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Robust Generalized Chinese-Remainder-Theorem-Based DOA Estimation for a Coprime Array

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ABSTRACT In this paper, we consider the problem of determining the direction of arrival (DOA) of two sources for a coprime array. The novelty of this paper is that we present an efficient algorithm to estimate two DOAs from their wrapped phases with errors. The proposed robust generalized Chinese remainder theorem is considerably less computational complex than the searching method while maintaining comparable estimation precision. Moreover, the largest range of DOA that leads to an unambiguous determination is obtained for a given coprime array. Numerical simulations are presented to verify the efficiency and the robustness of the proposed algorithm. The simulation results show that the proposed algorithm is much faster than the searching method while achieving a similar root-mean-square error.

INDEX TERMS Coprime array, DOA estimation, generalized Chinese remainder theorem, phase unwrapping.

I. INTRODUCTION

Direction-of-arrival (DOA) estimation using sensor arrays is common in various areas, such as radar, sonar, and wireless communications [1]–[3]. During past decades, numerous models have been proposed to estimate DOA from the sensor arrays, such as uniform linear arrays (ULA), uniform rectangular arrays, uniform (non-uniform) circular arrays, and nested array [4]. Recently, coprime arrays, a type of sparse array, have attracted substantial attention because the increased number of degrees-of-freedom in beamforming and DOA estimation [5], [6]. Moreover, the autocorrelation of signals can be estimated in a much denser spacing, and sinusoids can be estimated in a more efficient way [7].

For coprime arrays, the efficient methods for the ULAs are not directly applied due to the non-equal spacing between any two adjacent nodes. If we use the existing algorithms for arbitrary arrays, the performance may degrade and the computational complexity may increase as the array configuration changes. In fact, a coprime array can be decomposed into two coprime ULAs with different inter-element spacings. Consequently, the existing results for ULAs can

be applied to determine DOA in a more convenient way. However, DOA estimation for a coprime array is challenging. First, since the inter-element spacing is larger than half of the wavelength, phase ambiguity occurs in the two arrays. As a result, the DOA of the target cannot be uniquely determined. For the case of multiple targets, the determination process is much more complicated. One of the difficulties is that the correspondence between the detected phases in each ULA and the DOA is unknown because the detected phases are disordered and wrapped by 2π . Moreover, the detected phases in the ULAs always have errors in the presence of noise in the system.

Some approaches to determine multiple DOAs from a coprime array have been proposed in literature. In [8], a spatial-smoothing-based method was proposed; however, this type of method has high computational complexity. In [9], nonlinear coprime arrays are first decomposed into two ULAs; then, a two phase adaptive spectrum-search-based method is proposed to estimate the DOAs. A projection-based method was proposed in [10], where two DOAs are estimated from the wrapped phases in two ULAs by a

search-free process. However, no closed-form solution was provided. Most recently, a novel coprime array adaptive beamforming method is proposed in [11], where the DOA are determined by properly matching the super-resolution spatial spectra of the pair of uniform linear subarrays. In fact, the DOA estimation problem above can be viewed as the generalized Chinese remainder theorem (CRT). Specifically, the two DOAs can be viewed as two unknown numbers, the inter-element spacings can be viewed as moduli, and the detected phases from each ULA can be viewed as the residue sets. Hence, the problem is to determine multiple numbers from their residue sets, where the correspondence between the number and its remainder is unknown. This type of generalized CRT was first proposed in [12], in which conditions on the multiple numbers and the moduli were needed, and no determination algorithm was presented. Then, it was independently studied in [13] and [14]. Different results were obtained, and the reconstruction algorithms were presented based on the assumption that all the remainders are error-free. For the case of estimating a single number from its erroneous remainders, the problem is called robust CRT and is well studied in [15] and [16]. In this paper, we consider the problem of determining two DOAs from the erroneous detected phases for a coprime array. To obtain the optimal estimates of two DOAs, we first model the coprime array as two ULAs and give the largest dynamic range of the DOAs that leads to an unambiguous determination. Then, we propose the robust generalized CRT algorithm to determine the two DOAs from their wrapped phases with errors for the coprime array. Finally, we present the simulations to verify the efficiency and robustness of the proposed algorithm.

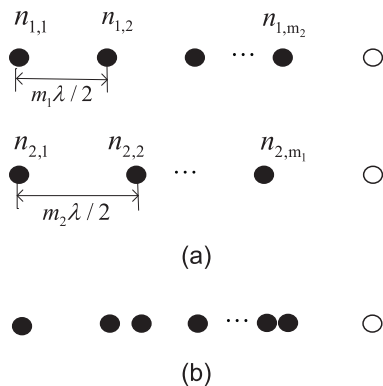


FIGURE 1. Illustration of a coprime array. (a) Two uniform linear arrays. (b) Coprime array formed by above ULAs.

II. COPRIME ARRAY SIGNAL MODEL

Let us consider the coprime array shown in FIGURE 1. The array is composed of two ULAs whose inter-element spacings are $m_2\lambda/2$ and $m_1\lambda/2$, where m_1 and m_2 are coprime, and λ is the wavelength of the carrier signal. Clearly, beyond the first element, the elements of the two subarrays are overlapped at the position $m_1m_2\lambda/2$, which is represented by hollow circles in the figure. If we align the two ULAs together

into one line, they form a coprime array with $m_1 + m_2 - 1$ elements. Conversely, a coprime array can be decomposed into two ULAs.

