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Analytical Performance Evaluation of MRC Receivers in Massive MIMO Systems

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ABSTRACT In recent years, massive or large-scale multiple-input multiple-output (MIMO) systems that rely on very large antenna arrays have become a hot topic of research in the field of wireless communications. This is in part due to the nearly optimum performance and relative simplicity of massive MIMO linear receivers and transmitters. This paper investigates the performance of maximum-ratio combining linear receivers in massive MIMO communication systems in terms of signal-to-interference-plus-noise ratio (SINR), bit error rate (BER), and outage probability. The probability density function (PDF) of SINR is analytically calculated for the first time and verified by simulation. Due to its complexity, the use of obtained analytical expression for performance evaluation purposes is prohibitive. Hence, this PDF is approximated by a gamma distribution and the resulting expression is used to evaluate the outage probability and the BER performance of the receiver.

INDEX TERMS Massive MIMO systems, maximum-ratio-combining (MRC) receivers, probability density function, signal-to-interference-plus-noise ratio.

I. INTRODUCTION

The volume of multimedia data traffic in mobile networks have been exponentially increasing in the past decade [1] and this trend is set to continue. As a result, the demand for fast and reliable communication systems will never stop increasing. One of the techniques that have been employed to simultaneously increase a communication system's data throughput and reliability is the use of multiple-input multiple-output (MIMO) communication systems that use multiple antennas at the transmitter and/or the receiver [2]–[4]. MIMO systems are widely used in modern communication systems such as the IEEE 802.11 Wi-Fi, the IEEE 802.16 WiMAX, and third and fourth generations of cellular networks [3]. In order to allow multiple users to simultaneously access a communication network, multiuser MIMO (MU-MIMO) systems emerged as an extension to MIMO systems. In a cellular MU-MIMO system, multiple users, each with either a single antenna or multiple antennas, simultaneously communicate with a multiple-antenna base station (BS) [4]. Although MU-MIMO systems can serve multiple users simultaneously, their practical implementation exhibits some drawbacks: since the users do not communicate among themselves, data cannot be coded at the users' side [1]. As a result, the BS must have perfect knowledge of the channel state information (CSI) in order to detect the conveyed symbols in the uplink transmission, i.e. when the users transmit their data to the BS [5]. Furthermore, during the downlink transmission (when the BS transmits data towards the users), the transmitter (BS) must make sure that each user only receives its intended symbols, which in turn necessitates the exact knowledge of CSI at the BS [6]. As a result, both in downlink and uplink of MU-MIMO transmission, the BS needs to have the exact channel information. Furthermore, in order to achieve high reliability and throughput, computationally complex techniques, such as multiuser maximum likelihood (ML) detection for the uplink [7] and dirty paper coding techniques for the downlink [8] should be employed. Large-scale MU-MIMO systems, widely known as massive MIMO systems are a special case of MU-MIMO in which the number of users and BS antennas are very large. Typically in a massive MIMO system, hundreds of BS antennas serve tens of single-antenna users simultaneously [9]. With large antenna arrays, conventional signal processing techniques such as ML detection become prohibitively complex. However it has been shown that when the number of BS antennas is very large compared to the number of active users, simple linear processing techniques achieve nearly optimal results [6]. In fact, when the number of BS antennas grows significantly larger than the number of users, the random channel vectors between the users and the

BS become pairwisely orthogonal, and the effect of small-scale fading can be averaged out [10]. As a result, even with simple linear signal processing techniques, such as maximum-ratio combining (MRC) in uplink or maximum-ratio transmission (MRT) in downlink, the effects of fast fading, intracell interference, and uncorrelated noise tend to disappear [5]. The reduced complexity of massive MIMO systems compared to MU-MIMO, and their maintaining of the benefits of MU-MIMO, make massive MIMO systems a natural choice for the future of wireless cellular communications [11].

In massive MIMO systems, simple linear detectors such as MRC, zero forcing (ZF), or minimum mean square error (MMSE) can achieve nearly optimal results [6]. MRC receivers maximize the received signal power while neglecting the interference from other users and is thus more suitable for lower transmission powers. ZF receivers mitigate the interference from other users while neglecting the effect of random noise, which makes them more efficient for large-power transmissions. MMSE receivers maximize the signal-to-interference-plus-noise ratio (SINR) at the BS and can yield nearly optimal results in both cases. MMSE and ZF receivers necessitate the computation of the pseudo inverse of the channel matrix, which renders them more complex than MRC receivers. Furthermore, if the channel is not well-conditioned, the performance of ZF receivers will significantly degrade [5], [12]. MRC receivers, on the other hand, use a very low-complexity algorithm, but yield poor results in interference-limited scenarios. However, when the number of BS antennas is at least one order of magnitude greater than the number of users, there are sufficient degrees of freedom to effectively mitigate the interference, even using MRC. In such cases the most efficient choice is to use lower transmission powers along with an MRC detector.

The analytical calculations of the probability density function (PDF) of SINR for ZF and MMSE receivers can be found in the literature: For a massive MIMO system with M BS antennas and K users, SINR_{ZF} is a chi-squared random variable with $2(M-K+1)$ degrees of freedom [13]. Wang *et al.* [14] have found a tight approximation for the distribution of SINR_{ZF} considering the channel approximation error. They used this approximation to obtain a closed form expression for outage probability and bit error rate (BER) of a ZF receiver. Ping Li et al. showed that the SINR of a MMSE receiver can be expressed as the sum of two independent random variables $\text{SINR}_{\text{MMSE}} = \text{SINR}_{\text{ZF}} + \tau$. Using the random matrix theory, they approximated the distribution of τ and used this approximation to obtain $\text{SINR}_{\text{MMSE}}$ and a closed form expression for BER under different conditions [15].

