IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received September 2, 2018, accepted October 3, 2018, date of publication October 11, 2018, date of current version November 9, 2018.

Digital Object Identifier 10.1109/ACCESS.2018.2875400

Group Decision-Making Model With Hesitant Multiplicative Preference Relations Based on Regression Method and Feedback Mechanism

MINGWEI LIN[1], QIANSHAN ZHAN[2], ZESHUI XU[3], (Senior Member, IEEE), AND RIQING CHEN[4]

[1]Key Laboratory of Meteorological Disaster, Ministry of Education/Joint International Research Laboratory of Climate and Environment Change/Collaborative Innovation Center on Forecast and Evaluation of Meteorological Disasters/School of Computer and Software, Nanjing University of Information Science and Technology, Nanjing 210044, China
[2]Digital Fujian Internet-of-Things Laboratory of Environmental Monitoring, Fujian Normal University, Fuzhou 350117, China
[3]Business School, Sichuan University, Chengdu 610064, China
[4]Digital Fujian Institute of the Big Data for Agriculture and Forestry, Fujian Agriculture and Forestry University, Fuzhou 350002, China

Corresponding author: Riqing Chen (riqing.chen@fafu.edu.cn)

This work was supported in part by the National Natural Science Foundation of China under Grant 61872086 and Grant 61502102, in part by the Natural Science Foundation of Fujian Province under Grant 2016J05149, and in part by the Distinguished Young Scientific Research Talents Plan in Universities of Fujian Province (2017).

ABSTRACT The hesitant multiplicative preference relation (HMPR) was initially put forward in 2013. Utilizing the HMPR, the decision makers can give some possible preference values from the Saaty's 1-9 scale for pairwise comparisons over alternatives. However, until now, there is little research on the consistency and consensus of HMPRs. In this paper, we focus on exploiting the regression method and the feedback mechanism to improve the consistency and consensus for HMPRs and developing an efficient group decision-making model with HMPRs. First, a regression method based on complete consistency is developed to reduce HMPRs to multiplicative preference relations (MPRs). After that, a novel consistency checking and revising method based on the threshold estimation method and the feedback mechanism is given to improve the consistency level of the reduced MPRs. Then, a group consensus index is put forward to calculate the deviation degree between the reduced MPR and the group MPR and it is utilized to develop a consensus reaching process based on the feedback mechanism to improve the group consensus level of the reduced MPRs. Next, a complete group decision-making model with HMPRs is developed to rank all the alternatives and select the best one. Finally, a numerical example with respect to the investment of shared bikes is presented to demonstrate the proposed group decision-making model and then we also compare our proposed model with the existing one.

INDEX TERMS Hesitant fuzzy set, consistency, consensus, hesitant multiplicative preference relation.

I. INTRODUCTION

Decision making refers to a process in which the individual decision makers (DMs) try to choose the best one from a set of alternatives or rank all of them [1]. However, because of the time pressure or the lack of knowledge, the individual decision makers may not be able to provide the reasonable decision-making results. In this case, it needs a group of decision makers to be involved in this process, which is called the group decision making [2]. Preference relations in the form of pairwise comparison matrices are important and efficient tools, which are commonly used by DMs to give their preference information for pairwise comparisons over alternatives [3]. Each element in a preference relation matrix represents a DM's preference information over two alternatives. Because there are different types of evaluation scales that are utilized by the DMs, the preference relations can be mainly categorized into three types: multiplicative preference relation [4], fuzzy preference relation (FPR) [5], and linguistic preference relation (LPR) [6].

Based on the non-uniformly distributed 1-9 scale, Saaty first put forward the analytic hierarchy process (AHP) [7], which has become an important decision-making method and has been broadly used in many fields such as military, medical treatment, environment, and economy. In the AHP, the DMs make the pairwise comparisons between any two alternatives and then select the preference values from the non-uniformly distributed 1-9 scale to give the preference information. All the preference information over any two alternatives form a pairwise comparison matrix, which is also called the multiplicative preference relation (MPR) [8]. Considering the fact that the DMs are usually uncertain when offering the preference information, Saaty and Vargas [9] also developed the concept of interval multiplicative preference relation to rank the alternatives .

The concept of MPR has been paid much attention by researchers [10]–[14]. For example, Chiclana *et al.* [10] devised some induced ordered weighted geometric operators, which are applied to aggregate MPRs. Zhang [11] introduced a method to estimate the unknown preference values in an incomplete MPR. Each element in the MPR only owns a membership degree expressing the preference intensity of an alternative over another one and cannot model the non-membership function. To extend the modeling capability of MPR, Xia *et al.* [12] gave the definition of intuitionistic MPR based on the concept of intuitionistic multiplicative sets (IMS) and devised some operational laws. Considering the hesitancy information, Xia *et al.* combined the concept of hesitant fuzzy sets [13] with the Saaty's 1-9 scale and developed the concept of hesitant multiplicative preference relation (HMPR) [14].

Consistency and consensus have great influence upon the decision-making results of group decision-making problems with various preference relations, such as MPRs, FPRs, and LPRs. The consistency refers to whether there are conflicts among the preferences in the preference relations given by the individual DMs. The consensus refers to the degree of agreement among the preference relations given by a group of DMs. The preference relations with low consistency or consensus will incur unreasonable decision-making results [15]. Because of the complexity of decision-making issues, time pressure, and lack of knowledge, DMs usually cannot provide the consistent preference relations. At the same time, DMs usually show diverse opinions, so the preference relations provided by them also cannot achieve acceptable group consensus level.

As for the consistency of various preference relations, three methods have been proposed to measure the degree of deviation between preference relations and their consistent preference relations. They are the consistency ratio offered by Saaty [7] based on eigenvector method, the geometric consistency index (GCI) based on a row geometric mean prioritization method [16], and the consistency index based on the distance measure [17]. To improve the unacceptable consistency level of various preference relations, a number of studies have been presented, which are mainly divided into two branches: programming models [18] and iterative models [19]. For example, Tang and Meng [20] put forward a linear goal programming model to achieve the triangular fuzzy MPRs with the acceptable consistency level from the inconsistent ones. He and Xu [21] introduced two automatic iterative algorithms to modify the unacceptably consistent LPRs until there are acceptable.

There also are many studies focusing on the consensus of various preference relations [22]–[26]. Existing studies use the distance or similarity between the individual preference relations and the group preference relation to measure the consensus for various preference relations. As a supplement, González-Arteaga *et al.* [22] used the Pearson correlation coefficient to develop a different definition of consensus for reciprocal preference relations. Based on the definition of consensus, many dynamic and iterative consensus reaching models have been devised to adjust the preference relations until they own the acceptable consensus level. The models are mainly divided into two types: automatic model [23] and feedback-based model [24]. Some studies developed a new concept called the consistency/consensus level, which considers the consistency level and consensus level at the same time and then developed the group decision-making models based on the consistency/consensus level to adjust the consistency and consensus simultaneously [25], [26].

However, the existing studies are dedicated to improving the consistency and consensus for MPRs, FPRs, and LPRs. They cannot be used to solve the group decision making problems with HMPRs. To fill the gap, Zhang and Wu [27] first normalized HMPRs by extending the shorter elements and then developed two automatic iterative algorithms to improve the consistency and consensus for HMPRs. To the best of our knowledge, it is the only work concerning the consistency and consensus of HMPRs. This work shows some drawbacks:

(1) Normalizing the HMPRs leads to the high amounts of computations;

(2) Two automatic iterative algorithms presented in [27] without the involvement of the DMs result in the deviation of HMPR from the DMs' original opinions;

(3) The threshold of the acceptable consistency level is usually determined by the experiences of the DMs and there is no any theoretical basis to support the given threshold.

Thus, as a supplement, we intend to improve the existing work from another point of view.

In this paper, we focus on the studies of the consistency and consensus of HMPRs by reducing HMPRs into MPRs. Our contributions are shown as follows:

(1) To reduce the computations, the concept of complete consistency and the error analysis method is introduced to put forward a regression method to reduce the HMPRs into MPRs before improving their consistency and consensus.

(2) A threshold estimation method is developed to obtain the threshold of the acceptable consistency level exploiting the probability theory. At the same time, a novel iterative consistency checking and revising method considering the feedback mechanism is devised to adjust the reduced MPR until its consistency level satisfies the threshold.

(3) An iterative consensus reaching process based on the feedback mechanism is proposed to improve the consensus level of the reduced MPRs. It returns the suggestions on the

modification of the inconsistent reduced MPRs to the DMs and uses the feedback mechanism to exploit the feedback information from DMs to modify the inconsistent reduced MPRs.

(4) A complete group decision-making model with the HMPRs is devised by combining the iterative consistency checking and revising method with the iterative consensus reaching process. Then it is applied into the investment of shared bikes and compared with the existing one to verify its effectiveness.

The rest of this paper is organized as follows: Section 2 gives brief introductions to the MPR, HMSs and HMPRs. Section 3 describes a regression method to reduce HMPRs into MPRs. Section 4 designs a consistency checking and revising method based on the feedback mechanism to get the acceptably consistent MPRs from inconsistent MPRs. In Section 5, an iterative consensus reaching process based on the feedback mechanism is proposed to adjust the group consensus level between the MPRs and the group MPR. A complete group decision-making model with HMPRs is put forward in Section 6. An illustrative example concerning the investment of shared bikes is demonstrated and then our proposed model is compared with the existing one [27] in Section 7. Finally, some conclusions are drawn in Section 8.

II. PRELIMINARIES

Definition 1 [7], [14]: Given a fixed finite set X, then a multiplicative preference relation (MPR) on X is defined as a reciprocal matrix $R = \left(r_{ij}\right)_{n\times n} \subset X \times X$ satisfying the following condition:

$$r_{ij} \cdot r_{ji} = 1, r_{ii} = 1, r_{ij} \in [1/9, 9], \quad \forall i,\ j = 1,\ 2,\ \ldots,\ n \tag{2.1}$$

The value of r_{ij} represents the intensity of the alternative x_i over the alternative x_j. $r_{ij} = 1$ means the indifference between x_i and x_j. If $r_{ij} > 1$, then it means that x_i is superior to x_j. If $r_{ij} < 1$, then it indicates that x_i is inferior to x_j.

Definition 2 [7], [14]: Given a MPR $R = \left(r_{ij}\right)_{n\times n}$, if it satisfies the following multiplicative transitivity:

$$r_{ij} = r_{ik} \cdot r_{kj}, \quad \forall i,\ k,\ j \in \{1,\ 2,\ \ldots,\ n\} \tag{2.2}$$

Then the MPR R is referred to as a complete consistent multiplicative preference relation.

Motivated by the concept of hesitant fuzzy sets and the Saaty's 1-9 scale, Xia *et al.* devised the concept of hesitant multiplicative sets (HMS) as follows [14]:

Definition 3 [14]: Let X be a fixed set, then a HMS on X is defined mathematically as:

$$H = \{\langle x, h(x)\rangle \,|x \in X\} \tag{2.3}$$

where $h(x) = \{\zeta \,|\zeta \in h(x)\}$ is a set of several possible values from the Saaty's 1-9 scale and it represents all the possible membership degrees of the element $x \in X$ to the set H.

For convenience, $h = h(x)$ is often called as a hesitant multiplicative element (HME). Then H is the set of all the HMEs.

Zhang and Wu [27] defined two operations for HMEs as follows:

Definition 4 [27], [28]: Let us assume that there exist two HMEs expressed as, $h_1 = \left\{\zeta_1^s | s = 1, 2, \ldots, \#h_1\right\}$ and $h_2 = \left\{\zeta_2^s | s = 1, 2, \ldots, \#h_2\right\}$ with their lengths $\#h_1 = \#h_2$, then $h_1 \otimes h_2 = \bigcup_{\zeta_1^{\delta(s)}\in h_1,\ \zeta_2^{\delta(s)}\in h_2} \left\{\zeta_1^{\delta(s)} \times \zeta_2^{\delta(s)}\right\}$, where $\zeta_1^{\delta(s)}$ and $\zeta_2^{\delta(s)}$ are the sth least elements in h_1 and h_2.

Another operation is defined as $h^\lambda = \bigcup_{\zeta^{\delta(s)}\in h} \left\{\left(\zeta^{\delta(s)}\right)^\lambda\right\}$, where $\lambda > 0$.

Definition 5 [27], [28]: Assume that there is a fixed set $X = \{x_1, x_2, \ldots, x_n\}$, then a HMPR is defined as a matrix $H = \left(h_{ij}\right)_{n\times n}$, where $h_{ij} = \left\{h_{ij}^s | s = 1, 2, \ldots, \#h_{ij}\right\}$ is a HME and it means all the possible preference values of x_i over x_j. For $i, j = 1, 2, \ldots, n$, h_{ij} satisfies that $h_{ij}^{\delta(s)} \times h_{ji}^{\delta(s)} = 1$, $h_{ii} = 1$, $\#h_{ij} = \#h_{ji}$ and $h_{ij}^{\delta(s)} < h_{ij}^{\delta(s+1)}$, $h_{ji}^{\delta(s+1)} < h_{ji}^{\delta(s)}$, where $h_{ij}^{\delta(s)}$ and $h_{ji}^{\delta(s)}$ are the sth least values in the HMEs h_{ij} and h_{ji}.

III. REGRESSION METHOD BASED ON COMPLETE CONSISTENCY FOR HMPRs

Motivated by the error analysis method to compute the consistency levels of FPRs [29], a regression method with the completely consistency is developed to reduce HMPRs into MPRs in this section.

Suppose that there is a HMPR $H = \left(h_{ij}\right)_{n\times n} \subset X \times X$, where $X = \{x_1, x_2, \ldots, x_n\}$ describes a set of alternatives. According to Definition 2, the possible preference values of the alternative x_i over the alternative x_j denoted as a HME $h_{ij}\ (i \neq j)$ can be estimated through an intermediate alternative $x_m\ (m \neq i, j)$:

$$h_{ij}^m = h_{im} \tilde{\times} h_{mj} \tag{3.1}$$

where h_{ij}^m is an estimated HME. The operations $\tilde{\times}$ and $\tilde{\div}$ are defined as follows:

Definition 6: Suppose that there are three HMEs h, h_1, and h_2, and a real number $a \in \left[1/9, 9\right]$, then it is defined as $h_1 \tilde{\times} h_2 = \bigcup_{r_1\in h_1, r_2\in h_2} \{r_1 \times r_2\}$ and $h \tilde{\div} a = \bigcup_{r\in h} \left\{\log_9 \frac{r}{a}\right\}$.

Before using Eq. (3.1) to estimate h_{ij}^m, all the alternatives $x_i\ (i = 1, 2, \ldots, n)$ should be divided into sets, which are defined as

(1) $\mathrm{A} = \{(i, j)\,| i, j \in \{1, 2, \ldots, n\} \wedge (i \neq j)\}$;
(2) $\mathrm{P^A} = \{(i, j) \in \mathrm{A}\}$;
(3) $\mathrm{C^A} = \left(\mathrm{P^A}\right)^c$;
(4) $\mathrm{M}_{ij}^m = \left\{m \neq i, j | (i, m), (m, j) \in \mathrm{C^A}\right\}$,

where A is a set containing all of the paired alternatives; $\mathrm{P^A}$ denotes a subset of A; $\mathrm{C^A}$ represents the complement set of $\mathrm{P^A}$; M_{ij}^m is a set composed of all the intermediate alternatives $x_m\ (m \neq i, j)$.

According to Eq. (3.1), all the estimated HMEs h_{ij}^m can be computed. To choose the optimal value from the HME h_{ij}, a geometric estimated preference value is calculated as

Algorithm 1 Regression Algorithm

Step 1. Choose a HME $h_{ij}(i \neq j)$, get the set M_{ij}^m, and compute $h_{ij}^m \left(m \in M_{ij}^m\right)$ using Eq. (3.1).

Step 2. Caulcuate the geometric estimated preference value h_{ij}^G using Eq. (3.2), and then get h_{ij}^* using Eqs. (3.3) and (3.4).

Step 3. Repeat Steps 1 and 2 until the optimal value of each HME in H has been got, turn to Step 4.

Step 4. Use all the obtained optimal values h_{ij}^* to form the reduced MPR $H^* = \left(h_{ij}^*\right)_{n\times n}$.

Step 5. End.

follows:

$$h_{ij}^G = \sqrt[\sum_{m\in M_{ij}^m}(\#h_{ij}^m)]{\prod_{m\neq i, m\neq j}^{\#h_{ij}^m} h_{ij}^m} \tag{3.2}$$

where $\#h_{ij}^m$ describes the total number of all the possible preference values in each h_{ij}^m.

Based on the degree of deviation between each possible value in the HME h_{ij} and its corresponding geometric estimated preference value h_{ij}^G, the error between them can be defined as follows:

Definition 7: Given any HME h_{ij} and its geometric estimated preference value h_{ij}^G, the error between them is mathematically computed as:

$$\varepsilon h_{ij} = \frac{1}{2}\left(\bigcup_{\varepsilon_{ij}\in\left(h_{ij}\tilde{\div}h_{ij}^G\right)} |\varepsilon_{ij}|\right) = \frac{1}{2}\left(\bigcup_{\varepsilon_{ij}\in\log_9\left(h_{ij}/h_{ij}^G\right)} |\varepsilon_{ij}|\right) \tag{3.3}$$

where εh_{ij} denotes the error, which is a set consists of several values. Obviously, each value in the error is in the interval [0, 1].

If there is a preference value $h_{ij}^* \in h_{ij}$ that satisfies the following condition:

$$\frac{1}{2}\left|\log_9 \frac{h_{ij}^*}{h_{ij}^G}\right| = \min(\varepsilon h_{ij}) \tag{3.4}$$

Then it is referred to as the optimal value. Following this principle and obtaining h_{ij}^* for all $i, j = 1, 2, \ldots, n;\ i \neq j$, the HMPR H can be reduced into a MPR $H^* = \left(h_{ij}^*\right)_{n\times n}$ that is known as a reduced MPR.