Suppose that there are K far-field narrowband sources located at $\boldsymbol{\theta} = [\theta_1, \theta_2, \dots, \theta_K]^T$, where $\theta_k \in (-\pi/2, \pi/2)$ and $(\cdot)^T$ denotes the transpose. Then the observation vector at the ℓ th snapshot can be represented as

$$\mathbf{x}(\ell) = \mathbf{A}\mathbf{s}(\ell) + \mathbf{w}(\ell), \quad \ell = 1, 2, \dots, L, \quad (1)$$

where $\mathbf{s}(\ell)$ is the vector of the source waveforms, $\mathbf{w}(\ell)$ is the additional Gaussian white noise, L is the number of snapshots, and $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_K]$ is the steering matrix composed of K steering vectors, where

$$\mathbf{a}_k = [e^{-j\frac{2\pi}{\lambda}p_1 \sin \theta_k}, \dots, e^{-j\frac{2\pi}{\lambda}p_{m_1+m_2-1} \sin \theta_k}]^T \quad (2)$$

and p_i is the position of the i -th sensor.

Note that the coprime array can be decomposed into a pair of ULAs. For two ULAs, we have the two steering vectors

$$\mathbf{a}_{k,1} = [1, e^{-j\frac{2\pi}{\lambda}m_2 \sin \theta_k}, \dots, e^{-j\frac{2\pi}{\lambda}(m_1-1)m_2 \sin \theta_k}]^T \quad (3)$$

and

$$\mathbf{a}_{k,2} = [1, e^{-j\frac{2\pi}{\lambda}m_1 \sin \theta_k}, \dots, e^{-j\frac{2\pi}{\lambda}(m_2-1)m_1 \sin \theta_k}]^T. \quad (4)$$

The received vector for each ULA at the ℓ -th snapshot can be represented as

$$\mathbf{x}_i(\ell) = \mathbf{A}_i\mathbf{s}(\ell) + \mathbf{w}(\ell), \quad i = 1, 2; \ell = 1, 2, \dots, L, \quad (5)$$

where $\mathbf{A}_i = [\mathbf{a}_{1,i}, \mathbf{a}_{2,i}, \dots, \mathbf{a}_{K,i}]$ are the steering matrices of the two ULAs. Now, we use the MUSIC algorithm to determine the DOA for each ULA. We herein assume the source number has been successfully estimated by the effective algorithms such as AIC criterion [17] or the second order statistic of the eigenvalues (SORTE) algorithm [18]. Generally speaking, the MUSIC spectrum is acquired by the following steps.

First, obtain the two sample covariance matrices from the two decomposed ULAs:

$$\mathbf{R}_i = \frac{1}{L} \sum_{\ell=1}^L \mathbf{x}_i(\ell)\mathbf{x}_i^H(\ell), \quad i = 1, 2, \quad (6)$$

where $(\cdot)^H$ denotes the Hermitian transpose. Then, apply eigen-decomposition to the sample covariance matrix above. Suppose that the eigenvalues of the i -th sample covariance matrix satisfy

$$\lambda_{1,i} \geq \dots \geq \lambda_{K,i} > \lambda_{K+1,i} = \dots = \lambda_{m_i,i} = \sigma^2, \quad (7)$$

and $\mathbf{e}_{1,i}, \mathbf{e}_{2,i}, \dots, \mathbf{e}_{m_i,i}$ are the corresponding eigenvectors. Then, we have

$$\mathbf{R}_i = \mathbf{E}_{s_i}\boldsymbol{\Lambda}_{s_i}\mathbf{E}_{s_i}^H + \mathbf{E}_{n_i}\boldsymbol{\Lambda}_{n_i}\mathbf{E}_{n_i}^H, \quad i = 1, 2, \quad (8)$$

where $\boldsymbol{\Lambda}_{s_i} = \text{diag}(\lambda_{1,i}, \lambda_{2,i}, \dots, \lambda_{K,i})$, $\boldsymbol{\Lambda}_{n_i} = \sigma^2\mathbf{I}_{m_i-K}$, $\mathbf{E}_{s_i} = [\mathbf{e}_{1,i}, \mathbf{e}_{2,i}, \dots, \mathbf{e}_{K,i}]$, and $\mathbf{E}_{n_i} = [\mathbf{e}_{K+1,i}, \mathbf{e}_{K+2,i}, \dots, \mathbf{e}_{m_i,i}]$.

Consequently, we can obtain the MUSIC spatial pseudo-power spectrum of the two ULAs as

$$P_i(\theta) = \frac{1}{\mathbf{a}_i^H(\theta)\mathbf{E}_{n_i}\mathbf{E}_{n_i}^H\mathbf{a}_i(\theta)}, \quad i = 1, 2 \quad (9)$$

where θ is the hypothetical direction with the range $(-\pi/2, \pi/2)$. Finally, the DOAs of the two ULAs can be determined by searching the peaks of the MUSIC spectrum.

Note that the inter-element spacing of the two ULAs is larger than the half-wavelength. Hence, the detected phases are ambiguous for the two ULAs. In other words, the detected phases are the remainders of the two DOAs wrapped by 2π .

Without considering the influence of noise in the systems, the wrapped phases of the two ULAs are assumed to be $\{\psi_{1,1}, \dots, \psi_{K,1}\}$ and $\{\psi_{1,2}, \dots, \psi_{K,2}\}$. According to [19], $\psi_{k,i}$ and the corresponding DOAs θ_k satisfy

$$\begin{cases} \sin \theta_1 = \sin \psi_{1,i} + \frac{2n_1}{m_i} \\ \sin \theta_2 = \sin \psi_{2,i} + \frac{2n_2}{m_i}, \end{cases} \quad (10)$$

where n_1 and n_2 are some unknown integers named folding integers.