For MRC receivers, the analytical expression for the PDF of SINR has not been calculated. In [16], SINR_{MRC} is approximated, only for high powers, by a random variable following a Fisher distribution. This approximation differs significantly from the actual distribution of SINR_{MRC} for lower transmission powers; when the MRC receivers are most beneficial. Ngo *et al.* [17] derived a lower bound for the capacity of massive MIMO systems, including the MRC receiver. In [18] the BER performance of MRC receivers is obtained by simulation.

In this paper, we derive an analytical expression for the exact distribution of SINR_{MRC} under Rayleigh flat fading conditions. Since this expression is too complicated to be used for analytical performance evaluation of massive MIMO systems; the obtained SINR_{MRC} is approximated by a gamma-distributed random variable for all transmission power values, under the assumption of large number of users and BS antennas. This assumption is advantageous to the high-power assumption of [16], because: i) by definition of massive MIMO systems, the number of BS antennas is very large. Therefore our proposed assumption is very realistic for all massive MIMO systems. And ii) the performance of MRC receivers is nearly optimal for lower transmission powers and degrades for higher powers. As a result the approximation in [16] cannot be used for MRC receivers in the optimal range of transmission power.

Using the proposed approximation, closed form expressions are obtained for outage probability and BER. Simulation results confirm the validity of derived expressions and proposed approximations.

A. PAPER ORGANIZATION

The rest of this paper is organized as follows: Section II presents the system model and preliminaries. Section III covers the derivation of the exact expression for the PDF of SINR_{MRC} and its approximation. In Section IV the performance of MRC receiver in massive MIMO systems is evaluated by calculating closed form expressions for outage probability and BER. The numerical results are provided in Section V and the concluding remarks are drawn in Section VI.

B. NOTATIONS

Throughout this paper, $(.)^T$ and $(.)^H$ respectively denote the transposed and conjugate transposed matrices of a matrix. $\|.\|$ is the L^2 norm (Euclidean norm) of a vector. $\Gamma(x) \triangleq \int_0^\infty t^{x-1}e^{-t}dt$ denotes the gamma function. $\mathbb{E}\{.\}$ is the mathematical expectation operator. $\mathcal{CN}(\mu, \sigma^2)$ represents a Gaussian distribution with mean μ and variance σ^2. $G(\alpha, \beta) \sim \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-\frac{x}{\beta}}$ and $G(\alpha, \beta, \zeta) \sim \frac{1}{\Gamma(\alpha)\beta^\alpha}(x-\zeta)^{\alpha-1}e^{-\frac{x-\zeta}{\beta}}$ respectively denote two-parameter and three-parameter (shifted) gamma distributions.

II. SYSTEM MODEL

A massive MIMO system containing one BS with M antennas and K single-antenna users is considered as depicted in Fig. 1. This paper addresses the uplink transmission where users send their data in the same time-frequency resource towards the BS. The signal is received by M BS antennas. The BS will use these M signals to estimate the likeliest user symbols. We assume that the transmission is subject to Rayleigh flat fading with $\mathbf{H} \in \mathbb{C}^{M\times K}$ denoting the channel matrix

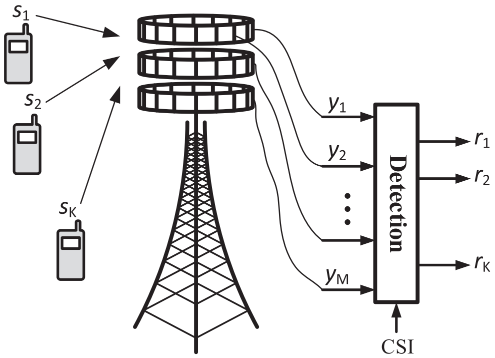

FIGURE 1. System model.

between the users and the BS. $h_{mk} = [\mathbf{H}]_{mk}$ is the channel coefficient between the kth user and the mth BS antenna. The channel coefficients are considered to be independent and identically distributed (i.i.d.) Gaussian random variables $h_{mk} \sim \mathcal{CN}(0, 1)$. The users simultaneously transmit their symbols. $\mathbf{s} = [s_1, s_2, \cdots, s_K]^T$ is the vector of transmitted user symbols with $\mathbb{E}\left\{|s_k|^2\right\} = 1$. The received signal at the BS, $\mathbf{y} \in \mathbb{C}^{M\times 1}$ is given by (1).

$$\mathbf{y} = \sqrt{p_u}\mathbf{H}\mathbf{s} + \mathbf{n}. \tag{1}$$

In (1), p_u and $\mathbf{n} \in \mathbb{C}^{M\times 1}$ respectively denote average transmission power and the noise vector containing i.i.d. Gaussian random variables with zero mean and unit variance. Using this definition, signal-to-noise ratio (SNR) will be equal to p_u.