Based on the completely consistency and above analysis, an algorithm that reduces a HMPR H into a MPR H^* is described in Algorithm 1.

Example 8: Let a HMPR be

$$H = \begin{Bmatrix} \{1\} & \{3, 5\} & \left\{{}^1/_7\right\} & \{5, 7\} \\ \left\{1/3, {}^1/_5\right\} & \{1\} & \left\{{}^1/_7, {}^1/_9\right\} & \{5\} \\ \{7\} & \{7, 9\} & \{1\} & \{1/3\} \\ \left\{1/5, {}^1/_7\right\} & \{1/5\} & \{3\} & \{1\} \end{Bmatrix}$$

Step 1: Choose the first HME h_{12}, and utilize Eq. (3.1) to compute all the estimated HMEs h_{ij}^m as:

$$h_{12}^3 = h_{13}\tilde{\times}h_{32} = \left\{1, {}^9/_7\right\}, h_{12}^4 = h_{14}\tilde{\times}h_{42} = \left\{1, {}^7/_5\right\}$$

Step 2: Use Eq. (3.2) to obtain the geometric estimated preference value as $h_{12}^G = \sqrt[4]{1 \times 1 \times \frac{9}{7} \times \frac{7}{5}} = 1.1583$.

Use Eqs. (3.3) and (3.4) to obtain the optimal value h_{12}^* from the HME h_{12} as follows:

$$\varepsilon h_{12} = \{0.2166, 0.3328\}$$

Thus, $\min(\varepsilon h_{12}) = 0.2166$ and then $h_{12}^* = 3$.

Step 3: Repeat Steps 1 and 2, and obtain $h_{13}^* = 1/7$, $h_{14}^* = 5$, $h_{23}^* = {}^1/_7$, $h_{24}^* = 5$, $h_{34}^* = 1/3$.

Step 4: Based on all the collected h_{ij}^* and Definition 1, the reduced MPR can be obtained as follows:

$$H^* = \begin{Bmatrix} \{1\} & \{3\} & \left\{{}^1/_7\right\} & \{5\} \\ \{1/3\} & \{1\} & \left\{{}^1/_7\right\} & \{5\} \\ \{7\} & \{7\} & \{1\} & \{1/3\} \\ \{1/5\} & \{1/5\} & \{3\} & \{1\} \end{Bmatrix}.$$

Step 5: End.

IV. CONSISTENCY CHECKING AND REVISING METHOD

A. CONSISTENCY INDEX

Based on Definition 2 and Eq. (3.1), the definition of an estimated geometric matrix for a reduced MPR could be given as follows:

Definition 9: Given a reduced MPR $R = \left(r_{ij}\right)_{n\times n}$, then its estimated geometric matrix (EGM) that is expressed by $E = \left(e_{ij}\right)_{n\times n}$ with its element e_{ij} satisfying

$$e_{ij} = \left(\bigotimes_{m=1, m\neq i,j}^{n} r_{ij}^m\right)^{\frac{1}{2(n-2)}} \tag{4.1}$$

where $n \geq 3$.

Theorem 10: Given a reduced MPR $R = \left(r_{ij}\right)_{n\times n}$, then its EGM $E = \left(e_{ij}\right)_{n\times n}$ is also a MPR.

Proof: It can be proven from two aspects:

(1) It should be proven that $e_{ij} \times e_{ji} = 1$ and $e_{ii} = 1$ for $\forall i, j \in \{1, 2, \ldots n\}$.

$$\begin{aligned} e_{ij} \times e_{ji} &= \left(\bigotimes_{m=1, m\neq i,j}^{n} r_{ij}^m\right)^{\frac{1}{2(n-2)}} \times \left(\bigotimes_{m=1, m\neq i,j}^{n} r_{ji}^m\right)^{\frac{1}{2(n-2)}} \\ &= \left(\bigotimes_{m=1, m\neq i,j}^{n} \left(r_{ij}^m \times r_{ji}^m\right)\right)^{\frac{1}{2(n-2)}} \\ &= \left(\bigotimes_{m=1, m\neq i,j}^{n} (1)\right)^{\frac{1}{2(n-2)}} = 1 \end{aligned}$$

If $i = j$, then $e_{ii} = \left(\bigotimes_{m=1, m\neq i}^{n} r_{ii}^m\right)^{\frac{1}{2(n-2)}} = 1$. It completes the proof.

(2) It should be proven that $e_{ij} \in [1/9, 9]$ for all the $i, j \in \{1, 2, \ldots n\}$.

$$e_{ij} = \left(\underset{m=1, m\neq i,j}{\overset{n}{\otimes}} r_{ij}^{m} \right)^{\frac{1}{2(n-2)}} = \left(\underset{m=1, m\neq i,j}{\overset{n}{\otimes}} \left(r_{im} \times r_{mj} \right) \right)^{\frac{1}{2(n-2)}}$$

$$= \left(\underbrace{\left(r_{i1} \times r_{1j} \right)^{\frac{1}{2}} \otimes \left(r_{i2} \times r_{2j} \right)^{\frac{1}{2}} \otimes \cdots \otimes \left(r_{in} \times r_{nj} \right)^{\frac{1}{2}}}_{(n-2)} \right)^{\frac{1}{n-2}}$$

Since $r_{im}, r_{mj} \in [1/9, 9]$, then $\left(r_{im} \times r_{mj} \right)^{\frac{1}{2}} \in [1/9, 9]$. Thus $e_{ij} = \left(\underset{m=1, m\neq i,j}{\overset{n}{\otimes}} r_{ij}^{m} \right)^{\frac{1}{2(n-2)}} \in [1/9, 9]$, which completes the proof.

Theorem 11: Given a reduced MPR $R = \left(r_{ij} \right)_{n\times n}$ with its EGM $E = \left(e_{ij} \right)_{n\times n}$, then the EGM E is a consistent MPR; if $r_{ij} = e_{ij}$ for $\forall i, j \in \{1, 2, \ldots, n\}$, then the reduced MPR $R = \left(r_{ij} \right)_{n\times n}$ is a consistent MPR.

Proof: Based on Definition 9 and Eq. (3.1), we have

$$\begin{aligned} r_{il} \otimes r_{lj} &= e_{il} \otimes e_{lj} \\ &= \left(\underset{m=1, m\neq i,l}{\overset{n}{\otimes}} r_{il}^{m} \right)^{\frac{1}{2(n-2)}} \otimes \left(\underset{m=1, m\neq l,j}{\overset{n}{\otimes}} r_{lj}^{m} \right)^{\frac{1}{2(n-2)}} \\ &= \left(\underset{m=1, m\neq i,l}{\overset{n}{\otimes}} \left(r_{im} \otimes r_{ml} \otimes r_{lm} \otimes r_{mj} \right) \right)^{\frac{1}{2(n-2)}} \\ &= \left(\underset{m=1, m\neq i,l}{\overset{n}{\otimes}} \left(r_{im} \otimes \left(r_{ml} \otimes r_{lm} \right) \otimes r_{mj} \right) \right)^{\frac{1}{2(n-2)}} \\ &= \left(\underset{m=1, m\neq i,l}{\overset{n}{\otimes}} \left(r_{im} \otimes \{1\} \otimes r_{mj} \right) \right)^{\frac{1}{2(n-2)}} \\ &= \left(\underset{m=1, m\neq i,l}{\overset{n}{\otimes}} r_{ij}^{m} \right)^{\frac{1}{2(n-2)}} = e_{ij} = r_{ij} \end{aligned}$$

According to Definition 2, the EGM is a consistent MPR, and the reduced MPR is also a consistent MPR if $r_{ij} = e_{ij}$, which completes the proof.

Based on Theorem 11, the EGM can be referred to as the consistent MPR of its reduced MPR.

Example 12: Suppose a reduced MPR $R = \left(r_{ij} \right)_{n\times n}$ as follows:

$$R = \left(r_{ij} \right)_{n\times n} = \begin{pmatrix} 1 & 3 & 1/9 & 5 \\ 1/3 & 1 & 1/7 & 1/5 \\ 9 & 7 & 1 & 1/3 \\ 1/5 & 5 & 3 & 1 \end{pmatrix}$$

Based on Eq. (3.1) and Definition 9, we have

$$h_{12}^{3} = h_{13} \otimes h_{32} = {}^{7}\!/_{9},\ h_{12}^{4} = h_{14} \otimes h_{42} = 25,$$

$$e_{12} = \left({}^{175}\!/_{9} \right)^{1/4}$$

$$h_{13}^{2} = h_{12} \otimes h_{23} = {}^{3}\!/_{7},\ h_{13}^{4} = h_{14} \otimes h_{43} = 15,$$

$$e_{13} = \left({}^{45}\!/_{7} \right)^{1/4}$$

$$h_{14}^{2} = h_{12} \otimes h_{24} = {}^{3}\!/_{5},\ h_{14}^{3} = h_{13} \otimes h_{34} = {}^{1}\!/_{27},$$

$$e_{14} = \left({}^{3}\!/_{135} \right)^{1/4}$$

$$h_{23}^{1} = h_{21} \otimes h_{13} = {}^{1}\!/_{27},\ h_{23}^{4} = h_{24} \otimes h_{43} = {}^{3}\!/_{5},$$

$$e_{23} = \left({}^{3}\!/_{135} \right)^{1/4}$$

$$h_{24}^{1} = h_{21} \otimes h_{14} = {}^{5}\!/_{3},\ h_{24}^{3} = h_{23} \otimes h_{34} = {}^{1}\!/_{21},$$

$$e_{14} = \left({}^{5}\!/_{63} \right)^{1/4}$$

$$h_{34}^{1} = h_{31} \otimes h_{14} = 45,\ h_{34}^{2} = h_{32} \otimes h_{24} = {}^{7}\!/_{5},$$

$$eh_{14} = \left({}^{315}\!/_{5} \right)^{1/4}$$

Then, the EGM $E = \left(e_{ij} \right)_{n\times n}$ of R can be constructed as follows:

$$E = \begin{Bmatrix} 1 & \left({}^{175}\!/_{9} \right)^{1/4} & \left({}^{45}\!/_{7} \right)^{1/4} & \left({}^{3}\!/_{135} \right)^{1/4} \\ \left({}^{9}\!/_{175} \right)^{1/4} & 1 & \left({}^{3}\!/_{135} \right)^{1/4} & \left({}^{5}\!/_{63} \right)^{1/4} \\ \left({}^{7}\!/_{45} \right)^{1/4} & \left({}^{135}\!/_{3} \right)^{1/4} & 1 & \left({}^{315}\!/_{5} \right)^{1/4} \\ \left({}^{135}\!/_{3} \right)^{1/4} & \left({}^{63}\!/_{5} \right)^{1/4} & \left({}^{5}\!/_{315} \right)^{1/4} & 1 \end{Bmatrix}$$

We use the Matlab software to draw "Figure of area" to offer a visible description of the inconsistent MPR R and its EGM E as shown in Fig. 1. It shows that the EGM performs more regularly than the inconsistent MPR.

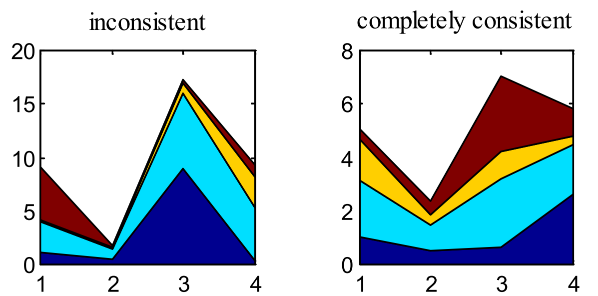


FIGURE 1. Areas of the MPR *R* and its EGM *E*.

Before giving the definition of a consistency index for a reduced MPR, a logarithmic distance between two reduced MPRs U and V is defined as follows:

Definition 13: Let $U = \left(u_{ij} \right)_{n\times n}$ and $V = \left(v_{ij} \right)_{n\times n}$ be two any reduced MPRs, then the logarithmic distance between U and V is defined as follows:

$$d(U, V) = \frac{1}{2n\,(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\left| \log_9 \frac{u_{ij}}{v_{ij}} \right| + \left| \log_9 \frac{u_{ji}}{v_{ji}} \right| \right) \tag{4.2}$$

Theorem 14: Suppose that there are two any reduced MPRs $U = \left(u_{ij} \right)_{n\times n}$ and $V = \left(v_{ij} \right)_{n\times n}$, then the logarithmic distance between U and V satisfies the following three properties:

(1) $0 \le d(U, V) \le 1$;
(2) $d(U, V) = 0$ if and only if $U = V$;
(3) $d(U, V) = d(V, U)$.

Proof:

For $\forall i = 1, 2, \ldots, n$ and $\forall j = 1, 2, \ldots, n$, we have

$$\left|\log_9 \frac{u_{ij}}{v_{ij}}\right| + \left|\log_9 \frac{u_{ji}}{v_{ji}}\right| = \left|\log_9 \frac{u_{ij}}{v_{ij}}\right| + \left|\log_9 \frac{v_{ij}}{u_{ij}}\right| = \left|\log_9 \frac{u_{ij}}{v_{ij}}\right| + \left|\log_9 \left(\frac{u_{ij}}{v_{ij}}\right)^{-1}\right| = 2\left|\log_9 \frac{u_{ij}}{v_{ij}}\right|$$

Since $u_{ij}, v_{ij} \in [1/9, 9]$, then $2\left|\log_9 \frac{u_{ij}}{v_{ij}}\right| \in [0, 4]$. Thus,

$$0 \le d(U, V) \le \frac{4}{2n(n-1)} \cdot \frac{n(n-1)}{2} \Leftrightarrow 0 \le d(U, V) \le 1.$$

According to the first property, if $d(U, V) = 0$, then $\frac{u_{ij}}{v_{ij}} = 1$. Thus, we have $u_{ij} = v_{ij} \Leftrightarrow U = V$.

According to Definition 13, we have

$$d(U, V) = \frac{1}{2n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\left|\log_9 \frac{u_{ij}}{v_{ij}}\right| + \left|\log_9 \frac{u_{ji}}{v_{ji}}\right|\right) = \frac{1}{2n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\left|\log_9 \frac{v_{ji}}{u_{ji}}\right| + \left|\log_9 \frac{v_{ij}}{u_{ij}}\right|\right) = d(V, U)$$

Based on Definition 13, the definition of a consistency index for a reduced MPR can be given as:

Definition 15: Given a reduced MPR $R = (r_{ij})_{n\times n}$ and its consistent EGM $E = (e_{ij})_{n\times n}$, then a consistency index of the MPR R is defined as $CI(R) = 1 - d(R, E)$, where $CI(R)$ denotes the consistency index of the reduced MPR $R = (r_{ij})_{n\times n}$.

Obviously, the consistency index also satisfies Theorem 14, so $0 \le CI(R) \le 1$ for the reduced MPR R. The larger the value of $CI(R)$ is, the closer to its consistent EGM E the reduced MPR R is.

In the practical applications, because of the large number of alternatives, the lack of knowledge, and time pressure, the MPRs often cannot be completely consistent. Through investigations, it can be found that MPRs with acceptable consistency can also get reasonable decision-making results [30]. Thus, we give the definition of acceptable consistency for MPRs as follows:

Definition 16: Given a reduced MPR $R = (r_{ij})_{n\times n}$ and a predefined threshold value $\bar{CI}$, if this reduced MPR R owns an acceptable consistency level, then it should satisfy $CI(R) \ge \bar{CI}$.

Theorem 17: Let $R_k = (r_{ij,k})_{n\times n}$ $(k = 1, 2, \ldots, K)$ be K MPRs given by K DMs and $G = (g_{ij})_{n\times n}$ be their group MPR, which is obtained utilizing Eq. (5.1), then we have $CI(G) \ge \min_{1\le k\le K} \{CI(R_k)\}$.

Proof: Let $\alpha = \frac{1}{2n(n-1)}$, then we have

$$\begin{aligned} CI(R_k) &= 1 - d(R_k, E_k) = 1 - \alpha \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(2\left|\log_9 \frac{r_{ij,k}}{e_{ij,k}}\right|\right) \\ &= 1 - \alpha \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(2\left|\log_9 r_{ij,k} - \log_9 e_{ij,k}\right|\right) \\ &= 1 - \alpha \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(2\left|\log_9 r_{ij,k} - \log_9 \left(\bigotimes_{m=1,m\ne i,j}^{n} r_{ij,k}^{m}\right)^{\frac{1}{2(n-2)}}\right|\right) \\ &= 1 - \alpha \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(2\left|\log_9 r_{ij,k} - \sum_{m=1,m\ne i,j}^{n} \log_9 \left(r_{ij,k}^{m}\right)^{\frac{1}{2(n-2)}}\right|\right) \end{aligned}$$

Let $\beta_{ij,k} = \log_9 r_{ij,k} - \sum_{m=1,m\ne i,j}^{n} \log_9 \left(r_{ij,k}^{m}\right)^{\frac{1}{2(n-2)}}$, then we have

$$CI(R_k) = 1 - \alpha \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(2\left|\beta_{ij,k}\right|\right) \ge \bar{CI} \Leftrightarrow \alpha \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(2\left|\beta_{ij,k}\right|\right) \le 1 - \bar{CI}$$

and

$$\alpha \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(2\left|\beta_{ij,k}\right|\right) = 1 - CI(R_k).$$

$$CI(G) = 1 - d(G, E) = 1 - \alpha \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(2\left|\log_9 g_{ij} - \sum_{m=1,m\ne i,j}^{n} \log_9 \left(g_{ij}^{m}\right)^{\frac{1}{2(n-2)}}\right|\right)$$

Since $\log_9 g_{ij} = \log_9 \left(\prod_{k=1}^{K} (r_{ij,k})^{\omega_k}\right) = \sum_{k=1}^{K} \omega_k \log_9 r_{ij,k}$ and

$$\begin{aligned} \sum_{m=1,m\ne i,j}^{n} \log_9 \left(g_{ij}^{m}\right)^{\frac{1}{2(n-2)}} &= \sum_{m=1,m\ne i,j}^{n} \log_9 \left(g_{im} \times g_{mj}\right)^{\frac{1}{2(n-2)}} \\ &= \sum_{m=1,m\ne i,j}^{n} \log_9 \left(\left(\prod_{k=1}^{K} (r_{im,k})^{\omega_k}\right) \times \left(\prod_{k=1}^{K} (r_{mj,k})^{\omega_k}\right)\right)^{\frac{1}{2(n-2)}} \\ &= \sum_{m=1,m\ne i,j}^{n} \log_9 \left(\left(\prod_{k=1}^{K} (r_{im,k} \times r_{mj,k})^{\omega_k}\right)\right)^{\frac{1}{2(n-2)}} \\ &= \sum_{m=1,m\ne i,j}^{n} \log_9 \left(\left(\prod_{k=1}^{K} \left(r_{ij,k}^{m}\right)^{\omega_k}\right)\right)^{\frac{1}{2(n-2)}} \\ &= \sum_{m=1,m\ne i,j}^{n} \left(\sum_{k=1}^{K} \omega_k \log_9 \left(r_{ij,k}^{m}\right)^{\frac{1}{2(n-2)}}\right), \end{aligned}$$

then

$$CI(G) = 1-\sum_{k=1}^{K}\omega_k\left(\alpha\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(2\left|\beta_{ij,k}\right|\right)\right)$$
$$= 1-\sum_{k=1}^{K}\omega_k\left(1-CI(R_k)\right)$$
$$= \sum_{k=1}^{K}\omega_k CI(R_k) \geq \min_{1\leq k\leq K}\{CI(R_k)\} \geq C\bar{I}$$

This result completes the proof of Theorem 17.

