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Linear-Type Discontinuous Control of Fixed-Deviation Stabilization and Synchronization for Fractional-Order Neurodynamic Systems With Communication Delays

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ABSTRACT This paper aims to feature new results on fixed-deviation stabilization and synchronization in neurodynamic systems of fractional order. This paper attempts to propose novel control algorithms that are based on linear-type discontinuous control, on the approach of auxiliary functions, and on fractional calculus to solve the problems of fixed-deviation stabilization and synchronization. This paper remains the focus of attention for stabilization and synchronization from the discontinuous control point of view; namely, the system under consideration is subject to discontinuous control. This paper shows that as long as some system-parameter-dependent conditions hold, the fixed-deviation stabilization and synchronization in fractional-order neurodynamic systems with communication delays can still be guaranteed, while good performance can be achieved even with heavy interference. Moreover, the fixed-deviation degree can be directly derived and estimated. Examples, together with their respective numerical simulations, show that the developing results are verifiable.

INDEX TERMS Fractional-order systems, communication delay, fixed-deviation dynamics, discontinuous control.

I. INTRODUCTION

Recently, fractional calculus has received profound research attention in various fields of physics and engineering [1]–[6]. Originally, fractional calculus was a natural extension of conventional integer-order integrals and derivatives. Currently, the phenomenon cannot be described in terms of integer-order dynamics; for example, synchronous oscillation and propagation and the forward problem of neural activities demonstrate non-integer order dynamics [2]–[4]. Considering the fact that the real phenomena may be best characterized as fractional-order systems, a much more common way to investigate fractional-order systems is through linear approximations. Together with frequency-domain techniques, a feasible linear approximation for fractional-order systems can be obtained [4]. Another method is to get a numerical solution in the time domain with respect to fractional-order systems, which follows some characteristics of the systems considerably. However, in light of the high-dimensional character and infinite memory of the fractional-order operator, it naturally makes one reflect on whether we can better handle fractional-order systems in theory.

For control systems, network delay is ubiquitous [7]–[14]. A control model without the consideration of network delay could barely capture actual message transfer delay in an asynchronous environment. To be more precise, if network-induced delay is inherent and time-varying, then time-varying communication delays should be introduced into network-based analysis and design [14]. The time-delay problem in control systems is a challenging issue, specifically when the time-varying communication delay is considered for fractional-order systems, and few studies have been dedicated to it. Two tricky issues for the current development of functional fractional-order systems are the proper consideration of the fractional dynamics and the handling of the emerging time-delay effect.

Deviation dynamics are of the most supreme importance to the evolutive characteristics of control systems [14]–[17]. Unlike the classic stability behaviour, where plant control can be achieved in a fully synchronous or asynchronous mode, in the proposed controlled model, the evolution behaviour of the system under consideration will emerge in a deviation mode. Realizing prescribed deviation dynamics as oscillatory mechanisms in response to rhythmicity needs to be investigated intensively. Chen *et al.* [14] yielded desired fixed-deviation stability with a globally uniformly ultimate bound evoked by a state-dependent switch in complex control systems and revealed a tradeoff between the evolutive track accuracy and damped fixed-deviation stability. As declared in [16], further bridging the gap between nonlinear analysis applied to complex control systems as well as some of the basic deviation-related analytical principles, have yet to be resolved.

From a theoretical viewpoint, discontinuous control is very attractive, since ideally, it exactly compensates for the inner or outer indeterminacy and allows for rapid convergence [18]–[25]. An approach reported in many publications is the construction of an auxiliary continuous system, which can arbitrarily approximate the controlled discontinuous system by applying the well-known analytical results of the control system theory. However, discontinuous systems can have undesirable chattering or rendezvous problems not reproducible in a continuous system. Generally, it is very hard to design a suitable auxiliary continuous system that satisfies the discontinuous system currently considered [22], [24]. In such situations, the accuracy of the approximation is likely to be challenged.

Based on the discussion above, we advance the problems of fixed-deviation stabilization and synchronization of fractional-order neurodynamic systems with communication delays. To analyse fixed-deviation dynamics of such a fractional-order system with communication delays, we employ an approach that is based on linear-type discontinuous control by making good use of sign functions and symbolic dynamics. Furthermore, we propose the corresponding fixed-deviation stabilization and synchronization results under an essentially more general algebraic condition. Indeed, the algebraic criteria only need the parameter information of the system itself and some free parameters. As one future direction, which is essentially an extension of the obtained results, one can formulate and provide some small gain results about fixed-deviation dynamics for other kinds of fractional-order systems underlying our theoretical analysis.

II. PRELIMINARIES

Let us start by reviewing the fractional derivative.

The Caputo fractional derivative ${}^{C}D^{q}_{t_0}(\cdot)$ of $\mathcal{K}(t) \in \mathcal{C}^{\mathscr{M}+1}([t_0, +\infty), \Re)$ with order $q > 0$ is defined as

$$ {}^{C}D^{q}_{t_0}\mathcal{K}(t) = \frac{1}{\Gamma(\mathscr{M}-q)}\int_{t_0}^{t}\frac{\mathcal{K}^{(\mathscr{M})}(s)}{(t-s)^{q-\mathscr{M}+1}}\mathrm{d}s, \quad t \geq t_0, $$

where $\Gamma(\cdot)$ is the Gamma function, $\mathscr{M}-1 < q < \mathscr{M}$, $\mathscr{M}$ is a positive integer, and t_0 is the initial time.

