} max\{(|r^j_{\mathcal{F}(e_l)}(x_p) - r^j_{\mathcal{G}(e_l)}(x_p)|)_{j\in k}\} = 1\\
&\qquad \text{and}\\
&\qquad \frac{1}{n}\sum_{p=1}^{n} max\{(|\omega^j_{\mathcal{F}(e)}(x_p) - \omega^j_{\mathcal{G}(e)}(x_p)|)_{j\in k}\} = 2\pi,\\
&\qquad \forall e \in \mathcal{A}, x \in U, j = 1,2,\ldots,k,\\
&\iff max\{(|r^j_{\mathcal{F}(e_l)}(x_p) - r^j_{\mathcal{G}(e_l)}(x_p)|)_{j\in k}\} = 1\\
&\qquad \text{and}\\
&\qquad max\{(|\omega^j_{\mathcal{F}(e_l)}(x_p) - \omega^j_{\mathcal{G}(e)}(x_p)|)_{j\in k}\} = 2\pi,\\
&\qquad \forall e_l \in \mathcal{E}, x \in U, j = 1,2,\ldots,k,\\
&\iff r^j_{\mathcal{F}(e_l)}(x) = 0, \quad r^j_{\mathcal{G}(e_l)}(x) = 1\text{or}\\
&\qquad r^j_{\mathcal{F}(e_l)}(x) = 1, \quad r^j_{\mathcal{G}(e_l)}(x) = 0\\
&\qquad \text{and } \omega^j_{\mathcal{F}(e_l)}(x) = 0, \quad \omega^j_{\mathcal{G}(e_l)}(x) = 2\pi \quad \text{or}\\
&\qquad \omega^j_{\mathcal{F}(e_l)}(x) = 2\pi, \ \omega^j_{\mathcal{G}(e_l)}(x) = 0.
\end{aligned}
$$

$(\hat{S}4)$ Since $(\mathcal{F},\mathcal{E}) \subseteq (\mathcal{G},\mathcal{E}) \subseteq (\mathcal{Q},\mathcal{E})$,

$$
\begin{aligned}
&\implies r^j_{\mathcal{F}(e_l)}(x_p) \le r^j_{\mathcal{G}(e_l)}(x_p) \le r^j_{\mathcal{Q}(e_l)}(x_p)\\
&\qquad \text{and}\\
&\qquad \omega^j_{\mathcal{F}(e_l)}(x_p) \le \omega^j_{\mathcal{G}(e_l)}(x_p) \le \omega^j_{\mathcal{Q}(e_l)}(x_p),\\
&\qquad \forall e_l \in \mathcal{E}, x \in U, j = 1,2,\ldots,k.\\
&\implies |r^j_{\mathcal{F}(e_l)}(x_p) - r^j_{\mathcal{Q}(e_l)}(x_p)| \le |r^j_{\mathcal{F}(e_l)}(x_p) - r^j_{\mathcal{G}(e_l)}(x_p)|\\
&\qquad \text{and}\\
&\qquad |\omega^j_{\mathcal{F}(e_l)}(x_p) - \omega^j_{\mathcal{Q}(e_l)}(x_p)| \le |\omega^j_{\mathcal{F}(e_l)}(x_p) - \omega^j_{\mathcal{G}(e_l)}(x_p)|,\\
&\qquad \forall e \in \mathcal{E}, x \in U, j = 1,2,\ldots,k.\\
&\implies 1 - \frac{1}{n}\sum_{p=1}^{n} max\{(|r^j_{\mathcal{F}(e_l)}(x_p) - r^j_{\mathcal{Q}(e_l)}(x_p)|)_{j\in k}\}\\
&\quad \le \ 1 - \frac{1}{n}\sum_{p=1}^{n} max\{(|r^j_{\mathcal{F}(e)}(x_p) - r^j_{\mathcal{G}(e_l)}(x_p)|)_{j\in k}\}\\
&\qquad \text{and}\\
&\qquad 2\pi - \frac{1}{n}\sum_{p=1}^{n} max\{(|\omega^j_{\mathcal{F}(e_l)}(x_p) - \omega^j_{\mathcal{Q}(e_l)}(x_p)|)_{j\in k}\}\\
&\quad \le \ 2\pi - \frac{1}{n}\sum_{p=1}^{n} max\{(|\omega^j_{\mathcal{F}(e_l)}(x_p) - \omega^j_{\mathcal{G}(e_l)}(x_p)|)_{j\in k}\}\\
&\implies \hat{S}_l^r((\mathcal{F},\mathcal{E}),(\mathcal{Q},\mathcal{E})) \le \hat{S}_l^r((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))\\
&\qquad \text{and}\\
&\qquad \hat{S}_l^{\omega}((\mathcal{F},\mathcal{E}),(\mathcal{Q},\mathcal{E})) \le \hat{S}_l^{\omega}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))
\end{aligned}
$$