Based on Theorem 17, two Corollaries can be got easily as follows:

Corollary 18: Let $R_k = \left(r_{ij,k}\right)_{n\times n}$ $(k = 1, 2, \ldots, K)$ be K MPRs given by K DMs and $G = \left(g_{ij}\right)_{n\times n}$ be their group MPR that is obtained using Eq. (5.1), then $CI(G) \geq C\bar{I}$ if $CI(R_k) \geq C\bar{I}$ for $\forall k = 1, 2, \ldots, K$.

Corollary 19: If $CI(R_k) = 1$ for $\forall k = 1, 2, \ldots, K$, then $CI(G) = 1$.

Based on Theorem 17, we can find that the consistency level of the group MPR is certainly higher than the lowest consistency level of all the individual MPRs. Corollary 18 implies that if all the individual MPRs satisfy acceptable consistency condition, then their group MPR is acceptably consistent. Corollary 19 describes that if all the individual MPRs are consistent, then their group MPR is consistent.

B. THRESHOLD ESTIMATION METHOD FOR CONSISTENCY INDEX

The threshold of consistency index for MPRs or HMPRs in the existing studies is determined by the experiences of the DMs. There is no any theoretical basis to support the reference value for the threshold. Moreover, the method for calculating the consistency index of the reduced MPRs in our paper is very different from that in the existing studies. Hence, the reference value for the threshold of consistency index of MPRs or HMPRs cannot be used in our paper. In this section, we develop a threshold estimation method to estimate the threshold of consistency index of the reduced MPRs.

Motivated by the definition of the consistency level of a hesitant fuzzy preference relation gave by Liu *et al.* [31], we first put forward a definition of consistency threshold of a MPR as follows:

Definition 20: Given a reduced MPR $R = \left(r_{ij}\right)_{n\times n}$ and its consistent EGM $E = \left(e_{ij}\right)_{n\times n}$. S that the density function of the distance $d(R, E)$between R and E is $g(x)$, then the consistency threshold $C\bar{I}$ of the reduced MPR R satisfies that $\int_0^{1-C\bar{I}} g(t)dt = \tau$, where τ is the confidence level and it is determined by the DMs.

The confidence level τ describes the probability of the distance $d(R, E)$ being in the interval $[0, 1-r]$. It can be seen that the confidence level τ influences the consistency threshold. The larger the confidence level is, the smaller the consistency threshold is. Hence, if the confidence level is set to be too high, then the consistency index of the reduced MPR would exceed the consistency threshold easily. Liu *et al.* [31] suggested that the confidence level is set to 20%.

Although it is very difficult for the DMs to provide the consistent MPRs in the practical application environments, the MPRs offered by DMs are repeatedly returned to them for adjusting. Hence, the distances between the MPRs and the consistent ones are trending to 0. Let a random variable $X = \left|\log_9 \frac{u_{ij}}{v_{ij}}\right| + \left|\log_9 \frac{u_{ji}}{v_{ji}}\right|$, then X will trend to 0. Assume that the variable X follows a normal distribution, then its density function is $g(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}}$.

Theorem 21: Let a random variable $Y = d(R, E)$, then its density function is

$$f_Y(y) = \frac{4}{\sqrt{2\pi}\sigma}e^{-\frac{8y^2}{\sigma^2}}$$

where $y \in [0, 1]$.

Proof: Since $X = \left|\log_9 \frac{u_{ij}}{v_{ij}}\right| + \left|\log_9 \frac{u_{ji}}{v_{ji}}\right|$, then we have

$$Y = d(R, E) = \frac{1}{2n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} X$$
$$= \frac{1}{2n(n-1)}\cdot\frac{n(n-1)}{2}X = \frac{X}{4}.$$

Let $F_X(x)$ and $F_Y(y)$ be the cumulative distribution functions of the random variables X and Y, then

$$F_Y(y) = P\{Y \leq y\} = P\left\{\frac{X}{4} \leq y\right\}$$
$$= P\{X \leq 4y\} = F_X(4y)$$

Taking the first derivative of $F_Y(y)$ with respect to y yields

$$f_Y(y) = f_X(4y)\cdot(4y)'$$
$$= \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{16y^2}{2\sigma^2}}\cdot 4 = \frac{4}{\sqrt{2\pi}\sigma}e^{-\frac{8y^2}{\sigma^2}}$$

which completes the proof of the theorem.

According to Definition 20 and Theorem 21, we have

$$\tau = \int_0^{1-C\bar{I}}\frac{4}{\sqrt{2\pi}\sigma}e^{-\frac{8y^2}{\sigma^2}}dy = \frac{4}{\sqrt{2\pi}\sigma}\int_0^{1-C\bar{I}} e^{-\left(\frac{\sqrt{8}\,y}{\sigma}\right)^2}dy$$
$$= \frac{4\sigma}{\sqrt{8}\cdot\sqrt{2\pi}\sigma}\int_0^{1-C\bar{I}} e^{-\left(\frac{\sqrt{8}y}{\sigma}\right)^2}d\frac{\sqrt{8}y}{\sigma}$$
$$= \frac{1}{\sqrt{\pi}}\int_0^{1-C\bar{I}} e^{-\left(\frac{\sqrt{8}y}{\sigma}\right)^2}d\frac{\sqrt{8}\,y}{\sigma}$$

Let $\eta = \frac{\sqrt{8}\,y}{\sigma}$, then we have $\tau = \frac{1}{\sqrt{\pi}}\int_0^{1-C\bar{I}} e^{-\eta^2}d\eta$. For a nonnegative random variable x, its error function can be defined as $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^{x} e^{-\eta^2}d\eta$ [32]. Then we can get $\tau = \frac{1}{\sqrt{\pi}}\int_0^{1-C\bar{I}} e^{-\eta^2}d\eta = \frac{1}{2}erf(1-C\bar{I})$.

When the value of the confidence level τ is given, the value of $1-C\bar{I}$ can be obtained by referring to the error function table and then the value of $C\bar{I}$ can be got. Here we give the value of $C\bar{I}$ with the different values of the confidence level as shown in Table 1.

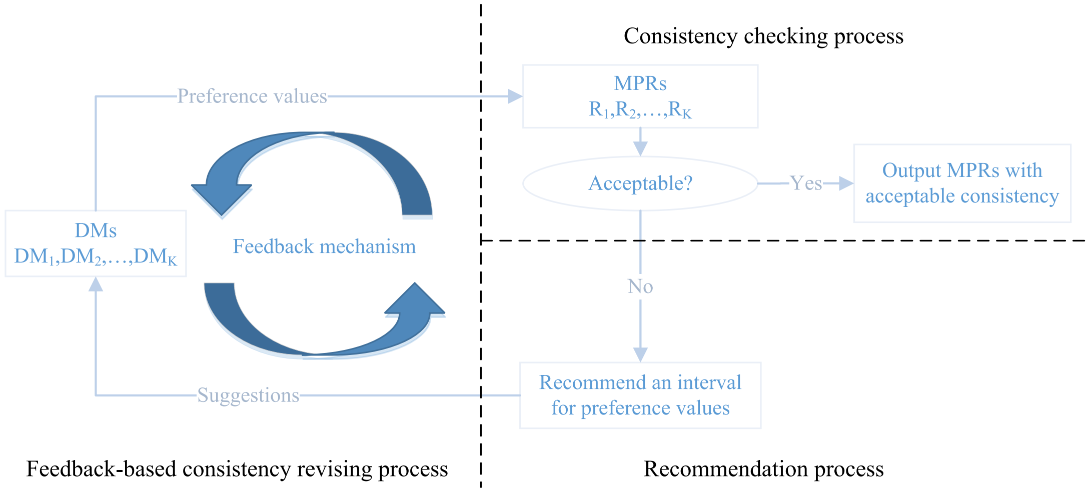


FIGURE 2. The flow chart of the consistency checking and revising algorithm.

TABLE 1. Values of the consistency threshold with different values of confidence level.

τ	0	5%	10%	15%	20%	25%
$1-\overline{CI}$	0	0.09	0.18	0.28	0.37	0.48
$\overline{CI}$	1	0.91	0.82	0.72	0.63	0.52

It can be seen that the value of the consistency threshold decreases with the increase of the value of the confidence level.

C. CONSISTENCY CHECKING AND REVISING ALGORITHM BASED ON FEEDBACK MECHANISM

In this subsection, based on Theorem 11, Definitions 15 and 16, we utilize the feedback mechanism to put forward a consistency checking and revising algorithm that checks and then revises the inconsistent reduced MPRs R with the involvement of the DMs until $CI(R) \geq \overline{CI}$.

Fig. 2 shows that this novel consistency checking and revising algorithm consists of three steps: the consistency checking process, recommendation process, and feedback-based consistency revising process. It can be seen that the consistency checking process is in charge of checking that whether a MPR satisfies the condition $CI(R) \geq \overline{CI}$ or the number of iterations exceeds a predefined threshold value. The recommendation process recommends an interval for the preference values to the DMs for guiding them to give suitable preference values. The consistency revising process reconstructs the MPR based on the feedback advices from the DMs.

Based on this feedback-based consistency and revising process shown in Algorithm 2, a theorem can be given as follows:

Algorithm 2 Consistency Checking and Revising Algorithm

Input: A reduced MPR, a threshold for consistence level denoted by $\overline{CI}$, and a predefined threshold for the number of iterations denoted as T $(T \geq 1)$.

Output: The number of iterations t, the updated MPR $R^{(t)} = (r_{ij})^{(t)}_{n\times n}$, and its $CI\left(R^{(t)}\right)$.

Step 1. Let $t = 0$, then $R^{(0)} = (r_{ij})^{(0)}_{n\times n}$;

Step 2. Using Definition 9, the EGM $E^{(t)} = \left(e_{ij}\right)^{(t)}_{n\times n}$ can be obtained;

Step 3. Utilize Definitions 13 and 15 to calculate the consistency index of $R^{(t)}$ denoted by $CI\left(R^{(t)}\right)$.
If $CI\left(R^{(t)}\right) \geq \overline{CI}$ or $t \geq T$, then turn to Step 5; otherwise, turn to the next step.

Step 4. Return $R^{(t)}$ to the DM and suggest he/she to offer an adjusted $R^{(t+1)} = (r_{ij})^{(t+1)}_{n\times n}$, which satisfies

$$r_{ij}^{(t+1)} \in \left(\min\left(r_{ij}^{(t)}, e_{ij}^{(t)}\right), \max\left(r_{ij}^{(t)}, e_{ij}^{(t)}\right)\right).$$

Let $t = t + 1$, then turn to Step 2;

Step 5. Output t, $R^{(t)} = (r_{ij})^{(t)}_{n\times n}$, its consistency index $CI\left(R^{(t)}\right)$.

Step 6. End.

Theorem 22: Let $R = \left(r_{ij}\right)_{n\times n}$ be an any inconsistent or unacceptable MPR, $R^{(t)}$ and $R^{(t+1)}$ be the updated MPRs after t and $t+1$ iterations of Step 4, and $\overline{CI} = \alpha$ be the threshold for the consistency index, then we can get that $CI\left(R^{(t)}\right) < CI\left(R^{(t+1)}\right)$ for each t and $\lim\limits_{x\to\infty} CI\left(R^{(t)}\right) \geq \alpha$.

Proof: According to Eq. (4.1), then we have

$$\begin{aligned}\left(e_{ij}\right)^{(t+1)} &= \left(\bigotimes_{m=1,m\neq i,j}^{n}\left(r_{ij}^{m}\right)^{(t+1)}\right)^{\frac{1}{2(n-2)}}\\ &= \left(\bigotimes_{m=1,m\neq i,j}^{n}\left(r_{im}\times r_{mj}\right)^{(t+1)}\right)^{\frac{1}{2(n-2)}} \qquad (4.3)\end{aligned}$$

Since

$$\begin{aligned}&\left(r_{im}\times r_{mj}\right)^{(t+1)}\\&\in\left[\min\left(\left(r_{im}\times r_{mj}\right)^{(t)},e_{ij}^{(t)}\right),\max\left(\left(r_{im}\times r_{mj}\right)^{(t)},e_{ij}^{(t)}\right)\right]\\&=\left[\min\left(\left(r_{ij}^{m}\right)^{(t)},e_{ij}^{(t)}\right),\max\left(\left(r_{ij}^{m}\right)^{(t)},e_{ij}^{(t)}\right)\right],\end{aligned}$$

then Eq. (4.3) can be transformed into

$$\begin{aligned}\left(e_{ij}\right)^{(t+1)}&\in\left[\begin{array}{l}\min\left(\left(\underset{m=1,m\neq i,j}{\overset{n}{\otimes}}\left(r_{ij}^{m}\right)^{(t)}\right)^{\frac{1}{2(n-2)}},e_{ij}^{(t)}\right),\\\max\left(\left(\underset{m=1,m\neq i,j}{\overset{n}{\otimes}}\left(r_{ij}^{m}\right)^{(t)}\right)^{\frac{1}{2(n-2)}},e_{ij}^{(t)}\right)\end{array}\right]\\&\Leftrightarrow\left(e_{ij}\right)^{(t+1)}\in\left[\min\left(e_{ij}^{(t)},e_{ij}^{(t)}\right),\max\left(e_{ij}^{(t)},e_{ij}^{(t)}\right)\right]\\&\Leftrightarrow\left(e_{ij}\right)^{(t+1)}=\left(e_{ij}\right)^{(t)}\end{aligned}$$

Thus,

$$\begin{aligned}\frac{(r_{ij})^{(t+1)}}{(e_{ij})^{(t+1)}}&=(r_{ij})^{(t+1)}\times(e_{ji})^{(t+1)}=(r_{ij})^{(t+1)}\times(e_{ji})^{(t)}\\&\in\left[\begin{array}{l}(e_{ji})^{(t)}\times\min\left((r_{ij})^{(t)},(e_{ij})^{(t)}\right),\\(e_{ji})^{(t)}\times\max\left((r_{ij})^{(t)},(e_{ij})^{(t)}\right)\end{array}\right]\\&=\left[\min\left(\frac{(r_{ij})^{(t)}}{(e_{ij})^{(t)}},1\right),\max\left(\frac{(r_{ij})^{(t)}}{(e_{ij})^{(t)}},1\right)\right]\end{aligned}$$

If $\frac{(r_{ij})^{(t)}}{(e_{ij})^{(t)}}<1$, then the above equation is transformed into

$$\begin{aligned}&\frac{(r_{ij})^{(t)}}{(e_{ij})^{(t)}}<\frac{(r_{ij})^{(t+1)}}{(e_{ij})^{(t+1)}}\Leftrightarrow2\left|\log_9\frac{(r_{ij})^{(t+1)}}{(e_{ij})^{(t+1)}}\right|<2\left|\log_9\frac{(r_{ij})^{(t)}}{(e_{ij})^{(t)}}\right|\\&\quad\Leftrightarrow CI\left(R^{(t)}\right)<CI\left(R^{(t+1)}\right)\end{aligned}$$

If $\frac{(r_{ij})^{(t)}}{(e_{ij})^{(t)}}>1$, then we have $CI\left(R^{(t)}\right)<CI\left(R^{(t+1)}\right)$.

Since $CI\left(R^{(t)}\right)\leq1$, then it indicates that the sequence $\left\{CI\left(R^{(t)}\right)\right\}$ is monotonically increasing and has an upper bound. Thus, we have $\lim_{t\to\infty}\left(CI\left(R^{(t)}\right)\right)=\sup\left\{CI\left(R^{(t)}\right)\right\}$.

Suppose that $\lim_{t\to\infty}\left(CI\left(R^{(t)}\right)\right)<\alpha$. We use Algorithm 2 to modify the MPR $R^{(t)}$, then the consistency index of the resulted MPR $R^{(t+1)}$ satisfies $CI\left(R^{(t)}\right)<CI\left(R^{(t+1)}\right)$ for $t\to\infty$. Hence, it indicates $CI\left(R^{(t+1)}\right)>\sup\left\{CI\left(R^{(t)}\right)\right\}$, which contradicts that $\lim_{t\to\infty}\left(CI\left(R^{(t)}\right)\right)=\sup\left\{CI\left(R^{(t)}\right)\right\}$. Thus, we have $\lim_{t\to\infty}\left(CI\left(R^{(t)}\right)\right)\geq\alpha$.