The Riemann-Liouville fractional derivative ${}^{RL}D^{q}_{t_0}(\cdot)$ of $\mathcal{K}(t) \in \mathcal{C}^{\mathscr{M}+1}([t_0, +\infty), \Re)$ with order $q > 0$ is defined as

$$ {}^{RL}D^{q}_{t_0}\mathcal{K}(t) = \frac{1}{\Gamma(\mathscr{M}-q)}\frac{\mathrm{d}^{\mathscr{M}}}{\mathrm{d}t^{\mathscr{M}}}\int_{t_0}^{t}\frac{\mathcal{K}(s)}{(t-s)^{q-\mathscr{M}+1}}\mathrm{d}s, \quad t \geq t_0, $$

where $\Gamma(\cdot)$ is the Gamma function, $\mathscr{M}-1 < q < \mathscr{M}$, $\mathscr{M}$ is a positive integer, and t_0 is the initial time.

Combining with the definitions of Caputo fractional derivative and Riemann-Liouville fractional derivative, then for $0 < q < 1$, $\mathcal{K}(t) \in \mathcal{C}^{1}([t_0, +\infty), \Re)$, the following relation is obvious:

$$ {}^{C}D^{q}_{t_0}\mathcal{K}(t) = {}^{RL}D^{q}_{t_0}\mathcal{K}(t) - \frac{\mathcal{K}(t_0)}{\Gamma(1-q)}(t-t_0)^{-q}. $$

The above equality also reveals the inner-related of Caputo fractional derivative and Riemann-Liouville fractional derivative.

Consider a fractional-order neurodynamic system with communication delay governed by

$$ \begin{aligned} {}^{C}D^{q}_{t_0}z_i(t) = -a_i(t)z_i(t) + \sum_{j=1}^{n} b_{ij}(t)f_j(z_j(t)) \\ + \sum_{j=1}^{n} c_{ij}(t)g_j(z_j(t-\theta_{ij}(t))) + u_i(t), \\ i = 1, 2, \cdots, n, \quad (1) \end{aligned} $$

where fractional order $0 < q < 1$, $z_i(t)$ denotes the ith neuron state, $a_i(t) > 0$ is the self-inhibition of a neuron i, $b_{ij}(t)$ and $c_{ij}(t)$ signify the connection weight and the delay connection weight, respectively, $f_j(\cdot)$ and $g_j(\cdot)$ represent the neuron outputs at times t and $t-\theta_{ij}(t)$, respectively, $\theta_{ij}(t)$ is a time-varying communication delay with $0 \leq \theta_{ij}(t) \leq \theta$ (where θ is a positive constant), and $u_i(t)$ is the external input of the ith neuron.

Let $\mathcal{C}_\theta = \mathcal{C}([-\theta, 0], \Re^n)$ be the Banach space of continuous functions mapping $[-\theta, 0]$ into $\Re^n$. In $\Re^n$, the vector norm $\|\cdot\|$ is defined as ∞-norm. For $\phi \in \mathcal{C}_\theta$, $\|\phi\|_{\mathcal{C}} = \sup_{-\theta \leq s \leq 0}\|\theta(s)\|$. $\mathrm{sign}(\cdot)$ is the sign function.

For (1), the initial values are endowed as

$$ z_i(t_0 + s) = \phi_i(s), \quad -\theta \leq s \leq 0, \; i = 1, 2, \cdots, n, \quad (2) $$

where $\phi(s) = (\phi_1(s), \phi_2(s), \cdots, \phi_n(s))^T \in \mathcal{C}([-\theta, 0], \Re^n)$.

Let $\phi \in \mathcal{C}_\theta$, $z(t, t_0, \phi)$ represent the solution of system (1) starting from the initial value (2).

In addition, in (1), for $i = 1, 2, \cdots, n$, the output functions $f_i(\cdot)$ and $g_i(\cdot)$ satisfy $f_i(0) = g_i(0) = 0$ and

$$ \begin{aligned} &\left|f_i(\mathcal{X}) - f_i(\widetilde{\mathcal{X}})\right| \leq l_i\left|\mathcal{X} - \widetilde{\mathcal{X}}\right|, \\ &\left|g_i(\mathcal{X}) - g_i(\widetilde{\mathcal{X}})\right| \leq \ell_i\left|\mathcal{X} - \widetilde{\mathcal{X}}\right|, \quad \forall \mathcal{X} \in \Re, \; \forall \widetilde{\mathcal{X}} \in \Re, \quad (3) \end{aligned} $$

where $l_i > 0$, $\ell_i > 0$.

Next, we introduce some concepts that are needed later.

Consider the following functional fractional-order differential system

$$ {}^{C}D^{q}_{t_0}z(t) = \mathcal{F}(t, z_t), \quad (4) $$

where $q > 0$, $\mathcal{F} : [t_0, +\infty) \times \mathcal{C}_\theta \to \Re^n$.