The BS can employ a multiuser ML detection to achieve optimal performances; however the computational complexity of this algorithm increases exponentially with the number of users [17]. In massive MIMO systems, the number of BS antennas and the number of users are large numbers and the number of BS antennas always exceeds the number of users. Under such conditions, linear receivers (MRC, ZF, and MMSE) have nearly optimal performances in spite of their simplicity [6]. Using linear receivers at the BS, the received signal is multiplied by a detection matrix $\mathbf{A} \in \mathbb{C}^{M\times K}$ forming K separate streams:

$$\mathbf{r} = \mathbf{A}^H\mathbf{y}. \tag{2}$$

Substituting (2) in (1) we obtain:

$$\mathbf{r} = \sqrt{p_u}\mathbf{A}^H\mathbf{H}\mathbf{s} + \mathbf{A}^H\mathbf{n}. \tag{3}$$

If the BS has the perfect CSI, the detection matrix for different receivers is given by [17]. Namely for MRC receiver, where interference from other users is neglected, we have:

$$\mathbf{A} = \mathbf{H} \quad \text{for MRC}. \tag{4}$$

If we denote the kth column of $\mathbf{A}$ and $\mathbf{H}$ respectively by $\mathbf{a}_k$ and $\mathbf{h}_k$, the kth element in the vector $\mathbf{r}$ can be obtained from [5]:

$$r_k = \sqrt{p_u}\mathbf{a}_k^H\mathbf{h}_k s_k + \sqrt{p_u}\sum_{\substack{i=1\\ i\neq k}}^{K}\mathbf{a}_k^H\mathbf{h}_i s_i + \mathbf{a}_k^H\mathbf{n}. \tag{5}$$

In this equation, $\sqrt{p_u}\mathbf{a}^H\mathbf{h}_k s_k$ is the desired signal, $\sqrt{p_u}\sum_{i=1,i\neq k}^{K}\mathbf{a}_k^H\mathbf{h}_i s_i$ is the interference caused by other users, and $\mathbf{a}_k^H\mathbf{n}$ denotes the additive noise. The SINR for the kth user is thus given by:

$$\gamma_k = \frac{p_u\left|\mathbf{a}_k^H\mathbf{h}_k\right|^2}{p_u\sum\limits_{\substack{i=1\\ i\neq k}}^{K}\left|\mathbf{a}_k^H\mathbf{h}_i\right|^2 + \|\mathbf{a}_k\|^2}. \tag{6}$$

The exact expression of the distribution of SINR for MRC receiver has not been analytically calculated. Furthermore, the approximation given in [16] yields acceptable results only for large p_u, whereas in practice, MRC receivers are especially interesting when p_u is small, such that the bottle neck of the system is the noise power. In the following section we will calculate the exact distribution of the SINR for MRC receivers. For simplicity we will denote SINR_{MRC} by SINR in the remainder of this paper.

III. PROBABILITY DENSITY FUNCTION OF SINR FOR MRC RECEIVERS

In this section we will derive analytical and approximate expressions for the PDF of the SINR of a massive MIMO system with an MRC receiver.

A. EXACT EXPRESSION FOR DISTRIBUTION OF SINR FOR MRC RECEIVERS

In an MRC receiver, the detection matrix $\mathbf{A}$ is equal to the channel coefficients matrix $\mathbf{H}$. By substituting $\mathbf{a}_k = \mathbf{h}_k$ in (6) and dividing the numerator and the denominator by $p_u\|\mathbf{h}_k\|^2$, the SINR for the kth user is obtained:

$$\gamma_k = \frac{\|\mathbf{h}_k\|^2}{\sum\limits_{i=1,i\neq k}^{K}\frac{\left|\mathbf{h}_k^H\mathbf{h}_i\right|^2}{\|\mathbf{h}_k\|^2} + \frac{1}{p_u}}. \tag{7}$$

Let us define $X \triangleq \|\mathbf{h}_k\|^2$, $z_{i,k} \triangleq \frac{|\mathbf{h}_k^H\mathbf{h}_i|^2}{\|\mathbf{h}_k\|^2}$, and $Z \triangleq \sum_{i=1,i\neq k}^{K} z_{i,k}$. Equation (7) then becomes:

$$\gamma_k = \frac{X}{\sum\limits_{i=1,i\neq k}^{K} z_{i,k} + \frac{1}{p_u}} = \frac{X}{Z + \frac{1}{p_u}}, \tag{8}$$

where $X \triangleq \|\mathbf{h}_k\|^2$ is the sum of M independent exponential random variables. Thus X has a gamma distribution $G(M, 1)$ with probability density function given by $f_X(x) = \frac{x^{M-1}}{\Gamma(M)}e^{-x}$.

Furthermore $z_{i,k} \triangleq \frac{|\mathbf{h}_k^H\mathbf{h}_i|^2}{\|\mathbf{h}_k\|^2}$ is an exponential random variable independent of X with $f_{z_{i,k}}(z_{i,k}) = e^{-z_{i,k}}$. Random variable Z is therefore the sum of $K-1$ independent exponential random variables and will follow $G(K-1, 1)$ gamma distribution with PDF given by $f_Z(z) = \frac{z^{k-2}}{\Gamma(K-1)}e^{-z}$ [16].

It results from the above discussion, that for a given value of transmission power p_u, the SINR will be a random variable γ_k which is the quotient of a random variable

$X \sim G(M, 1)$ divided by the sum of a random variable $Z \sim G(K-1, 1)$ plus a constant $\frac{1}{p_u}$. Choi *et al.* [16] neglected the constant $\frac{1}{p_u}$ and approximated SINR as $\gamma_k \approx \frac{X}{Z}$. It is obvious that this approximation holds only for small values of $\frac{1}{p_u}$, i.e. for high values of transmission power. In this paper we drop the large-power assumption and define the random variable $T \triangleq Z + \frac{1}{p_u}$ as the sum of a two-parameter gamma distributed random variable and a constant. Thus T has a three-parameter (shifted) gamma distribution $G(K-1, 1, \frac{1}{p_u})$ with probability distribution function given by:

$$f_T(t) = \frac{\left(t - \frac{1}{p_u}\right)^{K-2}}{\Gamma(K-1)} e^{-\left(t-\frac{1}{p_u}\right)}, \quad t > \frac{1}{p_u}. \tag{9}$$