It is worth mentioning that the initialized residues comes from the spatial spectrum searching by MUSIC algorithm. However, due to the characteristic of co-prime arrays, each group of the searched results occur spectrum aliasing, that is to say, there exist multiple spectrum peaks with respect to only one DOA. Actually, we can consider this problem from the perspective of roots of one polynomial related to the array steering vector. Taking one of the enlarged ULA of the co-prime array for example, i.e., m_1 -element ULA with $m_2\lambda/2$ spacing. If z_k is one root of the following polynomial (for ULA with $\lambda/2$ spacing),

$$b(z) = b_0 \prod_{i=1}^K (z - z_k)$$

then for the above enlarged array, the polynomial becomes

$$\bar{b}(z) = \bar{b}_0 \prod_{i=1}^K (z^M - \bar{z}_k)$$

and z_k is still a root of this new polynomial. More importantly, the $z_k e^{j\ell \frac{2\pi}{m_1}}$ for $\ell = 1, 2, \dots, m_1 - 1$ are all the roots. That is to say, the totally $m_1 \times K$ roots are periodic and the period is $2\pi/m_1$, and the roots in the range of $(0, 2\pi/m_1)$ are unique. The same conclusion is also suitable for the other enlarged array of the co-prime array except the period is $2\pi/m_2$.

Further, we should consider the allowed angular range determined by an enlarge array. As we know, the angular phase of $e^{j\pi \sin \theta}$ is $[-\pi, \pi)$, that is to say, $\sin \theta \in [-1, 1]$ or equivalently $\theta \in [-\pi/2, \pi/2]$ for avoiding angle ambiguity. However, for the enlarged array, it is changed as $e^{jm_i\pi \sin \theta}$, and consequently $\sin \theta \in [-1/m_i, 1/m_i]$, which means the observable angular range becomes narrow.

In real-word applications, the detected phases always contain error due to noise in the system. Let the erroneous wrapped phases in the two ULAs be $\{\tilde{\psi}_{1,1}, \dots, \tilde{\psi}_{K,1}\}$ and $\{\tilde{\psi}_{1,2}, \dots, \tilde{\psi}_{K,2}\}$, where

$$\tilde{\psi}_{k,i} = \psi_{k,i} + \Delta\psi_{k,i} \quad (11)$$

with errors $\Delta\psi_{k,i}$. Then the problem of determining multiple DOAs for the coprime array is changed to how to estimate $\{\theta_1, \dots, \theta_K\}$ from their erroneous detected phases $\{\tilde{\psi}_{1,1}, \dots, \tilde{\psi}_{K,1}\}$ and $\{\tilde{\psi}_{1,2}, \dots, \tilde{\psi}_{K,2}\}$ in the two ULAs, where $\tilde{\psi}_{k,i} \in (-\pi/2, \pi/2)$. In this paper, we consider the case of two DOAs.

III. ROBUST GENERALIZED CRT-BASED METHOD

A. MATHEMATICAL MODEL

We begin with the determination of the two DOAs $\{\theta_1, \theta_2\}$ from their error-free residue sets $\{\psi_{1,1}, \psi_{2,1}\}$ and $\{\psi_{1,2}, \psi_{2,2}\}$. In contrast to traditional CRT, the quantity of numbers to be determined is two rather than one. The main difficulty of the problem is that the correspondence between θ_i and the wrapped phase $\psi_{k,i}$ is unknown. Specifically, we have no way of knowing which angle in the residue set $\{\psi_{1,i}, \psi_{2,i}\}$ corresponds to the first DOA and which corresponds to the second DOA. Intuitively, the correspondence can be correctly determined by a combinatorial-based method; however, this process is time-consuming, especially when the number of remainders is large. Moreover, no closed-form solution is obtained. In fact, the problem can be modeled as a generalized CRT for two numbers, as shown below.

From (10), we obtain

$$\begin{cases} \sin \theta_1 \equiv \sin \psi_{1,i} \pmod{\frac{2}{m_i}} \\ \sin \theta_2 \equiv \sin \psi_{2,i} \pmod{\frac{2}{m_i}}, \end{cases} \quad (12)$$

where $i = 1, 2$. Let $\Gamma_1 = m_2, \Gamma_2 = m_1, \Gamma = \Gamma_1\Gamma_2$, and M be a positive integer which is chosen as needed. After multiplying $M\Gamma/2$ for both sides of (12), we have

$$\begin{cases} \frac{M\Gamma \sin \theta_1}{2} \equiv \frac{M\Gamma \sin \psi_{1,i}}{2} \pmod{M\Gamma_i} \\ \frac{M\Gamma \sin \theta_2}{2} \equiv \frac{M\Gamma \sin \psi_{2,i}}{2} \pmod{M\Gamma_i}. \end{cases} \quad (13)$$

Hence, (13) can be simplified as

$$\begin{cases} N_1 \equiv r_{1,i} \pmod{M\Gamma_i} \\ N_2 \equiv r_{2,i} \pmod{M\Gamma_i}, \end{cases} \quad (14)$$

where the two numbers are

$$N_k = \frac{M\Gamma \sin \theta_k}{2}, \quad k = 1, 2 \quad (15)$$

and the remainder of N_k modulo $M\Gamma_i$ is

$$r_{k,i} = \frac{M\Gamma \sin \psi_{k,i}}{2}, \quad k = 1, 2, i = 1, 2. \quad (16)$$

If $\theta_k \in (0, 2\pi/m_i)$, then we have $N_k > 0$ for $k = 1, 2$. Hence, the problem of determining the two DOAs $\{\theta_1, \theta_2\}$

of the coprime array from the wrapped phases $\{\psi_{1,1}, \psi_{2,1}\}$ and $\{\psi_{1,2}, \psi_{2,2}\}$ is equivalent to reconstructing the two positive numbers $\{N_1, N_2\}$ from their residue sets $\{r_{1,1}, r_{2,1}\}$ and $\{r_{1,2}, r_{2,2}\}$, which is the generalized CRT. When $\theta_k \in (-2\pi/m_i, 0)$, congruence (13) can also be solved by the generalized CRT after changing the two sides of the congruence to their absolute values. In the following, we consider only the case where $\theta_k \in (0, 2\pi/m_i)$. For convenience, the remainder of x modulo M is denoted as $\langle x \rangle_M$.