Definition 1: If, for any $\varphi \in \mathcal{C}_\theta$, $\tilde{\varphi} \in \mathcal{C}_\theta$, $\varsigma > 0$, when $\|\varphi - \tilde{\varphi}\|_{\mathcal{C}} \le \varsigma$, there exists a constant $\mathcal{T}(\varsigma) \ge 0$, such that for all $t \ge t_0 + \mathcal{T}(\varsigma)$,

$$\|z(t, t_0, \varphi) - z(t, t_0, \tilde{\varphi})\| \le \omega,$$

then system (4) is said to be globally uniformly fixed-deviation stable, where $\omega > 0$ is a constant (ω is also called a fixed-deviation degree).

Remark 1: According to Definition 1, it is easy to see that fixed-deviation stability is weaker than traditional Lyapunov stability. Actually, what fixed-deviation stability should refer to is that as long as the difference value between two different initial states of system discussed is limited in scope, the difference between final values for system trajectories starting from such two initial states will be kept at a fixed bound.

Remark 2: In Definition 1, let $\tilde{\varphi} = z^*$ (z^* denotes the equilibrium point of system discussed), then one can get $z(t, t_0, \tilde{\varphi}) = z^*$. If $\|z(t, t_0, \varphi) - z(t, t_0, \tilde{\varphi})\| \le \omega$ (or write $\|z(t, t_0, \varphi) - z^*\| \le \omega$) holds for any $\omega > 0$, then fixed-deviation stability in Definition 1 will evolve into global uniform asymptotical stability in Lyapunov sense.

Definition 2: The zero solution of system (1), where $u_i(t) = 0$ $(i = 1, 2, \cdots, n)$, is globally uniformly fixed-deviation stable, if for any $\phi \in \mathcal{C}_\theta$, $\varsigma > 0$, when $\|\phi\|_{\mathcal{C}} \le \varsigma$, there exists a constant $\mathcal{T}(\varsigma) \ge 0$, such that for all $t \ge t_0 + \mathcal{T}(\varsigma)$,

$$\|z(t, t_0, \phi)\| \le \omega,$$

where $\omega > 0$ is a constant.

Definition 3: System (1) is globally uniformly fixed-deviation stabilizable if the controlled system of (1) is globally uniformly fixed-deviation stable under some suitable feedback control strategy.

Consider the drive system

$$^{C}D^{q}_{t_0} z(t) = \mathcal{G}(t, z_t), \tag{5}$$

and the response system

$$^{C}D^{q}_{t_0} \tilde{z}(t) = \tilde{\mathcal{G}}(t, \tilde{z}_t, \mathcal{U}(t)), \tag{6}$$

where $q > 0$, $\mathcal{G} : [t_0, +\infty) \times \mathcal{C}_\theta \to \Re^n$, $\tilde{\mathcal{G}} : [t_0, +\infty) \times \mathcal{C}_\theta \times \Re^n \to \Re^n$, and $\mathcal{U}(t)$ is the designed controller, defining $e(t) = \tilde{z}(t) - z(t)$, we can conclude that the error dynamics of (5) and (6) are written in the form

$$^{C}D^{q}_{t_0} e(t) = \mathcal{J}(t, e_t), \tag{7}$$

where $\mathcal{J}(t, e_t) = \tilde{\mathcal{G}}(t, \tilde{z}_t, \mathcal{U}(t)) - \mathcal{G}(t, z_t)$.

Definition 4: The drive-response systems (5) and (6) are said to be globally uniformly fixed-deviation synchronized, if the error system (7) is globally uniformly fixed-deviation stable.

We end this section with two commonly used lemmas.

Lemma 1: Let $0 < q < 1$. If $\mathcal{K}(t) \in \mathcal{C}^1[t_0, +\infty]$, then

$$^{C}D^{q}_{t_0} |\mathcal{K}(t)| \le \mathrm{sign}(\mathcal{K}(t))\, {}^{C}D^{q}_{t_0} \mathcal{K}(t), \quad t \ge t_0,$$

where

$$^{C}D^{q}_{t_0} |\mathcal{K}(t)| = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{\frac{\mathrm{d}}{\mathrm{d}\tau} |\mathcal{K}(\tau)|}{(t-\tau)^q} \mathrm{d}\tau,$$
$$\frac{\mathrm{d}}{\mathrm{d}t} |\mathcal{K}(t)| = \mathrm{sign}(\mathcal{K}(t)) \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{K}(t).$$

Remark 3: Lemma 1 has built a necessary bridge between $|\mathcal{K}(t)|$ of Caputo fractional-order derivative and $\mathcal{K}(t)$ of Caputo fractional-order derivative.

For more detailed discussion about Lemma 1, see the "Theorem 2" in [Nonlinear Analysis: Hybrid Systems, vol. 16, pp. 104-121, May 2015, http://dx.doi.org/10.1016/j.nahs.2014.10.001], or the "Lemma 4.3" in [Neural Networks, vol. 68, pp. 78-88, Aug. 2015, http://dx.doi.org/10.1016/j.neunet.2015.04.006].