Thus SINR can be considered as the ratio of two independent random variables: $\gamma_k = \frac{X}{T}$ with $X \sim G(M, 1)$ and $T \sim G(K-1, 1, \frac{1}{p_u})$. Since the two random variables are independent, their joint probability density function is equal to $f_{XT}(x, t) = f_X(x) f_T(t)$:

$$f_{XT}(x, t) = \frac{x^{M-1} e^{-(x+t-\frac{1}{p_u})}}{\Gamma(M)\Gamma(K-1)} \left(t - \frac{1}{p_u}\right)^{K-2} \tag{10}$$

for $t > \frac{1}{p_u}$ and $x > 0$. Using the variable transformations, we can compute

$$f_{\gamma_k,T}(\gamma, t) = |J| f_{XT}(t\gamma, t) \tag{11}$$

where J is the Jacobian determinant of the transformation: $J = \frac{\partial(t\gamma)}{\partial\gamma} = t$. This yields:

$$f_{\gamma_k,T}(\gamma, t) = \frac{\gamma^{M-1} t^M e^{\frac{1}{p_u}} e^{-(\gamma+1)t}}{\Gamma(M)\Gamma(K-1)} \left(t - \frac{1}{p_u}\right)^{K-2} \tag{12}$$

for $t > \frac{1}{p_u}$ and $\gamma > 0$. By marginalizing T in (12) we obtain:

$$\begin{aligned} f_{\gamma_k}(\gamma) &= \int_{-\infty}^{+\infty} f_{\gamma_k,T}(\gamma, t) dt \\ &= \frac{\gamma^{M-1} e^{\frac{1}{p_u}}}{\Gamma(M)\Gamma(K-1)} \\ &\quad \times \int_{\frac{1}{p_u}}^{+\infty} t^M e^{-(\gamma+1)t} \left(t - \frac{1}{p_u}\right)^{K-2} dt. \end{aligned} \tag{13}$$

The integral in (13) can be solved using equation (3.383-4) in [19]:

$$\begin{aligned} f_{\gamma_k}(\gamma) = &\frac{e^{-\frac{\gamma-1}{2p_u}} \gamma^{M-1}}{\Gamma(M) p_u^{\frac{M+K-2}{2}} (\gamma+1)^{\frac{M+K}{2}}} \\ &\times W_{\frac{M-K+2}{2}, \frac{1-M-K}{2}} \left(\frac{\gamma+1}{p_u}\right) \end{aligned} \tag{14}$$

where $W_{\lambda,\mu}(z)$ denotes the Whittaker function. Considering the relation of Whittaker function to Laguerre polynomials, $L_n^{(\alpha)}(z)$ [20, eq. 13.18.17]:

$$W_{\frac{\alpha+2n+1}{2}, \frac{\alpha}{2}}(z) = (-1)^n n! e^{-\frac{z}{2}} z^{\frac{\alpha+1}{2}} L_n^{(\alpha)}(z), \tag{15}$$

we can further simplify (14) as:

$$f_{\gamma_k}(\gamma) = (-1)^M \frac{M e^{\frac{-\gamma}{p_u}} \gamma^{M-1}}{(\gamma+1)^{M+K-1}} L_M^{(1-K-M)}\left(\frac{1+\gamma}{p_u}\right). \tag{16}$$

Fig. 2 shows the simulation results for the PDF of the SINR of a MU-MIMO system for different values of p_u, as well as the theoretical result obtained by evaluation of (16). Also included for comparison is the approximation given

