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Integrated Virtual Laboratory in Engineering Mathematics Education: Fourier Theory

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ABSTRACT In this paper, we present a virtual learning laboratory environment for undergraduate mathematics education using an inquiry-based learning approach. The *Visible Thinking* pedagogical framework is also suggested to achieve a good complement to traditional lecture–tutorial systems. The virtual laboratory is implemented in an open-access *Java* interactive software. We demonstrate a viable instruction procedure, providing a set of virtual laboratory activities with real-world applications spanning signal processing, data science and analytics, sustainable infrastructure engineering, and theoretical physics. A preliminary study on a pilot cohort indicates that the proposed virtual laboratory can enhance students' learning. The virtual laboratory implementation is scalable and can be easily expanded in scope to other mathematical topics; transitioning to a tablet-based system for use in smart classrooms is also readily achieved. The *Java* interactive software is freely available on Open Science Framework.

INDEX TERMS Virtual laboratory, engineering mathematics, Fourier theory, interdisciplinary, education, data science, smart classroom.

I. INTRODUCTION

The lecture-and-tutorial mode of lesson delivery has its strengths as a form of didactic teaching. Its iterative nature between rapid content delivery in lectures and rigorous practice in tutorials with immediate lecturer feedback [1] allows for the elimination of misconceptions and the quick mastery of taught content. However, it has long been criticized for its tendency in turning students into passive learners, who acquire inert knowledge and are unable to apply them to solve real-world problems. Recent literature also suggests that current pedagogical methods do not fully equip students with the means of tackling non-routine problems, especially in mathematics [2], [3]. In these lecture-tutorial environments, students first learn new content during lectures and then attempt question sets in subsequent tutorials; there is little opportunity for students' discussion and experiential learning in this process [4]. It is also difficult to incorporate authentic learning experiences involving real-world applications. Notably, most modern applications involve computational and technological components that cannot be replicated in a pen-and-paper environment; furthermore, real-world problem solving is oftentimes complex, with group-work and access to interdisciplinary resources being more realistic provisions for students.

In other disciplines, such as the physical and biomedical sciences, lecture-tutorial courses are typically carried out in parallel with laboratory work to enable the application of learnt knowledge [5]–[8]. In engineering and computer science, hands-on project work is also common and has indeed been shown to be beneficial to students' learning outcomes [9]–[11]. In view of the general passivity in mathematics classrooms, there is a pressing need to explore alternative teaching approaches that would promote active learning, engage students in cognitive interaction, and nurture self-directed and independent learners [12]. Active learning involves higher-order thinking and metacognition by students [13]; opportunities must also be provided for students to meaningfully discuss, write, read and reflect on the content, ideas, and concerns of an academic subject [14], arguably lacking in our local context. The application of knowledge in realistic scenarios rather than word problem sums is a good avenue for introducing sophisticated problem-solving strategies and higher-order thinking, hence presenting a way of mitigating such shortcomings [15]–[17]. It is also paramount that students are given the opportunities to appreciate the real-world connections of the pitched content [17]–[20].

In this paper, we propose a virtual laboratory environment for mathematics education, as a way of providing a hands-on avenue for exposure into the real-world applications of learnt concepts to bring about active learning. The utilization of virtual laboratories, in which students are given freedom to explore and experiment, have already been suggested for chemistry [21], [22], physics [23]–[26] and engineering [27], [28]; the idea, however, has not been fully realized in mathematics education, with limited existing studies on interactive learning software [29], [30]. Our proposed virtual laboratory is implemented in the form of an interactive *Java* applet, which is by design platform-independent and widely portable; students are hence able to readily access such resources on their personal computers. We apply the virtual laboratory concept to the topics of Fourier series and Fourier transforms; in principle, the pedagogical tool of virtual laboratories is relevant to a wide range of mathematical disciplines, including geometry, multivariable and vector calculus, and differential equations, in which interactive visualization and free experimentation are greatly helpful for learning. Tackling real-world problem scenarios through the virtual laboratory also builds fundamental broad-ranging data science and data analysis skill-sets.

We present a theoretical overview of the selected topics in Section II, alongside recommendations for educators in conducting preparation for the virtual laboratory activities. The *Java* virtual laboratory application is then detailed in Section III. It is essential that the planned virtual laboratory work be integrated well into the course curriculum [31], [32]—to this end, we propose leveraging on the connect–extend–challenge *Making Thinking Visible* pedagogical routine framework (Section IV-A). Numerous application-based problem sets can be integrated with the interactive software, spanning related disciplines including signal processing, mechanical engineering, and theoretical physics (Section IV-B). A preliminary study on a class of second-year undergraduate students has also been carried out (Section V), with survey results indicating that the virtual laboratory activities are well-received and indeed achieve the objective of enhancing learning through experiential, authentic real-world applications.

II. THEORETICAL REVIEW

An overview of key concepts in Fourier series and Fourier transforms is first presented, alongside a curriculum progression that educators may adopt for a smooth review of the mathematical content to students. It is typical to teach these topics in a mathematically rigorous fashion in a lecture-tutorial setting; we are not proposing that this be replaced. We suggest instead that the review presented be utilized to ensure a strong foundation *after* adequate curriculum time, as preparation for virtual laboratory activities. This has been identified as an important implementation aspect of the virtual laboratory framework, and will be discussed further in Sections V and VI.

A. FOURIER SERIES

We begin from the concept of Fourier series; a natural extension then serves to introduce Fourier transforms within the established scaffolding. A Fourier series is a decomposition of a periodic function into a summation of sinusoidal constituents—specifically, the Fourier series expansion of a function $f(t)$ integrable on $[t_0, t_0 + T]$ and periodic with period T can be written as:

$$\begin{aligned} f(t) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \\ &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} A_n \sin(n\omega_0 t + \phi_n), \end{aligned} \tag{1}$$

where $\omega_0 = 2\pi/T$ is the fundamental frequency, and $A_n = (a_n^2 + b_n^2)^{1/2}$ and $\phi = \tan^{-1}(b_n/a_n)$. If a given waveform is defined only on the domain $[t_0, t_0 + T]$, it can be made periodic with period T by repeating it end-on-end. We call $a_0/2$ the average term, and $A_n \sin(n\omega_0\, t + \phi_n)$ the n^{th} harmonic. The amplitudes a_n and b_n are referred to as the Fourier coefficients.