Lemma 2: Let $0 < q < 1$. For any continuous differentiable function $\mathscr{L}(t) : [t_0, +\infty) \to [0, +\infty)$ and $\mathscr{M}(t) = (t - t_0 + \mathscr{P})^q \mathscr{L}(t)$, then

$$^{C}D^{q}_{t_0} \mathscr{M}(t) \le (t - t_0 + \mathscr{P})^q\, {}^{C}D^{q}_{t_0} \mathscr{L}(t) + \frac{1 + q - q^2}{\mathscr{P}^q\, \Gamma(2-q)} \overline{\mathscr{M}}(t), \quad t \ge t_0,$$

where $\mathscr{P} > 0$, $\overline{\mathscr{M}}(t) = \sup_{t_0 \le s \le t} \mathscr{M}(s)$.

Proof: Recalling the inner-related of Caputo fractional derivative and Riemann-Liouville fractional derivative, it follows that

$$\begin{aligned} ^{C}D^{q}_{t_0} \mathscr{M}(t) &= {}^{RL}D^{q}_{t_0} \mathscr{M}(t) - \frac{\mathscr{M}(t_0)}{\Gamma(1-q)} (t - t_0)^{-q} \\ &= {}^{RL}D^{q}_{t_0} \mathscr{M}(t) - \frac{\mathscr{P}^q \mathscr{L}(t_0)}{\Gamma(1-q)} (t - t_0)^{-q}. \end{aligned}$$

By mean of Leibniz rule for fractional differentiation,

$$\begin{aligned} ^{RL}D^{q}_{t_0} \mathscr{M}(t) &= {}^{RL}D^{q}_{t_0} \Big((t - t_0 + \mathscr{P})^q \mathscr{L}(t) \Big) \\ &\le (t - t_0 + \mathscr{P})^q\, {}^{RL}D^{q}_{t_0} \mathscr{L}(t) \\ &\quad + q^2 (t - t_0 + \mathscr{P})^{q-1}\, {}^{RL}D^{q-1}_{t_0} \mathscr{L}(t), \end{aligned}$$

then

$$\begin{aligned} &^{C}D^{q}_{t_0} \mathscr{M}(t) \\ &\le (t - t_0 + \mathscr{P})^q\, {}^{RL}D^{q}_{t_0} \mathscr{L}(t) \\ &\quad + q^2 (t - t_0 + \mathscr{P})^{q-1}\, {}^{RL}D^{q-1}_{t_0} \mathscr{L}(t) \\ &\quad - \frac{\mathscr{P}^q \mathscr{L}(t_0)}{\Gamma(1-q)} (t - t_0)^{-q} \\ &\le (t - t_0 + \mathscr{P})^q\, {}^{C}D^{q}_{t_0} \mathscr{L}(t) + (1 + 2q) \frac{\mathscr{L}(t_0)}{\Gamma(1-q)} \\ &\quad + q^2 (t - t_0 + \mathscr{P})^{q-1}\, {}^{RL}D^{q-1}_{t_0} \mathscr{L}(t) \\ &\le (t - t_0 + \mathscr{P})^q\, {}^{C}D^{q}_{t_0} \mathscr{L}(t) + (1 + 2q) \frac{\overline{\mathscr{M}}(t)}{\mathscr{P}^q\, \Gamma(1-q)} \\ &\quad + q^2 (t - t_0 + \mathscr{P})^{q-1}\, {}^{RL}D^{q-1}_{t_0} \mathscr{L}(t) \end{aligned}$$

for $t \ge t_0$.

On the other hand,

$$\begin{aligned}
&q^2(t-t_0+\mathscr{P})^{q-1}\,{}^{RL}D_{t_0}^{q-1}\mathscr{L}(t)\\
&=\frac{q^2}{\Gamma(1-q)}(t-t_0+\mathscr{P})^{q-1}\int_{t_0}^{t}(t-s)^{-q}\mathscr{L}(s)\mathrm{d}s\\
&\le\frac{q^2}{\mathscr{P}^q\,\Gamma(2-q)}\overline{\mathscr{M}}(t),
\end{aligned}$$

and hence,

$$\begin{aligned}
&{}^{C}D_{t_0}^{q}\mathscr{M}(t)\\
&\le(t-t_0+\mathscr{P})^q\,{}^{C}D_{t_0}^{q}\mathscr{L}(t)\\
&\quad+(1+2q)\frac{\overline{\mathscr{M}}(t)}{\mathscr{P}^q\,\Gamma(1-q)}\\
&\quad+\frac{q^2}{\mathscr{P}^q\,\Gamma(2-q)}\overline{\mathscr{M}}(t)\\
&=(t-t_0+\mathscr{P})^q\,{}^{C}D_{t_0}^{q}\mathscr{L}(t)\\
&\quad+(1+2q)(1-q)\frac{\overline{\mathscr{M}}(t)}{\mathscr{P}^q\,\Gamma(2-q)}\\
&\quad+\frac{q^2}{\mathscr{P}^q\,\Gamma(2-q)}\overline{\mathscr{M}}(t)\\
&=(t-t_0+\mathscr{P})^q\,{}^{C}D_{t_0}^{q}\mathscr{L}(t)+\frac{1+q-q^2}{\mathscr{P}^q\,\Gamma(2-q)}\overline{\mathscr{M}}(t)
\end{aligned}$$

for $t\ge t_0$.