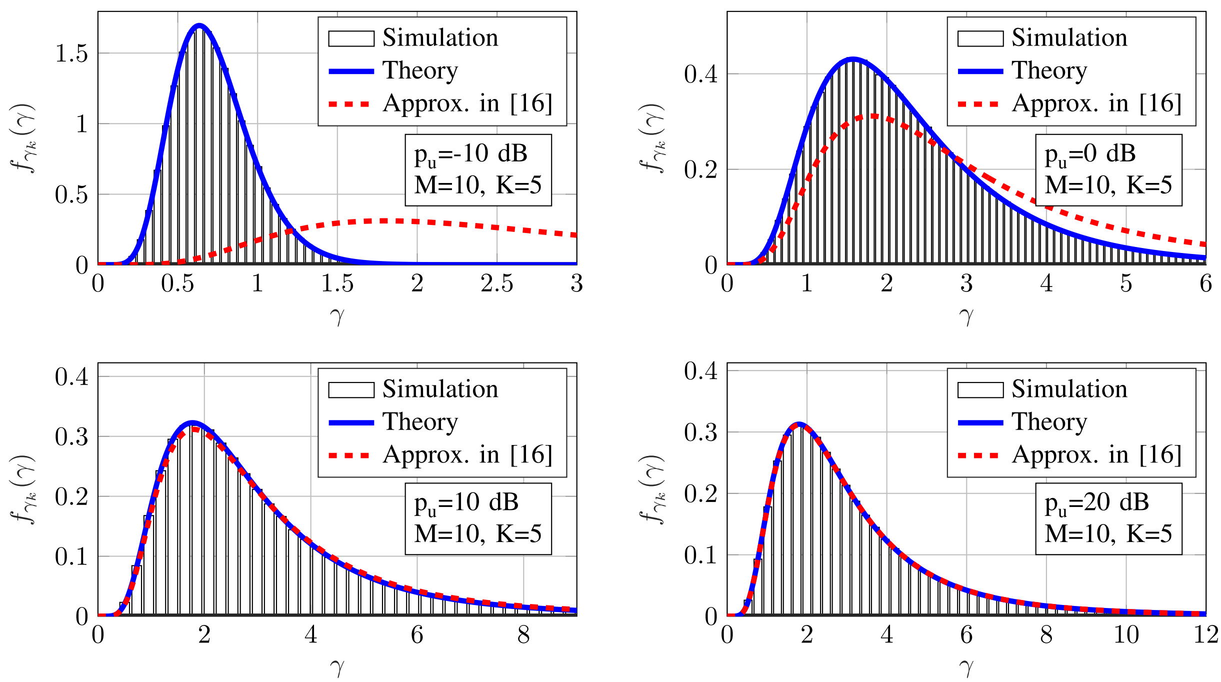

FIGURE 2. **PDF of the SINR in a system with a 10-antenna BS and 5 users for different values of signal strength.**

in [16]. It can be seen that the numerical results confirm the validity of (14). However, the approximative distribution of [16] is only valid for large p_u and significantly differs from the real distribution for smaller p_u.

Since (14) and (16) are complicated expressions, they cannot be practically used to derive analytical expressions for the performance of the system. It is therefore useful to have a simple approximate expression.

B. APPROXIMATION OF THE DISTRIBUTION OF SINR IN MRC RECEIVERS

We demonstrated that in general, for a MU-MIMO system with an MRC receiver, the probability density function is given by (14) or equivalently (16). The problem with (14) and (16) is that they are complicated equations and cannot be used to obtain closed form expressions for the performance measures of the system such as outage probability or BER. However, the expression in (16) can be simplified under certain conditions. For example in [16], the distribution of SINR for high values of SNR has been approximated by a Fisher distribution (the resulting expression can be obtained from (14) by letting $p_u \to \infty$). However in massive MIMO systems, MRC receivers are mostly used in lower SNRs where the approximation in [16] no longer holds. On the other hand, in massive MIMO systems, the number of users and BS antennas is very large with more BS antennas than active users.

For small values of p_u, the argument of Laguerre polynomial in (16) approaches infinity and the polynomial may be replaced with its highest order term, i.e. $(-1)^n \frac{1}{n!} z^n$ for $n = M$ and $z = (1+\gamma)/p_u$. Furthermore, under this assumption, the PDF of SINR is nonzero only for small values of γ, that is when $1+\gamma$ approaches 1. Under these conditions (16) becomes:

$$f_{\gamma_k}(\gamma) \approx \frac{\gamma^{M-1}}{\Gamma(M) p_u^M} e^{-\frac{\gamma}{p_u}}, \tag{17}$$

which is the PDF of a gamma distributed random variable. This, along with the fact that when $M > K \gg 1$, the form of SINR distribution resembles that of a gamma distribution, motivates us to approximate the PDF of SINR with a $G(\alpha, \beta)$ distribution:

$$f_{\gamma_k}(\gamma) \approx \frac{\gamma^{\alpha-1}}{\beta^\alpha \Gamma(\alpha)} e^{-\frac{\gamma}{\beta}}. \tag{18}$$

In order to find the parameters of the gamma distribution, we need to calculate the mean and the variance of the random variable γ_k. γ_k being the quotient of two independent random variables, its mean can be obtained by:

$$m_{\gamma_k} = \mathbb{E}\{X\} \mathbb{E}\left\{\left(Z + \frac{1}{p_u}\right)^{-1}\right\} \tag{19}$$

where

$$\begin{aligned}\mathbb{E}\{X\} &= \frac{1}{\Gamma(M)} \int_0^\infty x^M e^{-x} dx \\ &= \frac{\Gamma(M+1)}{\Gamma(M)} = M,\end{aligned} \tag{20}$$

and

$$\begin{aligned}&\mathbb{E}\left\{\left(Z + \frac{1}{p_u}\right)^{-1}\right\} \\ &= \frac{1}{\Gamma(K-1)} \int_0^\infty \left(z + \frac{1}{p_u}\right)^{-1} z^{K-2} e^{-z} dz \\ &= e^{\frac{1}{p_u}} E_{K-1}(\frac{1}{p_u}).\end{aligned} \tag{21}$$

with $E_n(z) \triangleq \int_1^\infty e^{-zt} t^{-n} dt$ denoting the generalized exponential integral function. The mean of SINR is then given by:

$$m_{\gamma_k} = M e^{\frac{1}{p_u}} E_{K-1}(\frac{1}{p_u}). \tag{22}$$

We can further simplify m_{γ_k} by using the following property of the generalized exponential function [21] and exploiting the fact that $K \gg 1$:

$$\frac{e^{-x}}{x+n} < E_n(x) \leq \frac{e^{-x}}{x+n-1}. \tag{23}$$

$$m_{\gamma_k} \approx \frac{M}{K-2+\frac{1}{p_u}}, \quad K \gg 1. \tag{24}$$

In order to calculate the variance of γ_k, using the independence of X and Z, we can write:

$$\sigma_{\gamma_k}^2 = \mathbb{E}\left\{X^2\right\} \mathbb{E}\left\{\left(Z + \frac{1}{p_u}\right)^{-2}\right\} - m_{\gamma_k}^2. \tag{25}$$

where m_{γ_k} is given by (22) and:

$$\begin{aligned}\mathbb{E}\left\{X^2\right\} &= \frac{1}{\Gamma(M)} \int_0^\infty x^2 x^{M-1} e^{-x} dx \\ &= \frac{\Gamma(M+2)}{\Gamma(M)} = M(M+1),\end{aligned} \tag{26}$$

and

$$\begin{aligned}&\mathbb{E}\left\{\left(Z + \frac{1}{p_u}\right)^{-2}\right\} \\ &= \frac{1}{\Gamma(K-1)} \int_0^\infty \left(z + \frac{1}{p_u}\right)^{-2} z^{K-2} e^{-z} dz \\ &= \frac{e^{\frac{1}{p_u}}\left(K-2+\frac{1}{p_u}\right) E_{K-2}(\frac{1}{p_u}) - 1}{K-2}.\end{aligned} \tag{27}$$