B. THE GENERALIZED CRT FOR TWO NUMBERS

Let us first recall the basic idea of the traditional CRT. According to [16], if integer N is less than the least common multiple (LCM) of the given moduli $M\Gamma_i$, then N can be reconstructed from its remainders r_i by the formula

$$N = MQ + r^c, \tag{17}$$

where

$$r^c = \langle r_i \rangle_M \tag{18}$$

and

$$Q \equiv \frac{r_i - r^c}{M} \text{ mod } \Gamma_i. \tag{19}$$

The reconstruction process for the generalized CRT for two numbers is more complicated. First, the dynamic range of $\{N_1, N_2\}$ should be determined from the given moduli, where the dynamic range represents the two numbers that can be uniquely determined from their residue sets. For example, we consider the determination of the two numbers $\{1, 2\}$ with moduli 2 and 3. Clearly, the residue sets of the two numbers modulo 2 and 3 are $\{0, 1\}$ and $\{1, 2\}$, respectively. If the two numbers are less than 6 (lcm of the moduli), then they cannot be uniquely determined because both candidates $\{1, 2\}$ and $\{4, 5\}$ are satisfied. For the case of pairwise coprime moduli Γ_1 and Γ_2 , the largest dynamic range is given in [14]. When the moduli are $M\Gamma_1$ and $M\Gamma_2$, we have the following result.

Theorem 1: If $\Gamma_i > 3$ for $i = 1, 2$, then the dynamic range of the two numbers for the modulus set $\mathcal{M} = \{M\Gamma_1, M\Gamma_2\}$ is

$$D_2(\mathcal{M}) = M(\Gamma_1 + \Gamma_2). \tag{20}$$

The proof of the theorem is similar to that of [14, Th. 1]; therefore, it is omitted.

According to Theorem 1, if both numbers are within the dynamic range $M(\Gamma_1 + \Gamma_2)$, i.e.,

$$N_k < M(\Gamma_1 + \Gamma_2), \quad k = 1, 2, \tag{21}$$

then $\{N_1, N_2\}$ can be uniquely determined from their residue sets $\{r_{1,1}, r_{2,1}\}$ and $\{r_{1,2}, r_{2,2}\}$ modulo $M\Gamma_1$ and $M\Gamma_2$, respectively. According to (15), we can obtain the dynamic range of two DOAs as

$$\theta_k < \arcsin \left[\frac{2(\Gamma_1 + \Gamma_2)}{\Gamma_1 \Gamma_2} \right], \quad k = 1, 2. \tag{22}$$

In other words, any two DOAs $\{\theta_1, \theta_2\}$ satisfying (22) can be uniquely determined by their wrapped phases $\{\psi_{1,1}, \psi_{2,1}\}$

and $\{\psi_{1,2}, \psi_{2,2}\}$. In the following, we suppose that the conditions described in (22) are always satisfied.

To reconstruct the two DOAs $\{\theta_1, \theta_2\}$, the two common remainders $\{r_1^c, r_2^c\}$ should first be determined from their residue sets $\{r_{1,1}, r_{2,1}\}$ and $\{r_{1,2}, r_{2,2}\}$. According to (18), the two common remainders can be obtained by

$$\{r_1^c, r_2^c\} = \{\langle r_{1,i} \rangle_M, \langle r_{2,i} \rangle_M\}, \quad i = 1, 2. \tag{23}$$

Then, the two integers $\{Q_1, Q_2\}$ can be determined from the residue sets $\{q_{1,1}, q_{2,1}\}$ and $\{q_{1,2}, q_{2,2}\}$ by using the generalized CRT for two integers proposed in [14], where

$$q_{k,i} = \frac{r_{k,i} - r_k^c}{M}, \quad k = 1, 2, \quad i = 1, 2. \tag{24}$$

Finally, the two numbers can be reconstructed as

$$\{N_1, N_2\} = \{MQ_1 + r_1^c, MQ_2 + r_2^c\}. \tag{25}$$

By (15), we can obtain the two DOAs

$$\{\theta_1, \theta_2\} = \left\{ \arcsin \frac{2N_1}{M\Gamma}, \arcsin \frac{2N_2}{M\Gamma} \right\}. \tag{26}$$

C. ROBUST GENERALIZED CRT-BASED ALGORITHM

Now, we present the basic idea of estimating the two DOAs $\{\theta_1, \theta_2\}$ from their erroneous residue sets $\{\tilde{\psi}_{1,1}, \tilde{\psi}_{2,1}\}$ and $\{\tilde{\psi}_{1,2}, \tilde{\psi}_{2,2}\}$.

First, the erroneous residue sets $\{\tilde{r}_{1,1}, \tilde{r}_{2,1}\}$ and $\{\tilde{r}_{1,2}, \tilde{r}_{2,2}\}$ are determined, where

$$\tilde{r}_{k,i} = \frac{M\Gamma \sin \tilde{\psi}_{k,i}}{2}, \quad k = 1, 2; \quad i = 1, 2. \tag{27}$$

