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Group Decision Making With Probabilistic Hesitant Multiplicative Preference Relations Based on Consistency and Consensus

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ABSTRACT The probabilistic hesitant multiplicative preference relations (PHMPRs) can model the opinions of the decision makers for pairwise comparisons over alternatives using some possible values from Saaty's 1–9 scale with the probability information. The existing study normalized the PHMPRs by adding additional preference values with the probability information into the shorter element before modifying the consistency and consensus. As a supplement, in this paper, we focus on dealing with the consistency and consensus for PHMPRs by means of multiplicative preference relations. We first present the definition of geometric consistency index for PHMPRs and develop an automatic iterative algorithm for checking and improving the geometric consistency index for PHMPRs. We also put forward the concept of geometric consensus degree for PHMPRs and then devise an automatic consensus reaching algorithm to modify the geometric consensus degree of PHMPRs. After that, a novel complete group decision making model with PHMPRs is put forward. Finally, an illustrative example is shown to verify the proposed model and we compare it with the existing study to show its superiority.

INDEX TERMS Hesitant multiplicative preference relation, probabilistic hesitant fuzzy preference relation, probabilistic hesitant multiplicative preference relation, hesitant fuzzy set, consistency.

I. INTRODUCTION

Decision making happens frequently in our daily life. For example, when people plan to buy a house, he or she should assess given alternatives of houses and then make a choice from them. With the increasing complexity of the decision making problems, a single person cannot make a reasonable result. In this case, it requires the involvement of a group of decision makers to complete the decision making processes [1], [2]. Preference relations (PRs) are an important tool of modeling the opinions of the decision makers for pairwise comparisons over a set of alternatives [3]. It has become the research spot in the decision making. Based on the methods used to model the preference information, the PRs could be divided into three types: linguistic PRs [4], fuzzy PRs [5], and multiplicative PRs [6].

The multiplicative PRs model the opinions of the decision makers using the preference values from Saaty's 1-9 scale [7], [8]. To model the uncertainty of the evaluation information, the concept of interval multiplicative PRs was studied [9]. To model the preference values containing the non-membership degree, the concept of intuitionistic multiplicative preference relations was proposed [10]. In some cases, decision makers may hesitate among some preference values when giving the preference information on the pairwise comparisons over the alternatives. To model this kind of hesitancy, the definition of hesitant multiplicative preference relations (HMPRs) was given [11], [12]. Bashir *et al.* [13] used the probability information to extend HMPRs into probabilistic hesitant multiplicative preference relations (PHMPRs) that allow decision makers to give some preference values from Saaty's 1-9 scale with the probability information. For example, when evaluating the intensity of the alternative x_i over the alternative x_j, an decision maker hesitates between 5 and 7 to provide his or her preference information. The probability assigned to 5 is 0.8, while the probability for 7 is 0.2. Thus, the intensity of the alternative x_i over the alternative x_j can be modeled as $\{5|0.8, 7|0.2\}$.

To better model the uncertainty under the hesitant fuzzy environment [34]–[36], the concept of probabilistic hesitant fuzzy preference relations (PHFPRs) has been devised [37]. The main difference between the PHMPRs and PHFPRs is the use of evaluation scale. The PHMPRs consist of several possible values from the Saaty's 1-9 scale along with their probability information, while the PHFPRs are composed of some possible values from the unit interval [0,1] associated with their probability information. The concept of PHFPRs has been paid attention by many scholars [38]–[40].

The consistency has a great impact on the decision results of the decision making models with PRs [14]. If there are contradictory preference information in the PRs, then the decision results will be unreasonable. Existing studies show that the decision making models with acceptably consistent PRs can obtain the reasonable decision results [15]. To make the inconsistent PRs be acceptably consistent, two main types of methods have been studied to adjust the inconsistent PRs, which are the programming ones [16] and the iterative ones [17]. However, these methods are designed to the fuzzy PRs, the multiplicative PRs [28]–[30], the hesitant multiplicative PRs [31]–[33], and the hesitant fuzzy PRs [41]–[43]. Hence, they cannot deal with the PHMPRs since PHMPRs contain probability information.

The consensus means the degree of agreement among a group of decision makers [18]. It also plays an important role on the decision results of group decision making problems with PRs [19]. In the group decision making process, a group of decision makers, coming from different professional fields, may have diverse opinions and then give different preference information. The serious differences among these preference information would degrade the quality of decision results. To solve it, many studies have been made to adjust the PRs with very low consensus [20]–[22].

However, the above studies do not solve the problems of consistency and consensus of PHMPRs. In this case, Bashir *et al.* [13] normalized the PHMPRs so that each PHME has the same number of elements and defined the distance between each PHMPR and its consistent PHMPR as the consistency index. They gave two iterative algorithms to modify the inconsistent PHMPRs and the PHMPRs with low consensus degree. As a supplement, in this paper, we also focus on the consistency and consensus of the PHMPRs. By means of the multiplicative preference relations, we utilize the logarithmic least squares model to obtain the priority vectors from the PHMPRs instead of computing the consistent PHMPRs and then develop an automatic consistency improving process to modify the inconsistent PHMPRs. At the same time, we also define the geometric consensus degree for the PHMPRs and develop an automatic consensus reaching process to adjust the acceptably consistent PHMPRs with the low consensus. After that, these two processes are combined to develop a complete group decision making model with the PHMPRs. Finally, an illustrative example is introduced to validate the proposed model and it is compared with the existing study.

The rest of this paper is structured as follows: Section II presents the definitions and the operational laws of hesitant multiplicative sets, probabilistic hesitant multiplicative sets, and probabilistic hesitant multiplicative preference relations. An automatic consistency improving process is proposed in Section III. In Section VI, an automatic consensus reaching process is given. In Section V, a complete group decision making model with PHMPRs is designed. In Section VI, an illustrative case is offered to verify the proposed model and it is also compared with the existing study in [13]. Finally, the conclusions are presented in Section VII.

II. PRELIMINARIES

To model the preference information that consists of some possible preference values from Saaty's 1-9 scale, Xia and Xu [12] gave the concept of hesitant multiplicative sets (HMSs), which can be described mathematically as:

Definition 1 [12]: A set X is given, then the mathematical expression of a HMS on X is

$$H = \{\langle x, h(x)\rangle \,|x \in X\}$$

where $h(x) = \{r|r \in [1/9, 9]\}$ is a set that consists of some possible values from the Saaty's 1-9 scale. It describes the possible membership degrees of the element x belonging to H. The element $h = h(x)$ is called a hesitant multiplicative element (HME) and all the HMEs form a HMS H.

Example 2: Assume that a decision maker evaluates a car with respect to its two attributes, which are the comfort and price. Then these attributes can be denoted as $X = \{x_1, x_2\}$. As for the comfort x_1, the decision maker hesitates between 5 and 7 when providing the preference information. Thus, $h(x_1) = \{5, 7\}$. When the price is assessed, he or she may hesitate between $1/3$ and $1/5$, then $h(x_2) = \{1/5, 1/3\}$.

However, if the expert shows the probability of 5 is 0.8 and that of 7 is 0.2, then the HMSs cannot model this case. To extend the modeling capability of HMSs, Bashir *et al.* [13] gave the concept of probabilistic hesitant multiplicative sets (PHMSs) as follows:

Definition 3 [13]: A reference set X is provided, then a probabilistic hesitant multiplicative set (PHMS) is defined as

$$H = \{\langle x, h(x)\rangle \,|x \in X\}$$

where $h(x) = \{r_l|p_l\}, \ l = 1, 2, \ldots, \#h$ with $r_l \in [1/9, 9]$ and $p_l \in [0, 1]$. $h(x)$ is composed of some possible membership degrees with their probabilities and it is considered as the probabilistic hesitant multiplicative element (PHME). The term p_l denotes the probability of the membership degree r_l.

To compare any two PHMEs, we put forward the score function and the deviation function for the PHMEs as follows:

Definition 4: Given a PHME $h = \{r_l | p_l\}$, $l = 1, 2, \ldots, \#h$, then the score function is computed as:

$$S(h) = \prod_{l=1}^{\#h} (r_l)^{p_l}$$

where $S(h)$ denotes the score value of the PHME h.

Definition 5: Given a PHME $h = \{r_l | p_l\}$, $l = 1, 2, \ldots, \#h$, then the variance function is computed as:

$$V(h) = \prod_{l=1}^{\#h} \left(\frac{r_l}{S(h)}\right)^{p_l}$$

where $V(h)$ denotes the variance value of the PHME h.

Based on Definitions 4 and 5, the comparison method for comparing two PHMEs h_1 and h_2 is developed as:

(1) If $S(h_1) > S(h_2)$, then $h_1 > h_2$;

(2) If $S(h_1) = S(h_2)$, then it needs to compare the variance values of h_1 and h_2;

1) If $V(h_1) > V(h_2)$, then $h_1 \leq h_2$;

2) If $V(h_1) = V(h_2)$, then $h_1 = h_2$.

To model the preference information though the pairwise comparisons on alternatives, Bashir *et al.* [13] presented the definition of probabilistic hesitant multiplicative preference relations (PHMPRs) as follows:

Definition 6 [13]: Let $X = \{x_1, x_2, \ldots, x_n\}$ denote a set of alternatives, then a PHMPR on X is defined as a matrix $H = (h_{ij})_{n\times n}$, where $h_{ij} = \left\{r_{ij}^l | p_{ij}^l\right\}$ is a PHME describing the intensity of the alternative x_i over the alternative x_j. For any i, j, each PHME satisfies the following conditions:

$$r_{ij}^{\rho(l)} r_{ji}^{\rho(l)} = 1, \quad p_{ij}^{\rho(l)} = p_{ji}^{\rho(l)}, \quad \#h_{ij} = \#h_{ji}, \quad h_{ii}^l = 1$$

and

$$r_{ij}^{\rho(l)} < r_{ij}^{\rho(l+1)}, \quad r_{ji}^{\rho(l+1)} < r_{ji}^{\rho(l)}$$

where $r_{ij}^{\rho(l)}$ denotes the lth least element in h_{ij}.

From Definition 6, some conclusions can be derived as follows:

(1) Each element in the PHMPR is a PHME;

(2) The preference values in a PHME of the upper triangular matrix are arranged according to the ascending order of the membership degree, while the preference values in a PHME of the lower triangular matrix are arranged according to the descending order of the membership degree.

III. CONSISTENCY IMPROVING PROCESS FOR PHMPRS

A. CONSISTENCY MEASURE

Firstly, we review the definition of multiplicative preference relations (MPRs) [23] as follows:

Definition 7 [23]: Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of alternatives, then a multiplicative preference relation (MPR) on X is a matrix $M = (m_{ij})_{n\times n} \subset X \times X$ with the element $m_{ij} \in \left[\frac{1}{9}, 9\right]$ and its element m_{ij} satisfies $m_{ij}m_{ji} = 1$ for any $i, j = 1, 2, \ldots, n$.

Inspired by the idea of expected fuzzy preference relation proposed by Wu *et al.* [24], we utilize the score function of PHMSs to give the definition of the geometric probabilistic hesitant multiplicative preference relations (GPHMPRs).

Definition 8: Given a PHMPR $H = \left(r_{ij}^l | p_{ij}^l\right)_{n\times n}$, then its geometric probabilistic hesitant multiplicative preference relation is defined as $E = (e_{ij})_{n\times n}$ with $e_{ij} = \prod_{l=1}^{\#h_{ij}} \left(r_{ij}^l\right)^{p_{ij}^l}$.

From Definition 8, a theorem can be obtained.

Theorem 9: Given a PHMPR $H = \left(r_{ij}^l | p_{ij}^l\right)_{n\times n}$ and its GPHMPR $E = (e_{ij})_{n\times n}$, then the GPHMPR E is a MPR.

Proof:

(1) According to Definition 8, we have

$$e_{ij}e_{ji} = \prod_{l=1}^{\#h_{ij}} \left(r_{ij}^l\right)^{p_{ij}^l} \prod_{l=1}^{\#h_{ji}} \left(r_{ji}^l\right)^{p_{ji}^l}$$

Since $\#h_{ij} = \#h_{ji}$, $r_{ij}^l r_{ji}^l = 1$, and $p_{ij}^l = p_{ji}^l$, then

$$e_{ij}e_{ji} = \prod_{l=1}^{\#h_{ij}} \left(r_{ij}^l r_{ji}^l\right)^{p_{ij}^l} = 1$$

(2) $e_{ii} = \prod_{l=1}^{\#h_{ii}} \left(r_{ii}^l\right)^{p_{ii}^l} = \prod_{l=1}^{\#h_{ii}} (1)^{p_{ii}^l} = 1$.

which completes the theorem.