A natural question then arises—how can these Fourier coefficients be computed? An important concept is that of orthogonal functions. For instance, the trigonometric functions $\sin(m\omega_0\, t)$ and $\sin(n\omega_0\, t)$ are orthogonal when $m \neq n$. The following identities hold for $m \neq 0$ and $n \neq 0$:

$$\begin{aligned} \int_{t_0}^{t_0+T} \sin(m\omega_0 t)\sin(n\omega_0 t)\, dt &= T\delta_{mn}/2 \\ \int_{t_0}^{t_0+T} \cos(m\omega_0 t)\cos(n\omega_0 t)\, dt &= T\delta_{mn}/2 \\ \int_{t_0}^{t_0+T} \sin(m\omega_0 t)\cos(n\omega_0 t)\, dt &= 0 \\ \int_{t_0}^{t_0+T} \sin(m\omega_0 t)\, dt &= 0 \\ \int_{t_0}^{t_0+T} \cos(m\omega_0 t)\, dt &= 0 \end{aligned} \tag{2}$$

A sieve can therefore be constructed from these orthogonality relations, such that the amplitude of a particular frequency component can be picked out. This leads us to the following method of computing the amplitude coefficients:

$$\begin{aligned} a_0 &= \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\, dt \\ a_n &= \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos(n\omega_0 t)\, dt \quad (n \geq 1) \\ b_n &= \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin(n\omega_0 t)\, dt \quad (n \geq 1) \end{aligned} \tag{3}$$

It can be observed all a_n terms will vanish for an odd function, and all b_n terms will vanish for an even function. Also notable are half-wave symmetries and quarter-wave symmetries, which also yield simplifications in the Fourier series expansions; these will be discussed in Section IV-B under the context of an inquiry-based learning example. It is recommended that students be exposed to symmetries within the Fourier scaffolding, to both reduce menial arithmetic and as a prelude for more advanced topics.

B. Fourier Transform

The complex exponential relations $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$ and $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$ immediately allow Eq. (1) to be expressed in the more compact form

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}, \tag{4}$$

$$c_n = \begin{cases} (a_n - ib_n)/2 & \text{for } n > 0 \\ a_0/2 & \text{for } n = 0 \\ (a_{||n||} + ib_{||n||})/2 & \text{for } n < 0. \end{cases} \tag{5}$$

The bounds of summation in Eq. (4) has been extended relative to the single-sided summation in Eq. (1) for $n \geq 0$—this is a natural consequence of the complex exponential trigonometric relations used in the derivation. At this point, a generalization into non-periodic functions can be considered. We define $\hat{f}(\xi)$ such that $\hat{f}(\xi) = c_n$ whenever $2\pi\xi = n\omega_0$. This allows us to compute $\hat{f}(\xi)$ in an analogous fashion as that presented in Eq. (3), thus yielding the following computation for $\hat{f}(\xi)$:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i\xi t}\, dt. \tag{6}$$

The above is known as the Fourier transform. If t is taken to represent time, then $f(t)$ is known as the time-domain representation, and $\hat{f}(t)$ is known as the frequency-domain representation. In general, a periodic function will contain only a discrete set of frequencies, hence the summation in Eq. (1) and (4); but a non-periodic function will contain a continuous range of frequency components. The inverse Fourier transform can also be written:

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i\xi t}\, d\xi. \tag{7}$$

There are a number of basic properties of Fourier transforms. Firstly, the Fourier transform is linear, in that if $h(x) = af(x) + bg(x)$, then $\hat{h}(\xi) = a\hat{f}(\xi) + b\hat{g}(\xi)$. Next, we have the time-translation property, in that if $h(x) = f(x - x_0)$, then $\hat{h}(\xi) = \exp[-2\pi i x_0 \xi]\hat{f}(\xi)$. Lastly, time-scaling also admits a simple formula, in that if $h(x) = f(x/a)$, then $\hat{h}(\xi) = |a|\hat{f}(a\xi)$.

As a prelude to the virtual laboratory activities (Section IV-B), it is suggested that lecturers point out the wide-ranging usefulness of Fourier transformations, for instance in electrical and electronic engineering, signal processing [33], theoretical physics, and data analytics. Observant students may also notice that the discussed mathematical framework does not carry well for discrete data, such as those collected via digital instruments. Educators may therefore wish to introduce the discrete Fourier transform (DFT) [34], where the integrals in Eq. (6) and Eq. (7) are replaced by analogous discrete sums. There exists a class of algorithms known as the fast Fourier transform (FFT) [35]–[37] that offers efficient computational transform performance. Students with computing science background may find details on the algorithm and computational considerations involved interesting; otherwise a brief overview will suffice. In courses covering Laplace transforms, connections may also be made to the Laplace transform on their mathematical similarities and areas of application.

III. JAVA INTERACTIVE APPLET

Our virtual laboratory environment is implemented in *Java* for platform independence, with the user interface built on the *JavaFX* platform. In principle, this allows educators to modify and extend the application via drag-and-drop development environments, such as *JavaFX Scene Builder* or existing toolkits [38], with minimal code. A tab-based interface is presented upon start-up, with the current implementation offering three distinct tabs—*Fourier series*, *Fourier transforms*, and *Fourier optics*—such a design allows easy expansion to cover a wider range of content. Screenshots of the interactive application are presented in Figure 1.

The *Fourier series* tab (Figure 1a) enables students to explore the Fourier series expansion of simple periodic functions, including square, triangle, and sawtooth waveforms. A real-time comparison graph displaying the Fourier series partial sum and the selected function is shown; a residual graph is also presented to highlight differences between the partial sum and the function. A *control panel* is provided, on which students may select or deselect harmonics to be included in the partial sum, and change their frequencies and amplitudes at will. A *shortcut panel* is also provided, to allow convenient selection of combinations of harmonics, for instance, summing up to the 7th or 21st harmonic. This tab can be used in activities exploring the mathematical fundamentals of Fourier series (Section IV-B.1).

The *Fourier transform* tab (Figures 1b and 1c) implements a fast Fourier transform algorithm. The user may import Waveform Audio (*.wav*) files into the software for analysis. The imported data is displayed in a graph for visualization; the discrete Fourier transform may then be run, whose results are presented in a zoomable and pannable graph. Exact coordinates of features of interest, for instance, peaks and valleys, on the graph may be obtained for analysis via cursor selection. This tab can be used in activities exploring data analysis and signal processing applications (Sections IV-B.2 and IV-B.3).

Lastly, the *Fourier optics* tab (Figure 1d) explores the application of Fourier transforms to the physical phenomenon of wave diffraction and interference, specifically demonstrating the *n*-slit interference pattern. Users may add additional slits at will, and independently modify the width, position, and illumination of each slit. The configuration of the slits, the associated aperture function, and the predicted interference pattern are displayed in real-time. This tab is used in activities exploring the applications of Fourier transforms in optical physics (Section IV-B.4).

It is notable that the *Java* interactive application presented here does not feature a three-dimensional virtual environment, unlike existing virtual laboratory environments for the physical sciences and engineering [21]–[28]; rather, the user interface is configured in a minimalistic flat layout.