V. CONSENSUS REACHING PROCESS

In the group decision-making problems [33], [34], there exists a group of DMs participating the process of evaluating the alternatives. Since the DMs are usually invited from different specialty fields and have different levels of knowledge, they have divergent opinions, which result in the low consensus among the reduced MPRs. Low consensus greatly influences on the results of group decision making problems [35]. Thus, in this section, we develop a consensus reaching process for the reduced MPRs. A group consensus index is first devised to measure the degree of agreement between each reduced MPR and their group MPR. Finally, a consensus reaching algorithm-based feedback mechanism is designed to modify the reduced MPR so as to reach a predefined consensus level.

A. GROUP CONSENSUS INDEX

Before providing the definition of group consensus index to measure the degree of agreement between each MPR and the group MPR, we define the group MPR as follows:

Definition 23: Let $R_k=\left(r_{ij,k}\right)_{n\times n}$ $(k=1,2,\ldots K)$ be K reduced MPRs given by K DMs, $\omega_k\in[0,1]$ be the weight of the kth DM satisfying $\sum_{k=1}^{K}\omega_k=1$, then the group MPR is obtained as:

$$G=\left(g_{ij}\right)_{n\times n}=\left(\overset{K}{\underset{k=1}{\otimes}}\left(r_{ij,k}\right)^{\omega_k}\right)_{n\times n}\tag{5.1}$$

where G denotes the obtained group MPR.

Theorem 24: Let $R_k=\left(r_{ij,k}\right)_{n\times n}$ $(k=1,2,\ldots K)$ be K reduced MPRs, $\omega=(\omega_1,\omega_2,\ldots,\omega_K)^T$ be the weight vector, then the group MPR $G=\left(g_{ij}\right)_{n\times n}$ is also a MPR.

Proof: Based on Eq. (5.1), we have

$$\begin{aligned}g_{ij}\times g_{ji}&=\left(\overset{K}{\underset{k=1}{\otimes}}\left(r_{ij,k}\right)^{\omega_k}\right)\times\left(\overset{K}{\underset{k=1}{\otimes}}\left(r_{ji,k}\right)^{\omega_k}\right)\\&=\overset{K}{\underset{k=1}{\otimes}}\left(r_{ij,k}\times r_{ji,k}\right)^{\omega_k}\\&=\overset{K}{\underset{k=1}{\otimes}}(1)^{\omega_k}=1,\end{aligned}$$

and $g_{ii}=\left(\overset{K}{\underset{k=1}{\otimes}}\left(r_{ii,k}\right)^{\omega_k}\right)=\left(\overset{K}{\underset{k=1}{\otimes}}(1)^{\omega_k}\right)=1.$

Because $r_{ij,k}\in[1/9,9]$ and $\omega_k\in[0,1]$, then we have

$$g_{ij}=\overset{K}{\underset{k=1}{\otimes}}\left(r_{ij,k}\right)^{\omega_k}\in[1/9,9].$$

According to Definition 1, the group MPR $G=\left(g_{ij}\right)_{n\times n}$ is a MPR, which completes the proof.

A closeness matrix (CM) between each reduced MPR R_k and group MPR $G=\left(g_{ij}\right)_{n\times n}$ is defined as follows:

Definition 25: Let $R_k=\left(r_{ij,k}\right)_{n\times n}$ $(k=1,2,\ldots K)$ be K MPRs and $G=\left(g_{ij}\right)_{n\times n}$ be group MPR, then the closeness matrix (CM) between $R_k=\left(r_{ij,k}\right)_{n\times n}$ and $G=\left(g_{ij}\right)_{n\times n}$ is constructed as follows:

$$CM\left(R_k,G\right)=\left(c_{ij,k}\right)_{n\times n}=\left(\frac{1}{2}\left|\log_9\frac{r_{ij,k}}{g_{ij}}\right|\right)_{n\times n}\tag{5.2}$$

where $c_{ij,k}\in[0,1]$.

Based on Definition 25, the definition of group consensus index for MPRs can be given as follows:

Definition 26: Let $R_k=\left(r_{ij,k}\right)_{n\times n}$ $(k=1,2,\ldots K)$ be K MPRs and $G=\left(g_{ij}\right)_{n\times n}$ be their group MPR, then the group consensus index matrix of $R_k=\left(r_{ij,k}\right)_{n\times n}$ is constructed as follows:

$$GCM\left(R_k\right)=\left(z_{ij,k}\right)_{n\times n}=\left(1-c_{ij,k}\right)_{n\times n}\tag{5.3}$$

where $z_{ij,k}\in[0,1]$.

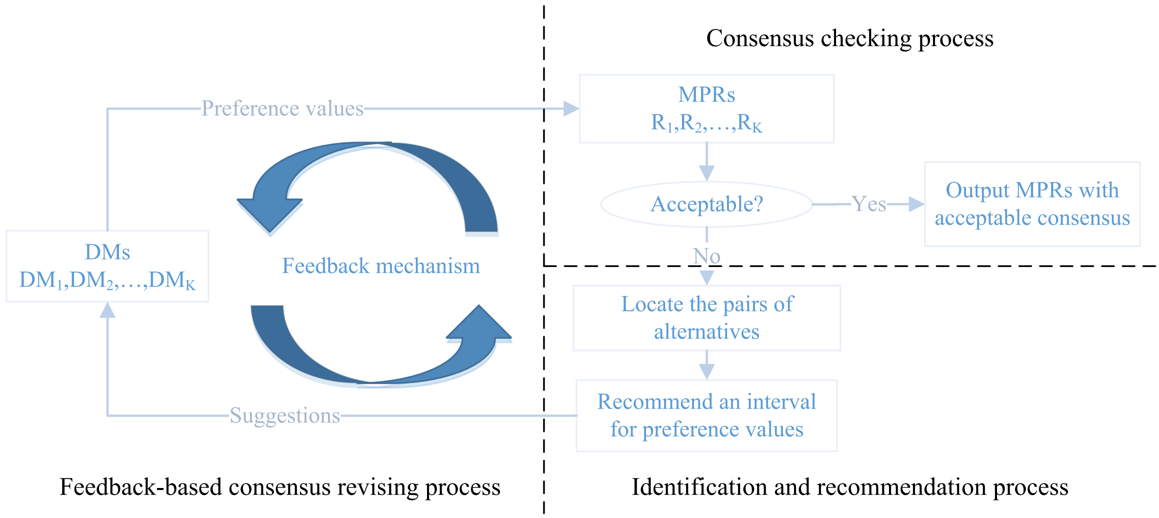


FIGURE 3. The process of consensus reaching algorithm based on feedback mechanism.

Then the group consensus index of the alternative $x_{i,k}$ in the MPR $R_k = \left(r_{ij,k}\right)_{n\times n}$ can be computed as:

$$GCI\left(x_{i,k}\right) = \sum_{j=1}^{n} z_{ij,k} \Big/ n \tag{5.4}$$

Therefore, the group consensus index of $R_k = \left(r_{ij,k}\right)_{n\times n}$ is calculated as follows:

$$GCI\left(R_k\right) = \sum_{i=1}^{n} GCI\left(x_{i,k}\right) \Big/ n = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} z_{ij,k} \tag{5.5}$$

where $GCI\left(R_k\right) \in [0, 1]$. If $GCI\left(R_k\right) = 1$, then it means that the kth DM shows full consensus with the group DM. The larger the value of $GCI\left(R_k\right)$ is, the closer to the group DM the kth DM is.

In the real decision-making problems, the DMs should determine a threshold $\overline{GCI}$ for the group consensus index in advance.

Definition 27: Let $R_k = \left(r_{ij,k}\right)_{n\times n} (k = 1, 2, \ldots K)$ be K reduced MPRs, $G = \left(g_{ij}\right)_{n\times n}$ be the group MPR, as well as $GCI\left(R_k\right)$ be the group consensus index of R_k, then R_k owns an acceptable consensus if $GCI\left(R_k\right) \geq \overline{GCI}$, where $\overline{GCI} \in [0, 1]$ denotes the threshold of the acceptable group consensus index.

In the practical group decision-making problems [36], the DMs, who give the MPRs with higher group consensus index, are assigned with larger weights. Based on this principle, the weight of each DM is defined as follows:

Definition 28: Let $R_k = \left(r_{ij,k}\right)_{n\times n} (k = 1, 2, \ldots K)$ be K reduced MPRs, then the weight ω_k of kth DM is computed as $\omega_k = \frac{GCI(R_k)}{\sum_{k=1}^{K} GCI(R_k)}$ $(k = 1, 2, \ldots, K)$ where $\sum_{k=1}^{K} \omega_k = 1$.

B. CONSENSUS REACHING ALGORITHM BASED ON FEEDBACK MECHANISM

Based on the above definitions, a new consensus reaching algorithm based on feedback mechanism is designed to make all the DMs achieve an acceptable consensus. This algorithm is composed of three parts, which are the consensus checking process, the identification and recommendation process, and the feedback-based consensus revising process, as shown in Fig. 3.

The consensus checking process is in charge of checking whether the group consensus index (GCI) of each MPR R_k satisfies $GCI\left(R_k\right) \geq \overline{GCI}$. If yes, it means that the MPR has an acceptable consensus level and it does not need to be revised further. Otherwise, it turns to the identification and recommendation process, which aims to locate all the pairs of alternatives whose group consensus indexes (GCI) satisfy the following equation:

$$L_k = \left\{ \begin{array}{l} \left(x_{i,k}, x_{j,k}\right) | GCI\left(x_{i,k}\right) < GCI\left(R_k\right) \wedge \\ z_{ij,k} < GCI\left(R_k\right) \end{array} \right\} \tag{5.6}$$

where L_k means the set of pairs of alternatives whose group consensus indexes are lower than that of R_k.

Then, it offers some suggestions to modify the preference value $r_{ij,k}$ for the pair of alternatives $\left(x_{i,k}, x_{j,k}\right) \in L_k$. The modification of the preference value $r_{ij,k}$ is suggested as:

S.i) If $\log_9 \frac{r_{ij,k}}{g_{ij}} > 0$, then the kth DM should decrease $r_{ij,k}$ to $r'_{ij,k} \in \left[\min\left(r_{ij,k}, g_{ij}\right), \max\left(r_{ij,k}, g_{ij}\right)\right]$;

S.ii) If $\log_9 \frac{r_{ij,k}}{g_{ij}} < 0$, then the kth DM should increase $r_{ij,k}$ to $r'_{ij,k} \in \left[\min\left(r_{ij,k}, g_{ij}\right), \max\left(r_{ij,k}, g_{ij}\right)\right]$.

S.iii) if $\log_9 \frac{r_{ij,k}}{g_{ij}} = 0$, then the kth DM does not need to modify $r_{ij,k}$.

The feedback-based consensus revising process uses the feedback advice on the preference value $r_{ij,k}$ from the kth DM to modify the MPR R_k whose group consensus index becomes larger.

According to the above discussions, a complete consensus reaching algorithm based on feedback mechanism is devised as follows:

Algorithm 3 Consensus Reaching Algorithm

Input: MPRs $R_k = \left(r_{ij,k}\right)_{n\times n}$ $(k = 1, 2, \ldots, K)$ offered by K DMs DM_k $(k = 1, 2, \ldots, K)$, the weight vector of DMs, $\omega = (\omega_1, \omega_2, \ldots, \omega_K)^T$, the predefined threshold $\overline{GCI}$ of acceptable group consensus index, the maximum number of iterations, $l_{\max} \geq 1$.

Output: The number of iterations, l, the modified MPRs $R_k^{(l)} = \left(r_{ij,k}^{(l)}\right)_{n\times n}$, their group MPR $G^{(l)} = \left(g_{ij}^{(l)}\right)_{n\times n}$, the group consensus indexes $GCI\left(R_k^{(l)}\right)$ for each MPR $R_k^{(l)}$ and $GCI\left(G^{(l)}\right)$ for the group MPR $G^{(l)}$.

Step 1. Let $l = 0$, then we have $R_k^{(0)} = \left(r_{ij,k}^{(0)}\right)_{n\times n}$ and $\omega^{(0)} = \left(\omega_1^{(0)}, \omega_2^{(0)}, \ldots, \omega_K^{(0)}\right)^T$;

Step 2. Use Definition 23 to fuse all the individual MPRs $R_k^{(l)} = \left(r_{ij,k}^{(l)}\right)_{n\times n}$ into their group MPR $G^{(l)} = \left(g_{ij}^{(l)}\right)_{n\times n}$, where $G^{(l)} = \left(g_{ij}^{(l)}\right)_{n\times n} = \left(\bigotimes_{k=1}^{K}\left(r_{ij,k}^{(l)}\right)^{\omega_k}\right)_{n\times n}$.

Step 3. Utilize Definition 25 as well as Eqs. (5.3), (5.4), and (5.5) to calculate the group consensus index matrix of $R_k^{(l)}$ denoted as $GCM\left(R_k^{(l)}\right)$, the group consensus index $GCI\left(x_{i,k}^{(l)}\right)$ of alternative $x_{i,k}^{(l)}$ in $R_k^{(l)}$, and the group consensus index $GCI\left(R_k^{(l)}\right)$. If $GCI\left(R_k^{(l)}\right) \geq \overline{GCI}$ for $\forall k = 1, 2, \ldots, K$ or $l \geq l_{\max}$, then turn to Step 7; otherwise, turn to Step 4.

Step 4. Locate the set of pairs of alternatives using Eq. (5.6) as follows:

$$L_k^{(l)} = \left\{\begin{array}{l}\left(x_{i,k}^{(l)}, x_{j,k}^{(l)}\right)|GCI\left(x_{i,k}^{(l)}\right) < GCI\left(R_k^{(l)}\right) \wedge \\ z_{ij,k}^{(l)} < GCI\left(R_k^{(l)}\right)\end{array}\right\}$$

Step 5. Use S.i, S.ii, and S.iii to give suggestions on the modification of preference values for $\left(x_{i,k}^{(l)}, x_{j,k}^{(l)}\right) \in L_k^{(l)}$.

Step 6. Adopt the feedback advices on preference values $\left(x_{i,k}^{(l)}, x_{j,k}^{(l)}\right) \in L_k^{(l)}$ to reconstruct $R_k^{(l+1)} = \left(r_{ij,k}^{(l+1)}\right)_{n\times n}$, we utilize Definition 28 to update the weight of each DM as $\omega_k^{(l+1)} = \frac{GCI\left(R_k^{(l)}\right)}{\sum_{k=1}^{K} GCI\left(R_k^{(l)}\right)}$ $(k = 1, 2, \ldots, K)$.

Let $l = l + 1$ and turn to Step 2.

Step 7. Output the number of iterations, l, the modified MPRs $R_k^{(l)} = \left(r_{ij,k}^{(l)}\right)_{n\times n}$ $(k = 1, 2, \ldots K)$, their group MPR $G^{(l)} = \left(g_{ij}^{(l)}\right)_{n\times n}$, the group consensus indexes $GCI\left(R_k^{(l)}\right)$ for the MPRs $R_k^{(l)}$.

Step 8. End.

Algorithm 3 is an iterative process and its convergence is discussed in Theorem 29.

Theorem 29: Let $R_k = \left(r_{ij,k}\right)_{n\times n}$ $(k = 1, 2, \ldots K)$ be MPRs, $\overline{GCI}$ denote the threshold for their group consensus index, $\left\{R_k^{(l)}\right\}$ denote a sequence of MPRs that are generated in Algorithm 3, and $GCI\left(R_k^{(l)}\right)$ be the group consensus index of $R_k^{(l)}$, then $GCI\left(R_k^{(l)}\right) < GCI\left(R_k^{(l+1)}\right)$ for $\forall k$.

Proof: According to Definition 23,

$$G^{(l+1)} = \left(g_{ij}^{(l+1)}\right)_{n\times n} = \left(\bigotimes_{k=1}^{K}\left(r_{ij,k}^{(l+1)}\right)^{\omega_k}\right)_{n\times n}.$$

Then we have $\frac{r_{ij,k}^{(l+1)}}{g_{ij}^{(l+1)}} = r_{ij,k}^{(l+1)} \times \prod_{t=1}^{K}\left(r_{ji,t}^{(l+1)}\right)^{\omega_k}$. Since $r_{ij,k}^{(l+1)} \in \left[\min\left(r_{ij,k}^{(l)}, g_{ij}^{(l)}\right), \max\left(r_{ij,k}^{(l)}, g_{ij}^{(l)}\right)\right]$, then the equation can be transformed into

$$\min\left(r_{ij,k}^{(l)}, g_{ij}^{(l)}\right) \times \prod_{t=1}^{K}\left(\min\left(r_{ji,t}^{(l)}, g_{ji}^{(l)}\right)\right)^{\omega_k}$$
$$\leq \frac{r_{ij,k}^{(l+1)}}{g_{ij}^{(l+1)}} \leq \max\left(r_{ij,k}^{(l)}, g_{ij}^{(l)}\right) \times \prod_{t=1}^{K}\left(\max\left(r_{ji,t}^{(l)}, g_{ji}^{(l)}\right)\right)^{\omega_k}$$

Since $g_{ji}^{(l)} = \prod_{t=1}^{K}\left(r_{ji,t}^{(l)}\right)^{\omega_k}$, then we have

$$\min\left(r_{ij,k}^{(l)}, g_{ij}^{(l)}\right) \times g_{ji}^{(l)} \leq \frac{r_{ij,k}^{(l+1)}}{g_{ij}^{(l+1)}} \leq \max\left(r_{ij,k}^{(l)}, g_{ij}^{(l)}\right) \times g_{ij}^{(l)}. \quad (5.7)$$

Assume that $r_{ij,k}^{(l)} < g_{ij}^{(l)}$, then

$$r_{ij,k}^{(l)} \times g_{ji}^{(l)} < \frac{r_{ij,k}^{(l+1)}}{g_{ij}^{(l+1)}} < g_{ij}^{(l)} \times g_{ji}^{(l)}$$
$$\Leftrightarrow \frac{r_{ij,k}^{(l)}}{g_{ij}^{(l)}} < \frac{r_{ij,k}^{(l+1)}}{g_{ij}^{(l+1)}} < 1$$
$$\Leftrightarrow 0 < c_{ij,k}^{(l+1)} = \frac{1}{2}\left|\log_9 \frac{r_{ij,k}^{(l+1)}}{g_{ij}^{(l+1)}}\right| < c_{ij,k}^{(l)} = \frac{1}{2}\left|\log_9 \frac{r_{ij,k}^{(l)}}{g_{ij}^{(l)}}\right|$$

Based on Definition 26, $z_{ij,k}^{(l)} < z_{ij,k}^{(l+1)} < 1$. According to Eqs. (5.4) and (5.5), then $GCI\left(R_k^{(l)}\right) < GCI\left(R_k^{(l+1)}\right)$. If $r_{ij,k}^{(l)} > g_{ij}^{(l)}$, it can be proven in the same way, which completes the proof.