Remark 4: The idea of proof about Lemma 2 is similar as [14, Lemma 3.1]. Nevertheless, some typing mistakes for mathematical terms exist in the original text [14]. Actually, more and more detailed explanation about Lemma 2, see the "Theorem 1" in [SpringerPlus, vol. 5, Article No. 1034, Jul. 2016, https://doi.org/10.1186/s40064-016-2374-3]. In this literature, the derivation process (7)-(11) in regard to "Theorem 1" in [SpringerPlus, vol. 5, Article No. 1034, Jul. 2016, https://doi.org/10.1186/s40064-016-2374-3] also builds a complete analysis.

III. MAIN RESULTS

In this section, we discuss the issues of fixed-deviation stabilization and synchronization in detail.

A. FIXED-DEVIATION STABILIZATION

For (1), the design of the linear-type discontinuous control is as follows

$$u_i(t)=\mathcal{P}_iz_i(t)+\mathcal{Q}_i\mathrm{sign}\big(z_i(t)\big),\quad i=1,2,\cdots,n,\tag{8}$$

where $\mathcal{P}_i$ and $\mathcal{Q}_i$ are control gains.

Under controller (8), system (1) is transformed into

$$\begin{aligned}
&{}^{C}D_{t_0}^{q}z_i(t)\\
&=-a_i(t)z_i(t)+\sum_{j=1}^{n}b_{ij}(t)f_j(z_j(t))\\
&\quad+\sum_{j=1}^{n}c_{ij}(t)g_j(z_j(t-\theta_{ij}(t)))\\
&\quad+\mathcal{P}_iz_i(t)+\mathcal{Q}_i\mathrm{sign}\big(z_i(t)\big),\quad i=1,2,\cdots,n.
\end{aligned}\tag{9}$$

Theorem 1: If there exist positive constants $\xi_i>0$ ($i\in\{1,2,\cdots,n\}$) and $\varrho>\theta\ge0$, for $i=1,2,\cdots,n$, $t\ge t_0$, and the following conditions

$$\begin{aligned}
&a_i(t)-|\mathcal{P}_i|-\frac{1+q-q^2}{\varrho^q\,\Gamma(2-q)}\\
&\quad-\frac{1}{\xi_i}\sum_{j=1}^{n}\Big[\big|b_{ij}(t)\big|\,l_j+\big|c_{ij}(t)\big|\,\ell_j\Big(\frac{\varrho}{\varrho-\theta}\Big)^q\Big]\xi_j>0
\end{aligned}\tag{10}$$

hold, then system (9) is globally uniformly fixed-deviation stable, that is, under the linear-type discontinuous control law (8), system (1) is globally uniformly fixed-deviation stabilizable, where the fixed-deviation degree is

$$\omega=\frac{\Xi\,\|\xi\|}{\Theta}\tag{11}$$

with

$$\begin{aligned}
\Theta=\min_{1\le i\le n}\Big(&a_i(t)-|\mathcal{P}_i|-\frac{1+q-q^2}{\varrho^q\,\Gamma(2-q)}\\
&-\frac{1}{\xi_i}\sum_{j=1}^{n}\Big[\big|b_{ij}(t)\big|\,l_j+\big|c_{ij}(t)\big|\,\ell_j\Big(\frac{\varrho}{\varrho-\theta}\Big)^q\Big]\xi_j\Big)
\end{aligned}$$

and

$$\Xi=\max_{1\le i\le n}\Big(\frac{|\mathcal{Q}_i|}{\xi_i}\Big).$$

Proof: To construct the auxiliary function

$$\mathcal{M}(t)=\max\Big\{\frac{|z_i(t)|}{\xi_i},\ i=1,2,\cdots,n\Big\}.$$

Let

$$\mathcal{W}(t)=(t-t_0+\varrho)^q\mathcal{M}(t)\quad\text{and}\ \overline{\mathcal{W}}(t)=\sup_{t_0-\varrho\le s\le t}\mathcal{W}(s).$$

Obviously, it follows that both of $\mathcal{M}(t)$ and $\mathcal{W}(t)$ are positive definite, $\overline{\mathcal{W}}(t)$ is monotone increasing in regard to t.

For any given $t\ge t_0$, there must be a $\nu\in\{1,2,\cdots,n\}$ such that

$$\mathcal{M}(t)=\frac{|z_\nu(t)|}{\xi_\nu}.$$

By Lemma 1, calculating the Caputo fractional derivative of $\mathcal{M}(t)$ along the trajectory of (9) yields