Then, the two common remainders $\{r_1^c, r_2^c\}$ are determined. According to [20], the precision of the common remainders is significant for the estimation of the two numbers. In contrast to the case of the error-free remainders, the two common remainders cannot be determined directly from the residue sets because the two sets, $\{\langle \tilde{r}_{1,1} \rangle_M, \langle \tilde{r}_{2,1} \rangle_M\}$ and $\{\langle \tilde{r}_{1,2} \rangle_M, \langle \tilde{r}_{2,2} \rangle_M\}$, may be different due to errors. Hence, we cannot determine the optimal estimates of the two common remainders. Note that all the erroneous common remainders $\langle \tilde{r}_{k,i} \rangle_M$ are obtained by modular operation. Therefore, distance in Euclidean space is an inappropriate measure of the errors. For example, the minimal distance between $\langle \tilde{r}_1 \rangle_M = 9$ and $\langle \tilde{r}_2 \rangle_M = 2$ for $M = 10$ is 3 when $\tilde{r}_1 = 9 + kM$ and $\tilde{r}_2 = 2 + (k + 1)M$ for some integer k , rather than 7 in Euclidean space. To describe this type of distance, we define the circular distance between two numbers x and y for a positive integer M as

$$d_M(x, y) \triangleq x - y - \left\lfloor \frac{x - y}{M} \right\rfloor M, \tag{28}$$

where $\lfloor x \rfloor$ denotes the rounding operation satisfying

$$-1/2 \leq x - \lfloor x \rfloor < 1/2. \tag{29}$$

To obtain the two optimal estimates $\{\hat{r}_1^c, \hat{r}_2^c\}$ from all the erroneous common remainders, we first let

$$\Omega \triangleq \{\langle \tilde{r}_{1,1} \rangle_M, \langle \tilde{r}_{2,1} \rangle_M, \langle \tilde{r}_{1,2} \rangle_M, \langle \tilde{r}_{2,2} \rangle_M\}. \tag{30}$$

Then, these erroneous common remainders can be sorted in increasing order as

$$\tilde{r}_{\zeta(1)}^c \leq \tilde{r}_{\zeta(2)}^c \leq \tilde{r}_{\zeta(3)}^c \leq \tilde{r}_{\zeta(4)}^c, \quad (31)$$

where $\tilde{r}_{\zeta(i)}^c \in \Omega$ and $\zeta(i)$ is a permutation of $\{1, 2, 3, 4\}$. Define

$$D_i \triangleq \begin{cases} \tilde{r}_{\zeta(i+1)}^c - \tilde{r}_{\zeta(i)}^c, & i = 1, 2, 3 \\ \tilde{r}_{\zeta(1)}^c - \tilde{r}_{\zeta(4)}^c + M, & i = 4. \end{cases} \quad (32)$$

By minimizing the summation of $D_i + D_{i+2}$ for $i = 1, 2$, two clusters Ω_1 and Ω_2 with the same elements can be obtain. To be specific, if $D_1 + D_3 < D_2 + D_4$, the two clusters can be described as

$$\Omega_1 = \{\tilde{r}_{\zeta(1)}^c, \tilde{r}_{\zeta(2)}^c\}, \quad \Omega_2 = \{\tilde{r}_{\zeta(3)}^c, \tilde{r}_{\zeta(4)}^c\}. \quad (33)$$

Otherwise, we have

$$\Omega_1 = \{\tilde{r}_{\zeta(2)}^c, \tilde{r}_{\zeta(3)}^c\}, \quad \Omega_2 = \{\tilde{r}_{\zeta(4)}^c, \tilde{r}_{\zeta(1)}^c\}. \quad (34)$$

Consequently, the two optimal estimates $\{\hat{r}_1^c, \hat{r}_2^c\}$ can be determined by

$$\{\hat{r}_1^c, \hat{r}_2^c\} \triangleq \left\{ \arg \min_{r \in \Omega_1, x \in [0, M)} \sum_{i=1}^2 d_M^2(r, x), \arg \min_{r_c \in \Omega_2, y \in [0, M)} \sum_{i=1}^2 d_M^2(r_c, y) \right\}. \quad (35)$$

After cancelling the proper estimate \hat{r}_1^c or \hat{r}_2^c from each remainder $\tilde{r}_{k,i}$, we obtain

$$\{\hat{q}_{1,i}, \hat{q}_{2,i}\} = \left\{ \left\lfloor \frac{\tilde{r}_{1,i} - \hat{r}_1^c}{M} \right\rfloor, \left\lfloor \frac{\tilde{r}_{2,i} - \hat{r}_2^c}{M} \right\rfloor \right\}, \quad i = 1, 2. \quad (36)$$

By using the generalized CRT for two integers proposed in [14], we can obtain the two integers $\{\hat{Q}_1, \hat{Q}_2\}$. Thus, the two estimates $\{\hat{N}_1, \hat{N}_2\}$ can be reconstructed by (25) after integers \hat{Q}_i and common remainders \hat{r}_i^c are properly matched. Finally, the two DOAs $\{\theta_1, \theta_2\}$ can be determined by (26).

We have the following result for the two estimates $\{\hat{r}_1^c, \hat{r}_2^c\}$.

Theorem 2: The two estimates $\{\hat{r}_1^c, \hat{r}_2^c\}$ in (35) can be simplified as

$$\{\hat{r}_1^c, \hat{r}_2^c\} = \begin{cases} \left\{ \left\lfloor \frac{\tilde{r}_{\zeta(1)}^c + \tilde{r}_{\zeta(2)}^c}{2} \right\rfloor, \left\lfloor \frac{\tilde{r}_{\zeta(3)}^c + \tilde{r}_{\zeta(4)}^c}{2} \right\rfloor \right\}, \\ \left\{ \left\lfloor \frac{\tilde{r}_{\zeta(2)}^c + \tilde{r}_{\zeta(3)}^c}{2} \right\rfloor, \left\lfloor \frac{\tilde{r}_{\zeta(4)}^c + \tilde{r}_{\zeta(1)}^c + M}{2} \right\rfloor \right\}_M, & \text{if } D_1 + D_3 < D_2 + D_4 \\ \text{otherwise.} \end{cases} \quad (37)$$

The proof of the theorem is in Appendix.