The definition of the group geometric probabilistic hesitant multiplicative preference relations (GGPHMPRs) is given as follows.

Definition 10: Let $H_k = (h_{ij,k})_{n\times n} = \left(r_{ij,k}^l | p_{ij,k}^l\right)_{n\times n}$, $k = 1, 2, \ldots, m$ be m PHMPRs and λ_k be their weights with $\lambda_k \in [0, 1]$ and $\sum_{k=1}^m \lambda_k = 1$, then the group geometric probabilistic hesitant multiplicative preference relation (GGPHMPR) denoted by $G = (g_{ij})_{n\times n}$ satisfies

$$g_{ij} = \prod_{k=1}^m (e_{ij,k})^{\lambda_k} = \prod_{k=1}^m \left(\prod_{l=1}^{\#h_{ij,k}} \left(r_{ij,k}^l\right)^{p_{ij,k}^l}\right)^{\lambda_k} \tag{3.1}$$

where $\#h_{ij,k}$ denotes the number of elements in $h_{ij,k}$.

According to the multiplicative transitivity of MPRs [23], the consistent GPHMPR is defined as:

Definition 11: Given a PHMPR $H = \left(r_{ij}^l | p_{ij}^l\right)_{n\times n}$ and its GPHMPR $E = (e_{ij})_{n\times n}$, if it satisfies

$$e_{ij} = e_{ik} \otimes e_{kj}, \quad i, k, j = 1, 2, \ldots, n \tag{3.2}$$

then E is called a consistent GPHMPR.

From (3.2), we can derive that

$$e_{ij} = \frac{w_i}{w_j}, \quad i, j = 1, 2, \ldots, n \tag{3.3}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ denotes the priority vector of the GPHMPR E and $\sum_{i=1}^n w_i = 1$, $w_i > 0$, $i = 1, 2, \ldots, n$.

Equation (3.3) can be transformed into $e_{ij}w_j = w_i$, which is equivalent to

$$\ln e_{ij} + \ln w_j = \ln w_i$$

Then the value of $\left|\ln e_{ij} + \ln w_j - \ln w_i\right|$ can be utilized to measure and define the consistency level of the PHMPR H. The smaller value the equation $\left|\ln e_{ij} + \ln w_j - \ln w_i\right|$ is, the more consistent the PHMPR is. If it equals to 0, the PHMPR is considered to be consistent.

Motivated by the definition of the geometric consistency index in [25], we give the concept of geometric consistency index (GCI) for PHMPRs.

Definition 12: Given a PHMPR $H = \left(r_{ij}^l | p_{ij}^l\right)_{n\times n}$, its GPHMPR $E = \left(e_{ij}\right)_{n\times n}$, the priority vector of the PHMPR H denoted by $w = (w_1, w_2, \ldots, w_n)^T$ with $\sum_{i=1}^{n} w_i = 1, w_i > 0$, then the geometric consistency index for the PHMPR H is computed as:

$$GCI(H) = GCI(E) = \frac{2}{(n-1)(n-2)} \sum_{i<j} \left(\ln e_{ij} + \ln w_j - \ln w_i\right)^2 \quad (3.4)$$

where $GCI(H)$ denotes the geometric consistency index of the PHMPR H.

From (3.4), it can be seen that the geometric consistency index $GCI(H) \geq 0$ for each PHMPR H. If $GCI(H) = 0$, then the PHMPR H is completely consistent. The larger value $GCI(H)$ is, the less consistent the PHMPR H is.

In the real applications, it is difficult for the decision makers to provide completely consistent PHMPRs. However, investigations show that the decision making models with acceptably consistent PHMPRs can also derive reasonable decision results [13].

Definition 13: Given a PHMPR $H = \left(r_{ij}^l | p_{ij}^l\right)_{n\times n}$ and a threshold expressed as $G\bar{CI}$ for the geometric consistency index, if the geometric consistency index $GCI(H)$ of the PHMPR H satisfies $GCI(H) \leq G\bar{CI}$, then it is called the PHMPR with the acceptable consistency or the acceptably consistent PHMPR.

More information about the threshold for the geometric consistency index can be obtained from [25].

Inspired by the method for deriving the priority vector presented in [26], we develop a logarithmic least squares model to derive the priority vector from the PHMPRs:

$$\text{(MOD 1)}\ Min\ J = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\ln e_{ij} + \ln w_j - \ln w_i\right)^2$$
$$\text{s.t.} \sum_{i=1}^{n} w_i = 1, \quad w_i \geq 0,\ i = 1, 2, \ldots, n$$

We make some transformations on J and get

$$\begin{aligned} J &= \sum_{i=1}^{n} \sum_{j=1}^{n} (\ln e_{ij} + \ln w_j - \ln w_i)^2 \\ &= \sum_{i<j}^{n} (\ln e_{ij} + \ln w_j - \ln w_i)^2 + \sum_{i>j}^{n} (\ln e_{ij} + \ln w_j - \ln w_i)^2 \\ &= \sum_{i<j}^{n} (\ln e_{ij} + \ln w_j - \ln w_i)^2 + \sum_{j>i}^{n} (-(\ln e_{ji} + \ln w_j - \ln w_i))^2 \\ &= \sum_{i<j}^{n} (\ln e_{ij} + \ln w_j - \ln w_i)^2 + \sum_{I<j}^{n} (\ln e_{ij} + \ln w_j - \ln w_i)^2 \\ &= 2\sum_{i<j}^{n} (\ln e_{ij} + \ln w_j - \ln w_i)^2 \end{aligned}$$

Thus, (MOD 1) can be rewritten as:

$$\text{(MOD 2)}\ Min\ J = 2\sum_{i<j}^{n} (\ln e_{ij} + \ln w_j - \ln w_i)^2$$
$$\text{s.t.} \sum_{i=1}^{n} w_i = 1, \quad w_i \geq 0,\ i = 1, 2, \ldots, n$$

which is equivalent to

$$\text{(MOD 3)}\ Min\ J = \sum_{i<j}^{n} (\ln e_{ij} + \ln w_j - \ln w_i)^2$$
$$\text{s.t.} \sum_{i=1}^{n} w_i = 1, \quad w_i \geq 0,\ i = 1, 2, \ldots, n$$

Referring to [26], the solution of (MOD 3) is

$$w_i = \frac{\left(\prod_{j=1}^{n} e_{ij}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n} \left(\prod_{j=1}^{n} e_{ij}\right)^{\frac{1}{n}}} \quad (3.5)$$

According to Definitions 10 and 11, a theorem about the priority vector of GGPHMPRs is given as follows:

Theorem 10: Let $H_k = \left(h_{ij,k}\right)_{n\times n} = \left(r_{ij,k}^l | p_{ij,k}^l\right)_{n\times n}$, $k = 1, 2, \ldots, m$ be m PHMPRs with the priority vectors $w^k = \left(w_1^k, w_2^k, \ldots, w_n^k\right)$, λ_k be their weights satisfying that $0 \leq \lambda_k \leq 1$ and $\sum_{k=1}^{m} \lambda_k = 1$, $E_k = \left(e_{ij,k}\right)_{n\times n}$ be their GPHMPRs, and $G = \left(g_{ij}\right)_{n\times n}$ be the GGPHMPR , then the priority vector of the GGPHMPR is $w_g = \left(w_{1,g}, w_{2,g}, \ldots, w_{n,g}\right)$ satisfying

$$w_{i,g} = \frac{\prod_{k=1}^{m} (w_i^k)^{\lambda_k}}{\sum_{i=1}^{n} \prod_{k=1}^{m} (w_i^k)^{\lambda_k}} \quad (3.6)$$

where $w_{i,g}$ denotes the ith element in the priority vector w_g.

Proof: According to Definition 10, we have

$$\frac{g_{ij}}{g_{ji}} = \frac{\prod_{k=1}^{m} \left(e_{ij,k}\right)^{\lambda_k}}{\prod_{k=1}^{m} \left(e_{ji,k}\right)^{\lambda_k}}$$

Then

$$w_{i,g} = \frac{\left(\prod_{j=1}^{n} g_{ij}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} g_{ij}\right)^{\frac{1}{n}}} = \frac{\left(\prod_{j=1}^{n}\prod_{k=1}^{m}\left(e_{ij,k}\right)^{\lambda_k}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n}\prod_{k=1}^{m}\left(e_{ij,k}\right)^{\lambda_k}\right)^{\frac{1}{n}}}$$

$$= \frac{\prod_{k=1}^{m}\left(\prod_{j=1}^{n}\left(e_{ij,k}\right)^{\lambda_k}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\prod_{k=1}^{m}\left(\prod_{j=1}^{n}\left(e_{ij,k}\right)^{\lambda_k}\right)^{\frac{1}{n}}}$$

$$= \frac{\prod_{k=1}^{m}\left(\prod_{j=1}^{n}\left(e_{ij,k}\right)^{\frac{1}{n}}\right)^{\lambda_k} \Big/ \sum_{i=1}^{n}\prod_{k=1}^{m}\left(e_{ij,k}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\prod_{k=1}^{m}\left(\prod_{j=1}^{n}\left(e_{ij,k}\right)^{\frac{1}{n}}\right)^{\lambda_k} \Big/ \sum_{i=1}^{n}\prod_{k=1}^{m}\left(e_{ij,k}\right)^{\frac{1}{n}}}$$

$$= \frac{\prod_{k=1}^{m}\left(\frac{\prod_{j=1}^{n}\left(e_{ij,k}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\prod_{k=1}^{m}\left(e_{ij,k}\right)^{\frac{1}{n}}}\right)^{\lambda_k}}{\sum_{i=1}^{n}\prod_{k=1}^{m}\left(\frac{\prod_{j=1}^{n}\left(e_{ij,k}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\prod_{k=1}^{m}\left(e_{ij,k}\right)^{\frac{1}{n}}}\right)^{\lambda_k}} = \frac{\prod_{k=1}^{m}\left(w_i^k\right)^{\lambda_k}}{\sum_{i=1}^{n}\prod_{k=1}^{m}\left(w_i^k\right)^{\lambda_k}}$$

which completes the proof of the theorem.

B. AUTOMATIC CONSISTENCY CHECKING AND IMPROVING PROCESS

In the real decision making problems, the perfectly consistent preference relations cannot be provided. Existing researches support that acceptably consistent preference relations can also derive the reasonable decision making results [27]. Thus, in this section, an automatic consistency improving process is put forward to check the geometric consistency index of a given PHMPR and then adjust the preference value until it is acceptably consistent. The basic idea of this algorithm is to obtain the priority vector from a given PHMPR using (MOD 3), form a complete consistent preference relation, and adjust the preference value of the given PHMPR so as to make it be closer to the complete consistent one. Based on this idea, this algorithm is developed as follows:

Three theorems could be derived from Algorithm 1 as

Theorem 15: Let $H = \left(r_{ij}^l | p_{ij}^l\right)_{n\times n}$ be an inconsistent PHMPR, $H^{(t)}$ and $H^{(t+1)}$ be two modified PHMPRs after t and $t+1$ iterations, then $GCI\left(H^{(t+1)}\right) < GCI\left(H^{(t)}\right)$.

Proof: According to (3.10), we have

$$GCI\left(H^{(t+1)}\right) = \frac{2}{(n-1)(n-2)}\sum_{i<j}\left(\ln e_{ij}^{(t+1)} + \ln w_j^{(t)} - \ln w_i^{(t)}\right)^2$$

$$= \frac{2}{(n-1)(n-2)} \times \sum_{i<j}\left(\ln\left(e_{ij}^{(t)}\right)^{1-\theta}\left(\frac{w_i^{(t)}}{w_j^{(t)}}\right)^{\theta} + \ln w_j^{(t)} - \ln w_i^{(t)}\right)^2$$

$$= \frac{2}{(n-1)(n-2)} \times \sum_{i<j}\left(\ln\left(e_{ij}^{(t)}\right)^{1-\theta} + \ln\left(w_i^{(t)}\right)^{\theta} - \ln\left(w_j^{(t)}\right)^{\theta} + \ln w_j^{(t)} - \ln w_i^{(t)}\right)^2$$

Algorithm 1 Consistency Checking and Improving Algorithm

Input: A given PHMPR $H = \left(r_{ij}^l | p_{ij}^l\right)_{n\times n}$, an optimized parameter $\theta \in [0, 1]$, the maximum number of iterations denoted by T, the threshold for geometric consistency index denoted by $G\bar{C}I$.