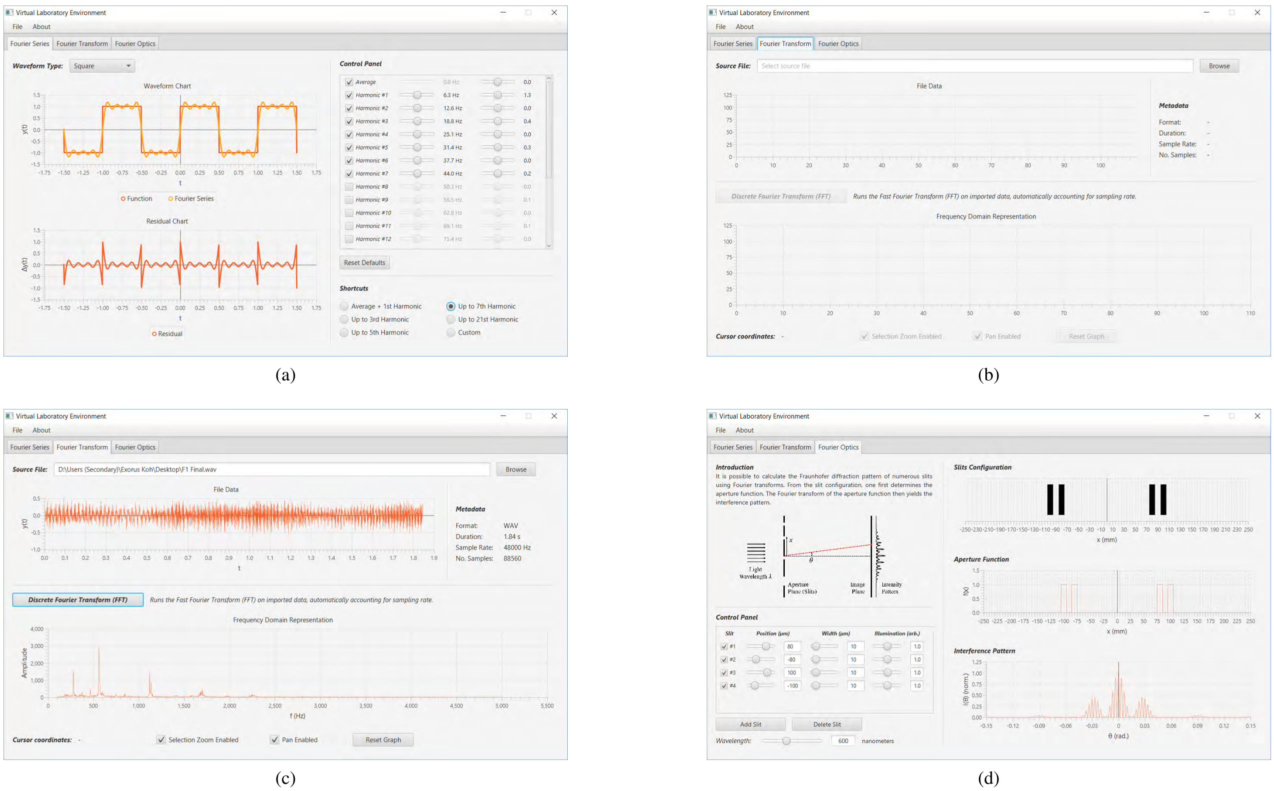


FIGURE 1. Screenshots of the interactive *Java* virtual laboratory application. The current implementation includes (a) a *Fourier series* tab to reinforce fundamental mathematical concepts of Fourier series expansions; a *Fourier transform* tab supporting the exploration of real-world applications, in (b) its initial state and (c) when a file has been imported; and (d) a *Fourier optics* tab that provides interactive visualization of multiple-aperture diffraction calculations.

This is intentional, due to the fundamental differences between mathematics and science education. In the physical sciences, the manipulation of lab equipment, techniques in handling materials, and the safe conduct of experiments are all critical facets to be learnt; but in the current context the learning emphasis is on the application of abstract mathematical concepts to real-world problems. A minimalistic user interface is clearer and more user-friendly for such objectives.

IV. INSTRUCTIONAL PROCEDURE

In this section, we present an overview of virtual laboratory course activities exploiting the interactivity of the applet for enhanced education outcomes. The *Visible Thinking* pedagogical framework is first detailed (Section IV-A), followed by a series of inquiry-based virtual laboratory activities, spanning areas of fundamental mathematics (Section IV-B.1), signal processing (Section IV-B.2), building engineering (Section IV-B.3), and theoretical physics (Section IV-B.4), thereby allowing educators to conveniently adopt the proposed framework for students of diverse backgrounds and fields.

A. VISIBLE THINKING FRAMEWORK

There are two primary challenges in the implementation of virtual laboratory sessions within mathematics. Firstly, because the virtual laboratory differs significantly from typical pen-and-paper coursework in required depth of thinking and interdisciplinary breadth, the activities have to be integrated well in the course structure, such that students are not overwhelmed. Secondly, connections to prior knowledge and real-world applications must be clearly communicated, so that students do not perceive the virtual laboratory activities as purposeless. We adapt from the *Visible Thinking* pedagogical paradigm [39]–[43] in addressing these challenges, in particular the *connect–extend–challenge* framework.

Current methods of formal schooling have been argued to hide key cognitive processes under a veil of invisibility, to both the educator and the student [39], [40], [44]. Efficiency in conveying factual information is maintained at an expense of emphasis on the reasoning and strategies employed in knowledge acquisition and problem-solving; where they are addressed, exam-oriented formulaic methods are often scaffolded, thus promoting low-level mechanistic skills in place of higher-order thinking. As a result, students rely on the recognition of standard patterns of problem presentation to find solutions [45]. Moreover, students may not be able to learn effectively from available resources, in particular expositions produced by educators and experts, for they lack a cognitive model in understanding the rationales and reasoning involved in the production of the work [46].

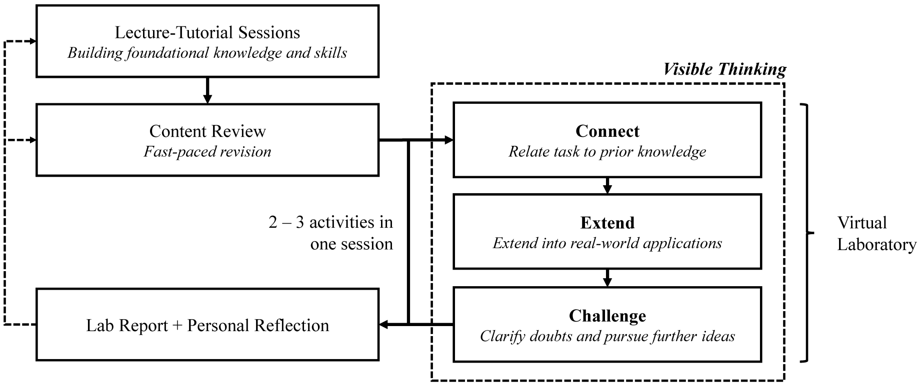


FIGURE 2. **Flowchart showing the process of virtual laboratory activities, with the modified connect–extend–challenge *Visible Thinking* framework integrated.**

The *Visible Thinking* paradigm advocates for educators to communicate explicitly the key cognitive processes in learning and problem-solving, such that students are able to construct cognitive models suitable for the discipline.