Theorem 2 shows that the two common remainders can be directly determined from the given erroneous common remainders. Compared with the searching method, it has a closed-form and hence the computational complexity is greatly reduced. Table 1 gives the robust generalized CRT-based algorithm discussed thus far, where the notation $\lceil x \rceil$ denotes the smallest integer not less than x .

TABLE 1. Robust generalized crt-based algorithm.

Robust Generalized CRT-Based DOA Estimation Algorithm
Step 1. Determine Ω as described in (30).
Step 2. Sort the erroneous common remainders as in (31).
Step 3. Determine the two optimal estimates as in (37).
Step 4. Compute $\hat{q}_{k,i}$ as (36) for $k = 1, 2; i = 1, 2$.
Step 5. Compute $\xi_1 \equiv q_1 \Gamma_2 \bar{\Gamma}_2 + q_2 \Gamma_1 \bar{\Gamma}_1 \pmod{\Gamma}$, where $q_i \equiv \hat{q}_{1,i} + \hat{q}_{2,i} \pmod{\Gamma_i}$ and $\bar{\Gamma}_i$ is the multiplicative inverse of Γ_i modulo Γ/Γ_i for $i = 1, 2$.
Step 6. Compute $\xi_2 \equiv (\hat{q}_{1,i} - s)(\hat{q}_{2,i} - s) \pmod{\Gamma_i}$, where $s = \max\{0, \lceil \xi_1 - 2\sqrt{\Gamma} \rceil\}$.
Step 7. Determine $\{Q_1, Q_2\}$ by solving the quadratic equation: $(x - s)^2 - (\xi_1 - 2s)(x - s) + \xi_2 = 0$.
Step 8. Determine the two numbers $\{N_1, N_2\}$ as (25).
Step 9. Determine the two DOAs $\{\theta_1, \theta_2\}$ as (26).

To show that the proposed algorithm is robust, we introduce a lemma below.

Lemma 1 [20]: Let $\tau = \max\{|\Delta r_{k,i}|, k = 1, 2; i = 1, 2\}$, where $\Delta r_{k,i}$ are the remainder errors. If $\tau < M/8$, then

$$|\hat{N}_k - N_k| \leq \tau, \quad k = 1, 2. \quad (38)$$

This lemma tells us that the estimates of $\{N_1, N_2\}$ are robust when the remainder error bound is less than $M/8$. From (16) and (27), we have

$$\begin{aligned} \Delta r_{k,i} &= \tilde{r}_{k,i} - r_{k,i} \\ &= \frac{M\Gamma}{2} (\sin \tilde{\psi}_{k,i} - \sin \psi_{k,i}). \end{aligned} \quad (39)$$

According to Lemma 1, when all the remainder errors satisfy

$$\frac{M\Gamma}{2} |\sin \tilde{\psi}_{k,i} - \sin \psi_{k,i}| < \frac{M}{8}, \quad (40)$$

the two estimates $\{\hat{N}_1, \hat{N}_2\}$ and thus $\{\hat{\theta}_1, \hat{\theta}_2\}$ are robust. Hence, the conditions of the robust estimation of the two angles $\{\theta_1, \theta_2\}$ are

$$|\sin \tilde{\psi}_{k,i} - \sin \psi_{k,i}| < \frac{1}{4\Gamma}, \quad k = 1, 2, i = 1, 2. \quad (41)$$

IV. SIMULATION RESULTS

In this section, we present some simulation results to verify the efficiency and the robustness of the proposed robust generalized CRT algorithm. In the simulations, the numbers of sensors in the two ULAs are $m_1 = 5$ and $m_2 = 7$. The two DOAs $\{\theta_1, \theta_2\}$ are set to $\{0.75, 0.55\}$ rad, which satisfy $\theta_k \in (0, 2\pi/m_i)$ and the largest dynamic range condition in (22). The additive noise is a complex additional white Gaussian random process with mean zero and variance σ^2 . The number of trials is 10000 for each signal-to-noise ratio (SNR).

For comparison purposes, we consider the searching method. For this method, we first consider all possible

remainders of $\{N_1, N_2\}$ modulo $M\Gamma_1$ and $M\Gamma_2$ from their residue sets. Since the remainders in each residue set are unordered, we have two cases, i.e.,

$$\{\tilde{r}_{1,1}, \tilde{r}_{1,2}\}, \quad \{\tilde{r}_{2,1}, \tilde{r}_{2,2}\} \quad (42)$$

and

$$\{\tilde{r}_{1,1}, \tilde{r}_{2,2}\}, \quad \{\tilde{r}_{2,1}, \tilde{r}_{1,2}\}. \quad (43)$$

Then, the two estimates $\{\hat{N}_1, \hat{N}_2\}$ are determined by the two minimization problems, which correspond to the two cases above. Take the first case as an example, $\{N_1, N_2\}$ can be determined by solving the following minimization problem:

$$\min_{N_1, N_2} \left\{ d_{M\Gamma_1}^2(N_1, \tilde{r}_{1,1}) + d_{M\Gamma_2}^2(N_2, \tilde{r}_{1,2}) + d_{M\Gamma_1}^2(N_1, \tilde{r}_{2,1}) + d_{M\Gamma_2}^2(N_2, \tilde{r}_{2,2}) \right\}. \quad (44)$$

After $\{\hat{N}_1, \hat{N}_2\}$ are determined by the searching method described above, the two angles $\{\theta_1, \theta_2\}$ are determined by (26).