Theorem 30: Let $R_k = \left(r_{ij,k}\right)_{n\times n}$ $(k = 1, 2, \ldots K)$ denote the reduced MPRs and G be their group MPR. Assume that $\left\{R_k^{(l)}\right\}$ and $\left\{G^{(l)}\right\}$ be the sequences of the reduced MPRs and their group MPRs that are obtained by Algorithm 3. If $\min_{1\leq k\leq K}\{CI\left(R_k\right)\} \geq CI$, then we have

$$\min_{1\leq k\leq K}\left\{CI\left(R_k^{(l+1)}\right)\right\} \geq \min_{1\leq k\leq K}\left\{CI\left(R_k^{(l)}\right)\right\} \geq CI.$$

Proof: For any $k = 1, 2, \ldots, K$, we have

$$r_{ij,k}^{(l+1)} \in \left(\min\left(r_{ij,k}^{(l)}, g_{ij}^{(l)}\right), \max\left(r_{ij,k}^{(l)}, g_{ij}^{(l)}\right)\right),$$

According to Theorem 17, then we have

$$CI\left(R_k^{(l+1)}\right) \geq \min\left\{CI\left(R_k^{(l)}\right), CI\left(G^{(l)}\right)\right\}.$$

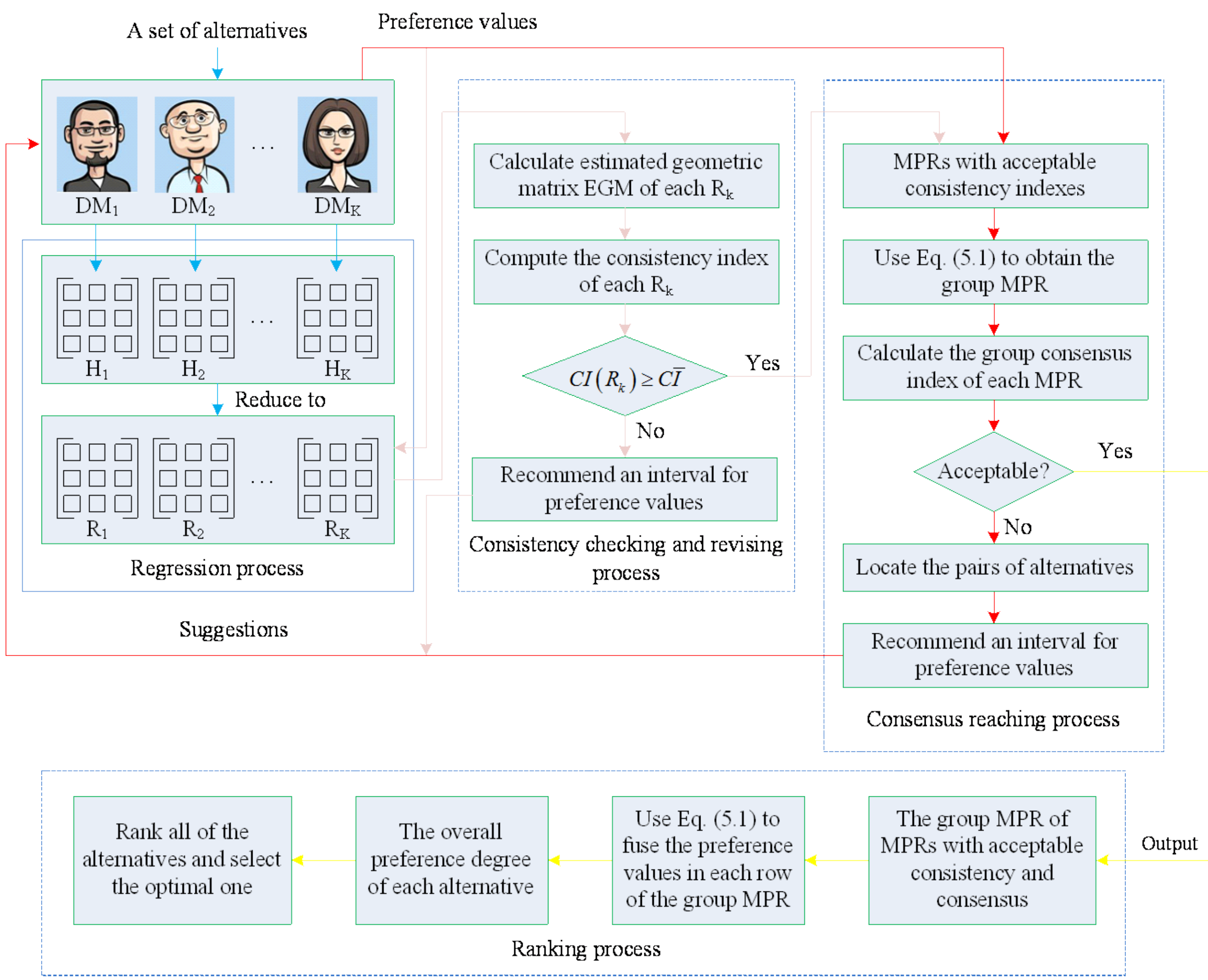


FIGURE 4. The flowchart of the complete group decision-making model.

Moreover, $CI\left(G^{(l)}\right) \geq \min_{1\leq k\leq K}\left\{CI\left(R_k^{(l)}\right)\right\}$, then we have

$$\min_{1\leq k\leq K}\left\{CI\left(R_k^{(l+1)}\right)\right\} \geq \min_{1\leq k\leq K}\left\{CI\left(R_k^{(l)}\right)\right\},$$

which implies that

$$\min_{1\leq k\leq K}\left\{CI\left(R_k^{(l+1)}\right)\right\} \geq \min_{1\leq k\leq K}\left\{CI\left(R_k^{(l)}\right)\right\} \geq \overline{CI}.$$

Theorem 30 implies that the consistency index of each MPR is still acceptable after it is updated using Algorithm 3. According to this theorem, it is feasible to put forward a complete group decision-making model that is composed of the consistency and consensus of HMPRs.

VI. COMPLETE GROUP DECISION-MAKING MODEL WITH HMPRs

In this section, a complete group decision-making model with HMPRs is put forward to rank all the alternatives in the complex decision-making problems.

As demonstrated in Fig. 4, the complete group decision-making model consists of four parts: the regression process, the consistency checking and revising process, the consensus reaching process, and also the ranking process, respectively. Theorem 30 shows that it is rational to place the consistency checking and revising process before the consensus reaching process in the complete group decision-making model. The methods for the regression process, consistency checking and revising process, and consensus reaching process have been shown in Sections 3, 4, and 5. The ranking process aims to rank all the alternatives and then select the optimal one. It first utilizes the multiplicative geometric (MG) operator (Eq. (5.1)) to aggregate all the preference values in each row of the group MPR and then we can obtain the overall preference degree of each alternative, x_i $(i = 1, 2, \ldots, n)$. Finally, all the alternatives can be ranked based on their overall preference degrees and then the optimal one can be selected.

According to Fig. 4, we combine Algorithms 1, 2, and 3 to develop a complete algorithm, which ranks all the alternatives and selects the optimal one from group decision-making problems with HMPRs.

Algorithm 4 Complete Group Decision-Making Algorithm

Input: HMPRs $H_k = (h_{ij,k})_{n\times n}$ $(k = 1, 2, \ldots, K)$ given by K DMs DM_k $(k = 1, 2, \ldots, K)$, the weight vector of DMs, $\omega = (\omega_1, \omega_2, \ldots, \omega_K)^T$, a threshold $\bar{CI}$ for the consistence index, a threshold $\bar{GCI}$ for the acceptable group consensus index, and the maximum number of iterations, $l_{\max} \geq 1$.

Output: The ranking of all the alternatives and the optimal one.

Step 1. Choose a HME $h_{ij,k}(i \neq j)$ from H_k, get the set $\mathrm{M}^m_{ij,k}$, and calculate $h^m_{ij,k}$ $\left(m \in \mathrm{M}^m_{ij,k}\right)$ using the following equation:

$$h^m_{ij,k} = h_{im,k} \tilde{\times} h_{mj,k} \tag{6.1}$$

Step 2. Utilize the following equation:

$$h^{\mathrm{G}}_{ij,k} = \sqrt[\sum_{m\in \mathrm{M}^m_{ij,k}} (\#h^m_{ij,k})]{\prod_{m\neq i, m\neq j}^{\#h^m_{ij,k}} h^m_{ij,k}} \tag{6.2}$$

to calculate the geometric estimated preference value $h^{\mathrm{G}}_{ij,k}$ and then obtain $h^*_{ij,k}$ satisfying the following equation:

$$\frac{1}{2}\left|\log_9 \frac{h^*_{ij}}{h^G_{ij}}\right| = \min(\varepsilon h_{ij}) \tag{6.3}$$

where

$$\varepsilon h_{ij} = \frac{1}{2}\left(\bigcup_{\varepsilon_{ij}\in\left(h_{ij}\tilde{\div}h^G_{ij}\right)} |\varepsilon_{ij}|\right) = \frac{1}{2}\left(\bigcup_{\varepsilon_{ij}\in\log_9\left(h_{ij}/h^G_{ij}\right)} |\varepsilon_{ij}|\right) \tag{6.4}$$

Step 3. Repeat Steps 1 and 2 until the optimal value of each HME in H_k has been obtained, then turn to Step 4.

Step 4. Use all the collected $h^*_{ij,k}$ $(i, j = 1, 2, \ldots, n; i \neq j)$ to form the reduced MPR $H^*_k = \left(h^*_{ij,k}\right)_{n\times n}$. Let $l = 0$, then $R^{(0)}_k = (r^{(0)}_{ij,k})_{n\times n} = H^*_k = \left(h^*_{ij,k}\right)_{n\times n}$.

Step 5. Use the following equation:

$$e_{ij,k} = \left(\bigotimes_{m=1, m\neq i,j}^{n} r^m_{ij,k}\right)^{\frac{1}{2(n-2)}} \tag{6.5}$$

to obtain the EGM $E^{(l)}_k = \left(e^{(l)}_{ij,k}\right)_{n\times n}$.

Step 6. Utilize the following equation:

$$\begin{aligned} CI\left(R^{(l)}_k\right) &= 1 - d\left(R^{(l)}_k, E^{(l)}_k\right) \\ &= 1 - \frac{1}{2n(n-1)} \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \\ &\quad \times \left(\left|\log_9 \frac{u^{(l)}_{ij,k}}{v^{(l)}_{ij,k}}\right| + \left|\log_9 \frac{u^{(l)}_{ji,k}}{v^{(l)}_{ji,k}}\right|\right) \end{aligned} \tag{6.6}$$

Algorithm 4 *(Continued.)* Complete Group Decision-Making Algorithm

to calculate the consistency index $CI\left(R^{(l)}_k\right)$ of $R^{(l)}_k$. If $CI\left(R^{(l)}_k\right) \geq \bar{CI}$ or $l \geq l_{\max}$, then turn to Step 9; otherwise, turn to the next step.

Step 7. Return $R^{(l)}_k$ to the kth DM and suggest he/she to offer an adjusted $R^{(l+1)}_k = (r_{ij,k})^{(l+1)}_{n\times n}$, which satisfies

$$r^{(l+1)}_{ij,k} \in \left(\min\left(r^{(l)}_{ij,k}, e^{(l)}_{ij,k}\right), \max\left(r^{(l)}_{ij,k}, e^{(l)}_{ij,k}\right)\right) \tag{6.7}$$

Let $l = l + 1$, then turn to Step 5;

Step 8. Utilize Definition 23 to aggregate all the MPRs $R^{(l)}_k = \left(r^{(l)}_{ij,k}\right)_{n\times n}$ $(k = 1, 2, \ldots K)$ into a group MPR $G^{(l)}$, where

$$G^{(l)} = \left(g^{(l)}_{ij}\right)_{n\times n} = \left(\bigotimes_{k=1}^{K}\left(r^{(l)}_{ij,k}\right)^{\omega_k}\right)_{n\times n} \tag{6.8}$$

Step 9. Utilize Eq. (5.3) to compute the group consensus index matrix $GCM\left(R^{(l)}_k\right) = \left(z^{(l)}_{ij,k}\right)_{n\times n}$ of $R^{(l)}_k$, where

$$\begin{aligned} GCM(R^{(l)}_k) &= \left(z^{(l)}_{ij,k}\right)_{n\times n} \\ &= \left(1 - c^{(l)}_{ij,k}\right)_{n\times n} = \left(\frac{1}{2}\left|\log_9 \frac{r^{(l)}_{ij,k}}{g^{(l)}_{ij}}\right|\right)_{n\times n} \end{aligned} \tag{6.9}$$

Utilize Eq. (5.4) to compute the group consensus index $GCI\left(x^{(l)}_{i,k}\right)$ of the alternative $x^{(l)}_{i,k}$ in $R^{(l)}_k$, where

$$GCI\left(x^{(l)}_{i,k}\right) = \left(\prod_{j=1}^{n} z^{(l)}_{ij,k}\right)^{\frac{1}{n}} \tag{6.10}$$

Utilize Eq. (5.5) to compute the group consensus index $GCI\left(R^{(l)}_k\right)$, where

$$GCI\left(R^{(l)}_k\right) = \left(\prod_{i=1}^{n}\prod_{j=1}^{n} z^{(l)}_{ij,k}\right)^{\frac{1}{n^2}} \tag{6.11}$$

If $GCI\left(R^{(l)}_k\right) \geq \bar{GCI}$ for $\forall k = 1, 2, \ldots, K$ or $l \geq l_{\max}$, then turn to Step 13; otherwise, turn to Step 11.

Step 10. Locate the set of pairs of alternatives using Eq. (5.6) as follows:

$$L^{(l)}_k = \left\{ \begin{array}{l} \left(x^{(l)}_{i,k}, x^{(l)}_{j,k}\right) | GCI\left(x^{(l)}_{i,k}\right) \\ < GCI\left(R^{(l)}_k\right) \wedge z^{(l)}_{ij,k} < GCI\left(R^{(l)}_k\right) \end{array} \right\} \tag{6.12}$$

Step 11. Use S.i, S.ii, and S.iii to offer suggestions on the modification of preference values for $\left(x^{(l)}_{i,k}, x^{(l)}_{j,k}\right) \in L^{(l)}_k$.

VII. AN ILLUSTRATIVE EXAMPLE AND COMPARISON ANALYSIS

In this section, we first give a numerical example to show the process of proposed group decision-making model. Then we

Algorithm 4 *(Continued.)* Complete Group Decision-Making Algorithm

Step 12. Use the feedback advices on preference values $\left(x_{i,k}^{(l)}, x_{j,k}^{(l)}\right) \in L_k^{(l)}$ to reconstruct $R_k^{(l+1)} = \left(r_{ij,k}^{(l+1)}\right)_{n\times n}$, and utilize Definition 28 to update the weights of the DMs as:

$$\omega_k^{(l+1)} = \frac{GCI\left(R_k^{(l)}\right)}{\sum_{k=1}^{K} GCI\left(R_k^{(l)}\right)} \quad (k = 1, 2, \ldots, K) \tag{6.13}$$

Let $l = l + 1$ and turn to Step 9.

Step 13. Use the MG operator (Eq. (5.1)):

$$\begin{aligned} g_{i*}^{(l)} &= \mathrm{MG}\left(g_{i1}^{(l)}, g_{i2}^{(l)}, \ldots g_{in}^{(l)}\right) \\ &= \bigotimes_{j=1}^{n} \left(g_{ij}^{(l)}\right)^{\frac{1}{n}} \quad (i = 1, 2, \ldots, n) \end{aligned} \tag{6.14}$$

to fuse all the preference values in each row of the group MPR $G^{(l)} = \left(g_{ij}^{(l)}\right)_{n\times n}$ and then get the overall preference degrees $g_{i*}^{(l)}$ of the alternatives x_i $(i = 1, 2, \ldots, n)$.

Step 14. Rank all the alternatives x_i $(i = 1, 2, \ldots, n)$ based on their overall preference degrees $g_{i*}^{(l)}$ $(i = 1, 2, \ldots, n)$ and then select the optimal one.

Step 15. End.

compare the proposed group decision-making model with the group decision-making model which was put forward by Zhang and Wu [27] to validate the effectiveness of our model.

A. AN ILLUSTRATIVE EXAMPLE

Example 31: In recent years, with the quick development of Internet Plus and mobile payment, the bicycle-sharing comes into people's daily lives. It becomes more and more popular and has been reported by The New York Times and Financial Times. A large number of shared bikes can be easily found in the streets or roads of cities and people can utilize their smart phones to scan the QR (Quick Response) code on the shared bikes to unlock them. The price of renting a shared bike is very low, about 1 RMB per hour. Moreover, people can park the shared bikes almost anywhere they like. Therefore, the emerging of bicycle-sharing makes people be very easy to travel and shared bikes have become one of the most popular means of the public transports in China. The 2017 Internet Trends Report delivered by Mary Meeker shows that the on-demand bike transportation in China is second only to cars and two-thirds of bike riders use bike-sharing programs three or more times per week.