Output: An acceptably consistent PHMPR $\hat{H}$.

Step 1. Let $t = 0$ and $H^{(t)} = H = \left(r_{ij}^l | p_{ij}^l\right)_{n\times n}$;

Step 2. Use Definition 8 to calculate the GPHMPR $E^{(t)} = \left(e_{ij}^{(t)}\right)_{n\times n}$ of $H^{(t)}$, which satisfies

$$e_{ij}^{(t)} = \prod_{l=1}^{\#h_{ij}}\left(\left(r_{ij}^l\right)^{(t)}\right)^{p_{ij}^l} \tag{3.7}$$

Step 3. Use (3.5) to calculate the priority vector of the GPHMPR $E^{(t)}$ as:

$$w_i^{(t)} = \frac{\left(\prod_{j=1}^{n} e_{ij}^{(t)}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} e_{ij}^{(t)}\right)^{\frac{1}{n}}} \tag{3.8}$$

Step 4. Calculate the geometric consistency index of the PHMPR $H^{(t)}$ as:

$$GCI\left(H^{(t)}\right) = GCI\left(E^{(t)}\right) = \frac{2}{(n-1)(n-2)} \times \sum_{i<j}\left(\ln e_{ij}^{(t)} + \ln w_j^{(t)} - \ln w_i^{(t)}\right)^2 \tag{3.9}$$

If $GCI(H) \le G\bar{C}I$, then turn to Step 6; otherwise, turn to the next step.

Step 5. Build the modified GPHMPR $E^{(t+1)} = \left(e_{ij}^{(t+1)}\right)_{n\times n}$, which satisfies

$$e_{ij}^{(t+1)} = \left(e_{ij}^{(t)}\right)^{(1-\theta)} \times \left(\frac{w_i}{w_j}\right)^{\theta} \tag{3.10}$$

Let $t = t + 1$, turn to Step 4;

Step 6. Let $\hat{H} = H^{(t)}$ and output $\hat{H}$;

Step 7. End.

$$= \frac{2}{(n-1)(n-2)} \times \sum_{i<j}\left(\ln\left(e_{ij}^{(t)}\right)^{1-\theta} - \ln\left(w_i^{(t)}\right)^{1-\theta} + \ln\left(w_j^{(t)}\right)^{1-\theta}\right)^2$$
$$= \frac{2(1-\theta)^2}{(n-1)(n-2)}\sum_{i<j}\left(\ln e_{ij}^{(t)} - \ln w_i^{(t)} + \ln w_j^{(t)}\right)^2$$
$$= (1-\theta)^2\, GCI\left(H^{(t)}\right) \le GCI\left(H^{(t)}\right)$$

which completes the proof of theorem.

Theorem 16: Let $H_k = \left(h_{ij,k}\right)_{n\times n} = \left(r_{ij,k}^l | p_{ij,k}^l\right)_{n\times n}$, $k = 1, 2, \ldots, m$ be m PHMPRs with the priority vectors $w^k = \left(w_1^k, w_2^k, \ldots, w_n^k\right)$, λ_k be the weights satisfying that $0 \le \lambda_k \le 1$ and $\sum_{k=1}^{m}\lambda_k = 1$, $E_k = \left(e_{ij,k}\right)_{n\times n}$ be the corresponding GPHMPRs, $G = \left(g_{ij}\right)_{n\times n}$ be their corresponding GGPHMPR with the priority vector $w_g = \left(w_{1,g}, w_{2,g}, \ldots, w_{n,g}\right)$, then $GCI\,(G) \le \max_k GCI\,(H_k)$.

Proof: According to Definition 12, we have

$$\max_k GCI\,(H_k) = \frac{2}{(n-1)(n-2)}\max_k\left\{\sum_{i<j}\left(\ln e_{ij,k} + \ln w_i^k + \ln w_j^k\right)^2\right\}$$

Let $\varepsilon_{ij}^k = \ln e_{ij,k} + \ln w_i^k - \ln w_j^k$, then

$$\max_k GCI(H^k) = \frac{2}{(n-1)(n-2)}\max_k\left\{\sum_{i<j}\left(\varepsilon_{ij}^k\right)^2\right\}$$

Similarly,

$$GCI\,(G) = \frac{2}{(n-1)(n-2)}\sum_{i<j}\left(\ln g_{ij} + \ln w_{j,g} - \ln w_{i,g}\right)^2$$
$$= \frac{2}{(n-1)(n-2)}\sum_{i<j}\left(\ln g_{ij} - \ln\frac{w_{i,g}}{w_{j,g}}\right)^2$$
$$= \frac{2}{(n-1)(n-2)}\sum_{i<j}\left(\ln\prod_{k=1}^{m}\left(e_{ij,k}\right)^{\lambda_k} - \ln\prod_{k=1}^{m}\left(\frac{w_i^k}{w_j^k}\right)^{\lambda_k}\right)^2$$
$$= \frac{2}{(n-1)(n-2)}\sum_{i<j}\sum_{k=1}^{m}(\lambda_k)^2\left(\ln e_{ij,k} - \ln\frac{w_i^k}{w_j^k}\right)^2$$
$$= \frac{2}{(n-1)(n-2)}\sum_{k=1}^{m}(\lambda_k)^2\sum_{i<j}\left(\varepsilon_{ij}^k\right)^2$$
$$\le \frac{2}{(n-1)(n-2)}\sum_{k=1}^{m}(\lambda_k)^2\max_k\left\{\sum_{i<j}\left(\varepsilon_{ij}^k\right)^2\right\}$$
$$\le \frac{2}{(n-1)(n-2)}\left(\sum_{k=1}^{m}\lambda_k\right)^2\max_k\left\{\sum_{i<j}\left(\varepsilon_{ij}^k\right)^2\right\}$$
$$= \frac{2}{(n-1)(n-2)}\max_k\left\{\sum_{i<j}\left(\varepsilon_{ij}^k\right)^2\right\}$$
$$= \max_k GCI\,(H_k)$$

which completes the proof Theorem 16.

Theorem 17: Let $H_k = \left(h_{ij,k}\right)_{n\times n} = \left(r_{ij,k}^l | p_{ij,k}^l\right)_{n\times n}$, $k = 1, 2, \ldots, m$ be m PHMPRs and $G = \left(g_{ij}\right)_{n\times n}$ be the GGPHMPR, if H_k is acceptably consistent for each k, then their GGPHMPR G is also acceptably consistent.

Proof: According to Theorem 16, we have

$$GCI\,(G) \le \max_k GCI\,(H_k)$$

Since $GCI\,(H_k) \le \overline{GCI}$, then

$$GCI\,(G) \le \max_k GCI\,(H_k) \le \overline{GCI}$$

which completes the proof the theorem.

To demonstrate the implementation process of Algorithm 1, an example is given as follows:

Example 18: Assume that an expert gives a PHMPR on the set of alternatives $X = \{x_1, x_2, x_3\}$ as:

$$H = \begin{Bmatrix} \{1\} & \{3|0.6, 5|0.4\} & \{3\} \\ \{1/3|0.6, 1/5|0.4\} & \{1\} & \{1/7|0.6, 1/3|0.4\} \\ \{1/3\} & \{7|0.4, 3|0.6\} & \{1\} \end{Bmatrix}$$

Step 1: Let $t = 0$ and $H^{(0)} = H = \left(r_{ij}^l | p_{ij}^l\right)_{n\times n}$;

Step 2: Use (3.7) to derive the GPHMPR $E^{(0)} = \left(e_{ij}^{(0)}\right)_{n\times n}$ of $H^{(0)}$ as:

$$E^{(0)} = \begin{Bmatrix} 1 & 3.68 & 3.00 \\ 0.27 & 1 & 0.24 \\ 0.33 & 4.21 & 1 \end{Bmatrix}$$

Step 3: Use (3.8) to calculate the priority vector of the GPHMPR $E^{(0)}$ as:

$$w = (0.5942, 0.1070, 0.2988)^T$$

Step 4: Use (3.9) to calculate the geometric consistency index of the PHMPR $H^{(0)}$ as:

$$GCI(H^{(0)}) = 0.5069$$

Referring to [25], we set $\overline{GCI} = 0.1573$. $GCI\left(H^{(0)}\right) \ge \overline{GCI}$, then we continue the process.

Step 5: Let $\theta = 0.6$ and then use (3.10) to construct the modified GPHMPR $E^{(1)} = \left(e_{ij}^{(1)}\right)_{n\times n}$ as:

$$E^{(1)} = \begin{Bmatrix} 1 & 4.71 & 2.34 \\ 0.21 & 1 & 0.30 \\ 0.43 & 3.29 & 1 \end{Bmatrix}$$

Let $t = 1$, turn to the next step;

Step 6: Use (3.9) to calculate the geometric consistency index of the PHMPR $H^{(1)}$ as:

$$GCI(H^{(1)}) = 0.0811 < \overline{GCI}$$

Step 7: Let $\hat{H} = H^{(1)}$ and output $\hat{H}$.

IV. CONSENSUS REACHING PROCESS FOR PHMPRS

During the real group decision making problems, the experts from the different professional fields usually have the diverse opinions and then provide PHMPRs with very low consensus. The consensus also plays an important role on the decision-making results. To obtain reasonable decision-making results, in this section, an iterative consensus reaching process based on geometric consensus is put forward to adjust the PHMPRs until they satisfy the acceptable consensus condition.

We first give the definition of geometric consensus degree for PHMPRs as follows:

Definition 19: Let $H_k = (h_{ij,k})_{n\times n} = \left(r^l_{ij,k}|p^l_{ij,k}\right)_{n\times n}$, $k = 1, 2, \ldots, m$ be m PHMPRs with the weight λ_k meeting $0 \le \lambda_k \le 1$ and $\sum_{k=1}^{m} \lambda_k = 1$, $E_k = (e_{ij,k})_{n\times n}$ be the GPHMPRs, and $G = (g_{ij})_{n\times n}$ be their corresponding GGPHMPR with the priority vector $w_g = (w_{1,g}, w_{2,g}, \ldots, w_{n,g})^T$, then the geometric consensus degree of H_k is computed as:

$$GCD(H_k) = GCD(E_k) = \frac{2}{(n-1)(n-2)} \sum_{i<j} \left(\ln e_{ij,k} + \ln w_{j,g} - \ln w_{i,g}\right)^2 \tag{4.1}$$

where $GCD(H_k)$ means the geometric consensus degree of H_k. As presented in Definition 19, it can be seen that the geometric consensus degree of H_k can be considered as the closeness degree between its GPHMPR and GGPHMPR.

Based on Definition 19, an algorithm for the consensus reaching process is put forward to modify the PHMPRs as follows:

Algorithm 2 Consensus Reaching Algorithm

Input: m PHMPRs $H_k = (h_{ij,k})_{n\times n} = \left(r^l_{ij,k}|p^l_{ij,k}\right)_{n\times n}$ with the weight λ_k satisfying $0 \le \lambda_k \le 1$ and $\sum_{k=1}^{m} \lambda_k = 1$, a parameter $\theta \in [0, 1]$, the threshold for geometric consensus degree that is expressed by $\overline{GCD}$, the maximum number of iterations denoted by T.

Output: The PHMPRs having the geometric consensus degrees less than or equal to $\overline{GCD}$.