$$\Longrightarrow \hat{S}^r_l((\mathcal{F},\mathcal{E}),(\mathcal{Q},\mathcal{E})) + \hat{S}^\omega_l((\mathcal{F},\mathcal{E}),(\mathcal{Q},\mathcal{E})) \leq \hat{S}^r_l((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) + \hat{S}^\omega_l((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))$$

$$\Longrightarrow \frac{1}{2m}\sum_{l=1}^{m}[\hat{S}^r_l((\mathcal{F},\mathcal{E}),(\mathcal{Q},\mathcal{E})) + \frac{\hat{S}^\omega_l((\mathcal{F},\mathcal{E}),(\mathcal{Q},\mathcal{E}))}{2\pi}]$$

$$\leq \frac{1}{2m}\sum_{l=1}^{m}[\hat{S}^r_l((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) + \frac{\hat{S}^\omega_l((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))}{2\pi}]$$

$$\Longrightarrow \hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{Q},\mathcal{E})) \leq \hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))$$

Similarly, it can be proven that

$$\hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{Q},\mathcal{E})) \leq \hat{S}((\mathcal{G},\mathcal{E}),(\mathcal{Q},\mathcal{E})).$$

This completes the proof of Theorem 7.2.

Example 29: Let $U = \{x_1, x_2\}$, $\mathcal{E} = \{e_1, e_2\}$. Suppose that $(\mathcal{F},\mathcal{E})$ and $(\mathcal{G},\mathcal{E})$ are two complex multi-fuzzy soft sets of dimension three over U defined as follows.

$$\begin{aligned}\mathcal{F}(e_1) &= \{x_1/(0.6e^{i(0.3\pi)}, 0.4e^{i(0.7\pi)}, 0.4e^{i(1.5\pi)}),\\ &\quad x_2/(0.1e^{i(\pi)}, 0.0e^{i(0.3\pi)}, 0.7e^{i(0.8\pi)})\},\\ \mathcal{F}(e_2) &= \{x_1/(0.8e^{i(\pi)}, 0.4e^{i(0.6\pi)}, 0.8e^{i(0.5\pi)}),\\ &\quad x_2/(0.7e^{i(0.3\pi)}, 0.1e^{i(1.1\pi)}, 0.5e^{i(0.5\pi)})\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}(e_1) &= \{x_1/(0.3e^{i(\pi)}, 0.5e^{i(0\pi)}, 0.2e^{i(1.0\pi)}),\\ &\quad x_2/(0.33e^{i(\pi)}, 0.3e^{i(\pi)}, 0.3e^{i(1.2\pi)})\},\\ \mathcal{G}(e_2) &= \{x_1/(0.4e^{i(0.5\pi)}, 0.4e^{i(0.1\pi)}, 0.4e^{i(0.8\pi)}),\\ &\quad x_2/(0.7e^{i(0.3\pi)}, 0.3e^{i(1.1\pi)}, 0.6e^{i(0.5\pi)})\}.\end{aligned}$$