The variance of SINR is therefore obtained from:

$$\begin{aligned}\sigma_{\gamma_k}^2 = M(M+1)&\frac{e^{\frac{1}{p_u}}(K-2+\frac{1}{p_u})E_{K-2}(\frac{1}{p_u}) - 1}{K-2} \\ &- \left(M e^{\frac{1}{p_u}} E_{K-1}(\frac{1}{p_u})\right)^2.\end{aligned} \tag{28}$$

Equation (28) can be further simplified by using the following property of the generalized exponential function [21]:

$$n E_{n+1}(x) = e^{-x} - x E_n(x). \tag{29}$$

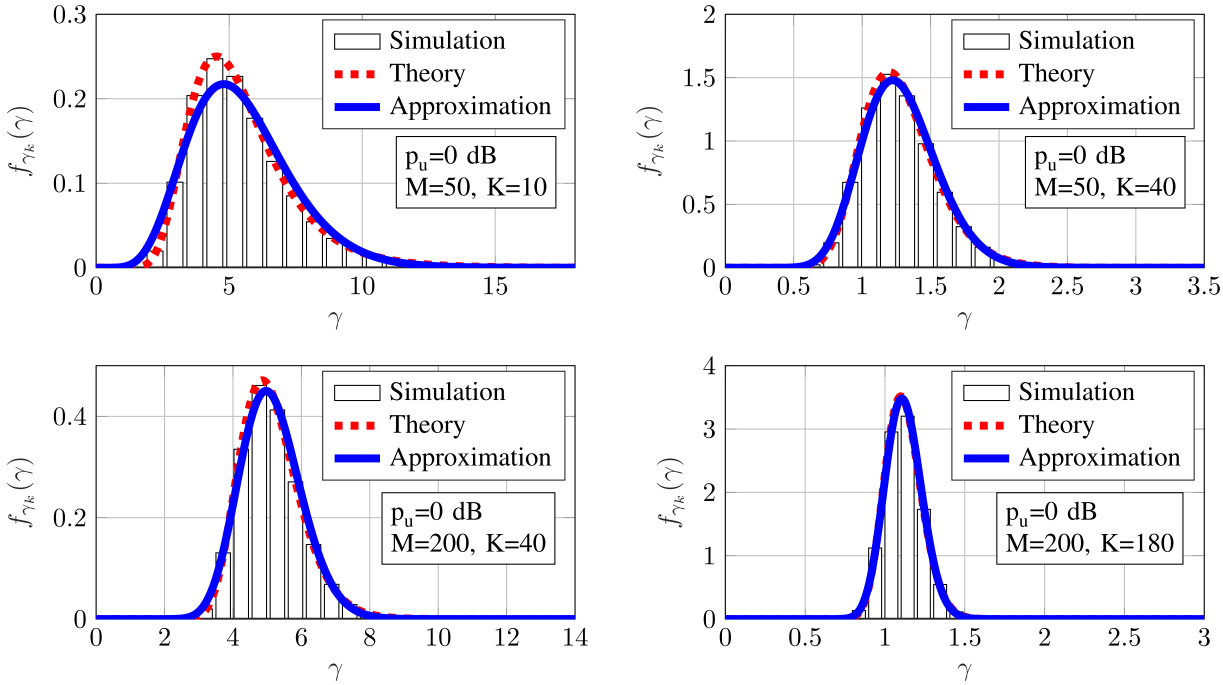

FIGURE 3. PDF of the SINR in a system with $p_u = 0$ dB and for different values of M and K.

Using (29), it can be demonstrated that

$$\frac{e^x(n+x)E_n(x)-1}{n} = e^x E_n(x) - e^x E_{n+1}(x) \approx \frac{1}{n+x-1} - \frac{1}{n+x}. \quad (30)$$

where the second line follows from (23). The variance of SINR in (28) then becomes:

$$\sigma^2_{\gamma_k} \approx M(M+1)\left(\frac{1}{K-3+\frac{1}{p_u}} - \frac{1}{K-2+\frac{1}{p_u}}\right) - \left(\frac{M}{K-2+\frac{1}{p_u}}\right)^2. \quad (31)$$

Note that (24) and (31) contain only basic arithmetic operators. Using the mean and variance of SINR in (24) and (31), we can determine the parameters of gamma distribution:

$$\beta = \frac{\sigma^2_{\gamma_k}}{m_{\gamma_k}}, \quad (32)$$

$$\alpha = \frac{m^2_{\gamma_k}}{\sigma^2_{\gamma_k}}. \quad (33)$$

Thus, for all massive MIMO systems (provided that $M > K \gg 1$), we can use a gamma distribution $G(\alpha, \beta)$ with α and β given by (32) and (33) for the SINR, in order to evaluate the performance of the system. Fig. 3 shows the simulated distribution of the SINR of a MU-MIMO system with MRC receiver with $p_u = 0$ dB for different values of M and K. The figure also depicts theoretical distribution of SINR obtained by (14) and its approximation given by (18). It can be seen that the derived theoretical expression follows exactly the simulation result. However the value given by approximation in (18) differs from the simulation for smaller number of users. For $K \gg 1$ the approximation of (18) is shown to be a very good approximation of the real distribution of the SINR.