$$\begin{aligned}
&{}^{C}D_{t_0}^{q}\mathcal{M}(t)\\
&=\frac{1}{\xi_\nu}\,{}^{C}D_{t_0}^{q}\,|z_\nu(t)|\\
&\le\frac{\mathrm{sign}\big(z_\nu(t)\big)}{\xi_\nu}\,{}^{C}D_{t_0}^{q}z_\nu(t)\\
&\le\frac{-(a_\nu(t)-|\mathcal{P}_\nu|)}{\xi_\nu}z_\nu(t)+\frac{1}{\xi_\nu}\sum_{j=1}^{n}\big|b_{\nu j}(t)\big|\,l_j\,\big|z_j(t)\big|\\
&\quad+\frac{1}{\xi_\nu}\sum_{j=1}^{n}\big|c_{\nu j}(t)\big|\,\ell_j\,\big|z_j(t-\theta_{\nu j}(t))\big|+\frac{|\mathcal{Q}_\nu|}{\xi_\nu}\\
&\le-(a_\nu(t)-|\mathcal{P}_\nu|)\mathcal{M}(t)+\frac{1}{\xi_\nu}\sum_{j=1}^{n}\big|b_{\nu j}(t)\big|\,l_j\xi_j\mathcal{M}(t)\\
&\quad+\frac{1}{\xi_\nu}\sum_{j=1}^{n}\big|c_{\nu j}(t)\big|\,\ell_j\xi_j\mathcal{M}(t-\theta_{\nu j}(t))+\frac{|\mathcal{Q}_\nu|}{\xi_\nu}.
\end{aligned}\tag{12}$$

Using Lemma 2, from (12), then

$$
\begin{aligned}
&{}^{C}D_{t_0}^{q}\mathcal{W}(t)\\
&\le (t-t_0+\varrho)^q\,{}^{C}D_{t_0}^{q}\mathcal{M}(t)+\frac{1+q-q^2}{\varrho^q\,\Gamma(2-q)}\overline{\mathcal{W}}(t)\\
&\le\Big[-(a_\nu(t)-|\mathcal{P}_\nu|)+\frac{1}{\xi_\nu}\sum_{j=1}^{n}\big|b_{\nu j}(t)\big|\,l_j\xi_j\Big]\mathcal{W}(t)\\
&\quad+\frac{1}{\xi_\nu}\sum_{j=1}^{n}\big|c_{\nu j}(t)\big|\,\ell_j\xi_j\Big(\frac{t-t_0+\varrho}{t-\theta_{\nu j}(t)-t_0+\varrho}\Big)^q\mathcal{W}(t-\theta_{\nu j}(t))\\
&\quad+(t-t_0+\varrho)^q\frac{|\mathcal{Q}_\nu|}{\xi_\nu}+\frac{1+q-q^2}{\varrho^q\,\Gamma(2-q)}\overline{\mathcal{W}}(t)\\
&\le\Big[-(a_\nu(t)-|\mathcal{P}_\nu|)+\frac{1}{\xi_\nu}\sum_{j=1}^{n}\big|b_{\nu j}(t)\big|\,l_j\xi_j\Big]\mathcal{W}(t)\\
&\quad+\frac{1}{\xi_\nu}\sum_{j=1}^{n}\big|c_{\nu j}(t)\big|\,\ell_j\xi_j\Big(\frac{t-t_0+\varrho}{t-\theta_{\nu j}(t)-t_0+\varrho}\Big)^q\sup_{t_0-\varrho\le s\le t}\mathcal{W}(s)\\
&\quad+(t-t_0+\varrho)^q\frac{|\mathcal{Q}_\nu|}{\xi_\nu}+\frac{1+q-q^2}{\varrho^q\,\Gamma(2-q)}\overline{\mathcal{W}}(t)\\
&=\Big[-(a_\nu(t)-|\mathcal{P}_\nu|)+\frac{1}{\xi_\nu}\sum_{j=1}^{n}\big|b_{\nu j}(t)\big|\,l_j\xi_j\Big]\mathcal{W}(t)\\
&\quad+\frac{1}{\xi_\nu}\sum_{j=1}^{n}\big|c_{\nu j}(t)\big|\,\ell_j\xi_j\Big(\frac{t-t_0+\varrho}{t-\theta_{\nu j}(t)-t_0+\varrho}\Big)^q\overline{\mathcal{W}}(t)\\
&\quad+(t-t_0+\varrho)^q\frac{|\mathcal{Q}_\nu|}{\xi_\nu}+\frac{1+q-q^2}{\varrho^q\,\Gamma(2-q)}\overline{\mathcal{W}}(t).
\end{aligned}
$$

Note that $\dfrac{\varrho+\mathscr{X}}{-\theta_{\nu j}(t)+\varrho+\mathscr{X}}$ is monotone non-increasing with respect to $\mathscr{X}\ge 0$, hence

$$\frac{t-t_0+\varrho}{t-\theta_{\nu j}(t)-t_0+\varrho}\le\frac{\varrho}{-\theta_{\nu j}(t)+\varrho},$$

on the other hand,

$$\frac{\varrho}{-\theta_{\nu j}(t)+\varrho}\le\frac{\varrho}{-\theta+\varrho},$$

whereupon,

$$\frac{t-t_0+\varrho}{t-\theta_{\nu j}(t)-t_0+\varrho}\le\frac{\varrho}{-\theta+\varrho},$$