The *connect–extend–challenge* framework is a routine designed to provide accessible starting grounds for students in tackling new problems, and to promote higher-order cognition and further learning once the student has gained confidence. The *connect* stage relates the presented information to existing knowledge that the student possesses, thereby serving to initiate the learning cycle. We next adapt the *extend* and *challenge* phases into our mathematics education context. The *extend* phase tasks students with the application of their knowledge, in the process elucidating new ideas and concepts; it is found that contextualizing this phase with real-life applications allows for a more robust discussion. Lastly, in the *challenge* phase, students clarify their remaining doubts and explore any questions that they may have in mind. Based on our pilot study elsewhere, it has been noticed that students often do not actively pursue the *challenge* phase; therefore prompt questions have been provided for scaffolding. Figure 2 presents the modified *Visible Thinking* pedagogical paradigm, integrated into the virtual laboratory coursework to promote active learning.

Utilization of the *connect–extend–challenge* routine in the entire course will acclimatize students to its structure and yield maximal effect. It has been established in existing literature [39] that it is crucial in *Visible Thinking* to communicate the processes of the task to students, situate the tasks in authentic contexts, and vary the diversity of the contexts; the virtual laboratory environment is inherently suited in all three aspects.

B. Inquiry-Driven Learning Examples

1) FOURIER SERIES

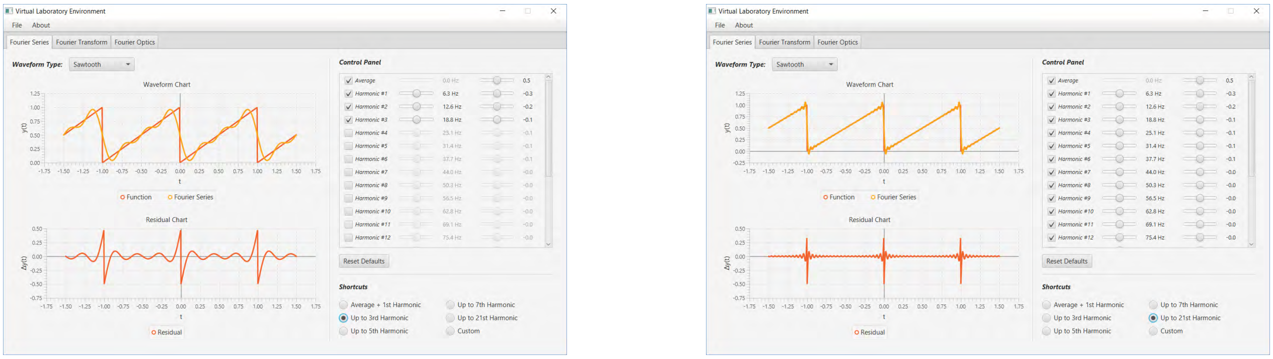
Using the *Fourier Series* tab of the virtual laboratory application (Section III), lecturers may facilitate an interactive exploration of the mathematical fundamentals of Fourier series expansions. Students may investigate and visualize the expansions of the square, triangle, and sawtooth waveforms as a starting point. The *control panel* allows independent modification of the amplitudes and frequencies of the summed sinusoids; students may hence verify that the Fourier series expansion indeed gives a good match to the waveforms, as a qualitative verification of the learnt mathematical techniques (Section II-A). Educators may consider the problem set in Table 1 to guide students in their learning.

TABLE 1. **A suggested problem set to guide the learning of students on Fourier series (Section IV-B.1), to be used in conjunction with the virtual laboratory application.**

Type	Question
Connect	When the number of summed harmonics is increased, does the partial sum representation match the original function better or worse?
Extend	The square wave and triangle wave contain only odd harmonics. The *sawtooth* wave, however, contains both odd and even harmonics. Explain why.
Extend	Derive the Fourier series for the waveforms shown in the *Java* applet.
Challenge	Near discontinuities on the square and sawtooth wave, the partial sum representation deviates considerably from the original functions. Can we still say that Eq. (1) is an exact equality? Explain.

A virtual environment aids greatly in the teaching of Fourier series, for it presents direct interactive graphical visualizations of the Fourier series expansions, largely unachievable in traditional pen-and-paper approaches. In particular, students easily observe that the approximation to the original function improves as the number of sinusoidal terms in the Fourier series partial sum increases (Figure 3). It is then intuitive that an equality can be achieved in the limit of an infinite sum.

This, however, is complicated by the Gibbs phenomenon [47], [48], referring to the significant deviations observed near jump discontinuities. Such deviations converge to a finite value as the number of terms in the partial sum is increased. In other words, for periodic functions with jump discontinuities, *point-wise* but *non-uniform* convergence



(a) (b)

FIGURE 3. Screenshots of the *Fourier series* tab of the virtual laboratory application, showing (a) the sawtooth waveform with up to the $n = 3$ harmonic included in the Fourier series partial sum, and (b) up to $n = 21$ harmonic included. The improvement in approximating the sawtooth function can be clearly observed.

is achieved. In the limit of an infinite sum these deviations are constrained to exist only at the points of discontinuities; in practice, a partial sum of sufficient number of terms will squeeze the deviations into such small widths that their significance may be reasonably neglected. Lecturers may wish to discuss the Gibbs phenomenon as a *challenge* for students; additional reading may include the possible methods to reduce the Gibbs deviations, such as using Lanczos sigma factors [48], [49]. The Gibbs phenomenon is a significant source of ringing artifacts in signal processing [50], [51], for instance, in magnetic resonance imaging applications [52], [53].

Concepts of symmetry may also be discussed, in particular *half-wave* and *quarter-wave* symmetries, the former responsible for the vanishing even harmonics in the square and triangle waves. Half-wave symmetry is present when $f(t) = -f(t - T/2)$. Clearly this can only be satisfied by sinusoids of the form $\sin((2k + 1)\omega_0 t + \phi)$, $k \in \mathbb{Z}$, hence $a_n = b_n = 0$ for all even n in the Fourier series. The symmetry condition also implies a vanishing average value. On the other hand, a quarter-wave symmetric function possesses half-wave symmetry and is even-symmetric about the midpoints of its positive and negative half-cycles; such a function can always be made even or odd via translation.