In the next section we will use this approximation to evaluate some performance measures of massive MIMO systems.

IV. PERFORMANCE EVALUATION OF MASSIVE MIMO SYSTEMS WITH MRC RECEIVER

In this section we will use the approximate expression for SINR given by (18) to evaluate the performance of massive MIMO systems. Two principal performance parameters are considered: outage probability and BER.

A. OUTAGE PROBABILITY

Outage probability is an important measure of a communication system's performance when the system is subject to fading. It is defined as the probability that the instantaneous value of SINR falls under a given threshold [14]:

$$P_{\text{outage}}(\gamma_{th}) = \Pr(\gamma \le \gamma_{th}) = \int_0^{\gamma_{th}} f_{\gamma_k}(\gamma)d\gamma. \quad (34)$$

Since the SINR follows a gamma distribution, the outage probability is given by:

$$P_{\text{outage}}(\gamma_{th}) = \frac{\gamma(\alpha, \frac{\gamma_{th}}{\beta})}{\Gamma(\alpha)}, \quad (35)$$

where $\gamma(s,x) = \int_0^x t^{s-1}e^{-t}dt$ is the lower incomplete gamma function.

B. BIT ERROR RATE

Another important measure of a communication system's performance is its BER. In a slow flat fading communication channel, the BER can be calculated from the following integral [22]:

$$P_b = \int_0^\infty P_{b,\mathrm{AWGN}}(\gamma) f_\gamma(\gamma) d\gamma, \tag{36}$$

where $P_{b,\mathrm{AWGN}}$ is the BER for the additive white Gaussian noise (AWGN) channel. We consider a quadrature amplitude modulation (QAM), with $P_{b,\mathrm{AWGN}}$ given by [23]:

$$P_{b,\mathrm{AWGN}}(\gamma) \approx \frac{4\left(\sqrt{m}-1\right)}{\sqrt{m}\log_2(m)} Q(\sqrt{\frac{3\gamma}{m-1}}), \tag{37}$$

where m denotes the modulation order. Substituting (18) and (37) in (36) yields:

$$P_b \approx \frac{4\left(\sqrt{m}-1\right)}{\beta^\alpha \Gamma(\alpha)\sqrt{m}\log_2(m)} \int_0^\infty \gamma^{\alpha-1} e^{-\frac{\gamma}{\beta}} Q(\sqrt{\frac{3\gamma}{m-1}}) d\gamma. \tag{38}$$

The integral in (38) can be computed by numerical methods. We can also use the simpler approximation of $P_{b,\mathrm{AWGN}}$ given in [24] to obtain an approximate closed form expression for BER:

$$P_{b,\mathrm{AWGN}}(\gamma) \approx 0.2 e^{-\frac{1.5\gamma}{m-1}}. \tag{39}$$

Using this simpler approximation of $P_{b,\mathrm{AWGN}}$ instead of that given by (37), a closed form approximation for BER is obtained:

$$P_b \approx \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty 0.2 e^{-\frac{1.5\gamma}{m-1}} \gamma^{\alpha-1} e^{-\frac{\gamma}{\beta}} d\gamma = \frac{0.2}{\left(\frac{1.5\beta}{m-1}+1\right)^\alpha}. \tag{40}$$

V. NUMERICAL RESULTS

In this section we provide Monte Carlo simulation results in order to verify the proposed equations.[1] Fig. 3 showed the simulated distribution of the SINR of a massive MIMO system with $p_u = 0$ dB and for different values of M and K. The figure also includes the theoretical distribution obtained from (16) and the approximated distribution obtained from (18) with α and β given by (32) and (33). It can be seen that while the exact expression of (16) coincides perfectly with simulation results, the approximated distribution of (18) deviates from simulation results especially for smaller values of K.

Fig. 4 depicts the same distribution, this time for a MU-MIMO system with a 10-antenna BS serving 5 single-antenna users using MRC detection. Transmission power varies between −10 dB to 20 dB. Again, it can be seen that the exact expression of (16) always yields the correct distribution of SINR, regardless of values of M, K, and p_u. It can be seen furthermore that even for moderate values of K, if the transmission power is small enough, the approximation in (18)

[1] All Matlab codes to reproduce simulation results can be downloaded from https://github.com/hmeghdadi/massive_mimo1