and then

$$
\begin{aligned}
&{}^{C}D_{t_0}^{q}\mathcal{W}(t)\\
&\quad\le\Big[-(a_\nu(t)-|\mathcal{P}_\nu|)+\frac{1+q-q^2}{\varrho^q\,\Gamma(2-q)}\Big]\mathcal{W}(t)\\
&\qquad+\frac{1}{\xi_\nu}\sum_{j=1}^{n}\Big[\big|b_{\nu j}(t)\big|\,l_j+\big|c_{\nu j}(t)\big|\,\ell_j\Big(\frac{\varrho}{\varrho-\theta}\Big)^q\Big]\xi_j\mathcal{W}(t)\\
&\qquad+(t-t_0+\varrho)^q\frac{|\mathcal{Q}_\nu|}{\xi_\nu}\\
&\quad\le-\Theta\mathcal{W}(t)+(t-t_0+\varrho)^q\,\Xi,
\end{aligned}
\tag{13}
$$

when $\overline{\mathcal{W}}(t)=\mathcal{W}(t)$ for $t\ge t_0$, where

$$
\begin{aligned}
\Theta\triangleq\min_{1\le i\le n}\Big(&a_i(t)-|\mathcal{P}_i|-\frac{1+q-q^2}{\varrho^q\,\Gamma(2-q)}\\
&-\frac{1}{\xi_i}\sum_{j=1}^{n}\Big[\big|b_{ij}(t)\big|\,l_j+\big|c_{ij}(t)\big|\,\ell_j\Big(\frac{\varrho}{\varrho-\theta}\Big)^q\Big]\xi_j\Big)
\end{aligned}
$$

and

$$\Xi\triangleq\max_{1\le i\le n}\Big(\frac{|\mathcal{Q}_i|}{\xi_i}\Big).$$

In the following, for any $t\ge t_0$, we are going to consider three cases to prove global uniform fixed-deviation stabilization.

According to the definition $\overline{\mathcal{W}}(t)=\sup_{t_0-\varrho\le s\le t}\mathcal{W}(s)$, that is, $\overline{\mathcal{W}}(t)$ is the maximum value of $\mathcal{W}(s)$ in $[t_0-\varrho,t]$, then three situations may be happen: (1) The maximum value is just at time t_0; (2) The maximum value is mutative, which changes with time; (3) The maximum value is just at one moment in (t_0,t).

Case 1: $\overline{\mathcal{W}}(s)>\mathcal{W}(s)$ for any $t_0<s\le t$. Then, it follows that

$$\overline{\mathcal{W}}(t)=\overline{\mathcal{W}}(t_0),\quad\forall t\ge t_0.$$

Consequently,

$$
\begin{aligned}
\|z(t)\|&\le\|\xi\|\,\mathcal{M}(t)=\frac{\|\xi\|}{(t-t_0+\varrho)^q}\mathcal{W}(t)\\
&\le\frac{\|\xi\|}{(t-t_0+\varrho)^q}\overline{\mathcal{W}}(t)\\
&=\frac{\|\xi\|}{(t-t_0+\varrho)^q}\overline{\mathcal{W}}(t_0)\\
&\le\frac{\|\xi\|\,\varrho^q}{(t-t_0+\varrho)^q\xi_{min}}\|\phi\|_{\mathcal{C}}\\
&\le\frac{\|\xi\|\,\varrho^q\varsigma}{(t-t_0+\varrho)^q\xi_{min}},
\end{aligned}
$$

when $\|\phi\|_{\mathcal{C}}\le\varsigma$, where $\xi_{min}=\min_{1\le i\le n}\{\xi_i\}$.

Case 2: $\overline{\mathcal{W}}(t)=\mathcal{W}(t)$. One can conclude that for $t\ge t_0$,

$$
\begin{aligned}
&\int_{t_0}^{t}\frac{\overline{\mathcal{W}}'(s)-\mathcal{W}'(s)}{(t-s)^q}\mathrm{d}s\\
&\quad=\lim_{s\to t_-}\frac{\overline{\mathcal{W}}(s)-\mathcal{W}(s)}{(t-s)^q}\\
&\qquad-\frac{\overline{\mathcal{W}}(t_0)-\mathcal{W}(t_0)}{(t-t_0)^q}-q\int_{t_0}^{t}\frac{\overline{\mathcal{W}}(s)-\mathcal{W}(s)}{(t-s)^{q+1}}\mathrm{d}s\\
&\quad=\lim_{s\to t_-}\frac{1}{q}\Big[\overline{\mathcal{W}}'(s)-\mathcal{W}'(s)\Big](t-s)^{1-q}(-1)\\
&\qquad-\frac{\overline{\mathcal{W}}(t_0)-\mathcal{W}(t_0)}{(t-t_0)^q}-q\int_{t_0}^{t}\frac{\overline{\mathcal{W}}(s)-\mathcal{W}(s)}{(t-s)^{q+1}}\mathrm{d}s\\
&\quad=-\frac{\overline{\mathcal{W}}(t_0)-\mathcal{W}(t_0)}{(t-t_0)^q}-q\int_{t_0}^{t}\frac{\overline{\mathcal{W}}(s)-\mathcal{W}(s)}{(t-s)^{q+1}}\mathrm{d}s\\
&\quad\le 0,
\end{aligned}
$$

which implies that

$$ {}^{C}D_{t_0}^{q}\overline{\mathcal{W}}(t) \le {}^{C}D_{t_0}^{q}\mathcal{W}(t), \quad t \ge t_0. \tag{14}$$