2) DOPPLER RADAR

Using the *Fourier transform* tab of the virtual laboratory application, various real-world application examples may be introduced to students. One interesting interdisciplinary example involving basic signal processing is that of Doppler radars. The frequency components of sound emitted by the object of interest during approach and departure can be analyzed, and the velocity of the object calculated. We apply such a technique to Formula One racing in this example.

A video of a race car traversing pass at high speed is provided (Figure 4); the Doppler shift from high to low in the audible pitch of the engine is easily observed. Two audio files are also provided, one recorded during the approach of the vehicle, and one during the departure of the vehicle.

TABLE 2. A suggested problem set to guide the learning of students on the *Doppler Radar* virtual laboratory activity (Section IV-B.2).

Type	Question
Connect	Review the Fourier transform. What can you tell about a signal from its frequency-domain spectrum? Discuss the significance of valleys and peaks on the spectrum. Run the discrete Fourier transform on both audio files, and record the frequencies of the most significant peaks.
Extend	To an approximation, we may take the vehicle and the observer to be co-linear at all times. The perceived sound frequency due to the Doppler effect for a stationary observer is given by $f = cf_0/(c + v_s)$, where f_0 is the emitted frequency, $c = 343$ m/s is the speed of sound, and v_s is the velocity of the source. Determine the speed of the car.
Challenge	Active Doppler radars measure speed in a similar fashion, by sending out microwaves of known frequency and measuring the Doppler shift in the reflected signal. Such a signal must be finite in duration. Explain why there is an inherent limitation in the accuracy of Doppler radars.

Lecturers may consider the problem set in Table 2 to guide students in this virtual laboratory activity.

To promote higher-order thinking, lecturers may wish to discuss the last sub-problem in greater detail, for it leads to intriguing insights that generalize beyond introductory Fourier theory. In the current context of the Doppler radar, there is an intrinsic trade-off between the measurable precision in an object's position and its velocity; more generally, there is a trade-off in the variance of a quantity and its Fourier transform. A qualitative conceptualization of such a property is not difficult to achieve. An active Doppler radar may measure the position of an object by sending a signal pulse of finite duration Δt, and measuring the time elapsed before the reflected signal is received. The uncertainty in the measurement of the time elapsed is approximately proportional to Δt. At the same time, the radar may measure the velocity of the object by analyzing the Doppler shift in the reflected signal. As the signal is finite in duration, there must be some spread in its frequency components, resulting in an uncertainty of Δf.

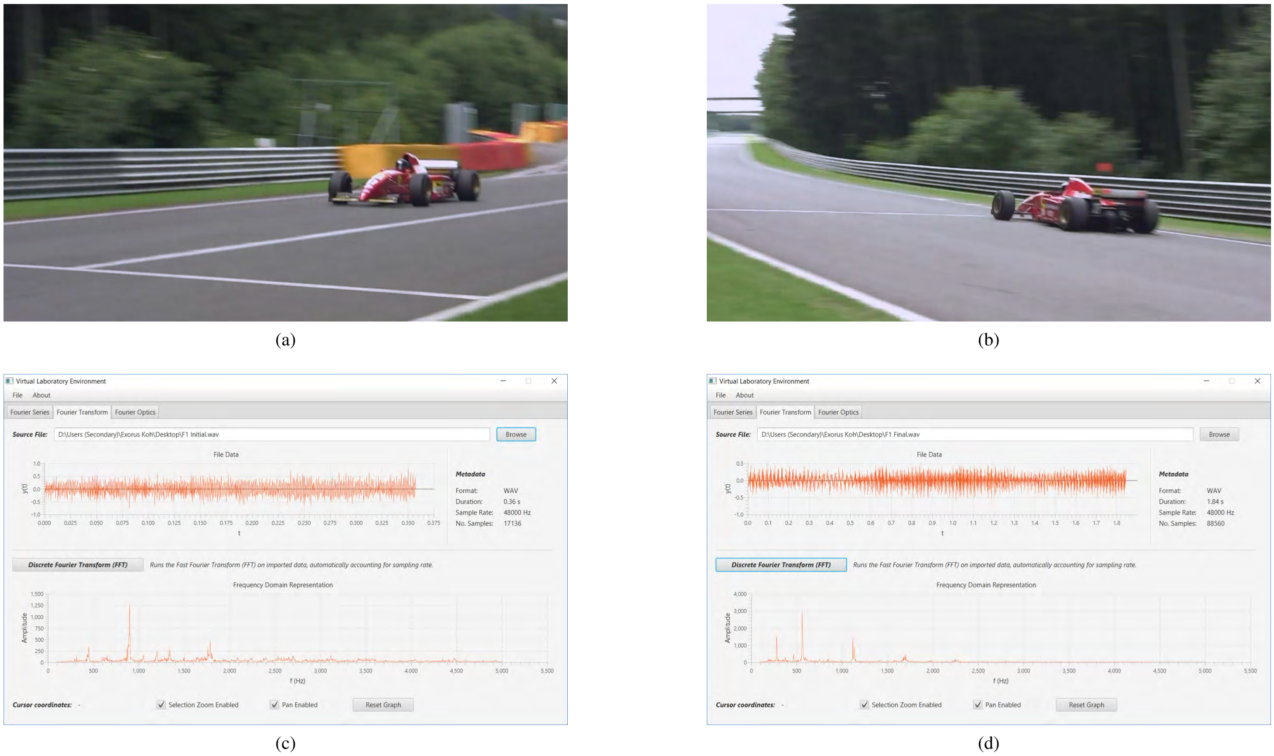


FIGURE 4. Snapshots of a video of a Formula One race car pass-by, on (a) approach and (b) departure with respect to a stationary observer, used in the *Doppler Radar* virtual laboratory activity (Section IV-B.2). Fast Fourier transform results of the recorded sound as computed within the virtual laboratory are also shown, for (c) approach and (d) departure, with most significant peaks at approximately 899.3 Hz and 561.5 Hz respectively. Such a Doppler shift corresponds to a co-linear velocity of 285.5 km/h. Users may zoom and pan the graph on the interactive application to observe the spectrum in greater clarity.