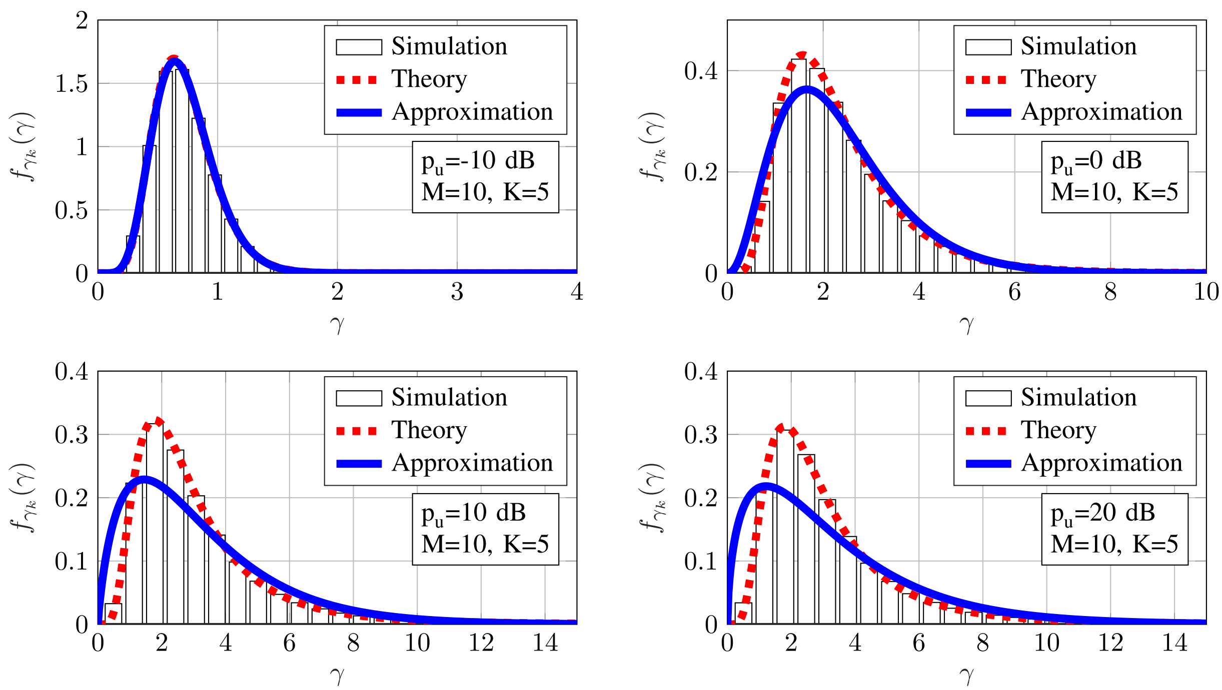

FIGURE 4. PDF of the SINR in a system with $M = 10$ and $K = 5$ for different values of p_u.

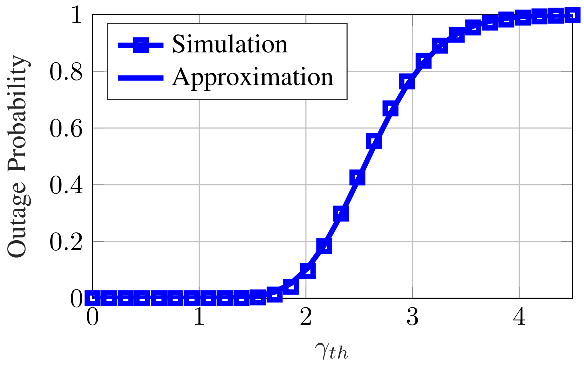

FIGURE 5. Outage probability for a system with $M = 100$ and $K = 40$.

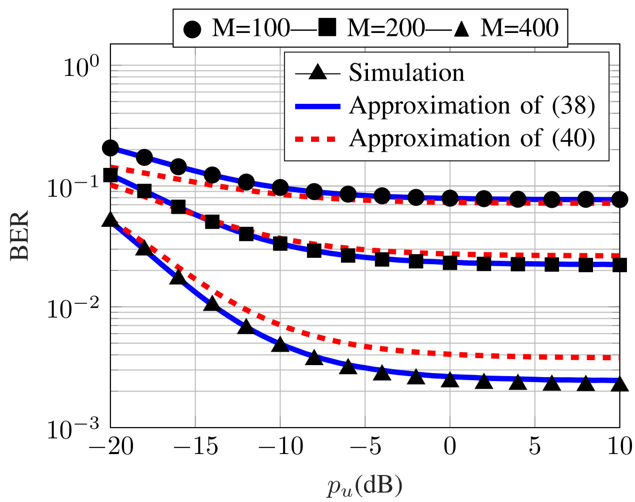

FIGURE 6. BER of a system with $K = 50$ and $M = 100, 200, 400$.

is still valid. It is also worth noting that for MRC receivers increasing the transmission power beyond a certain limit has no effect on the distribution of the SINR and cannot enhance the system performance. As a result, the approximation of [16] which holds only for large values of SNR, cannot be used to evaluate the performance of massive MIMO systems with MRC receivers where using powerful mobile transmitters is not practical and is inefficient. Fig. 5 shows the outage probability of a massive MIMO system with 100 BS antennas and 40 users. It can be seen that (35) is a very tight approximation of the outage probability. In Fig. 6, the 4-QAM BER of a massive MIMO system with 50 users is plotted for 100, 200, and 400 BS antennas. The figure shows simulated BER as well as the approximative expressions obtained from (38) and (40). Both theoretical curves use the gamma distribution of (18) for the PDF of SINR. However, while the solid-line curve uses the tighter approximation of (37) for $P_{b,\text{AWGN}}$, the dashed-line curve uses the less exact approximation of (39) for $P_{b,\text{AWGN}}$. The former results in a better evaluation of the system but necessitates the integral in (38) to be computed numerically. The simpler approximation of (39) leads to the closed-form expression of (40), but is a less accurate prediction of the BER.

VI. CONCLUSION

In this paper, an exact expression for the PDF of SINR of an MRC receiver for a MU-MIMO system is derived and verified by simulation. This expression is approximated for all massive MIMO systems (i.e. when $M > K \gg 1$), resulting in a gamma distribution with relatively simple expressions for its parameters. The approximate PDF is used to predict the outage probability and the BER of MRC receivers.

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