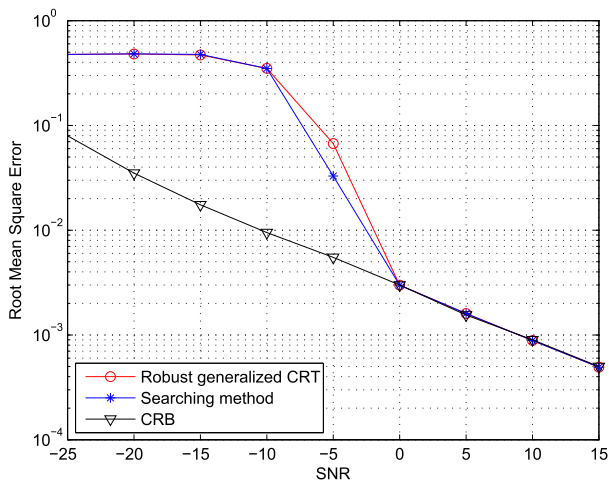


FIGURE 2. RMSE versus SNR.

FIGURE 2 shows that the root-mean-square error (RMSE) of the two angles depends on SNR, where the number of the snapshots is 20. The RMSE of $\{\theta_1, \theta_2\}$ is defined as

$$\theta_{RMSE} = E \left\{ \sqrt{\frac{1}{2} \sum_{k=1}^2 (\hat{\theta}_k - \theta_k)^2} \right\}, \quad (45)$$

where $E\{\cdot\}$ denotes the mean over all trails and $\hat{\theta}_k$ represents the DOA estimates. The Cramér-Rao bound (CRB) given in [21] shows that the proposed robust generalized CRT method has slightly worse performance than the searching method when the SNR is between -10 dB and 0 dB. When the SNR is greater than 0 dB, the two methods have nearly equivalent performance, which is in agreement with the theory of CRB. When the SNR is larger than -10 dB, the two algorithms have worse performances than the CRB, because the remainder errors are large and beyond the error correction ability.

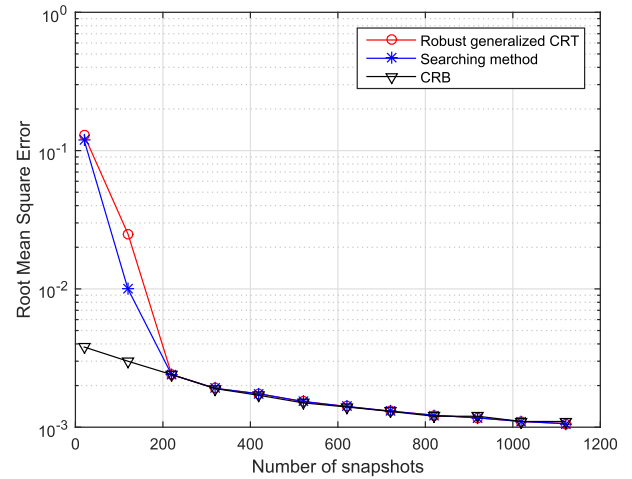


FIGURE 3. RMSE versus number of snapshots.

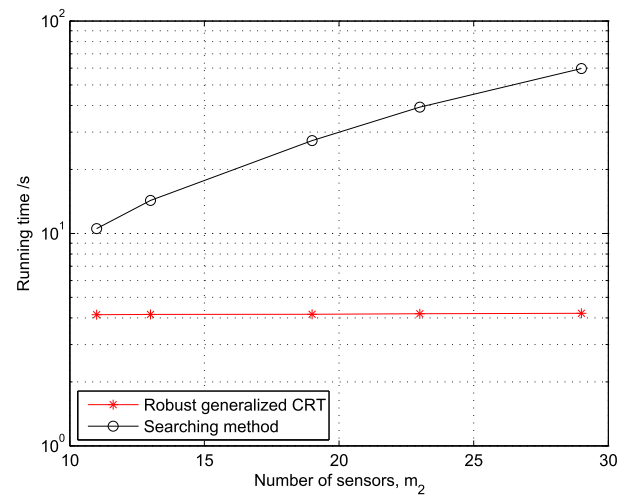


FIGURE 4. Running time versus number of sensors.

FIGURE 3 shows the dependence of the RMSEs of two angles $\{\theta_1, \theta_2\}$ on the number of snapshots when the SNR is set to -10 dB. The proposed robust generalized CRT algorithm has slightly worse performance than the searching method when the number of snapshots is between 20 and 200. When the number of snapshots is greater than 200, the two methods have nearly identical performance.

Finally, we consider the computational complexity of the two algorithms. For our proposed robust generalized CRT method, the computational complexity is $\mathcal{O}(12K)$. For the searching method, the computational complexity is $\mathcal{O}(m_1^2 m_2^2)$, where K is the number of sources. Clearly, the computational complexity of our proposed method is much lower than that of the searching method. Another merit of our proposed method is that the computational complexity is independent of the number of sensors. FIGURE 4 shows the curves of running time versus the number of sensors when the number of sensors $m_1 = 7$ and the number of sensors m_2 is 11, 13, 19, 23, 29, respectively. The number of snapshots is 400 for each case. Simulation result shows that the proposed algorithm has nearly equivalent running time for different

coprime arrays. By contrast, the running time increases with increasing number of sensors for the searching method.

V. CONCLUSION

In this paper, we proposed a robust generalized CRT algorithm to determine two sources for a coprime array, where the array is viewed as two ULAs. Compared with that of the searching method, the proposed method has significantly low computational complexity because aimless searching is avoided. Moreover, the proposed method has a closed-form solution. Simulations are presented to verify the efficiency and the robustness of the proposed algorithm.

**APPENDIX
PROOF OF THEOREM 2**

Proof: We have two cases below.

Case I: $D_1 + D_3 < D_2 + D_4$.