To enhance precision in the measurement of position, the signal can be made shorter in duration, such that Δt is reduced. This naturally results in an inflation of Δf, as can be observed in the time-scaling mathematical property of Fourier transforms (Section II-B). The precision in the measurement of velocity is therefore compromised. On the other hand, precision in the measurement of velocity can be enhanced by using a longer signal, such that Δf is reduced; this inevitably leads to a larger Δt. There is therefore an inherent trade-off between the two. To substantiate this qualitative argument, a simple worked example may be given, for instance, by considering a Gaussian signal pulse of variance σ^2. The signal $f(t)$ and its Fourier transform $\mathcal{F}_\xi\{f\}$ are

$$
\begin{aligned}
f(t) &= C_1 e^{-t^2/2\sigma^2}, \\
\mathcal{F}_\xi\{f\} &= C_2 e^{-2\pi^2\xi^2\sigma^2},
\end{aligned}
\tag{8}
$$

where C_1 and C_2 are normalization constants. The transform of a Gaussian signal of variance σ^2 therefore produces a Gaussian frequency-domain representation of variance $1/4\pi^2\sigma^2$, implying $\operatorname{var} f \operatorname{var} \mathcal{F}_\xi\{f\} = 1/4\pi^2$. The Heisenberg uncertainty principle is well-known in quantum physics [54], [55], which indicates that $\sigma_x\sigma_p \geq \hbar/2$, where σ denotes the standard deviation, and x and p denotes the position and momentum of the examined particle respectively. The uncertainty principle and the precision trade-off discussed here are indeed related; in quantum mechanics, position and momentum are related by a Fourier transform. In general, similar uncertainty principle analogues also appear in signal processing and data analytics, where Fourier transforms are utilized. Such an insight may interest engineering, physics, and applied maths students.

3) EARTHQUAKE ANALYSIS

Vibration and structural analysis are of great relevance to mechanical and civil engineering. The interactive virtual laboratory application can be used to perform power spectrum analysis on recorded earthquake vibrations, thereby demonstrating the usefulness of Fourier transforms in building engineering. An extract of vibrational data of the 2011 Honshu earthquake [56] is provided, which can be loaded into the interactive software (Figure 5). Lecturers may play the audio file as a demonstration—a low-pitched hum with no distinctive pattern can be heard. The problem set in Table 3 can be considered to guide the learning of students.

In this context, a simplistic vibrational model for the building structure is considered, in that it has a readily excited natural frequency close to a significant frequency component of the earthquake. It is reasonably assumed that the building is underdamped, and that power transfer

FIGURE 5. **Audio waveform and FFT results of the recorded vibrational data for the *Earthquake Analysis* virtual laboratory activity (Section IV-B.3), as seen on the interactive application. Significant vibrations can be observed in the 100 Hz–320 Hz displayed range, corresponding to approximately 6 Hz–20 Hz after accounting for the blue-shifting in the data recording process. The most significant peak is at approximately 143.5/16 ≈ 8.97 Hz.**

TABLE 3. **A suggested problem set to guide the learning of students on the *Earthquake Analysis* virtual laboratory activity (Section IV-B.3).**

Type	Question
Connect	The audio file has been blue-shifted by a factor of 16. Lossy audio file compression utilize Fourier transforms to remove audio components outside of the frequency range of human hearing. Explain why the blue-shifting is necessary in this context. Speculate how the audio frequency components are filtered during compression.
Extend	A building with sub-par construction in the affected zone has a natural frequency of 9.0 Hz. Analyze the seismic vibration data via Fourier transform and predict whether this building will collapse.
Challenge	Seismic tomography is a subsurface imaging technique exploiting the detection and analysis of seismic waves—it has, for instance, enabled us to understand the layered structure of the Earth. Explain how Fourier analysis of seismic signals may yield information on substrate density, thickness, and attenuation properties. Recall that Fourier transforms yield both the amplitude and *phase* of frequency components, and that multiple seismic sensors can be used.

between the earthquake driving vibrations and the building is efficient, leading to a collapse. However, modern buildings in earthquake-prone zones typically employ a range of structural reinforcements and isolation techniques to improve earthquake resilience, such as the usage of damping bearings [57], [58], shear walls that exploit buckling behavior to absorb energy [59], and tuned mass dampers [60].

4) FOURIER OPTICS

Fourier transforms are also greatly applicable in the modelling of optical phenomena. The field of Fourier optics focuses on such analyses [61], [62], and is particularly successful in the description of diffraction, interference, and defocusing. In this example, an introductory prelude on the application of Fourier optics in multi-aperture diffraction and interference is demonstrated, which may be of considerable interest to mathematical courses catering to physics and engineering students.

Consider a plane $\mathcal{P}_0$ containing an arbitrary arrangement of apertures, described by an aperture function $A(x_0, y_0)$ (Figure 6a). The aperture function takes on a non-zero value where there is an aperture and is zero everywhere else; specifically, if each aperture $\mathcal{A}_j$ receives illumination h_j, then $A(x_0, y_0) = h_j$ whenever $(x_0, y_0) \in \mathcal{A}_j$ and is zero otherwise. Placed z distance away is the image plane $\mathcal{P}$ with coordinate system (x, y). For an incident plane wave and ignoring the phase factors, the Fresnel-Kirchoff diffraction integral expresses the electric field E on $\mathcal{P}$ for $x_0 \ll z$ and $y_0 \ll z$:

$$E(x, y) \propto \iint A(x_0, y_0) \times \exp\left[\frac{2\pi i}{\lambda}\left(\frac{x_0^2+y_0^2}{2z} - \frac{2xx_0+2yy_0}{2z}\right)\right] dx_0 dy_0 \quad (9)$$

With $x_0^2/\lambda z \ll 1$ and $y_0^2/\lambda z \ll 1$, the system is reduced into the Fraunhofer regime:

$$E(x, y) \propto \iint A(x_0, y_0) \exp\left[-\frac{2\pi i}{\lambda z}(xx_0+yy_0)\right] dx_0\, dy_0 \quad (10)$$

The Fraunhofer diffraction equation is mathematically equivalent to the Fourier transform of the aperture function $A(x_0, y_0)$, from position variables (x_0, y_0) to the conjugate variables $(x/\lambda z, y/\lambda z)$—this forms the theoretical basis for this virtual laboratory exploration activity. For simplicity, a one-dimensional arrangement of slits $\mathcal{A}_j$ each with position d_j, width a_j, and illumination h_j is considered. A schematic of this system is shown in Figure 6b. The aperture function is then:

$$A(x_0) = \sum_j h_j \operatorname{rect}\left(\frac{x_0 - d_j}{a_j}\right) \quad (11)$$

Denoting the Fourier transform of frequency ξ as $\mathcal{F}_\xi\{\cdot\}$, it is deduced that $E(x) = \mathcal{F}_\zeta\{A\}$ with $\zeta = x/\lambda z$. The usefulness of such a method lies in that the Fourier transforms of a wide variety of functions are well-established, and therefore the electric field is easily evaluated. It is, for instance, known that $\mathcal{F}_\xi\{\operatorname{rect} u\} = \operatorname{sinc} \xi$. Exploiting the properties of Fourier transforms (Section II-B), the following can be easily derived:

$$\begin{aligned} E(x) \propto \mathcal{F}_\zeta\{A\} &= \sum_j \mathcal{F}_\zeta\left\{h_j \operatorname{rect}\left(\frac{x_0 - d_j}{a_j}\right)\right\} \\ &= \sum_j \left[a_j h_j \exp\left(-\frac{2\pi i x d_j}{\lambda z}\right) \operatorname{sinc}\left(\frac{\pi x a_j}{\lambda z}\right)\right] \end{aligned} \quad (12)$$