Step 1. Let $t = 0$ and $H^{(t)}_k = H_k$;

Step 2. Use Definition 10 to compute the group geometric PHMPR (GGPHMPR) $G^{(t)} = \left(g^{(t)}_{ij}\right)_{n\times n}$ as:

$$g^{(t)}_{ij} = \prod_{k=1}^{m} \left(e^{(t)}_{ij,k}\right)^{\lambda_k} \tag{4.2}$$

Step 3. Calculate the priority vector of the GGPHMPR $G^{(t)}$ as:

$$w^{(t)}_{i,g} = \frac{\left(\prod_{j=1}^{n} g^{(t)}_{ij}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} g^{(t)}_{ij}\right)^{\frac{1}{n}}} \tag{4.3}$$

Step 4. Utilize Definition 19 to calculate the geometric consensus degree of each PHMPR as:

$$GCD\left(H^{(t)}_k\right) = \frac{2}{(n-1)(n-2)} \sum_{i<j} \left(\ln e^{(t)}_{ij,k} + \ln w^{(t)}_{j,g} - \ln w^{(t)}_{i,g}\right)^2 \tag{4.4}$$

If $GCD\left(H^{(t)}_k\right) \le \overline{GCD}$ or $t > T$, turn to Step 6; otherwise, turn to the next step.

Step 5. Construct the modified PHMPR $H^{(t+1)}_k = \left(e^{(t+1)}_{ij,k}\right)_{n\times n}$ as

$$e^{(t+1)}_{ij,k} = \left(e^{(t)}_{ij,k}\right)^{(1-\theta)} \times \left(\frac{w^{(t)}_{i,g}}{w^{(t)}_{j,g}}\right)^{\theta} \tag{4.5}$$

Let $t = t + 1$, turn to Step 2.

Step 6. Let $\hat{H} = H^{(t)}$ and output $\hat{H}$;

Step 7. End.

where $w^{(t)}_g = \left(w^{(t)}_{1,g}, w^{(t)}_{2,g}, \ldots, w^{(t)}_{n,g}\right)$ denotes the priority vector of GGPHMPR $G^{(t)}$. Hence, $\frac{w^{(t)}_{i,g}}{w^{(t)}_{j,g}}$ denotes the element of the complete consistent preference relation of the GGPHMPR $G^{(t)}$ denoted by $\tilde{G}^{(t)}$.

Two theorems can be derived from Algorithm 2.

Theorem 20: Let $H_k = (h_{ij,k})_{n\times n} = \left(r^l_{ij,k}|p^l_{ij,k}\right)_{n\times n}$, $k = 1, 2, \ldots, m$ be m PHMPRs, $E_k = (e_{ij,k})_{n\times n}$ be the GPHMPRs, $G = (g_{ij})_{n\times n}$ be their GGPHMPR, $\left\{E^{(t)}_k\right\}$ and $\left\{G^{(t)}\right\}$ be the sequences of the GPHMPRs and their GGPHMPRs that are produced by the process of Algorithm 2. If $\max_{1\le k\le m} \{GCI(H_k)\} \le \overline{GCI}$, then $\max_{1\le k\le m}\left\{GCI\left(H^{(t+1)}_k\right)\right\} \le \max_{1\le k\le m}\left\{GCI\left(H^{(t)}_k\right)\right\} \le \overline{GCI}$.

Proof: According to Algorithm 2, the GPHMPR $E^{(t+1)}_k$ that is obtained after t + 1 iterations should satisfy

$$e^{(t+1)}_{ij,k} = \left(e^{(t)}_{ij,k}\right)^{(1-\theta)} \times \left(\frac{w^{(t)}_{i,g}}{w^{(t)}_{j,g}}\right)^{\theta}$$

Then,

$$e_{ij,k}^{(t+1)} \in \left\{ \min\left(e_{ij,k}^{(t)}, \frac{w_{i,g}^{(t)}}{w_{j,g}^{(t)}} \right), \max\left(e_{ij,k}^{(t)}, \frac{w_{i,g}^{(t)}}{w_{j,g}^{(t)}} \right) \right\}.$$

According to Theorem 16, we have

$$GCI\left(H_k^{(t+1)}\right) = GCI\left(E_k^{(t+1)}\right) \leq \max\left\{ GCI\left(H_k^{(t)}\right), GCI\left(\tilde{G}^{(t)}\right) \right\}.$$

Since $GCI\left(\tilde{G}^{(t)}\right) = 0$, then

$$\max\left\{ GCI\left(H_k^{(t+1)}\right) \right\} = \max\left\{ GCI\left(E_k^{(t+1)}\right) \right\} \leq \max\left\{ GCI\left(H_k^{(t)}\right) \right\} \leq \bar{GCI}$$

which completes the proof.

Theorem 21: Let $H_k = \left(h_{ij,k}\right)_{n\times n} = \left(r_{ij,k}^{l} | p_{ij,k}^{l}\right)_{n\times n}$, $k = 1, 2, \ldots, m$ be m PHMPRs, $E_k = \left(e_{ij,k}\right)_{n\times n}$ be their GPHMPRs, $G = \left(g_{ij}\right)_{n\times n}$ be their GGPHMPR, $\left\{E_k^{(t)}\right\}$ and $\left\{G^{(t)}\right\}$ be the sequences of the GPHMPRs and their GGPHMPRs that are produced by Algorithm 2, then $GCD\left(H_k^{(t+1)}\right) \leq GCD\left(H_k^{(t)}\right)$.

Proof: $e_{ij,k}^{(t+1)} = \left(e_{ij,k}^{(t)}\right)^{(1-\theta)} \times \left(\frac{w_{i,g}^{(t)}}{w_{j,g}^{(t)}}\right)^{\theta}$ and

$$w_{i,k}^{(t+1)} = \frac{\left(\prod_{j=1}^{n} e_{ij,k}^{(t+1)}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} e_{ij,k}^{(t+1)}\right)^{\frac{1}{n}}},$$

so we have

$$w_{i,k}^{(t+1)} = \frac{\left(w_{i,k}^{(t)}\right)^{1-\theta}\left(w_{i,g}^{(t)}\right)^{\theta}}{\sum_{i=1}^{n}\left(\left(w_{i,k}^{(t)}\right)^{1-\theta}\left(w_{i,g}^{(t)}\right)^{\theta}\right)}$$

Hence,

$$\begin{aligned}
&w_{i,g}^{(t+1)} \\
&= \frac{\prod_{k=1}^{m}\left(w_{i,k}^{(t+1)}\right)^{\lambda_k}}{\sum_{i=1}^{n}\prod_{k=1}^{m}\left(w_{i,k}^{(t+1)}\right)^{\lambda_k}} = \frac{\prod_{k=1}^{m}\left(\left(w_{i,k}^{(t)}\right)^{1-\theta}\left(w_{i,g}^{(t)}\right)^{\theta}\right)^{\lambda_k}}{\sum_{i=1}^{n}\prod_{k=1}^{m}\left(\left(w_{i,k}^{(t)}\right)^{1-\theta}\left(w_{i,g}^{(t)}\right)^{\theta}\right)^{\lambda_k}} \\
&= \frac{\prod_{k=1}^{m}\left(\left(w_{i,k}^{(t)}\right)^{1-\theta}\left(\prod_{k=1}^{m}\left(w_{i,k}^{(t)}\right)^{\lambda_k}\right)^{\theta}\right)^{\lambda_k}}{\sum_{i=1}^{n}\prod_{k=1}^{m}\left(\left(w_{i,k}^{(t)}\right)^{1-\theta}\left(\prod_{k=1}^{m}\left(w_{i,k}^{(t)}\right)^{\lambda_k}\right)^{\theta}\right)^{\lambda_k}} \\
&= \frac{\prod_{k=1}^{m}\left(\left(w_{i,k}^{(t)}\right)^{\lambda_k}\right)^{1-\theta}\prod_{k=1}^{m}\left(\left(w_{i,k}^{(t)}\right)^{\lambda_k}\right)^{\theta}}{\sum_{i=1}^{n}\prod_{k=1}^{m}\left(\left(w_{i,k}^{(t)}\right)^{\lambda_k}\right)^{1-\theta}\prod_{k=1}^{m}\left(\left(w_{i,k}^{(t)}\right)^{\lambda_k}\right)^{\theta}} \\
&= \frac{\prod_{k=1}^{m}\left(w_{i,k}^{(t)}\right)^{\lambda_k}}{\sum_{i=1}^{n}\prod_{k=1}^{m}\left(w_{i,k}^{(t)}\right)^{\lambda_k}} = w_{i,g}^{(t)}
\end{aligned}$$

$$\begin{aligned}
&GCD\left(H_k^{(t+1)}\right) \\
&= GCD\left(E_k^{(t+1)}\right) \\
&= \frac{2}{(n-1)(n-2)} \\
&\quad \times \sum_{i<j}\left(\ln\left(\left(e_{ij,k}^{(t)}\right)^{1-\theta} \times \left(\frac{w_{i,g}^{(t)}}{w_{j,g}^{(t)}}\right)^{\theta}\right) + \ln w_{j,g}^{(t+1)} - \ln w_{i,g}^{(t+1)} \right)^2 \\
&= \frac{2}{(n-1)(n-2)} \\
&\quad \times \sum_{i<j}\left(\ln\left(e_{ij,k}^{(t)}\right)^{1-\theta} + \ln\left(w_{i,g}^{(t)}\right)^{\theta} - \ln\left(w_{j,g}^{(t)}\right)^{\theta} + \ln w_{j,g}^{(t)} - \ln w_{i,g}^{(t)} \right)^2 \\
&= \frac{2(1-\theta)^2}{(n-1)(n-2)} \\
&= \sum_{i<j}\left(\ln\left(e_{ij,k}^{(t)}\right)^{1-\theta} + \ln\left(w_{j,g}^{(t)}\right)^{1-\theta} - \ln\left(w_{i,g}^{(t)}\right)^{1-\theta} \right)^2 \\
&= (1-\theta)^2\, GCD\left(H_k^{(t)}\right) \leq GCD\left(H_k^{(t)}\right)
\end{aligned}$$

which completes the proof of the theorem.

V. GROUP DECISION MAKING MODEL BASED ON PHMPRS

As shown in Theorem 20, it can be seen that the acceptably consistent PHMPRs, that are modified by Algorithm 2, are still acceptably consistent. Hence, Algorithms 1 and 2 can be combined to design a complete group decision making model to deal with the consistency and consensus of PHMPRs and make reasonable decisions.

Prior to designing the group decision making model, the score function of the GGPHMPR $G = \left(g_{ij}\right)_{n\times n}$ is defined as follows:

Definition 22: Given a GGPHMPR $G = \left(g_{ij}\right)_{n\times n}$, then the score function of the alternative x_i is defined as:

$$s\left(x_i\right) = \prod_{j=1}^{n}\left(g_{ij}\right)^{\frac{1}{n}}$$

where $s\left(x_i\right)$ denotes the score function of the alternative x_i.

As demonstrated in Figure 1, the complete group decision making model is composed of three components that are the consistency improving process, consensus reaching process,

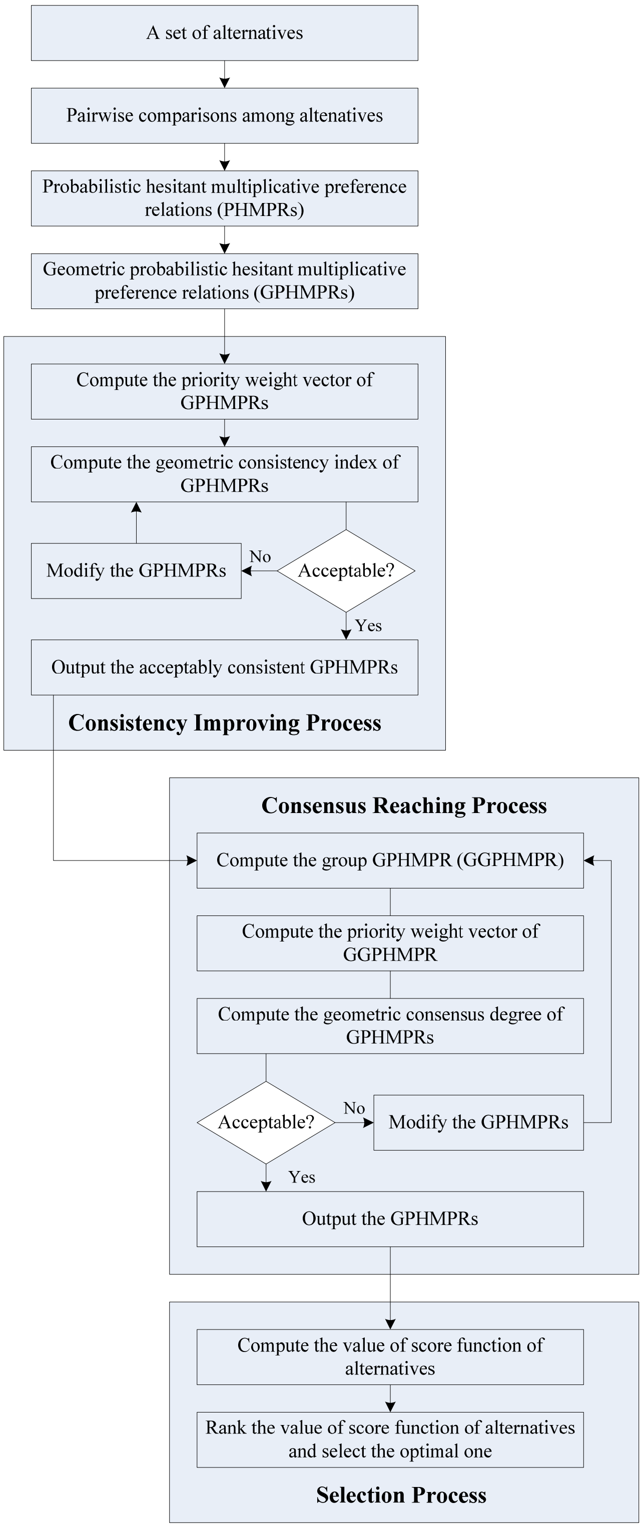

FIGURE 1. **A complete group decision making model with PHMPRs.**

and selection process. The former two processes have been described in Section III and Section IV. Based on Definition 22, the selection process is designed to rank the alternatives based on their values of score function and select the optimal one with the highest value. The above group decision making model is implemented using Algorithm 3 listed as follows.