Since $\sum_{i=1}^4 D_i = M$, we obtain

$$D_1 + D_3 < M/2.$$

Otherwise, we have $D_2 + D_4 > D_1 + D_3 \geq M/2$ and hence $\sum_{i=1}^4 D_i > M$, which is a contradiction. Since $D_i > 0$, we have

$$D_1 < M/2, D_3 < M/2.$$

By the definition of D_i in (32), we have

$$\tilde{r}_{S(2)}^c - \tilde{r}_{S(1)}^c < M/2 \quad \text{and} \quad \tilde{r}_{S(4)}^c - \tilde{r}_{S(3)}^c < M/2.$$

Let $f(x) = \sum_{i=1}^2 d_M^2(\tilde{r}_{S(i)}^c, x)$. Then, the local minimum point of $f(x)$ can be obtained by $f'(x) = 0$. Hence, the optimal estimate $x = \hat{r}_1^c$ satisfies

$$f'(x)|_{x=\hat{r}_1^c} = 0.$$

That is,

$$\sum_{i=1}^2 (\tilde{r}_{S(i)}^c - x - k_i M) \Big|_{x=\hat{r}_1^c} = 0$$

for some $k_i \in \{-1, 0, 1\}$. This leads to

$$\hat{r}_1^c \in \left\{ \left\langle \frac{\tilde{r}_{S(1)}^c + \tilde{r}_{S(2)}^c}{2} - \frac{k'M}{2} \right\rangle_M, k' = 0, 1 \right\}.$$

It is not difficult to find that

$$f\left(\left\langle \frac{\tilde{r}_{S(1)}^c + \tilde{r}_{S(2)}^c}{2} \right\rangle_M\right) < f\left(\left\langle \frac{\tilde{r}_{S(1)}^c + \tilde{r}_{S(2)}^c}{2} - \frac{M}{2} \right\rangle_M\right).$$

Hence,

$$\hat{r}_1^c = \frac{\tilde{r}_{S(1)}^c + \tilde{r}_{S(2)}^c}{2}.$$

Similarly, we can obtain the optimal estimate \hat{r}_2^c of $\Omega_2 = \{\tilde{r}_{S(3)}^c, \tilde{r}_{S(4)}^c\}$ as

$$\hat{r}_2^c = \frac{\tilde{r}_{S(3)}^c + \tilde{r}_{S(4)}^c}{2}.$$

Therefore,

$$\{\hat{r}_1^c, \hat{r}_2^c\} = \left\{ \frac{\tilde{r}_{S(1)}^c + \tilde{r}_{S(2)}^c}{2}, \frac{\tilde{r}_{S(3)}^c + \tilde{r}_{S(4)}^c}{2} \right\}.$$

Case II: $D_1 + D_3 \geq D_2 + D_4$.

In this case, we have

$$D_2 + D_4 < M/2$$

and

$$\Omega_1 = \{\tilde{r}_{S(2)}^c, \tilde{r}_{S(3)}^c\}, \quad \Omega_2 = \{\tilde{r}_{S(4)}^c, \tilde{r}_{S(1)}^c\}.$$

Similar to the proof above, we can obtain the optimal estimate \hat{r}_1^c of the cluster Ω_1 as

$$\hat{r}_1^c = \frac{\tilde{r}_{S(2)}^c + \tilde{r}_{S(3)}^c}{2}.$$

Next, we give the optimal estimate \hat{r}_2^c of Ω_2 . Note that

$$\tilde{r}_{S(4)}^c - \tilde{r}_{S(1)}^c = D_1 + D_2 + D_3 > M/2.$$

Hence,

$$\tilde{r}_{S(1)}^c + M - \tilde{r}_{S(4)}^c < M/2. \tag{46}$$

On the other hand, from $\tilde{r}_{S(4)}^c - \tilde{r}_{S(1)}^c < M$, we obtain

$$\tilde{r}_{S(1)}^c + M - \tilde{r}_{S(4)}^c > 0. \tag{47}$$

According to (46) and (47), we have

$$d_M(\tilde{r}_{S(4)}^c, \tilde{r}_{S(1)}^c) = \tilde{r}_{S(1)}^c + M - \tilde{r}_{S(4)}^c.$$

Let $g(x) = d_M^2(\tilde{r}_{S(4)}^c, x) + d_M^2(\tilde{r}_{S(1)}^c, x)$. Then, the minimal point of $x = \hat{r}_2^c$ satisfies

$$g'(x)|_{x=\hat{r}_2^c} = 0.$$

That is,

$$\left[(\tilde{r}_{S(4)}^c - y - k_4 M) + (\tilde{r}_{S(1)}^c - y - k_1 M) \right] \Big|_{x=\hat{r}_2^c} = 0$$

for some $k_1, k_4 \in \{-1, 0, 1\}$. Hence,

$$\hat{r}_2^c \in \left\{ \left\langle \frac{\tilde{r}_{S(4)}^c + \tilde{r}_{S(1)}^c}{2} - \frac{k'M}{2} \right\rangle_M, k' = 0, -1 \right\}.$$

It is not difficult to find that

$$g\left(\left\langle \frac{\tilde{r}_{S(4)}^c + \tilde{r}_{S(1)}^c + M}{2} \right\rangle_M\right) < g\left(\left\langle \frac{\tilde{r}_{S(4)}^c + \tilde{r}_{S(1)}^c}{2} \right\rangle_M\right).$$

This leads to

$$\hat{r}_2^c = \left\langle \frac{\tilde{r}_{S(4)}^c + \tilde{r}_{S(1)}^c + M}{2} \right\rangle_M.$$

Therefore,

$$\{\hat{r}_1^c, \hat{r}_2^c\} = \left\{ \frac{\tilde{r}_{S(2)}^c + \tilde{r}_{S(3)}^c}{2}, \left\langle \frac{\tilde{r}_{S(4)}^c + \tilde{r}_{S(1)}^c + M}{2} \right\rangle_M \right\}.$$

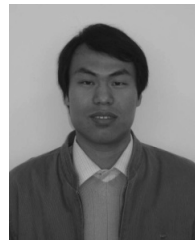
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