There are many different types of bike-sharing brands in Chinese market, such as mobike, UBIKE, OFO, Hellobike, bluegogo, UniBike, and youon. Assume that there exists an investment company that plans to invest one of bike-sharing brands. There are four possible alternatives to be considered, which are listed as: (1) x_1 is mobike; (2) x_2 is bluegogo; (3) x_3 is OFO; (4) x_4 is youon. Four DMs DM_k $(k = 1, 2, 3, 4)$ are called to compare these four types of bike-sharing brands concerning the criterion growth analysis by using the Saaty's 1-9 scale and the weight vector is $\omega = (1/4, 1/4, 1/4, 1/4)$. The DMs provide their quantitative assessment information in the form of HMPRs as follows:

$$H_1 = \begin{Bmatrix} \{1\} & \{3,5\} & \{1/9,1/7\} & \{5\} \\ \{1/3,1/5\} & \{1\} & \{1/5\} & \{7\} \\ \{9,7\} & \{5\} & \{1\} & \{3\} \\ \{1/5\} & \{1/7\} & \{1/3\} & \{1\} \end{Bmatrix},$$

$$H_2 = \begin{Bmatrix} \{1\} & \{1/3\} & \{1/9,1/7\} & \{3\} \\ \{3\} & \{1\} & \{1/7,1/5\} & \{5\} \\ \{9,7\} & \{7,5\} & \{1\} & \{1/7\} \\ \{1/3\} & \{1/5\} & \{7\} & \{1\} \end{Bmatrix},$$

$$H_3 = \begin{Bmatrix} \{1\} & \{3,5\} & \{1/5\} & \{3\} \\ \{1/3,1/5\} & \{1\} & \{1/7\} & \{9,7\} \\ \{5\} & \{7\} & \{1\} & \{1/9\} \\ \{1/3\} & \{1/9,1/7\} & \{9\} & \{1\} \end{Bmatrix},$$

$$H_4 = \begin{Bmatrix} \{1\} & \{3\} & \{1/9\} & \{5,7\} \\ \{1/3\} & \{1\} & \{1/7,1/5\} & \{7\} \\ \{9\} & \{7,5\} & \{1\} & \{3\} \\ \{1/5,1/7\} & \{1/7\} & \{1/3\} & \{1\} \end{Bmatrix}$$

To depict the process of Algorithm 4, the involved steps are shown as follows:

Step 1: Choose the HME $h_{12,1}$ from R_1, achieve the set $\mathrm{M}_{12,1}^{m} = \{3, 4\}$, use Eq. (6.1) to calculate $h_{12,1}^{m}\left(m \in \mathrm{M}_{12,1}^{m}\right)$ as $h_{12,1}^{3} = h_{13,1} \tilde{\times} h_{32,1} = \left\{\frac{5}{9}, \frac{5}{7}\right\}$, $h_{12,1}^{4} = h_{14,1} \tilde{\times} h_{42,1} = \left\{\frac{5}{7}\right\}$.

Step 2: Use Eq. (6.2) to compute the geometric estimated preference value $h_{12,1}^{\mathrm{G}}$ as:

$$h_{12,1}^{\mathrm{G}} = \sqrt[3]{\frac{5}{9} \times \frac{5}{7} \times \frac{5}{7}} = 0.6569.$$

Then, we use Eqs. (6.3) and (6.4) to get the optimal value h_{12}^{*} from the HME h_{12} as follows:

$$\varepsilon h_{12,1} = \frac{1}{2}\left\{ \bigcup_{\varepsilon_{12,1} \in \log_9\left(h_{12,1}/h_{12,1}^{G}\right)} |\varepsilon_{12,1}| \right\} = \{0.3456, 0.4619\}.$$

Thus, $\min(\varepsilon h_{12,1}) = 0.3456$. Then $h_{12,1}^{*} = 3$.

Step 3: Repeat Steps 1 and 2 until the optimal value of each HME in H_k has been obtained.

$h_{13,1}^{*} = 1/7$, $h_{21,1}^{*} = 1/3$, $h_{13,2}^{*} = 1/7$, $h_{23,2}^{*} = 1/5$, $h_{12,3}^{*} = 3$, $h_{23,3}^{*} = 7$, $h_{14,4}^{*} = 5$, $h_{23,4}^{*} = 1/5$

Step 4: Use all the collected $h_{ij,k}^{*}$ $(i, j = 1, 2, \ldots, n; i \neq j)$ to form the reduced MPR $H_k^{*} = \left(h_{ij,k}^{*}\right)_{4\times 4}$. Let $l = 0$, then $R_k^{(0)} = (r_{ij,k}^{(0)})_{4\times 4} = H_k^{*} = \left(h_{ij,k}^{*}\right)_{4\times 4}$.

$$R_1^{(0)} = H_1^{*} = \begin{Bmatrix} \{1\} & \{3\} & \{1/7\} & \{5\} \\ \{1/3\} & \{1\} & \{1/5\} & \{7\} \\ \{7\} & \{5\} & \{1\} & \{3\} \\ \{1/5\} & \{1/7\} & \{1/3\} & \{1\} \end{Bmatrix},$$

$$R_2^{(0)} = H_2^* = \left\{ \begin{array}{cccc} \{1\} & \{1/3\} & \{1/7\} & \{3\} \\ \{3\} & \{1\} & \{1/5\} & \{5\} \\ \{7\} & \{5\} & \{1\} & \{1/7\} \\ \{1/3\} & \{1/5\} & \{7\} & \{1\} \end{array} \right\},$$

$$R_3^{(0)} = H_3^* = \left\{ \begin{array}{cccc} \{1\} & \{3\} & \{1/5\} & \{3\} \\ \{1/3\} & \{1\} & \{1/7\} & \{7\} \\ \{5\} & \{7\} & \{1\} & \{1/9\} \\ \{1/3\} & \{1/7\} & \{9\} & \{1\} \end{array} \right\},$$

$$R_4^{(0)} = H_4^* = \left\{ \begin{array}{cccc} \{1\} & \{3\} & \{1/9\} & \{5\} \\ \{1/3\} & \{1\} & \{1/5\} & \{7\} \\ \{9\} & \{5\} & \{1\} & \{3\} \\ \{1/5\} & \{1/7\} & \{1/3\} & \{1\} \end{array} \right\}.$$

Step 5: Use Eq. (6.5) to compute the EGM, namely the consistent MPR, of each reduced $R_k^{(0)}$ as:

$$E_1^{(0)} = \left\{ \begin{array}{cccc} \{1\} & \{0.8452\} & \{1\} & \{1.7321\} \\ \{1.1832\} & \{1\} & \{0.5774\} & \{1\} \\ \{1\} & \{1.7321\} & \{1\} & \{5.9161\} \\ \{0.5774\} & \{1\} & \{0.1690\} & \{1\} \end{array} \right\},$$

$$E_2^{(0)} = \left\{ \begin{array}{cccc} \{1\} & \{0.8091\} & \{1.0878\} & \{0.4295\} \\ \{1.2359\} & \{1\} & \{1.9680\} & \{0.7121\} \\ \{0.9193\} & \{0.5081\} & \{1\} & \{4.7867\} \\ \{2.3286\} & \{1.4043\} & \{0.2089\} & \{1\} \end{array} \right\},$$

$$E_3^{(0)} = \left\{ \begin{array}{cccc} \{1\} & \{0.8801\} & \{1.8444\} & \{0.8265\} \\ \{1.1362\} & \{1\} & \{1.4317\} & \{0.3549\} \\ \{0.5422\} & \{0.6585\} & \{1\} & \{5.2068\} \\ \{1.2099\} & \{2.8173\} & \{0.1921\} & \{1\} \end{array} \right\},$$

$$E_4^{(0)} = \left\{ \begin{array}{cccc} \{1\} & \{0.7937\} & \{1\} & \{1.6266\} \\ \{1.2599\} & \{1\} & \{0.5422\} & \{1\} \\ \{1\} & \{1.8444\} & \{1\} & \{6.2997\} \\ \{0.6148\} & \{1\} & \{0.1587\} & \{1\} \end{array} \right\}.$$

Step 6: Utilize Eq. (6.6) to compute the consistency index $CI\left(R_k^{(0)}\right)$ of $R_k^{(0)}$:

$$CI\left(R_1^{(0)}\right) = 0.6982, \quad CI\left(R_2^{(0)}\right) = 0.5218,$$
$$CI\left(R_3^{(0)}\right) = 0.4739, \quad CI\left(R_4^{(0)}\right) = 0.6839.$$

The value of $C\bar{I}$ is set to 0.85 in this paper. Because $CI\left(R_k^{(0)}\right) < 0.85$, each $R_k^{(0)}$ should be modified in the next step.

Step 7: Return $R_k^{(0)}$ to the kth DM and suggest he/she to provide an adjusted MPR $R_k^{(1)} = (r_{ij,k})_{4\times4}^{(1)}$ using Eq. (6.7) as:

$$R_1^{(1)} = \left\{ \begin{array}{cccc} 1 & 1.2761 & 0.8286 & 2.3856 \\ 0.7836 & 1 & 0.5019 & 2.2000 \\ 1.2069 & 1.9925 & 1 & 5.3329 \\ 0.4192 & 0.4545 & 0.1875 & 1 \end{array} \right\},$$

$$R_2^{(1)} = \left\{ \begin{array}{cccc} 1 & 0.6188 & 0.7098 & 1.4577 \\ 1.6160 & 1 & 1.2608 & 2.4273 \\ 1.4089 & 0.7932 & 1 & 2.9292 \\ 0.6860 & 0.4120 & 0.3414 & 1 \end{array} \right\},$$

$$R_3^{(1)} = \left\{ \begin{array}{cccc} 1 & 1.3041 & 1.5155 & 1.2612 \\ 0.7668 & 1 & 1.1738 & 1.6840 \\ 0.6599 & 0.8519 & 1 & 4.1877 \\ 0.7929 & 0.5938 & 0.2388 & 1 \end{array} \right\},$$

$$R_4^{(1)} = \left\{ \begin{array}{cccc} 1 & 1.2351 & 0.8222 & 2.3013 \\ 0.8097 & 1 & 0.4738 & 2.2000 \\ 1.2162 & 2.1108 & 1 & 5.6398 \\ 0.4345 & 0.4545 & 0.1773 & 1 \end{array} \right\}.$$

Step 8: Use Eq. (6.5) to compute the EGM, namely the consistent MPR, of each reduced $R_k^{(1)}$ as:

$$E_1^{(1)} = \left\{ \begin{array}{cccc} 1 & 1.1567 & 0.7316 & 1.8767 \\ 0.8645 & 1 & 0.7194 & 1.4956 \\ 1.3668 & 1.3900 & 1 & 1.8848 \\ 0.5328 & 0.6686 & 0.5305 & 1 \end{array} \right\},$$

$$E_2^{(1)} = \left\{ \begin{array}{cccc} 1 & 0.7625 & 0.7894 & 1.3293 \\ 1.3114 & 1 & 0.9874 & 1.7174 \\ 1.2668 & 1.0128 & 1 & 1.4101 \\ 0.7523 & 0.5823 & 0.7092 & 1 \end{array} \right\},$$

$$E_3^{(1)} = \left\{ \begin{array}{cccc} 1 & 0.9916 & 0.8240 & 1.9322 \\ 1.0084 & 1 & 0.8268 & 1.4766 \\ 1.2136 & 1.2095 & 1 & 1.0453 \\ 0.5176 & 0.6772 & 0.9567 & 1 \end{array} \right\},$$

$$E_4^{(1)} = \left\{ \begin{array}{cccc} 1 & 1.1608 & 0.6990 & 1.8840 \\ 0.8615 & 1 & 0.7139 & 1.4938 \\ 1.4306 & 1.4008 & 1 & 1.8987 \\ 0.5308 & 0.6694 & 0.5267 & 1 \end{array} \right\}.$$

Step 9: Utilize Eq. (6.6) to compute the consistency index $CI\left(R_k^{(1)}\right)$ of $R_k^{(1)}$:

$$CI\left(R_1^{(1)}\right) = 0.9147, \quad CI\left(R_2^{(1)}\right) = 0.9344,$$
$$CI\left(R_3^{(1)}\right) = 0.8794, \quad CI\left(R_4^{(1)}\right) = 0.9124$$

Because $CI\left(R_k^{(1)}\right) \geq 0.85$ for all the $k = 1, 2, 3, 4$, all the reduced MPRs $R_k^{(1)}$ satisfy the condition of acceptable consistency.

The Matlab software is used to draw "Figure of area" as shown in Figs. 5-8, which gives a visible description of the inconsistent MPRs $R_k^{(0)}$ $(k = 1, 2, 3, 4)$, the acceptably consistent MPRs $R_k^{(1)}$ $(k = 1, 2, 3, 4)$, the EGM, namely the consistent MPRs, $E_k^{(1)}$ $(k = 1, 2, 3, 4)$. Figs. 5-8 depict that the more consistent the MPRs are, the more regular their "Figure of area" perform.

Step 10: Utilize Eq. (6.8) to fuse all the reduced MPRs $R_k^{(1)} = \left(r_{ij,k}^{(1)}\right)_{4\times4}$ into a group MPR $G^{(1)}$, where

$$G^{(1)} = \left\{ \begin{array}{cccc} 1 & 0.8915 & 0.9185 & 1.5347 \\ 1.1217 & 1 & 1.0205 & 2.1328 \\ 1.0888 & 0.9799 & 1 & 3.6965 \\ 0.6516 & 0.4689 & 0.2705 & 1 \end{array} \right\}$$

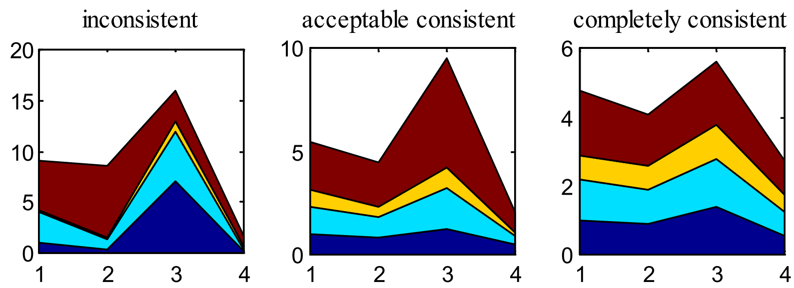


FIGURE 5. Areas of $R_1^{(0)}$, $R_1^{(1)}$, $E_1^{(1)}$.

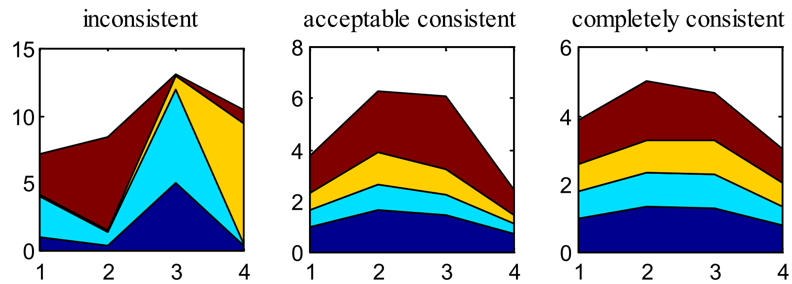


FIGURE 6. Areas of $R_2^{(0)}$, $R_2^{(1)}$, $E_2^{(1)}$.

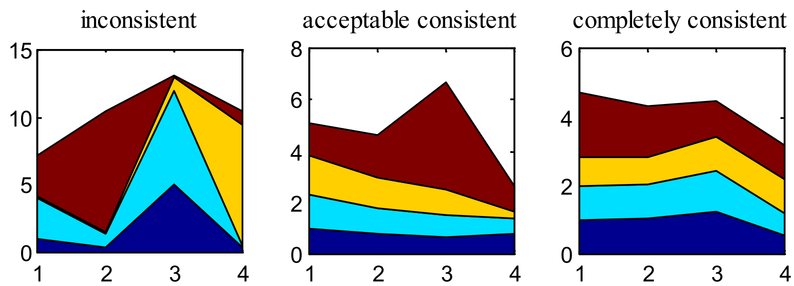


FIGURE 7. Areas of $R_3^{(0)}$, $R_3^{(1)}$, $E_3^{(1)}$.

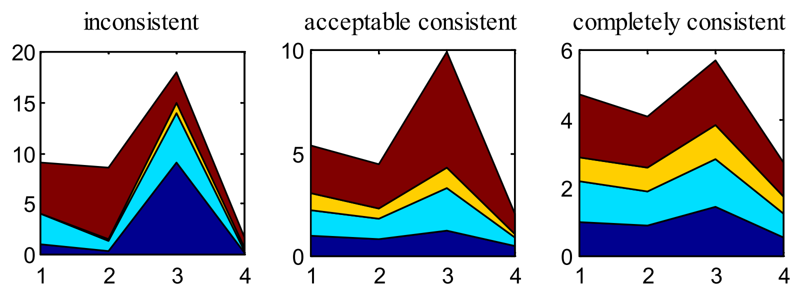


FIGURE 8. Areas of $R_4^{(0)}$, $R_4^{(1)}$, $E_4^{(1)}$.