Then we have

$$\begin{aligned}&\hat{S}^r_1((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))\\ &= 1 - \frac{1}{n}\sum_{p=1}^{n} max\{(|r^j_{\mathcal{F}(e_1)}(x_p) - r^j_{\mathcal{G}(e_1)}(x_p)|)_{j\in k}\},\\ &= 1 - \frac{1}{2}\left[\begin{array}{l} max(|0.6-0.3|, |0.4-0.5|, |0.4-0.2|)\\ +max(|0.1-0.33|, |0-0.3|, |0.7-0.3|)\end{array}\right]\\ &= 0.65.\end{aligned}$$

Similarly $\hat{S}^r_2((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) = 0.7$.
Then, we calculate $\hat{S}^\omega_1((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))$,

$$\begin{aligned}&\hat{S}^\omega_1((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))\\ &= 2\pi - \frac{1}{n}\sum_{p=1}^{n} max\{(|\omega^j_{\mathcal{F}(e_1)}(x_p) - \omega^j_{\mathcal{G}(e_1)}(x_p)|)_{j\in k}\},\\ &= 2\pi - \frac{1}{2}\left[\begin{array}{l} max(|0.3\pi-\pi|, |0.7\pi-0\pi|, |1.5\pi-1.0\pi|)\\ +max(|\pi-\pi|, |0.3\pi-0.7\pi|, |0.8\pi-1.2\pi|)\end{array}\right]\\ &= 1.3\pi.\end{aligned}$$

Similarly $\hat{S}^r_2((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) = 1.75\pi$.
Hence the degree of similarity between $(\mathcal{F},\mathcal{E})$ and $(\mathcal{F},\mathcal{E})$ is given by:

$$\begin{aligned}&\hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))\\ &= \frac{1}{2m}\sum_{l=1}^{m}[\hat{S}^r_l((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E})) + \frac{\hat{S}^\omega_l((\mathcal{F},\mathcal{E}),(\mathcal{G},\mathcal{E}))}{2\pi}],\\ &= \frac{1}{2\times 2}[(0.65 + \frac{1.3\pi}{2\pi}) + (0.7 + \frac{1.75\pi}{2\pi})]\\ &= 0.719.\end{aligned}$$

Theorem 30: Let $\hat{S}$ be a similarity measure of $\mathcal{CM}^k\mathcal{FSS}s$ as defined in Definition 25. Define

$$\hat{E}(\mathcal{F},\mathcal{E}) = \hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{F},\mathcal{E})^c)$$

Then $\hat{E}(\mathcal{F},\mathcal{E})$ is an entropy of $\mathcal{CM}^k\mathcal{FSS}s$.

Proof: It is sufficient to show that $\hat{E}(\mathcal{F},\mathcal{E})$ satisfies the four axioms given in Definition 25.

$(\hat{E}1)$ For any $(\mathcal{F},\mathcal{E}) \in \mathcal{CM}^{\mathrm{k}}\mathcal{FSS}(U)$, we have $\hat{E}(\mathcal{F},\mathcal{E}) = 0$

$$\begin{aligned}&\Longleftrightarrow \hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{F},\mathcal{E})^c) = 0\\ &\Longleftrightarrow r^j_{\mathcal{F}(e_l)}(x) = 1, \quad r^j_{\mathcal{F}^c(e_l)}(x) = 0\\ &\qquad \text{and } \omega^j_{\mathcal{F}(e_l)}(x) = 2\pi, \quad \omega^j_{\mathcal{F}^c(e_l)}(x) = 0,\\ &\qquad \forall e_l \in \mathcal{E}, x \in U, j = 1,2,\ldots,k.\\ &\Longleftrightarrow r^j_{\mathcal{F}(e_l)}(x) = 1 \text{ and } \omega^j_{\mathcal{F}(e_l)}(x) = 2\pi,\\ &\qquad \forall e_l \in \mathcal{E}, x \in U, j = 1,2,\ldots,k.\end{aligned}$$