Algorithm 3 Complete Group Decision Making Algorithm

Input: m PHMPRs $H_k = (h_{ij,k})_{n\times n} = \left(r_{ij,k}^{l}|p_{ij,k}^{l}\right)_{n\times n}$ with their weights λ_k satisfying that $0 \le \lambda_k \le 1$ and $\sum_{k=1}^{m} \lambda_k = 1$, a parameter $\theta_1 \in [0, 1]$, a parameter $\theta_2 \in [0, 1]$, the threshold for the geometric consistency index expressed by $GC\bar{I}$, the threshold for geometric consensus degree denoted by $GC\bar{D}$, the maximum number of iterations denoted by T.

Output: The ranking of alternatives and the optimal one.

Step 1. Let $t = 0$ and $H_k^{(t)} = H_k$;

Step 2. Utilize Definition 8 to compute the GPHMPR $E_k^{(t)} = \left(e_{ij,k}^{(t)}\right)_{n\times n}$ of $H_k^{(t)}$, which satisfies

$$e_{ij,k}^{(t)} = \prod_{l=1}^{\#h_{ij}} \left(\left(r_{ij,k}^{l}\right)^{(t)}\right)^{p_{ij,k}^{l}} \tag{5.1}$$

Step 3. Use (3.5) to calculate the priority vector of the GPHMPR $E_k^{(t)}$ as:

$$w_{i,k}^{(t)} = \frac{\left(\prod_{j=1}^{n} e_{ij,k}^{(t)}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} e_{ij,k}^{(t)}\right)^{\frac{1}{n}}} \tag{5.2}$$

Step 4. Calculate the geometric consistency index of the PHMPR $H_k^{(t)}$ as:

$$\begin{aligned} GCI\left(H_k^{(t)}\right) &= GCI\left(E_k^{(t)}\right) \\ &= \frac{2}{(n-1)(n-2)} \sum_{i<j} \left(\ln e_{ij,k}^{(t)} + \ln w_{j,k}^{(t)} - \ln w_{i,k}^{(t)}\right)^2 \end{aligned} \tag{5.3}$$

If $GCI\left(H_k^{(t)}\right) \le GC\bar{I}$, then turn to Step 6; otherwise, turn to the next step.

Step 5. Build the modified GPHMPR $E_k^{(t+1)} = \left(e_{ij,k}^{(t+1)}\right)_{n\times n}$, which satisfies

$$e_{ij,k}^{(t+1)} = \left(e_{ij,k}^{(t)}\right)^{(1-\theta)} \times \left(\frac{w_{i,k}}{w_{j,k}}\right)^{\theta} \tag{5.4}$$

Let $t = t + 1$, turn to Step 4;

Step 6. Use Definition 10 to compute the group geometric PHMPR (GGPHMPR) $G^{(t)} = \left(g_{ij}^{(t)}\right)_{n\times n}$ as:

$$g_{ij}^{(t)} = \prod_{k=1}^{m} \left(e_{ij,k}^{(t)}\right)^{\lambda_k} \tag{5.5}$$

Step 7. Calculate the priority vector of GGPHMPR $G^{(t)}$ as:

$$w_{i,g}^{(t)} = \frac{\left(\prod_{j=1}^{n} g_{ij}^{(t)}\right)^{\frac{1}{n}}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} g_{ij}^{(t)}\right)^{\frac{1}{n}}} \tag{5.6}$$

Step 8. Utilize Definition 19 to calculate the geometric consensus degree of each PHMPR as:

$$GCD\left(H_k^{(t)}\right) = \frac{2}{(n-1)(n-2)} \sum_{i<j} \left(\ln e_{ij,k}^{(t)} + \ln w_{j,g}^{(t)} - \ln w_{i,g}^{(t)}\right)^2 \tag{5.7}$$

If $GCD\left(H_k^{(t)}\right) \le GC\bar{D}$ or $t > T$, then we turn to Step 10; otherwise, turn to the next step.

Step 9. Build the modified PHMPR $H_k^{(t+1)} = \left(e_{ij,k}^{(t+1)}\right)_{n\times n}$ as:

$$e_{ij,k}^{(t+1)} = \left(e_{ij,k}^{(t)}\right)^{(1-\theta)} \times \left(\frac{w_{i,g}^{(t)}}{w_{j,g}^{(t)}}\right)^{\theta} \tag{5.8}$$

Let $t = t + 1$, turn to Step 6.

Step 10. Compute the score value of each alternative using the GGPHMPR $G^{(t)} = \left(g_{ij}^{(t)}\right)_{n\times n}$;

Step 11. Rank all the alternatives according to their score values and select the optimal one;

Step 12. End.

$$H_1 = \begin{Bmatrix} \{1\} & \{3|0.6, 5|0.4\} & \{5, 7\} & \{3\} \\ \{1/3|0.6, 1/5|0.4\} & \{1\} & \{5\} & \{7\} \\ \{1/5, 1/7\} & \{1/5\} & \{1\} & \{3\} \\ \{1/3\} & \{1/7\} & \{1/3\} & \{1\} \end{Bmatrix}$$

$$H_2 = \begin{Bmatrix} \{1\} & \{1/3\} & \{1/7|0.3, 1/5|0.7\} & \{1/3\} \\ \{3\} & \{1\} & \{5|0.4, 7|0.6\} & \{3\} \\ \{7|0.3, 5|0.7\} & \{1/5|0.4, 1/7|0.6\} & \{1\} & \{1/3\} \\ \{3\} & \{1/3\} & \{3\} & \{1\} \end{Bmatrix}$$

$$H_3 = \begin{Bmatrix} \{1\} & \{1/7|0.8,1/3|0.2\} & \{1/5\} & \{1/3\} \\ \{7|0.8, 3|0.2\} & \{1\} & \{3\} & \{3|0.4, 5|0.6\} \\ \{5\} & \{1/3\} & \{1\} & \{1/3\} \\ \{3\} & \{1/3|0.4,1/5|0.6\} & \{3\} & \{1\} \end{Bmatrix}$$

$$H_4 = \begin{Bmatrix} \{1\} & \{1/3\} & \{1/7, 1/5\} & \{1/5|0.6,1/3|0.4\} \\ \{3\} & \{1\} & \{3\} & \{5\} \\ \{7, 5\} & \{1/3\} & \{1\} & \{1/3\} \\ \{5|0.6, 3|0.4\} & \{1/5\} & \{3\} & \{1\} \end{Bmatrix}$$

VI. ILLUSTRATIVE EXAMPLE AND COMPARISON ANALYSIS

A. ILLUSTRATIVE EXAMPLE

Example 23: In recent years, more and more parents pay attention to the English education for kids. In the education markets, there are many kinds of English education courses provided by companies. Assume that there are four popular English education brands, which are VIPKID, Cinostar, EF Education, and GIRAFFE expressed by $X = \{x_1, x_2, x_3, x_4\}$. To select an appropriate suit of English education courses for Chinese kids from these four English education brands, four experts with their weight vector $w = (0.25, 0.25, 0.25, 0.25)$ are invited to assess these four English education brands and give their evaluation information in the form of PHMPRs as H_1, H_2, H_3, H_4, as shown at the top of this page.

To show the practical application processes of Algorithm 3, this example is implemented as follows:

Step 1: Let $t = 0$ and $H_k^{(0)} = H_k$ for each k;

Step 2: Use (5.1) to derive the GPHMPR $E_k^{(0)} = \left(e_{ij,k}^{(0)}\right)_{n\times n}$ of $H_k^{(0)}$ as:

$$E_1^{(0)} = \begin{Bmatrix} 1.0 & 3.6801 & 5.9161 & 3.0 \\ 0.2717 & 1.0 & 5.0 & 7.0 \\ 0.1690 & 0.2 & 1.0 & 3.0 \\ 0.3333 & 0.1429 & 0.3333 & 1.0 \end{Bmatrix},$$

$$E_2^{(0)} = \begin{Bmatrix} 1.0 & 0.3333 & 0.1808 & 0.3333 \\ 3.0 & 1.0 & 6.1182 & 3.0 \\ 5.5311 & 0.1634 & 1.0 & 0.3333 \\ 3.0 & 0.3333 & 3.0 & 1.0 \end{Bmatrix},$$

$$E_3^{(0)} = \begin{Bmatrix} 1.0 & 0.1692 & 0.2 & 0.3333 \\ 5.9088 & 1.0 & 3.0 & 4.0760 \\ 5.0 & 0.3333 & 1.0 & 0.3333 \\ 3.0 & 0.2453 & 3.0 & 1.0 \end{Bmatrix},$$

$$E_4^{(0)} = \begin{Bmatrix} 1.0 & 0.3333 & 0.1690 & 0.2453 \\ 3.0 & 1.0 & 3.0 & 5.0 \\ 5.9161 & 0.3333 & 1.0 & 0.3333 \\ 4.0760 & 0.2 & 3.0 & 1.0 \end{Bmatrix}$$

Step 3: Use (5.2) to calculate the priority vectors of the GPHMPRs $E_k^{(0)}$ as:

$$\begin{aligned} w_1 &= (0.5152, 0.3182, 0.1023, 0.0643)^T, \\ w_2 &= (0.0730, 0.5282, 0.1437, 0.2552)^T, \\ w_3 &= (0.0612, 0.5476, 0.1622, 0.2290)^T, \\ w_4 &= (0.0674, 0.5095, 0.1771, 0.2460)^T \end{aligned}$$

Step 4: Use (5.3) to calculate the geometric consistency indexes of the PHMPRs $H_k^{(0)}$ as:

$$GCI\left(H_1^{(0)}\right) = 3.2174, \quad GCI\left(H_2^{(0)}\right) = 3.3842,$$
$$GCI\left(H_3^{(0)}\right) = 1.9871, \quad GCI\left(H_4^{(0)}\right) = 3.8619$$

According to [25], the value of $\overline{GCI}$ is set to 0.04. Since $GCI\left(H_k^{(0)}\right) > \overline{GCI}$, then we turn to the next step:

Step 5: Let $\theta = 0.9$, then we use (5.4) to construct the modified GPHMPR $E_k^{(1)} = \left(e_{ij,k}^{(1)}\right)_{n\times n}$ as:

$$E_1^{(1)} = \begin{Bmatrix} 1.0 & 1.7573 & 5.1192 & 7.2601 \\ 0.5690 & 1.0 & 3.2630 & 5.1222 \\ 0.1953 & 0.3065 & 1.0 & 1.6941 \\ 0.1377 & 0.1952 & 0.5903 & 1.0 \end{Bmatrix},$$

$$E_2^{(1)} = \begin{Bmatrix} 1.0 & 0.1509 & 0.4582 & 0.2905 \\ 6.6260 & 1.0 & 3.8688 & 2.1481 \\ 2.1822 & 0.2585 & 1.0 & 0.5342 \\ 3.4427 & 0.4655 & 1.8719 & 1.0 \end{Bmatrix},$$

$$E_3^{(1)} = \begin{Bmatrix} 1.0 & 0.1165 & 0.3543 & 0.2733 \\ 8.5822 & 1.0 & 3.3373 & 2.5227 \\ 2.8227 & 0.2996 & 1.0 & 0.6568 \\ 3.6587 & 0.3964 & 1.5224 & 1.0 \end{Bmatrix},$$

$$E_4^{(1)} = \begin{Bmatrix} 1.0 & 0.1452 & 0.3511 & 0.2712 \\ 6.8873 & 1.0 & 2.8886 & 2.2620 \\ 2.8482 & 0.3462 & 1.0 & 0.6667 \\ 3.6878 & 0.4421 & 1.5000 & 1.0 \end{Bmatrix}$$

Let $t = 1$, then we turn to the next step;

Step 6: Use (5.3) to calculate the geometric consistency indexes of the PHMPRs $H_k^{(1)}$ as:

$$GCI\left(H_1^{(1)}\right)=0.0322,\quad GCI\left(H_2^{(1)}\right)=0.0338,$$
$$GCI\left(H_3^{(1)}\right)=0.0199,\quad GCI\left(H_4^{(1)}\right)=0.0386$$

Since $GCI\left(H_k^{(1)}\right)<\bar{GCI}$ for each k, then we turn to the next step;

Step 7: Use (5.5) to calculate GGPHMPR $G^{(1)}=\left(g_{ij}^{(1)}\right)_{n\times n}$ as:

$$G^{(1)}=\begin{Bmatrix}1.0 & 0.2588 & 0.7350 & 0.6288\\ 3.8673 & 1.0 & 3.3214 & 2.8149\\ 1.3606 & 0.3011 & 1.0 & 0.7934\\ 1.5904 & 0.3552 & 1.2604 & 1.0\end{Bmatrix}$$

Step 8: Utilize (5.6) to calculate the priority vector of GGPHMPR $G^{(1)}$ as:

$$w_g^{(1)}=(0.1248{,}0.5201{,}0.1602{,}0.1949)^T$$

Step 9: Use (5.7) to calculate the geometric consensus degree of each PHMPR as:

$$GCD\left(H_1^{(1)}\right)=4.7855,\quad GCD\left(H_2^{(1)}\right)=0.4613,$$
$$GCD\left(H_3^{(1)}\right)=0.6412,\quad GCD\left(H_4^{(1)}\right)=0.5699$$

Step 10: The value of $\bar{GCD}$ is set to 0.05. Because the geometric consensus degree of each PHMPR is higher than 0.05, then we turn to the next step;

Step 11: Utilize (5.8) to construct the modified PHMPR $H_k^{(2)}=\left(e_{ij,k}^{(2)}\right)_{n\times n}$ as:

$$E_1^{(2)}=\begin{Bmatrix}1.0 & 0.2927 & 0.9402 & 0.8161\\ 3.4160 & 1.0 & 3.2485 & 2.8486\\ 1.0636 & 0.3078 & 1.0 & 0.8836\\ 1.2253 & 0.3511 & 1.1317 & 1.0\end{Bmatrix},$$
$$E_2^{(2)}=\begin{Bmatrix}1.0 & 0.2290 & 0.7386 & 0.5915\\ 4.3665 & 1.0 & 3.3043 & 2.6115\\ 1.3539 & 0.3026 & 1.0 & 0.7873\\ 1.6905 & 0.3829 & 1.2702 & 1.0\end{Bmatrix},$$
$$E_3^{(2)}=\begin{Bmatrix}1.0 & 0.2232 & 0.7199 & 0.5880\\ 4.4810 & 1.0 & 3.2558 & 2.6538\\ 1.3892 & 0.3071 & 1.0 & 0.8037\\ 1.7008 & 0.3768 & 1.2442 & 1.0\end{Bmatrix},$$
$$E_4^{(2)}=\begin{Bmatrix}1.0 & 0.2281 & 0.7192 & 0.5875\\ 4.3834 & 1.0 & 3.2091 & 2.6250\\ 1.3904 & 0.3116 & 1.0 & 0.8049\\ 1.7022 & 0.3810 & 1.2424 & 1.0\end{Bmatrix}$$

Let $t=2$, then we turn to the next step.

Step 12: Utilize (5.5) to compute the group geometric PHMPR (GGPHMPR) $G^{(2)}=\left(g_{ij}^{(2)}\right)_{n\times n}$ as:

$$G^{(2)}=\begin{Bmatrix}1.0 & 0.2417 & 0.7744 & 0.6390\\ 4.1372 & 1.0 & 3.2543 & 2.6830\\ 1.2914 & 0.3073 & 1.0 & 0.8190\\ 1.5649 & 0.3727 & 1.2209 & 1.0\end{Bmatrix}$$

Step 13: Utilize (5.6) to calculate the priority vector of GGPHMPR $G^{(2)}$ as:

$$w_g^{(2)}=(0.1248{,}0.5201{,}0.1602{,}0.1949)^T$$

Step 14: Use (5.7) to calculate the geometric consensus degree of each PHMPR as:

$$GCD\left(H_1^{(2)}\right)=0.0479,\quad GCD\left(H_2^{(2)}\right)=0.0046,$$
$$GCD\left(H_3^{(2)}\right)=0.0064,\quad GCD\left(H_4^{(2)}\right)=0.0057$$

Since $GCD\left(H_k^{(2)}\right)<\bar{GCD}$ for each k, then we turn to the next step.

Step 15: Use Definition 22 to calculate the score value of each alternative as:

$$s(x_1)=0.5880,\quad s(x_2)=2.4516,$$
$$s(x_3)=0.7551,\quad s(x_4)=0.9186$$

Step 16: According to their score values, the alternatives are ranked as:

$$x_2\succ x_4\succ x_3\succ x_1$$

B. COMPARISON ANALYSIS

To verify the effectiveness of our model, the model proposed by Bashir *et al.* [13] is introduced to process Example 23 as follows:

Step 1: Normalize all the PHMPRs into NPHMPRs as $H_1^{(0)},H_2^{(0)},H_3^{(0)},H_4^{(0)}$, as shown at the next page.

Step 2: Calculate their consistency indexes as:

$$CI\left(H_1^{(0)}\right)=1.1976,\quad CI\left(H_2^{(0)}\right)=1.2524,$$
$$CI\left(H_3^{(0)}\right)=1.2162,\quad CI\left(H_4^{(0)}\right)=1.2236$$

Step 3: The threshold for the consistency index in [13] is set to 1.01. Hence, all the PHMPRs are inconsistent. After two iterations, acceptably consistent PHMPRs are obtained as $H_1^{(2)},H_2^{(2)},H_3^{(2)},H_4^{(2)}$, as shown at the next page.

Their completely consistent PHMPRs are $\tilde{H}_1^{(2)},\tilde{H}_2^{(2)},\tilde{H}_3^{(2)},\tilde{H}_4^{(2)}$, as shown at the next page.

In this step, their consistency indexes are

$$CI\left(H_1^{(2)}\right)=1.0081,\quad CI\left(H_2^{(2)}\right)=1.0078,$$
$$CI\left(H_3^{(2)}\right)=1.0068,\quad CI\left(H_4^{(2)}\right)=1.0083$$

Step 5: Utilize the PHMWG operator in [13] to obtain the group PHMPR as $H_g^{(2)}$, as shown at the top of the page 13.

$$H_1^{(0)} = \left\{ \begin{array}{cccc} \{1|0.5,\ 1|0.5\} & \{3|0.6,\ 5|0.4\} & \{5|0.5,\ 7|0.5\} & \{3|0.5,\ 3|0.5\} \\ \{1/3|0.6,\ 1/5|0.4\} & \{1|0.5,\ 1|0.5\} & \{5|0.5,\ 5|0.5\} & \{7|0.5,\ 7|0.5\} \\ \{1/5|0.5, 1/7|0.5\} & \{1/5|0.5, 1/5|0.5\} & \{1|0.5,\ 1|0.5\} & \{3|0.5,\ 3|0.5\} \\ \{1/3|0.5, 1/3|0.5\} & \{1/7|0.5,\ 1/7|0.5\} & \{1/3|0.5,\ 1/3|0.5\} & \{1|0.5,\ 1|0.5\} \end{array} \right\},$$

$$H_2^{(0)} = \left\{ \begin{array}{cccc} \{1|0.5,\ 1|0.5\} & \{1/3|0.5,\ 1/3|0.5\} & \{1/7|0.3,\ 1/5|0.7\} & \{1/3|0.5,\ 1/3|0.5\} \\ \{3|0.5,\ 3|0.5\} & \{1|0.5,\ 1|0.5\} & \{5|0.4,\ 7|0.6\} & \{3|0.5,\ 3|0.5\} \\ \{7|0.3,\ 5|0.7\} & \{1/5|0.4,\ 1/7|0.6\} & \{1|0.5,\ 1|0.5\} & \{1/3|0.5,\ 1/3|0.5\} \\ \{3|0.5,\ 3|0.5\} & \{1/3|0.5,\ 1/3|0.5\} & \{3|0.5,\ 3|0.5\} & \{1|0.5,\ 1|0.5\} \end{array} \right\},$$

$$H_3^{(0)} = \left\{ \begin{array}{cccc} \{1|0.5,\ 1|0.5\} & \{1/7|0.8, 1/3|0.2\} & \{1/5|0.5,\ 1/5|0.5\} & \{1/3|0.5,\ 1/3|0.5\} \\ \{7|0.8,\ 3|0.2\} & \{1|0.5,\ 1|0.5\} & \{3|0.5,\ 3|0.5\} & \{3|0.4,\ 5|0.6\} \\ \{5|0.5,\ 5|0.5\} & \{1/3|0.5,\ 1/3|0.5\} & \{1|0.5,\ 1|0.5\} & \{1/3|0.5,\ 1/3|0.5\} \\ \{3|0.5,\ 3|0.5\} & \{1/3|0.4, 1/5|0.6\} & \{3|0.5,\ 3|0.5\} & \{1|0.5,\ 1|0.5\} \end{array} \right\},$$

$$H_4^{(0)} = \left\{ \begin{array}{cccc} \{1|0.5,\ 1|0.5\} & \{1/3|0.5,\ 1/3|0.5\} & \{1/7|0.5,\ 1/5|0.5\} & \{1/5|0.6, 1/3|0.4\} \\ \{3|0.5,\ 3|0.5\} & \{1|0.5,\ 1|0.5\} & \{3|0.5,\ 3|0.5\} & \{5|0.5,\ 5|0.5\} \\ \{7|0.5,\ 5|0.5\} & \{1/3|0.5,\ 1/3|0.5\} & \{1|0.5,\ 1|0.5\} & \{1/3|0.5,\ 1/3|0.5\} \\ \{5|0.6,\ 3|0.4\} & \{1/5|0.5,\ 1/5|0.5\} & \{3|0.5,\ 3|0.5\} & \{1|0.5,\ 1|0.5\} \end{array} \right\}$$

$$H_1^{(2)} = \left\{ \begin{array}{cccc} \{1.00|0.52,\ 1.00|0.48\} & \{1.51|0.54,\ 2.16|0.46\} & \{4.46|0.51,\ 6.02|0.49\} & \{6.68|0.51,\ 8.08|0.49\} \\ \{0.66|0.53,\ 0.47|0.47\} & \{1.00|0.53,\ 1.00|0.47\} & \{3.16|0.51,\ 3.29|0.49\} & \{4.78|0.51,\ 5.36|0.49\} \\ \{0.22|0.51,\ 0.17|0.49\} & \{0.32|0.51,\ 0.30|0.49\} & \{1.00|0.50,\ 1.00|0.50\} & \{1.63|0.50,\ 1.73|0.50\} \\ \{0.15|0.51,\ 0.12|0.49\} & \{0.21|0.51,\ 0.19|0.49\} & \{0.57|0.50,\ 0.61|0.50\} & \{1.00|0.50,\ 1.00|0.50\} \end{array} \right\},$$