The Fresnel-Kirchoff condition of $x \ll z$ implies $x/z \approx \sin\theta$. The light intensity at the image plane is then given by:

$$\begin{aligned} I(\theta) &= |E|^2 \\ &\propto \left|\sum_j \left[a_j h_j \exp\left(-\frac{2\pi i d_j \sin\theta}{\lambda}\right) \operatorname{sinc}\left(\frac{\pi a_j \sin\theta}{\lambda}\right)\right]\right|^2 \end{aligned} \quad (13)$$

The *Fourier Optics* tab in the virtual laboratory software implements such a theoretical framework. Students may add

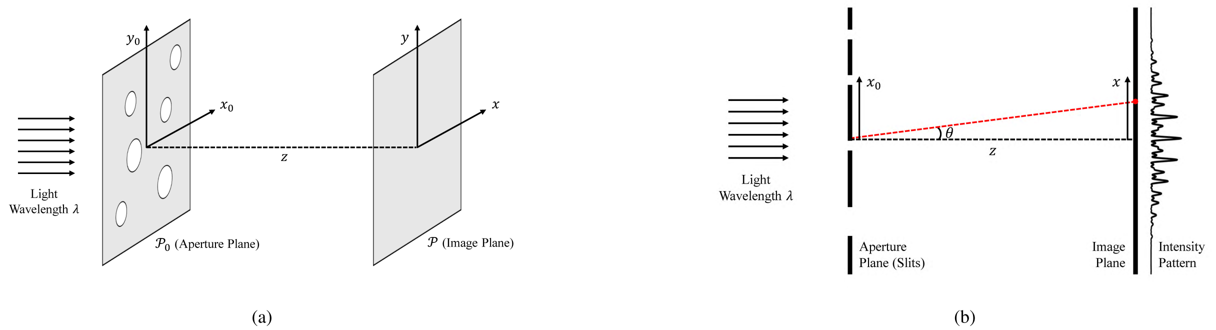


FIGURE 6. (a) Schematic of a general multiple-aperture optical system with a two-dimensional aperture plane, and (b) the multiple-slit system examined in the virtual laboratory.

TABLE 4. A suggested problem set to guide the learning of students on the *Fourier Optics* virtual laboratory activity (Section IV-B.4).

Type	Question
Connect	Using the interactive software, observe the diffraction pattern for a single slit. Does the solution make sense?
Extend	The Young's double-slit experiment, consisting of two slits of width a placed at $\pm d/2$ from the origin, carries great historical value in elucidating the wave-like properties of light and matter particles. Using the theoretical framework discussed, in particular Eq. (12) and Eq. (13), derive the intensity pattern for the double-slit experiment. Compare your solution with that derived in freshman physics class.
Challenge	As the number of regularly-spaced slits is increased, the intensity pattern exhibits sharper peaks, with vanishing intensity everywhere else—observe this effect in the virtual laboratory. Such an aperture configuration is known as a *diffraction grating*. Reproduce this result analytically.

an arbitrary number of slits into the system, each with independently variable position, width, and illumination; the aperture function and predicted intensity pattern are computed and displayed in real-time. Lecturers may consider the problem set in Table 4 to guide students in their investigations.

Lecturers may note that substituting the double-slit configuration into Eq. (13) yields $I(\theta) \propto \cos^2(\pi d \sin\theta/\lambda)\,\mathrm{sinc}^2(\pi a \sin\theta/\lambda)$ where d is the distance between the slits and a is the width of the slits, which matches the intensity pattern typically taught in freshman physics class. The cosine term can be interpreted as manifesting from the interference between light from the two slits, while the sinc term is due to the diffraction of light at the slits. There is an exceedingly wide range of advanced applications of Fourier optics, including telescope and interferometer design [63]–[65], laser optics [66], rendering engines [67], plasmonics [68], and quantum systems [69].

V. CASE-STUDY

As a pilot programme, we have implemented the proposed virtual laboratory learning environment in a second-year engineering mathematics course of $N = 101$ students majoring in the Sustainable Infrastructure Engineering degree in Spring 2018. A study was conducted to assess students' perception of the effectiveness of the virtual laboratory environment, as well as their overall reception response.

The teaching methodology and virtual laboratory activities had been discussed in Section IV—in particular, the *Visible Thinking* framework, and the emphasis on exposing students to a wide range of real-world applications. In consideration of the student demographics (engineering mathematics), the first three activities were included in full in the virtual laboratory session (Sections IV-B.1, IV-B.2 and IV-B.3), with the remaining activity on Fourier optics left as optional worked examples. The class was divided into two batches, each attending an identically-conducted virtual laboratory session spanning an hour. The interactive application and supporting materials were distributed to students in advance for self-reading and preparation, and a comprehensive revision (Section II) had been carried out in prior lectures. The sessions were held during scheduled tutorial timeslots, so that students need not spend additional extracurricular time. It was also communicated clearly that the virtual laboratory coursework, as a pilot programme, would not be included in the calculation of module grades.

At the conclusion of the virtual laboratory session, an anonymous voluntary survey implementing a five-point Likert scale rating was conducted. The survey questions and results are presented in Tables 5 and 6. Students may also submit voluntary free-response feedback. Notable survey responses and the implications of these results are discussed in the next section. In general, the cohort comprised students of diverse mathematical aptitude, therefore making this a good case study on the utilization of virtual laboratory activities in mathematics.

VI. DISCUSSION

The evaluation survey results (Table 5) indicate that the student cohort is in general supportive of the inclusion of the virtual laboratory activities as part of coursework. Student responses to all six Likert rating survey questions are positive, with average scores and 95% confidence intervals ≥ 3.75 throughout. In particular, survey questions Q2, Q3 and Q7 stand out with the highest mean Likert ratings of 4.07. These responses reflect that the virtual laboratory activities were perceived to be greatly beneficial at enabling students to apply their learnt knowledge to authentic problems, in the process promoting deep connections between

TABLE 5. Survey questions and results as evaluated by the pilot cohort. The survey implements a 5-point Likert scale rating across 6 questions, where a score of 5 represents a stance of "strongly agree", and a score of 1 represents "strongly disagree". The frequency counts of the students' responses were collated, from which the average scores and confidence intervals were calculated. Response rate is 75/101 ≈ 74.3%.