Step 11: Utilize Eq. (6.9) to compute the group consensus index matrix $GCM\left(R_k^{(1)}\right) = \left(z_{ij,k}^{(1)}\right)_{4\times 4}$ of $R_k^{(1)}$, where

$$GCM(R_1^{(1)}) = \begin{Bmatrix} 1 & 0.9184 & 0.9766 & 0.8996 \\ 0.9184 & 1 & 0.8385 & 0.9929 \\ 0.9766 & 0.8385 & 1 & 0.9166 \\ 0.8996 & 0.9929 & 0.9166 & 1 \end{Bmatrix},$$

$$GCM(R_2^{(1)}) = \begin{Bmatrix} 1 & 0.9169 & 0.9414 & 0.9883 \\ 0.9169 & 1 & 0.9519 & 0.9706 \\ 0.9414 & 0.9519 & 1 & 0.9471 \\ 0.9883 & 0.9706 & 0.9471 & 1 \end{Bmatrix},$$

$$GCM(R_3^{(1)}) = \begin{Bmatrix} 1 & 0.9134 & 0.8860 & 0.9553 \\ 0.9134 & 1 & 0.9681 & 0.9462 \\ 0.8860 & 0.9681 & 1 & 0.9716 \\ 0.9553 & 0.9462 & 0.9716 & 1 \end{Bmatrix},$$

$$GCM(R_4^{(1)}) = \begin{Bmatrix} 1 & 0.9258 & 0.9748 & 0.9078 \\ 0.9258 & 1 & 0.8254 & 0.9929 \\ 0.9748 & 0.8254 & 1 & 0.9039 \\ 0.9078 & 0.9929 & 0.9038 & 1 \end{Bmatrix}.$$

Utilize Eq. (6.10) to compute the group consensus index $GCI\left(x_{i,k}^{(1)}\right)$ of the alternative $x_{i,k}^{(1)}$ in $R_k^{(1)}$, where

$$GCI(x_{1,1}^{(1)}) = 0.9315, GCI(x_{2,1}^{(1)}) = 0.9166,$$
$$GCI(x_{3,1}^{(1)}) = 0.9106, GCI(x_{4,1}^{(1)}) = 0.9364$$
$$GCI(x_{1,2}^{(1)}) = 0.9488, GCI(x_{2,2}^{(1)}) = 0.9465,$$
$$GCI(x_{3,2}^{(1)}) = 0.9468, GCI(x_{4,2}^{(1)}) = 0.9686$$
$$GCI(x_{1,3}^{(1)}) = 0.9183, GCI(x_{2,3}^{(1)}) = 0.9426,$$
$$GCI(x_{3,3}^{(1)}) = 0.9419, GCI(x_{4,3}^{(1)}) = 0.9577$$
$$GCI(x_{1,4}^{(1)}) = 0.9362, GCI(x_{2,4}^{(1)}) = 0.9147,$$
$$GCI(x_{3,4}^{(1)}) = 0.9014, GCI(x_{4,4}^{(1)}) = 0.9349$$

Utilize Eq. (6.11) to compute the group consensus index $GCI\left(R_k^{(1)}\right)$, where

$$GCI_1^{(1)} = 0.92,\ GCI_2^{(1)} = 0.95,$$
$$GCI_3^{(1)} = 0.94,\ GCI_4^{(1)} = 0.92$$

Because $GCI\left(R_2^{(1)}\right) > 0.95$, $R_2^{(1)}$ does not need to be modified. $GCI\left(R_1^{(1)}\right)$, $GCI\left(R_3^{(1)}\right)$, and $GCI\left(R_4^{(1)}\right)$ are lower than 0.95, turn to the next step.

Step 12: Locate the set of pairs of alternatives using Eq. (6.12) as follows:

$$L_1^{(1)} = \begin{Bmatrix} \left(x_{2,1}^{(1)}, x_{1,1}^{(1)}\right), \left(x_{2,1}^{(1)}, x_{3,1}^{(1)}\right), \\ \left(x_{3,1}^{(1)}, x_{2,1}^{(1)}\right), \left(x_{3,1}^{(1)}, x_{4,1}^{(1)}\right) \end{Bmatrix}$$
$$L_3^{(1)} = \left\{\left(x_{1,3}^{(1)}, x_{2,3}^{(1)}\right), \left(x_{1,3}^{(1)}, x_{3,3}^{(1)}\right)\right\}$$
$$L_4^{(1)} = \left\{\left(x_{2,4}^{(1)}, x_{3,4}^{(1)}\right), \left(x_{3,4}^{(1)}, x_{2,4}^{(1)}\right), \left(x_{3,4}^{(1)}, x_{4,4}^{(1)}\right)\right\}$$

Step 13: Use S.i, S.ii, and S.iii to offer suggestions on the modification of preference values for $\left(x_{i,k}^{(1)}, x_{j,k}^{(1)}\right) \in L_k^{(1)}$ and use the feedback on preference values from the DMs to construct $R_k^{(2)} = \left(r_{ij,k}^{(2)}\right)_{4\times 4}$, where

$$R_1^{(2)} = \begin{Bmatrix} 1 & 0.8915 & 0.8286 & 2.3856 \\ 1.1217 & 1 & 1.0205 & 2.2000 \\ 1.2069 & 0.9799 & 1 & 3.6965 \\ 0.4192 & 0.4545 & 0.2705 & 1 \end{Bmatrix},$$
$$R_3^{(2)} = \begin{Bmatrix} 1 & 0.8915 & 0.9185 & 1.2612 \\ 1.1217 & 1 & 1.1738 & 1.6840 \\ 1.0888 & 0.8519 & 1 & 4.1877 \\ 0.7929 & 0.5938 & 0.2388 & 1 \end{Bmatrix},$$
$$R_4^{(2)} = \begin{Bmatrix} 1 & 1.2350 & 0.8222 & 2.3013 \\ 0.8097 & 1 & 1.0205 & 2.2000 \\ 1.2162 & 0.9799 & 1 & 3.6965 \\ 0.4345 & 0.4545 & 0.2705 & 1 \end{Bmatrix}.$$

Utilize Eq. (6.13) to update the weight of DMs as:

$$\omega^{(2)} = (0.2471, 0.2548, 0.2515, 0.2466)^T$$

Step 14: Utilize Eq. (6.8) to fuse all the reduced MPRs $R_k^{(2)} = \left(r_{ij,k}^{(2)}\right)_{4\times 4}$ $(k = 1, 2, 3, 4)$ with the weight vector $\omega^{(2)}$ into a group MPR $G^{(1)} = \left(g_{ij}^{(1)}\right)_{4\times 4}$, where

$$G^{(2)} = \begin{Bmatrix} 1 & 0.8803 & 0.8159 & 1.7767 \\ 1.1360 & 1 & 1.1156 & 2.1092 \\ 1.2257 & 0.8964 & 1 & 3.5947 \\ 0.5628 & 0.4741 & 0.2782 & 1 \end{Bmatrix}$$

Step 15: Utilize Eqs. (6.9), (6.10), (6.11) to compute the group consensus index $GCI\left(R_k^{(2)}\right)$, where

$$GCI_1^{(2)} = 0.98,\quad GCI_2^{(2)} = 0.96,$$
$$GCI_3^{(2)} = 0.97,\quad GCI_4^{(2)} = 0.97$$

Because $GCI\left(R_k^{(2)}\right) > 0.95$ for $\forall k = 1, 2, 3, 4$, then it turns to the next step.

Step 16: Utilize Eq. (6.14) to aggregate all the preference values in each row of the group MPR $G^{(2)} = \left(g_{ij}^{(2)}\right)_{4\times 4}$ and obtain the overall preference degree $g_{i*}^{(2)}$ $(i = 1, 2, 3, 4)$ of each alternative x_i.

$$g_{1*}^{(2)} = 1.06,\ g_{2*}^{(2)} = 1.28,$$
$$g_{3*}^{(2)} = 1.41,\ g_{4*}^{(2)} = 0.52$$

Step 17: Rank all the alternatives x_i $(i = 1, 2, 3, 4)$ based on their overall preference degrees $g_{i*}^{(2)}$ $(i = 1, 2, 3, 4)$ and then select the optimal one.

$$g_{3*}^{(2)} > g_{2*}^{(2)} > g_{1*}^{(2)} > g_{4*}^{(2)}$$

Thus, the best one is x_3 (OFO).

B. COMPARISON ANALYSIS

To the best of our knowledge, currently, only Zhang and Wu [27] developed a group decision-making model with HMPRs taking consistency and consensus into account. Therefore, we will compare our proposed model with the model which was designed by Zhang and Wu [27] to show the advantages of our model.

We utilize the group decision-making model, which was devised by Zhang and Wu [27] to deal with HMPRs in Example 31. The parameters are set to: $C\bar{I} = 1.01$, $\delta = 0.1$, $\eta = 0.2$, the weight vector of all the DMs, $\lambda = (0.1, 0.5, 0.3, 0.1)$, $GC\bar{I} = 1.05$, which are derived from Example 31 in the work of Zhang and Wu [27]. We try to implement the group decision-making model that was proposed by Zhang and Wu [27] using the Java language on the MyEclipse development software. We first normalize all the HMPRs in Example 31 as:

$$H_1^{(0)} = \begin{Bmatrix} \{1, 1\} & \{3, 5\} & \{1/9, 1/7\} & \{5, 5\} \\ \{1/3, 1/5\} & \{1, 1\} & \{1/5, 1/5\} & \{7, 7\} \\ \{9, 7\} & \{5, 5\} & \{1, 1\} & \{3, 3\} \\ \{1/5, 1/5\} & \{1/7, 1/7\} & \{1/3, 1/3\} & \{1, 1\} \end{Bmatrix},$$

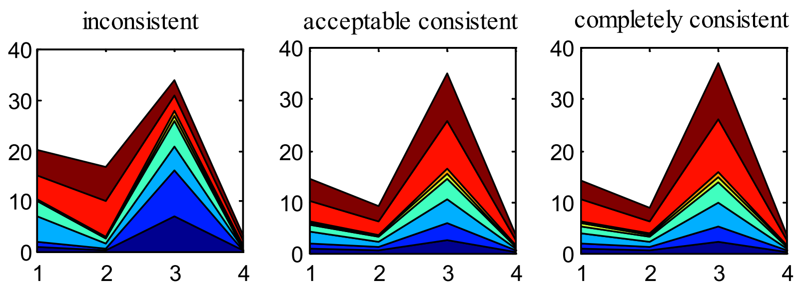


FIGURE 9. Areas of $H_1^{(0)}, H_1^{(1)}, \tilde{H}_1^{(1)}$.

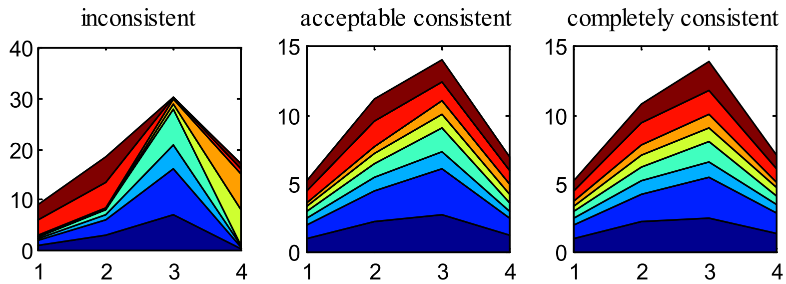


FIGURE 10. Areas of $H_2^{(0)}, H_2^{(1)}, \tilde{H}_2^{(1)}$.

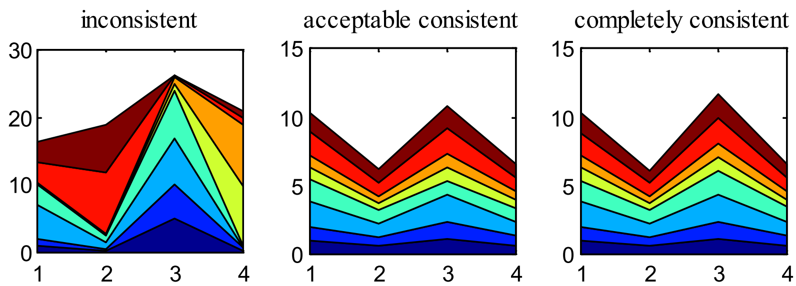


FIGURE 11. Areas of $H_3^{(0)}, H_3^{(2)}, \tilde{H}_3^{(2)}$.

$$H_2^{(0)} = \begin{Bmatrix} \{1,1\} & \{1/3,1/3\} & \{1/9,1/7\} & \{3,3\} \\ \{3,3\} & \{1,1\} & \{1/7,1/5\} & \{5,5\} \\ \{9,7\} & \{7,5\} & \{1,1\} & \{1/7,1/7\} \\ \{1/3,1/3\} & \{1/5,1/5\} & \{7,7\} & \{1,1\} \end{Bmatrix},$$

$$H_3^{(0)} = \begin{Bmatrix} \{1,1\} & \{3,5\} & \{1/5,1/5\} & \{3\} \\ \{1/3,1/5\} & \{1,1\} & \{1/7,1/7\} & \{7,9\} \\ \{5,5\} & \{7,7\} & \{1,1\} & \{1/9,1/9\} \\ \{1/3,1/3\} & \{1/7,1/9\} & \{9,9\} & \{1,1\} \end{Bmatrix},$$

$$H_4^{(0)} = \begin{Bmatrix} \{1,1\} & \{3,3\} & \{1/9,1/9\} & \{5,7\} \\ \{1/3,1/3\} & \{1,1\} & \{1/7,1/5\} & \{7\} \\ \{9,9\} & \{7,5\} & \{1,1\} & \{3\} \\ \{1/5,1/7\} & \{1/7,1/7\} & \{1/3,1/3\} & \{1,1\} \end{Bmatrix}.$$

We can get acceptably consistent HMPRs and complete consistent HMPRs of H_1, H_2, H_4 after only one iteration and the acceptably consistent HMPR and complete consistent HMPR of H_3 after two iterations, which are listed as $H_1^{(1)}$, $H_2^{(1)}$, $H_3^{(2)}$, $H_4^{(1)}$, $\tilde{H}_1^{(1)}$, $\tilde{H}_2^{(1)}$, $\tilde{H}_3^{(2)}$, and $\tilde{H}_4^{(1)}$, as shown at the top of the next page, where the HMPRs $H_1^{(1)}$, $H_2^{(1)}$, $H_4^{(1)}$ as well as $H_3^{(2)}$ are acceptably consistent, and $\tilde{H}_1^{(1)}$, $\tilde{H}_2^{(1)}$, $\tilde{H}_4^{(1)}$ and $\tilde{H}_3^{(2)}$ are complete consistent HMPRs.

We draw "Figure of area" to give a visible description of the inconsistent HMPRs $H_k^{(0)}$ $(k = 1, 2, 3, 4)$, the acceptably consistent HMPRs $H_k^{(1)}$ $(k = 1, 2, 4)$ and $H_3^{(2)}$, and the complete consistent HMPRs $\tilde{H}_k^{(1)}$ $(k = 1, 2, 4)$ and $\tilde{H}_3^{(2)}$. Figs. 9-12 show that the more consistent the HMPR is, the more regular its "Figure of area" performs.

Through running the java code, it can be found that all the acceptably consistent HMPRs can get the acceptable group consensus level after one iteration and their group HMPR is obtained as $G^{(1)}$, as shown at the top of the next page.

Then, the HMG operator and the score function that were proposed in Zhang and Wu [27] are used to compute the sores of all the alternatives x_i $(i = 1, 2, 3, 4)$ as:

$$s(x_1) = 0.8552, \quad s(x_2) = 0.9758,$$
$$s(x_3) = 1.7970, \quad s(x_4) = 0.6669.$$

$$H_1^{(1)} = \begin{Bmatrix} \{1,1\} & \{1.4863, 2.0829\} & \{0.2987, 0.3847\} & \{3.7547, 4.4571\} \\ \{0.6728, 0.4801\} & \{1,1\} & \{0.2379, 0.2244\} & \{2.9161, 2.5995\} \\ \{3.3484, 2.5995\} & \{4.2043, 4.4571\} & \{1,1\} & \{9.5897, 9.0625\} \\ \{0.2663, 0.2244\} & \{0.3430, 0.3847\} & \{0.1043, 0.1103\} & \{1,1\} \end{Bmatrix}$$

$$H_2^{(1)} = \begin{Bmatrix} \{1,1\} & \{0.4604, 0.4516\} & \{0.2986, 0.3699\} & \{0.8081, 0.8551\} \\ \{2.1721, 2.2142\} & \{1,1\} & \{0.5960, 0.7589\} & \{1.6552, 1.7853\} \\ \{3.3484, 2.7035\} & \{1.6778, 1.3177\} & \{1,1\} & \{1.6020, 1.4036\} \\ \{1.2374, 1.1694\} & \{0.6042, 0.5601\} & \{0.6242, 0.7125\} & \{1,1\} \end{Bmatrix}$$

$$H_3^{(2)} = \begin{Bmatrix} \{1,1\} & \{1.5348, 1.8666\} & \{0.8132, 0.9228\} & \{1.4422, 1.7416\} \\ \{0.6516, 0.5357\} & \{1,1\} & \{0.5339, 0.5007\} & \{0.9582, 0.9586\} \\ \{1.2297, 1.0836\} & \{1.8729, 1.9972\} & \{1,1\} & \{1.6885, 1.7969\} \\ \{0.6934, 0.5742\} & \{1.0437, 1.0432\} & \{0.5922, 0.5565\} & \{1,1\} \end{Bmatrix}$$

$$H_4^{(1)} = \begin{Bmatrix} \{1,1\} & \{1.6032, 1.6032\} & \{0.2769, 0.3221\} & \{3.7547, 4.5180\} \\ \{0.6238, 0.6238\} & \{1,1\} & \{0.1977, 0.2379\} & \{2.7035, 3.1454\} \\ \{3.6117, 3.1042\} & \{5.0590, 4.2043\} & \{1,1\} & \{10.3439, 10.3439\} \\ \{0.2663, 0.2213\} & \{0.3699, 0.3179\} & \{0.0967, 0.0967\} & \{1,1\} \end{Bmatrix}$$

$$\tilde{H}_1^{(1)} = \begin{Bmatrix} \{1,1\} & \{1.3747, 1.8898\} & \{0.3333, 0.4295\} & \{3.6371, 4.4006\} \\ \{0.7274, 0.5292\} & \{1,1\} & \{0.2425, 0.2272\} & \{2.6458, 2.3286\} \\ \{3, 2.3286\} & \{4.1241, 4.4006\} & \{1,1\} & \{10.9114, 10.2470\} \\ \{0.2749, 0.2272\} & \{0.3780, 0.4295\} & \{0.0916, 0.0976\} & \{1,1\} \end{Bmatrix}$$