$$H_2^{(2)} = \left\{ \begin{array}{cccc} \{1.00|0.46,\ 1.00|0.54\} & \{0.14|0.46,\ 0.14|0.54\} & \{0.42|0.45,\ 0.55|0.55\} & \{0.27|0.47,\ 0.29|0.53\} \\ \{7.23|0.46,\ 7.23|0.54\} & \{1.00|0.47,\ 1.00|0.53\} & \{3.15|0.45,\ 4.06|0.55\} & \{1.98|0.48,\ 2.15|0.52\} \\ \{2.35|0.45,\ 1.83|0.55\} & \{0.32|0.45,\ 0.25|0.55\} & \{1.00|0.44,\ 1.00|0.56\} & \{0.53|0.46,\ 0.62|0.54\} \\ \{3.70|0.47,\ 3.40|0.53\} & \{0.51|0.48,\ 0.47|0.52\} & \{1.89|0.46,\ 1.60|0.54\} & \{1.00|0.48,\ 1.00|0.52\} \end{array} \right\},$$

$$H_3^{(2)} = \left\{ \begin{array}{cccc} \{1.00|0.55,\ 1.00|0.45\} & \{0.11|0.55,\ 0.12|0.45\} & \{0.26|0.53,\ 0.33|0.47\} & \{0.17|0.52,\ 0.23|0.48\} \\ \{8.77|0.55,\ 8.26|0.45\} & \{1.00|0.54,\ 1.00|0.46\} & \{2.75|0.52,\ 2.99|0.48\} & \{1.89|0.52,\ 1.98|0.48\} \\ \{3.76|0.53,\ 3.05|0.47\} & \{0.36|0.53,\ 0.33|0.47\} & \{1.00|0.51,\ 1.00|0.49\} & \{0.64|0.50,\ 0.73|0.50\} \\ \{3.76|0.53,\ 3.05|0.47\} & \{0.53|0.53,\ 0.36|0.47\} & \{1.56|0.51,\ 1.37|0.49\} & \{1.00|0.50,\ 1.00|0.50\} \end{array} \right\},$$

$$H_4^{(2)} = \left\{ \begin{array}{cccc} \{1.00|0.52,\ 1.00|0.48\} & \{0.13|0.51,\ 0.16|0.49\} & \{0.31|0.51,\ 0.41|0.49\} & \{0.23|0.53,\ 0.33|0.47\} \\ \{7.49|0.51,\ 6.19|0.49\} & \{1.00|0.50,\ 1.00|0.50\} & \{2.78|0.50,\ 3.00|0.50\} & \{2.16|0.51,\ 2.42|0.49\} \\ \{3.27|0.51,\ 2.42|0.49\} & \{0.36|0.50,\ 0.33|0.50\} & \{1.00|0.50,\ 1.00|0.50\} & \{0.66|0.51,\ 0.69|0.49\} \\ \{4.29|0.53,\ 3.00|0.47\} & \{0.46|0.51,\ 0.41|0.49\} & \{1.51|0.51,\ 1.45|0.49\} & \{1.00|0.52,\ 1.00|0.48\} \end{array} \right\}$$

$$\tilde{H}_1^{(2)} = \left\{ \begin{array}{cccc} \{1.00|0.52,\ 1.00|0.48\} & \{1.40|0.52,\ 1.97|0.48\} & \{4.40|0.51,\ 5.92|0.49\} & \{7.30|0.51,\ 9.02|0.49\} \\ \{0.71|0.52,\ 0.51|0.48\} & \{1.00|0.52,\ 1.00|0.48\} & \{3.01|0.51,\ 3.14|0.49\} & \{4.58|0.51,\ 5.21|0.49\} \\ \{0.23|0.51,\ 0.17|0.49\} & \{0.33|0.51,\ 0.32|0.49\} & \{1.00|0.51,\ 1.00|0.49\} & \{1.52|0.51,\ 1.66|0.49\} \\ \{0.14|0.51,\ 0.11|0.49\} & \{0.22|0.51,\ 0.19|0.49\} & \{0.66|0.51,\ 0.60|0.49\} & \{1.00|0.51,\ 1.00|0.49\} \end{array} \right\},$$

$$\tilde{H}_2^{(2)} = \left\{ \begin{array}{cccc} \{1.00|0.46,\ 1.00|0.54\} & \{0.14|0.46,\ 0.14|0.54\} & \{0.43|0.45,\ 0.55|0.55\} & \{0.27|0.47,\ 0.29|0.53\} \\ \{7.30|0.46,\ 7.30|0.54\} & \{1.00|0.47,\ 1.00|0.53\} & \{3.13|0.46,\ 4.03|0.54\} & \{1.97|0.47,\ 2.14|0.53\} \\ \{2.33|0.45,\ 1.81|0.55\} & \{0.32|0.46,\ 0.25|0.54\} & \{1.00|0.45,\ 1.00|0.55\} & \{0.53|0.46,\ 0.63|0.54\} \\ \{3.71|0.47,\ 3.41|0.53\} & \{0.51|0.47,\ 0.47|0.53\} & \{1.88|0.46,\ 1.59|0.54\} & \{1.00|0.47,\ 1.00|0.53\} \end{array} \right\},$$

$$\tilde{H}_3^{(2)} = \left\{ \begin{array}{cccc} \{1.00|0.54,\ 1.00|0.46\} & \{0.11|0.54,\ 0.12|0.46\} & \{0.26|0.53,\ 0.32|0.47\} & \{0.16|0.53,\ 0.23|0.48\} \\ \{8.70|0.54,\ 8.48|0.46\} & \{1.00|0.53,\ 1.00|0.47\} & \{2.70|0.52,\ 2.94|0.48\} & \{1.88|0.52,\ 1.96|0.48\} \\ \{3.88|0.53,\ 3.14|0.47\} & \{0.37|0.52,\ 0.34|0.48\} & \{1.00|0.52,\ 1.00|0.48\} & \{0.63|0.51,\ 0.71|0.49\} \\ \{6.07|0.53,\ 4.32|0.48\} & \{0.53|0.52,\ 0.51|0.48\} & \{1.59|0.51,\ 1.40|0.49\} & \{1.00|0.51,\ 1.00|0.49\} \end{array} \right\},$$

$$\tilde{H}_4^{(2)} = \left\{ \begin{array}{cccc} \{1.00|0.52,\ 1.00|0.48\} & \{0.12|0.51,\ 0.12|0.49\} & \{0.33|0.51,\ 0.45|0.49\} & \{0.24|0.52,\ 0.33|0.48\} \\ \{8.29|0.51,\ 6.71|0.49\} & \{1.00|0.51,\ 1.00|0.49\} & \{2.76|0.51,\ 3.00|0.49\} & \{1.97|0.51,\ 2.24|0.49\} \\ \{3.01|0.51,\ 2.24|0.49\} & \{0.36|0.51,\ 0.33|0.49\} & \{1.00|0.51,\ 1.00|0.48\} & \{0.71|0.51,\ 0.74|0.49\} \\ \{4.21|0.52,\ 3.00|0.48\} & \{0.51|0.51,\ 0.45|0.49\} & \{1.40|0.51,\ 1.34|0.49\} & \{1.00|0.52,\ 1.00|0.48\} \end{array} \right\}$$

$$H_g^{(2)} = \begin{Bmatrix} \{1.00|0.51, 1.00|0.49\} & \{0.22|0.51, 0.28|0.49\} & \{0.63|0.50, 0.82|0.50\} & \{0.52|0.51, 0.67|0.49\} \\ \{4.47|0.51, 3.62|0.49\} & \{1.00|0.51, 1.00|0.49\} & \{3.05|0.50, 3.21|0.50\} & \{2.56|0.50, 2.65|0.50\} \\ \{1.60|0.50, 1.22|0.50\} & \{0.33|0.50, 0.31|0.50\} & \{1.00|0.49, 1.00|0.51\} & \{0.81|0.49, 0.83|0.51\} \\ \{1.94|0.51, 1.52|0.49\} & \{0.39|0.50, 0.38|0.50\} & \{1.23|0.49, 1.21|0.51\} & \{1.00|0.50, 1.00|0.50\} \end{Bmatrix}$$

$$H_1^{(3)} = \begin{Bmatrix} \{1.00|0.51, 1.00|0.49\} & \{0.27|0.52, 0.34|0.48\} & \{0.76|0.50, 1.00|0.50\} & \{0.67|0.51, 0.84|0.49\} \\ \{3.69|0.52, 2.94|0.48\} & \{1.00|0.51, 1.00|0.49\} & \{3.07|0.50, 3.20|0.50\} & \{2.76|0.50, 2.81|0.50\} \\ \{1.31|0.50, 1.00|0.50\} & \{0.33|0.50, 0.31|0.50\} & \{1.00|0.49, 1.00|0.51\} & \{0.89|0.49, 0.87|0.51\} \\ \{1.50|0.51, 1.18|0.49\} & \{0.36|0.50, 0.36|0.50\} & \{1.12|0.49, 1.15|0.51\} & \{1.00|0.50, 1.00|0.50\} \end{Bmatrix},$$

$$H_2^{(3)} = \begin{Bmatrix} \{1.00|0.51, 1.00|0.49\} & \{0.21|0.51, 0.26|0.49\} & \{0.60|0.49, 0.78|0.51\} & \{0.48|0.51, 0.61|0.49\} \\ \{4.69|0.51, 3.88|0.49\} & \{1.00|0.50, 1.00|0.50\} & \{3.06|0.49, 3.28|0.51\} & \{2.50|0.50, 2.59|0.50\} \\ \{1.66|0.49, 1.27|0.51\} & \{0.33|0.49, 0.30|0.51\} & \{1.00|0.48, 1.00|0.52\} & \{0.78|0.49, 0.80|0.51\} \\ \{2.07|0.51, 1.65|0.49\} & \{0.40|0.50, 0.39|0.50\} & \{1.29|0.49, 1.24|0.51\} & \{1.00|0.50, 1.00|0.50\} \end{Bmatrix},$$

$$H_3^{(3)} = \begin{Bmatrix} \{1.00|0.52, 1.00|0.48\} & \{0.20|0.52, 0.25|0.48\} & \{0.57|0.50, 0.74|0.50\} & \{0.46|0.51, 0.59|0.49\} \\ \{4.90|0.52, 3.93|0.48\} & \{1.00|0.51, 1.00|0.49\} & \{3.04|0.50, 3.16|0.50\} & \{2.49|0.51, 2.57|0.49\} \\ \{1.74|0.50, 1.34|0.50\} & \{0.33|0.50, 0.32|0.50\} & \{1.00|0.49, 1.00|0.51\} & \{0.80|0.50, 0.81|0.50\} \\ \{2.17|0.51, 1.69|0.49\} & \{0.40|0.51, 0.39|0.49\} & \{1.25|0.50, 1.24|0.50\} & \{1.00|0.50, 1.00|0.50\} \end{Bmatrix},$$

$$H_4^{(3)} = \begin{Bmatrix} \{1.00|0.51, 1.00|0.49\} & \{0.21|0.51, 0.26|0.49\} & \{0.58|0.50, 0.76|0.50\} & \{0.48|0.51, 0.61|0.49\} \\ \{4.71|0.51, 3.82|0.49\} & \{1.00|0.51, 1.00|0.49\} & \{3.02|0.50, 3.19|0.50\} & \{2.52|0.50, 2.63|0.50\} \\ \{1.72|0.50, 1.31|0.50\} & \{0.33|0.50, 0.31|0.50\} & \{1.00|0.49, 1.00|0.51\} & \{0.80|0.50, 0.81|0.50\} \\ \{2.10|0.51, 1.63|0.49\} & \{0.40|0.50, 0.38|0.50\} & \{1.25|0.50, 1.24|0.50\} & \{1.00|0.50, 1.00|0.50\} \end{Bmatrix}$$

$$H_g^{(3)} = \begin{Bmatrix} \{1.00|0.51, 1.00|0.49\} & \{0.22|0.51, 0.28|0.49\} & \{0.63|0.50, 0.82|0.50\} & \{0.52|0.51, 0.67|0.49\} \\ \{4.47|0.51, 3.62|0.49\} & \{1.00|0.51, 1.00|0.49\} & \{3.05|0.50, 3.21|0.50\} & \{2.56|0.50, 2.65|0.50\} \\ \{1.60|0.50, 1.22|0.50\} & \{0.33|0.50, 0.31|0.50\} & \{1.00|0.49, 1.00|0.51\} & \{0.81|0.49, 0.83|0.51\} \\ \{1.94|0.51, 1.52|0.49\} & \{0.39|0.50, 0.38|0.50\} & \{1.23|0.49, 1.21|0.51\} & \{1.00|0.50, 1.00|0.50\} \end{Bmatrix}$$

Step 6: The consensus degrees of these four PHMPRs are computed as:

$$GCI\left(H_1^{(2)}\right) = 2.42, \quad GCI\left(H_2^{(2)}\right) = 1.14,$$
$$GCI\left(H_3^{(2)}\right) = 1.22, \quad GCI\left(H_4^{(2)}\right) = 1.10$$

The consensus degree in [13] is set to be 1.1. Hence, these PHMPRs should be modified. After 1 iteration, the PHMPRs with the consensus degree lower than 1.1 can be obtained as $H_1^{(3)}$, $H_2^{(3)}$, $H_3^{(3)}$, $H_4^{(3)}$, as shown at the top of this page, as well as the following GGPHMPR, $H_g^{(3)}$, as shown at the top of this page.