	Survey Question	Score Count					Mean	95% C.I.	
		1	2	3	4	5		Lower	Upper
Q1	These virtual laboratory activities have enhanced my understanding of Fourier series and Fourier transform concepts.	0	1	17	39	18	3.99	3.82	4.15
Q2	These virtual laboratory activities have given me the opportunity to apply the concepts learnt during the course to solve real-world problems.	0	0	13	44	18	4.07	3.92	4.21
Q3	These virtual laboratory activities have allowed me to make deeper connections between course concepts and real-world applications.	0	1	13	41	20	4.07	3.90	4.23
Q4	These virtual laboratory activities have stimulated my interest in learning engineering mathematics.	0	5	14	35	21	3.96	3.76	4.16
Q6	The inclusion of such virtual laboratory activities will give me more confidence to apply what I have learnt in the module to solve/approximate the solution to other practical problems.	1	2	17	35	20	3.95	3.75	4.14
Q7	These virtual laboratory activities have encouraged me to think more deeply about how these mathematical concepts are used in industry.	0	2	15	34	24	4.07	3.88	4.25

theoretical concepts and practical application. Students generally were appreciative of such a learning environment, and were motivated to extend their understanding of mathematics into real-world industrial and scientific applications. These being the key focal points of the virtual laboratory learning environment, it can be concluded that the proposed implementation is successful to a good extent.

The virtual laboratory activities were also perceived to have reinforced the students' understanding of the examined concepts (Likert rating 3.99), which were previously taught in a pen-and-paper lecture-tutorial environment. The activities also appear to have stimulated students' interest in pursuing further education in mathematics (Likert rating 3.96). In direct support of the ratings, the free-response section of the survey reflects that the virtual laboratory was well-received. A significant number of students were positive about the activities; as a way of illustration, we quote from a few students:

- "The activities include real-life examples which are not featured in traditional lecture-tutorials... allows us to apply the formulae learnt";
- "It makes me more interested in how we can apply what we have learnt onto real-life situations. Next time when we walk around we can confidently say, *oh this equipment makes use of these concepts to operate*";
- "Applications are pretty eye-opening";
- 'It helped me connect what I learned in class to real life. Very few modules let us understand why we are learning the content";
- "It is definitely more hands-on and interesting compared to just doing lectures and tutorials";
- "It is the first time I am able to appreciate mathematics. Everything seemed so abstract and murky until now. The interactive software helped a lot with visualization. More modules should be taught this way";
- "Brings back memories... where we characterized vibrational modes of antibubbles using DFT. This is like a mini-research task—very lively and fascinating".

The reception of the virtual laboratory programme is evidently generally positive. However, when inquired on their preference between traditional lecture-tutorial programmes

TABLE 6. Survey results on student preference between traditional lecture-tutorial and the proposed virtual laboratory activities, indicating an approximately even split. Students were given the hypothetical context of the replacement of tutorials with virtual laboratory learning sessions. Response rate is 75/101 ≈ 74.3%.

	Survey Question	Count	
		Lecture-Tutorial	Virtual Lab
Q5	In your opinion, is traditional lecture-tutorial style or current virtual laboratory activities more beneficial for the learning of our students?	37	38

and virtual laboratory activities, the student population displayed an approximately even split (Table 6), reflecting a largely neutral stance. An analysis on their open-ended responses indicate two primary reasons for the observed neutrality. Firstly, there is an immense concern on the final course examination, and the virtual laboratory activities were perceived to be inferior to traditional tutorials in preparing for the assessment. Secondly, the pilot class may not have been sufficiently prepared in their mathematical foundations to tackle the virtual laboratory effectively. Students have indicated that "it was difficult to follow the virtual lab if we had not revised the lectures and tutorials". While review sessions had been carried out, not all students may have had adequate time to internalize the foundational content. This is plausibly connected to the comparatively poor response with regard to confidence in practical problem-solving (Likert rating 3.95).

These findings suggest plausible routes for improvement in future virtual laboratory implementations. An overly exam-oriented assessment environment has been shown here to hinder opportunities to participate in these learning activities; educators may hence consider restructuring grade weightages to reduce emphasis on written assessments. An alternative, as suggested by students in our pilot programme, is to implement the virtual laboratory *as a separate module*, in similar fashion to science courses. For such a solution to be viable, the content coverage of the virtual laboratory has to be expanded, perhaps in a way that a single virtual laboratory module may support multiple theoretical courses. Educators should also note the importance of laying firm foundations before the virtual laboratory activities, which demand higher-order inquiry-driven thinking.

The lecture-tutorial system and the proposed virtual laboratory environment ought to be considered complementary, and should not be seen as mutually exclusive; a mixture of both should be implemented for the best educational outcome. The two learning environments carry distinct advantages and shortcomings—in particular, the former excels at efficiently communicating fundamental concepts and drilling foundational skills, whilst the latter enables the application of learnt knowledge in authentic contexts and promotes deeper learning depth. Indeed, a significant number of students have explicitly indicated that they would prefer a near-equal balance of tutorials and virtual laboratory sessions.

VII. CONCLUSION

In this study, we have proposed a virtual laboratory learning environment in mathematics education. Our *Java* interactive application is readily adopted by educators and students; an accompanying review of introductory Fourier theory to facilitate instruction and a set of inquiry-based learning virtual laboratory activities integrated with the software have also been presented, spanning related disciplines of data science, signal processing, mechanical and civil engineering, and physics. The wide-ranging interdisciplinary links within these real-world application problems enable students of diverse backgrounds to be engaged effectively, and fundamental broad-ranging data analysis skills to be built. We have also suggested that a modified connect–extend–challenge *Visible Thinking* pedagogical framework be integrated into the virtual laboratory environment, thereby creating a structured, systematic approach to inquiry-based learning in authentic contexts. Results from our pilot programme indicate that the virtual laboratory implementation is well-received by students, with considerable perceived benefits in enhancing conceptual understanding, improving exposure and confidence in the application of learnt knowledge, and in stimulating interest in learning mathematics.

By incorporating hands-on application-based interactive environments in which students may conduct free exploration, the shortcomings of traditional lecture-tutorial systems can be addressed. Such an improvement may take the form of seminar-style course structures, comprising integrated lectures, tutorials, and virtual laboratory sessions. Future work may involve the expansion of the current implementation to span other topics in mathematics, such as geometry and calculus. Additional interdisciplinary examples may also be included, for instance electrocardiogram analysis in biomedical engineering [70], [71]. The *Java* interactive application was designed to be lightweight, hence enabling a feasible transition to an app-based platform, suitable for deployment on tablets and smartphones—this is a suitable step into the smart classroom paradigm [72]–[74]. It is our hope that educators can make use of our exposition in improving mathematics education.

VIII. VIRTUAL LABORATORY

The *Java* interactive software and test examples are freely available on Open Science Framework (OSF): VirtualLaboratory.jar of https://goo.gl/uk2Xjz.

IX. COMPETING INTERESTS

The authors declare no competing interests.

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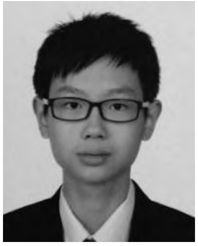

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