$$\tilde{H}_2^{(1)} = \begin{Bmatrix} \{1,1\} & \{0.4772, 0.4671\} & \{0.3333, 0.4111\} & \{0.6985, 0.7438\} \\ \{2.0956, 2.1407\} & \{1,1\} & \{0.6985, 0.8801\} & \{1.4639, 1.5923\} \\ \{3.0, 2.4323\} & \{1.4316, 1.1362\} & \{1,1\} & \{2.0956, 1.8092\} \\ \{1.4316, 1.3444\} & \{0.6831, 0.6280\} & \{0.4772, 0.5527\} & \{1,1\} \end{Bmatrix}$$

$$\tilde{H}_3^{(2)} = \begin{Bmatrix} \{1,1\} & \{1.5244, 1.8481\} & \{0.8248, 0.9372\} & \{1.4316, 1.7321\} \\ \{0.6560, 0.5411\} & \{1,1\} & \{0.5411, 0.5071\} & \{0.9391, 0.9372\} \\ \{1.2124, 1.0670\} & \{1.8481, 1.9720\} & \{1,1\} & \{1.7356, 1.8481\} \\ \{0.6985, 0.5774\} & \{1.0648, 1.0670\} & \{0.5762, 0.5411\} & \{1,1\} \end{Bmatrix}$$

$$\tilde{H}_4^{(1)} = \begin{Bmatrix} \{1,1\} & \{1.4953, 1.4953\} & \{0.3064, 0.3626\} & \{3.6371, 4.3035\} \\ \{0.6687, 0.6687\} & \{1,1\} & \{0.2049, 0.2425\} & \{2.4323, 2.8779\} \\ \{3.2633, 2.7580\} & \{4.8797, 4.1241\} & \{1,1\} & \{11.8690, 11.8690\} \\ \{0.2749, 0.2324\} & \{0.4111, 0.3475\} & \{0.0842, 0.0842\} & \{1,1\} \end{Bmatrix}$$

$$G^{(1)} = \begin{bmatrix} \{1,1\} & \{0.8416, 0.9142\} & \{0.4003, 0.4818\} & \{1.3072, 1.4747\} \\ \{1.1883, 1.0938\} & \{1,1\} & \{0.4711, 0.5281\} & \{1.5615, 1.6278\} \\ \{2.4981, 2.0754\} & \{2.1228, 1.8937\} & \{1,1\} & \{2.3455, 2.2242\} \\ \{0.7649, 0.6781\} & \{0.6405, 0.6143\} & \{0.4264, 0.4496\} & \{1,1\} \end{bmatrix}$$

Finally, we rank all of the alternatives $x_i\ (i = 1, 2, 3, 4)$ according to their scores as:

$$s(x_3) > s(x_2) > s(x_1) > s(x_4)$$

Thus, the optimal one is x_3 (OFO), which shows that our proposed group decision-making model can obtain the same result as the group decision-making model that was devised by Zhang and Wu [27]. It can validate the effectiveness of our proposed group decision-making model.

In the following part, we will compare the proposed group decision-making model with that is presented in Zhang and Wu [27] and describe the differences between them:

(1) The group decision-making model proposed by Zhang *et al.* normalizes all the HMPRs so that all of the HMEs have the same number of elements, while our proposed decision-making model introduces the regression method based on the complete consistency to reduce each HMPR into the MPR, which can decrease the computation complexity efficiently.

(2) The threshold of the consistency index in Zhang and Wu [27] is determined by the experiences of the decision makers, which is lack of the theoretical basis to support it. While we develop a new threshold estimation method to calculate the consistency threshold under different confidence levels using the probability theory.

(3) From the above comparison analysis, we can find that our proposed group decision-making model can reduce the number of iterations compared with that of Zhang *et al.*

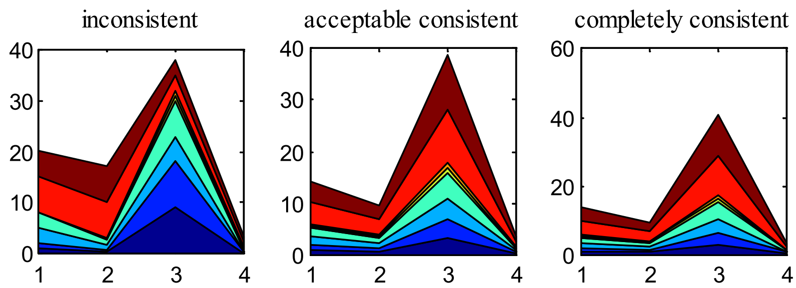


FIGURE 12. Areas of $H_4^{(0)}, H_4^{(1)}, \tilde{H}_4^{(1)}$.

(4) The group decision-making model proposed by Zhang *et al*. modifies the consistency and the consensus of HMPRs without the involvements of the DMs, while our proposed decision-making model exploits the feedback mechanism to modify the consistency and the consensus, which can avoid the loss of information and then prevent from resulting in the incorrect decision-making results.

VIII. CONCLUSIONS

In this paper, we have developed an efficient group decision-making model to solve HMPRs considering the consistency and consensus. To reduce the computations in the decision making process, we have introduced the concept of complete consistency and error analysis method to design a regression method to reduce HMPRs into MPRs. Then, based on the logarithmic distance, we have defined the consistency index to measure the consistency level of an inconsistent reduced MPR, developed a threshold estimation method to estimate the consistency threshold for the reduced MPRs using the probability theory, and proposed a feedback mechanism to put forward a consistency checking and revising algorithm to modify the reduced MPRs until they satisfy the acceptable consistency.

At the same time, we have defined the group consensus index of HMPRs and then exploited a feedback mechanism to design a consensus reaching algorithm to check whether the group consensus index of each reduced MPR is enough and revise the reduced MPRs with the involvement of the DMs.

Finally, we combined the regression method, consistency checking and revising method, and the consensus reaching process to develop a complete group decision-making model with HMPRs and provided a practical example concerning the investment of shared bikes to demonstrate our proposed group decision-making model. At the same time, we have used the group decision-making model developed by Zhang and Wu [27] to solve HMPRs in Example 31 and performed a detailed comparison analysis between our model and the one of Zhang and Wu [27], which validates the effectiveness of our model and its advantages over the Zhang *et al*.'s model.

In the future, we will intend to use the local outlier factor algorithm and kernel density estimation function to develop a novel method, which can accurately estimate the thresholds of consistency index and group consensus index instead of being determined by experiences of the DMs.

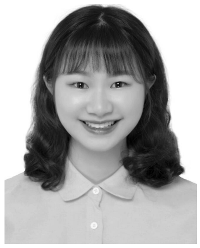
REFERENCES

[1] S. Danjuma, T. Herawan, M. A. Ismail, H. Chiroma, A. I. Abubakar, and A. M. Zeki, "A review on soft set-based parameter reduction and decision making," *IEEE Access*, vol. 5, pp. 4671–4689, 2017.
[2] C.-L. Fan, Y. Song, Q. Fu, L. Lei, and X. Wang, "New operators for aggregating intuitionistic fuzzy information with their application in decision making," *IEEE Access*, vol. 6, pp. 27214–27238, 2018.
[3] D. Liang, Y. Zhang, Z. Xu, and A. P. Darko, "Pythagorean fuzzy Bonferroni mean aggregation operator and its accelerative calculating algorithm with the multithreading," *Int. J. Intell. Syst.*, vol. 33, no. 3, pp. 615–633, 2018.
[4] F. Meng and X. Chen, "An approach to incomplete multiplicative preference relations and its application in group decision making," *Inf. Sci.*, vol. 309, pp. 119–137, Jul. 2015.
[5] S.-M. Chen and S.-J. Niou, "Fuzzy multiple attributes group decision-making based on fuzzy preference relations," *Expert Syst. Appl.*, vol. 38, no. 4, pp. 3865–3872, 2011.
[6] P. K. Kwok and H. Y. K. Lau, "A modified consensus-building methodology for reaching a group decision using minimum costs," *IEEE Access*, vol. 6, pp. 3509–3523, 2017.
[7] T. L. Saaty, *The Analytical Hierarchy Process*. New York, NY, USA: McGraw-Hill, 1980.
[8] T. L. Saaty, "A scaling method for priorities in hierarchical structures," *J. Math. Psychol.*, vol. 15, no. 3, pp. 234–281, 1977.
[9] T. L. Saaty and L. G. Vargas, "Uncertainty and rank order in the analytic hierarchy process," *Eur. J. Oper. Res.*, vol. 32, no. 1, pp. 107–117, 1987.
[10] F. Chiclana, E. Herrera-Viedma, F. Herrera, and S. Alonso, "Induced ordered weighted geometric operators and their use in the aggregation of multiplicative preference relations," *Int. J. Intell. Syst.*, vol. 19, no. 3, pp. 233–255, 2004.
[11] H. Zhang, "Group decision making based on incomplete multiplicative and fuzzy preference relations," *Appl. Soft Comput.*, vol. 48, pp. 735–744, Nov. 2016.
[12] M. Xia, Z. Xu, and H. Liao, "Preference relations based on intuitionistic multiplicative information," *IEEE Trans. Fuzzy Syst.*, vol. 21, no. 1, pp. 113–133, Feb. 2013.
[13] V. Torra, "Hesitant fuzzy sets," *Int. J. Intell. Syst.*, vol. 25, no. 6, pp. 529–539, 2010.
[14] M. Xia and Z. Xu, "Managing hesitant information in GDM problems under fuzzy and multiplicative preference relations," *Int. J. Uncertainty, Fuzziness Knowl.-Based Syst.*, vol. 21, no. 6, pp. 865–897, 2013.
[15] S.-M. Chen, S.-H. Cheng, and T.-E. Lin, "Group decision making systems using group recommendations based on interval fuzzy preference relations and consistency matrices," *Inf. Sci.*, vol. 298, pp. 555–567, Mar. 2015.
[16] J. Aguaron and J. M. Moreno-Jimenez, "The geometric consistency index: Approximated thresholds," *Eur. J. Oper. Res.*, vol. 147, no. 1, pp. 137–145, 2003.
[17] H. Gao, G. W. Wei, and Y. H. Huang, "Dual hesitant bipolar fuzzy Hamacher prioritized aggregation operators in multiple attribute decision making," *IEEE Access*, vol. 6, pp. 11508–11522, 2018.
[18] Y. Xu and H. Wang, "Eigenvector method, consistency test and inconsistency repairing for an incomplete fuzzy preference relation," *Appl. Math. Model.*, vol. 37, no. 7, pp. 5171–5183, 2013.

[19] Z. Xu and M. Xia, "Iterative algorithms for improving consistency of intuitionistic preference relations," *J. Oper. Res. Soc.*, vol. 65, no. 5, pp. 708–722, 2014.
[20] J. Tang and F. Meng, "A consistency-based method to decision making with triangular fuzzy multiplicative preference relations," *Int. J. Fuzzy Syst.*, vol. 19, no. 5, pp. 1317–1332, 2017.
[21] Y. He and Z. S. Xu, "Error analysis methods for group decision making based on hesitant fuzzy preference relation," *Int. J. Intell. Syst.*, vol. 31, no. 11, pp. 1104–1128, 2016.
[22] T. González-Arteaga, R. de Andrés Calle, and F. Chiclana, "A new measure of consensus with reciprocal preference relations: The correlation consensus degree," *Knowl. Based Syst.*, vol. 107, pp. 104–116, Sep. 2016.
[23] Y. Zhang, Z. Xu, and H. Liao, "A consensus process for group decision making with probabilistic linguistic preference relations," *Inf. Sci.*, vol. 414, pp. 260–275, Nov. 2017.
[24] Z. Wu and J. Xu, "A consensus model for large-scale group decision making with hesitant fuzzy information and changeable clusters," *Inf. Fusion*, vol. 41, pp. 217–231, May 2018.
[25] E. Herrera-Viedma, S. Alonso, F. Chiclana, and F. Herrera, "A consensus model for group decision making with incomplete fuzzy preference relations," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 5, pp. 863–877, Oct. 2007.
[26] Z. Wu and J. Xu, "A consistency and consensus based decision support model for group decision making with multiplicative preference relations," *Decis. Support Syst.*, vol. 52, pp. 757–767, Feb. 2012.
[27] Z. Zhang and C. Wu, "A decision support model for group decision making with hesitant multiplicative preference relations," *Inf. Sci.*, vol. 282, pp. 136–166, Oct. 2014.
[28] Z. Zhang, "Some hesitant multiplicative aggregation operators and their application in group decision making with hesitant multiplicative preference relations," *Int. J. Fuzzy Syst.*, vol. 18, no. 2, pp. 177–197, 2016.
[29] B. Zhu and Z. Xu, "Regression methods for hesitant fuzzy preference relations," *Technol. Econ. Develop. Economy*, vol. 19, pp. S214–S227, Dec. 2013.
[30] M. Lin, Z. Xu, Y. Zhai, and Z. Yao, "Multi-attribute group decision-making under probabilistic uncertain linguistic environment," *J. Oper. Res. Soc.*, vol. 69, no. 2, pp. 157–170, 2018.
[31] H. Liu, Z. Xu, and H. Liao, "The multiplicative consistency index of hesitant fuzzy preference relation," *IEEE Trans. Fuzzy Syst.*, vol. 24, no. 1, pp. 82–93, Feb. 2016.
[32] L. C. Andrews, *Special Functions of Mathematics for Engineers*. New York, NY, USA: McGraw-Hill, 1997.
[33] H. Garg, "A new generalized pythagorean fuzzy information aggregation using Einstein operations and its application to decision making," *Int. J. Intell. Syst.*, vol. 31, no. 9, pp. 886–920, 2016.
[34] E. Szmidt and J. Kacprzyk, "A consensus-reaching process under intuitionistic fuzzy preference relations," *Int. J. Intell. Syst.*, vol. 18, no. 7, pp. 837–852, 2003.
[35] N. Rehman, A. Ali, M. I. Ali, and C. Park, "SDMGRS: Soft dominance based multi granulation rough sets and their applications in conflict analysis problems," *IEEE Access*, vol. 6, pp. 31399–31416, 2018.
[36] M. J. del Moral, F. Chiclana, J. M. Tapia, and E. Herrera-Viedma, "A comparative study on consensus measures in group decision making," *Int. J. Intell. Syst.*, vol. 33, no. 8, pp. 1624–1638, 2018.

MINGWEI LIN received the B.S. degree in software engineering and the Ph.D. degree in computer science and technology from Chongqing University, Chongqing, China, in 2009 and 2014, respectively.

From 2015 to 2016, he was a Lecturer with the Faculty of Software, Fujian Normal University, Fuzhou, China, where he is currently an Associate Professor. He has published over 20 research papers as the first author in international journals and conference proceedings. His research interests include decision making and preference relations. He was a recipient of the CSC-IBM Chinese Excellent Student Scholarship in 2012.



QIANSHAN ZHAN is currently pursuing the B.S. degree in software engineering. Her research interests include decision making and preference relations.

ZESHUI XU (M'08–SM'09) received the Ph.D. degree in management science and engineering from Southeast University, Nanjing, China, in 2003. From 2003 to 2005, he was a Post-Doctoral Researcher with the School of Economics and Management, Southeast University. From 2005 to 2007, he was a Post-Doctoral Researcher with the School of Economics and Management, Tsinghua University, Beijing, China. He is currently a Professor with the Business School, Sichuan University, Chengdu, and also with the College of Sciences, PLA University of Science and Technology, Nanjing. His current research interests include information fusion, group decision making, computing with words, and aggregation operators. He is a Distinguished Young Scholar of the National Natural Science Foundation of China and the Chang Jiang Scholars of the Ministry of Education of China. He is a member of the Advisory Boards of the Knowledge-Based Systems and Granular Computing, and also a member of Editorial Boards of over 40 professional journals. He has contributed over 650 journal articles to professional journals. He has been selected as the Thomson Reuters Highly Cited Researcher (in the fields of computer science) (2014–2017) and Engineering [(2014) and (2016–2017)], respectively, included in The World's Most Influential Scientific Minds 2014–2017, and also the Most Cited Chinese Researchers (Ranked first in Computer Science, 2014-2016, Released by Elsevier). His h-index is 107. He has authored the books *Uncertain Multiple Attribute Decision Making: Methods and Applications* (Springer, 2015), *Intuitionistic Fuzzy Information Aggregation: Theory and Applications* (Science Press and Springer, 2012), *Linguistic Decision Making: Theory and Methods* (Science Press and Springer, 2012), *Intuitionistic Fuzzy Preference Modeling and Interactive Decision Making* (Springer, 2013), *Intuitionistic Fuzzy Aggregation and Clustering* (Springer, 2013), *Hesitant Fuzzy Sets Theory* (Springer, 2014), *Hesitant Fuzzy Methods for Multiple Criteria Decision Analysis* (Springer, 2016), *Hesitant Fuzzy Decision Making Methodologies and Applications* (Springer, 2017), and *Intuitionistic Fuzzy Calculus* (Springer, 2017). He is currently an Associate Editor of the IEEE TRANSACTIONS ON FUZZY SYSTEMS, INFORMATION SCIENCES, the *International Journal of Fuzzy Systems*, the *International Journal Machine Leaning and Cybernetics*, and *Fuzzy Optimization and Decision Making*.

RIQING CHEN received the B.Eng. degree in communication engineering from Tongji University, Shanghai, China, in 2001, the M.Sc. degree in communications and signal processing from Imperial College London, U.K., in 2004, and the Ph.D. degree in engineering science from the University of Oxford, U.K., in 2010. Since 2014, he has been a Full Professor with the Faculty of Computer and Information Sciences, Fujian Agriculture and Forestry University, Fuzhou, China. His research interests include big data and visualization, cloud computing, consumer electronics, flash memory, and wireless sensor networking.