Their consensus degrees are updated as:

$$GCI\left(H_1^{(3)}\right) = 1.0071, \quad GCI\left(H_2^{(3)}\right) = 1.0033,$$
$$GCI\left(H_3^{(3)}\right) = 1.0006, \quad GCI\left(H_4^{(3)}\right) = 1.0003$$

Step 7: Use the PHMG operator in [13] to aggregate each row of the group PHMPR $H_g^{(3)}$ as:

$$H_g^1 = \{0.52|0.51, 0.62|0.49\}, \ H_g^2 = \{2.43|0.51, 2.35|0.49\},$$
$$H_g^3 = \{0.81|0.49, 0.74|0.51\}, \ H_g^4 = \{0.98|0.50, 0.92|0.50\}$$

Step 8: Use the score function defined in [13] to compute the score value of each alternative as:

$$s(x_1) = 0.75, \quad s(x_2) = 1.55, \ s(x_3) = 0.88, \ s(x_4) = 0.97$$

Step 9: According to the score values, all the alternatives are ranked as:

$$x_2 \succ x_4 \succ x_3 \succ x_1$$

From the rankings of all the alternatives in our model and the model proposed by Bashir *et al.* [13], it can be seen that our model is effective.

We introduce the concept of the confidence level to make the comparison analysis between our group decision making model and the model proposed by Bashir *et al.* [13] using the probability theory.

Definition 24: Given a random variable X and a threshold $\bar{GCI}$ for geometric consistency index, then the confidence level for the geometric consistency index should satisfy the following equation:

$$LCI = P\left\{X \leq \bar{GCI}\right\}$$

where the term LCI denotes the confidence level for the geometric consistency index. It can be easily seen that the confidence level has a great impact on the threshold. The higher the confidence level is, the higher the threshold is. It means that most PHMPRs could reach the threshold easily.

Definition 25: Let X be a random variable and $\bar{GCD}$ be the threshold for the geometric consensus degree, the confidence level for the geometric consensus degree should satisfy

$$LCD = P\left\{X \leq GC\bar{D}\right\}$$

where the term LCD denotes the confidence level for the geometric consensus degree. Similar to LCI, the higher the value of LCD is, the higher the value of $GC\bar{D}$ is.

In our model, the geometric consistency index (GCI) for a PHMPR H is computed as:

$$\begin{aligned} GCI(H) &= GCI(E) \\ &= \frac{2}{(n-1)(n-2)} \sum_{i<j} \left(\ln e_{ij} + \ln w_j - \ln w_i\right)^2 \end{aligned}$$

Assume that a random variable $X = (\ln e_{ij} + \ln w_j - \ln w_i)^2$. Because of the iterative process for modifying the PHMPRs in Algorithm 2, the value of the random variable X tends to 0. Hence, let us assume that the random variable X follows the normal distribution having a mean of zero and a variance of σ_1^2, which is denoted by $X \sim N\left(0, \sigma_1^2\right)$. Since the linear combination of n random variables which follow the normal distribution still follows the normal distribution, we have

$$E\left(\sum_{i<j} X\right) = 0$$

and

$$D\left(\sum_{i<j} X\right) = \frac{n(n-1)}{2}\sigma_1^2$$

Since $E(CX) = CE(X)$ and $D(CX) = C^2 D(X)$, then

$$E\left(\frac{2}{(n-1)(n-2)} \sum_{i<j} X\right) = 0$$

and

$$D\left(\frac{2}{(n-1)(n-2)} \sum_{i<j} X\right) = \frac{2n}{(n-1)(n-2)^2}\sigma_1^2$$

Let a random variable $Y = GCI(H)$, then

$$Y \sim N\left(0, \frac{2n}{(n-1)(n-2)^2}\sigma_1^2\right).$$

The distribution function of the random variable Y is $F(y) = P\{Y \leq y\} = \frac{(n-2)\sqrt{n-1}}{2\sqrt{n\pi}\sigma_1} \int_{-\infty}^{y} e^{-\frac{(n-1)(n-2)^2 t^2}{4n\sigma_1^2}} dt$. In this paper, the value of σ_1 is set to 0.2, then $Y \sim N(0, 0.0267)$. In Example 23, $GC\bar{I}$ is set to 0.04. Based on Definition 24, we have

$$LCI = P\{0 \leq Y \leq 0.04\} = F(0.04) - F(0) = 9.67\%$$

Similar to the computation process of LCI, when the value of $GC\bar{D}$ is set to 0.05, then we have

$$LCD = P\{0 \leq Y \leq 0.05\} = F(0.05) - F(0) = 12.02\%$$

In [13], the consistency index of PHMPRs is computed as:

$$\begin{aligned} CI(H) = &\frac{1}{dn^2} \sum\nolimits_{i=1}^{n} \sum\nolimits_{j=1}^{n} \sum\nolimits_{s=1}^{d} \left(\left(\bar{h}_{ij}^{(t)}\right)^{\sigma(s)} \times \left(\tilde{h}_{ji}^{(t)}\right)^{\sigma(s)}\right) \\ &+ \frac{2}{nd(n+1)} \\ &\times \sum\nolimits_{i=1}^{n} \sum\nolimits_{j\geq i}^{n} \sum\nolimits_{s=1}^{d} \left(\left(\bar{p}_{ij}^{(t)}\right)^{\sigma(s)} \times \left(\tilde{p}_{ji}^{(t)}\right)^{\sigma(s)}\right) \end{aligned}$$

The above equation can be further simplified as:

$$\begin{aligned} &CI(H) \\ &= \frac{2}{nd(n+1)} \sum\nolimits_{i=1}^{n} \sum\nolimits_{j\geq i}^{n} \sum\nolimits_{s=1}^{d} \left(\left(\bar{h}_{ij}^{(t)}\right)^{\sigma(s)} \times \left(\tilde{h}_{ji}^{(t)}\right)^{\sigma(s)}\right) \\ &\quad + \frac{2}{nd(n+1)} \sum\nolimits_{i=1}^{n} \sum\nolimits_{j\geq i}^{n} \sum\nolimits_{s=1}^{d} \\ &\quad \times \left(\left|\left(\bar{p}_{ij}^{(t)}\right)^{\sigma(s)} - \left(\tilde{p}_{ji}^{(t)}\right)^{\sigma(s)}\right|\right) \\ &= \frac{2}{nd(n+1)} \sum\nolimits_{i=1}^{n} \sum\nolimits_{j\geq i}^{n} \sum\nolimits_{s=1}^{d} \\ &\quad \times \left(\left(\bar{h}_{ij}^{(t)}\right)^{\sigma(s)} \times \left(\tilde{h}_{ji}^{(t)}\right)^{\sigma(s)} + \left|\left(\bar{p}_{ij}^{(t)}\right)^{\sigma(s)} - \left(\tilde{p}_{ji}^{(t)}\right)^{\sigma(s)}\right|\right) \end{aligned}$$

Let a random variable $R = \left(\bar{h}_{ij}^{(t)}\right)^{\sigma(s)} \times \left(\tilde{h}_{ji}^{(t)}\right)^{\sigma(s)} + \left|\left(\bar{p}_{ij}^{(t)}\right)^{\sigma(s)} - \left(\tilde{p}_{ji}^{(t)}\right)^{\sigma(s)}\right|$, then the value of R tends to 1. Hence, let us assume that the random variable R follows the normal distribution with the mean of 1 and the variance of σ_2^2, which is expressed by $R \sim N\left(1, \sigma_2^2\right)$. Namely, $E(R) = 1$ and $D(R) = \sigma_2^2$. Since the linear combination of n random variables obeying the normal distribution can also follow the normal distribution, then

$$E(CI(H)) = E\left(\frac{2}{nd(n+1)} \sum\nolimits_{i=1}^{n} \sum\nolimits_{j\geq i}^{n} \sum\nolimits_{s=1}^{d} R\right) = 1$$

and

$$\begin{aligned} D(CI(H)) &= D\left(\frac{2}{nd(n+1)} \sum\nolimits_{i=1}^{n} \sum\nolimits_{j\geq i}^{n} \sum\nolimits_{s=1}^{d} R\right) \\ &= \frac{2}{nd(n+1)}\sigma_2^2 \end{aligned}$$

Let a random variable $S = CI(H)$, then $S \sim N\left(1, \frac{2}{nd(n+1)}\sigma_2^2\right)$ and its distribution function is

$$F(s) = P\{S \leq s\} = \frac{\sqrt{nd(n+1)}}{2\sqrt{\pi}\sigma_2} \int_{-\infty}^{s} e^{-\frac{nd(n+1)(t-1)^2}{4\sigma_2^2}} dt$$

To perform fair comparison, the value of σ_2 is also set to 0.2, then $S \sim N(1, 0.0013)$. In [13], the value of $C\bar{I}$ is 1.01, then

$$LCI = P\{0 \leq S \leq 1.01\} = F(1.01) - F(0) = 0.1092.$$

Similar to LCI, the value of $C\bar{R}$ in [13] is set to 1.10, then

$$LCD = P\{0 \leq S \leq 1.10\} = F(1.10) - F(0) \approx 0.9972.$$

Based on the above analysis, the performance comparison between them is summarized in Table 1.

TABLE 1. Comparison analysis between our model and that in [13].

Studies	Iteration count for consistency	Confidence level LCI	Iteration count for consensus	Confidence level LCD
Our model	1 time	9.67%	1 time	12.02%
Model in [13]	2 times	10.92%	1 time	99.72%

As listed in Table I, it can be seen that our model requires one time when improving the consistency index. However, it takes the model proposed by Bashir *et al.* [13] two times to update the PHMPRs. During the consensus reaching process, both our model and the model proposed by Bashir *et al.* [13] requires only one time when revising the consensus degree. As discussed before, the higher the confidence levels for the consistency index and the consensus degree are, the higher the thresholds for the consistency index and the consensus degree are. Namely, the PHMPRs can reach the conditions for the consistency and consensus more easily. Even if the confidence levels for consistency index and consensus degree in our model is lower than that in [13], it takes our model less times to obtain the PHMPRs with the acceptably consistent index and consensus degree. It indicates that our model can perform better than the model proposed by Bashir *et al.* [13].

VII. CONCLUSIONS

In this paper, we have analyzed the existing group decision making model with PHMPRs proposed by Bashir *et al.* [13]. As a supplement to it, we put forward a novel group decision making model with the PHMPRs. Firstly, by means of the multiplicative preference relations (MPRs), the PHMPRs are transformed into the multiplicative preference relations. Then, based on the priority weights of the PHMPRs, the geometric consistency index is given to measure the consistency index of the PHMPRs and an automatic consistency checking and improving process is designed to revise the PHMPRs. After that, the geometric consensus degree based on the priority weight is designed to measure the consensus degree of the PHMPRs and it is utilized to develop a consensus reaching process for PHMPRs. Combining the consistency improving process and consensus reaching process, a complete group decision making model is presented to deal with PHMPRs and then make decisions. Finally, an example is provided to demonstrate the implementation process of our model and the probability theory is utilized to perform the comparison analysis between our model and the model that is proposed by Bashir *et al.* [13].

In the near future, we plan to introduce the uncertainty to extend the PHMPRs and study their features.

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