$(\hat{E}2)$ For any $(\mathcal{F},\mathcal{E}) \in \mathcal{CM}^{\mathrm{k}}\mathcal{FSS}(U)$, we have $\hat{E}(\mathcal{F},\mathcal{E}) = 1$

$$\begin{aligned}&\Longleftrightarrow \hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{F},\mathcal{E})^c) = 1\\ &\Longleftrightarrow (\mathcal{F},\mathcal{E}) = (\mathcal{F},\mathcal{E})^c,\\ &\Longleftrightarrow r^j_{\mathcal{F}(e_l)}(x) = r^j_{\mathcal{F}^c(e_l)}(x) \text{ and } \omega^j_{\mathcal{F}(e_l)}(x) = \omega^j_{\mathcal{F}^c(e_l)}(x),\\ &\qquad \forall e \in \mathcal{E}, x \in U, j = 1,2,\ldots,k.\\ &\Longleftrightarrow r^j_{\mathcal{F}(e_l)}(x) = 1 - r^j_{\mathcal{F}(e_l)}(x) \text{ and}\\ &\qquad \omega^j_{\mathcal{F}(e_l)}(x) = 2\pi - \omega^j_{\mathcal{F}(e_l)}(x),\\ &\qquad \forall e \in \mathcal{E}, x \in U, j = 1,2,\ldots,k.\\ &\Longleftrightarrow r^j_{\mathcal{F}(e_l)}(x) = 0.5 \text{ and } \omega^j_{\mathcal{F}(e_l)}(x) = \pi,\\ &\qquad \forall e_l \in \mathcal{E}, x \in U, j = 1,2,\ldots,k.\end{aligned}$$

$(\hat{E}3)$ For any $(\mathcal{F},\mathcal{E}) \in \mathcal{CM}^{\mathrm{k}}\mathcal{FSS}(U)$,

$$\begin{aligned}\hat{E}((\mathcal{F},\mathcal{E})^c) &= \hat{S}((\mathcal{F},\mathcal{E})^c, ((\mathcal{F},\mathcal{E})^c)^c)\\ &= \hat{S}((\mathcal{F},\mathcal{E})^c, (\mathcal{F},\mathcal{E}))\\ &= \hat{S}((\mathcal{F},\mathcal{E}), (\mathcal{F},\mathcal{E})^c)\\ &= \hat{E}(\mathcal{F},\mathcal{E}).\end{aligned}$$

$(\hat{E}4)$ $\forall e \in \mathcal{A}, x \in U, j = 1,2,\ldots,k.$ when $(\mathcal{F},\mathcal{E}) \subseteq (\mathcal{G},\mathcal{E})$ and $r^j_{\mathcal{F}(e_l)}(x) \leq r^j_{\mathcal{G}(e_l)}(x)$, $\omega^j_{\mathcal{F}(e_l)}(x) \leq \omega^j_{\mathcal{G}(e_l)}(x)$,

$$\begin{aligned}&\Longrightarrow 1 - r^j_{\mathcal{F}(e_l)}(x) \geq 1 - r^j_{\mathcal{G}(e_l)}(x),\\ &\qquad \text{and } 2\pi - \omega^j_{\mathcal{F}(e_l)}(x) \geq 2\pi - \omega^j_{\mathcal{G}(e_l)}(x),\\ &\Longrightarrow |r^j_{\mathcal{F}(e_l)}(x) - [1 - r^j_{\mathcal{F}(e_l)}(x)]|\\ &\leq |r^j_{\mathcal{G}(e_l)}(x) - [1 - r^j_{\mathcal{G}(e_l)}(x)]|\\ &\qquad \text{and } |\omega^j_{\mathcal{F}(e_l)}(x) - [2\pi - \omega^j_{\mathcal{F}(e_l)}(x)]|\\ &\leq |\omega^j_{\mathcal{G}(e_l)}(x) - [2\pi - \omega^j_{\mathcal{G}(e_l)}(x)]|,\\ &\Longrightarrow |r^j_{\mathcal{F}(e_l)}(x) - r^j_{\mathcal{F}^c(e_l)}(x)| \leq |r^j_{\mathcal{G}(e_l)}(x) - r^j_{\mathcal{G}^c(e_l)}(x)|\\ &\qquad \text{and}\\ &\qquad |\omega^j_{\mathcal{F}(e_l)}(x) - \omega^j_{\mathcal{F}^c(e_l)}(x)| \leq |\omega^j_{\mathcal{G}(e_l)}(x) - \omega^j_{\mathcal{G}^c(e_l)}(x)|,\end{aligned}$$

$$\Longrightarrow 1 - |r^j_{\mathcal{F}(e_l)}(x) - r^j_{\mathcal{F}^c(e_l)}(x)| \geq 1$$
$$-|r^j_{\mathcal{G}(e_l)}(x) - r^j_{\mathcal{G}^c(e_l)}(x)|$$
and
$$2\pi - |\omega^j_{\mathcal{F}(e_l)}(x) - \omega^j_{\mathcal{F}^c(e_l)}(x)| \geq 2\pi$$
$$-|\omega^j_{\mathcal{G}(e_l)}(x) - \omega^j_{\mathcal{G}^c(e_l)}(x)|,$$

Thuse, $\hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{F},\mathcal{E})^c) \geq \hat{S}((\mathcal{G},\mathcal{E}),(\mathcal{G},\mathcal{E})^c)$.
So we have $\hat{E}(\mathcal{F},\mathcal{E}) \geq \hat{E}(\mathcal{G},\mathcal{E})$.
This completes the proof of Theorem 30.

Theorem 31: Let $\hat{S}$ be a similarity measure of $\mathcal{CM}^k\mathcal{FSS}s$ as defined in Definition 25. Define

$$\hat{E}(\mathcal{F},\mathcal{E}) = \frac{\hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{F},\mathcal{E})^c)}{2 - \hat{S}((\mathcal{F},\mathcal{E}),(\mathcal{F},\mathcal{E})^c)}$$

then $\hat{E}(\mathcal{F},\mathcal{E})$ is an entropy of $\mathcal{CM}^k\mathcal{FSS}s$.

Proof: It is similar to the proof of Theorem 30.

VIII. CONCLUSION

A new mathematical tool to model the information or data which is observed repeatedly over a period of time is developed. The complex multi-fuzzy soft set is established by incorporating the features of both complex fuzzy set and multi-fuzzy soft set. In addition, the complex multi-fuzzy soft set theory also extended the complex fuzzy soft set theory and complex intuitionistic fuzzy soft set theory. We then defined some fundamental operations on the CMFSS such as union and intersection. The basic properties and other relevant laws pertaining to the concept of CMFSSs were also discussed. A new general framework of CMFSS for dealing with uncertainty in decision making has thus been proposed and its associated algorithm constructed. This algorithm is then applied to determine the degree and the total time of the influence of the economic factors on the sectors that promotes the Malaysian economy and then deduced results that could help in making the decision in determining the most important factor. This new tool will provide a significant addition to existing theories for handling uncertainties, imprecision and vagueness of information by adding a crucial aspect, which is the time factor to measure not only the degree of influence on the economic factors but also the time of this influence, where time plays a vital role in the process of decision making. The structure of the CMFSS is also rehabilitated to describe the periodic data/information that have uncertain data where the amplitude terms represent the uncertainty and the phase terms represent the periodicity semantic. Thus, the CMFSS may provide a theoretical framework to represent problems with uncertainty and periodicity simultaneously in the field of engineering, medical, physics, automobiles, defense and security, and other fields. This new interpretation of the phase terms opens avenue for many applications in the field of physics and other natural sciences where phase terms may also represent the distance, temperature, pressure or any variable that affects and interacts with its corresponding amplitude terms in the decision process. Finally, beginning from our belief on the importance of the concepts of entropy and similarity measure, we have introduced axiomatic definitions of entropy and similarity measure of CMFSSs. Moreover, we proposed new formulas to calculate the entropy and the similarity measure of CMFSS, and proved some theorems that the similarity measure and the entropy of CMFSS can be transformed into each other based on their axiomatic definitions. These measures will be useful to handle several realistic uncertainty problems such as problems in social, economic system, approximate reasoning, image processing, game theory, and others.

ACKNOWLEDGMENT

We are thankful to Professor Kamarulzaman Ibrahim of Universiti Kebangsaan Malaysia who contributed in the improvement of our manuscript.

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