Next, we show that

$$\mathcal{M}(t) \le \frac{\Xi}{\Theta}, \quad t \ge t_0. \tag{15}$$

Otherwise, by virtue of (13) and (14), for $t \ge t_0$,

$$\begin{aligned} {}^{C}D_{t_0}^{q}\overline{\mathcal{W}}(t) &\le {}^{C}D_{t_0}^{q}\mathcal{W}(t) \\ &\le -\Theta\mathcal{W}(t) + (t - t_0 + \varrho)^q\ \Xi \\ &= -\Theta(t - t_0 + \varrho)^q\ \mathcal{M}(t) + (t - t_0 + \varrho)^q\ \Xi \\ &< 0. \end{aligned}$$

Recall that $\overline{\mathcal{W}}(t)$ is monotone increasing in regard to t, so $\overline{\mathcal{W}}'(t) \ge 0$, then

$$ {}^{C}D_{t_0}^{q}\overline{\mathcal{W}}(t) = \frac{1}{\Gamma(1-q)}\int_{t_0}^{t}\frac{\overline{\mathcal{W}}'(s)}{(t-s)^q}\mathrm{d}s \ge 0, \quad t \ge t_0,$$

which is a contradiction. Thus, it follows that (15) is true. Then we get

$$\|z(t)\| \le \|\xi\|\,\mathcal{M}(t) \le \frac{\Xi\,\|\xi\|}{\Theta}, \quad t \ge t_0.$$

Case 3: $\overline{\mathcal{W}}(\breve{t}) = \mathcal{W}(\breve{t})$, $t_0 \le \breve{t} < t$, and $\overline{\mathcal{W}}(s) > \mathcal{W}(s)$, $\forall\, s \in (\breve{t}, t]$.

Applying a similar argument as that used in Case 1, we finally obtain that

$$\mathcal{M}(\breve{t}) \le \frac{\Xi}{\Theta}$$

and

$$\begin{aligned} \mathcal{W}(t) &< \overline{\mathcal{W}}(t) = \overline{\mathcal{W}}(\breve{t}) = \mathcal{W}(\breve{t}) \\ &= (\breve{t} - t_0 + \varrho)^q\mathcal{M}(\breve{t}) \le (\breve{t} - t_0 + \varrho)^q\frac{\Xi}{\Theta}. \end{aligned}$$

Hence, for $t \ge t_0$, one has

$$\begin{aligned} \|z(t)\| &\le \|\xi\|\,\mathcal{M}(t) = \frac{\|\xi\|}{(t - t_0 + \varrho)^q}\mathcal{W}(t) \\ &\le \frac{\|\xi\|}{(t - t_0 + \varrho)^q}(\breve{t} - t_0 + \varrho)^q\frac{\Xi}{\Theta} \\ &\le \frac{\Xi\,\|\xi\|}{\Theta}. \end{aligned}$$

In summary, let

$$\mathcal{T}(\varsigma) = \max\left\{\left[\left(\frac{\Theta\ \varsigma}{\Xi\ \xi_{min}}\right)^{\frac{1}{q}} - 1\right]\varrho,\ 0\right\},$$

then

$$\|z(t)\| \le \frac{\Xi\,\|\xi\|}{\Theta} \triangleq \omega$$

for all $t \ge t_0 + \mathcal{T}(\varsigma)$, when $\|\phi\|_{\mathcal{C}} \le \varsigma$. Therefore, it can be deduced that system (9) is globally uniformly fixed-deviation stable, that is, under the linear-type discontinuous control law (8), system (1) is globally uniformly fixed-deviation stabilizable.

Corollary 1: If there are $\xi_i > 0$ ($i \in \{1, 2, \cdots, n\}$), for $i = 1, 2, \cdots, n$, $t \ge t_0$, and the following conditions

$$a_i(t) - |\mathcal{P}_i| - \frac{1}{\xi_i}\sum_{j=1}^{n}\Big[\big|b_{ij}(t)\big|\, l_j + \big|c_{ij}(t)\big|\,\ell_j\Big]\xi_j > 0 \tag{16}$$

hold, then system (9) is globally uniformly fixed-deviation stable, that is, under the linear-type discontinuous control law (8), system (1) is globally uniformly fixed-deviation stabilizable.

Proof: Let

$$\begin{aligned} \mathscr{H}_i(\chi) &= a_i(t) - |\mathcal{P}_i| - \frac{1+q-q^2}{\chi^q\,\Gamma(2-q)} \\ &\quad - \frac{1}{\xi_i}\sum_{j=1}^{n}\Big[\big|b_{ij}(t)\big|\, l_j + \big|c_{ij}(t)\big|\,\ell_j\Big(\frac{\chi}{\chi-\theta}\Big)^q\Big]\xi_j \end{aligned} \tag{17}$$

for $i = 1, 2, \cdots, n$, $t \ge t_0$, $\chi > \theta$, then

$$\begin{aligned} \lim_{\chi\to+\infty}\mathscr{H}_i(\chi) &= a_i(t) - |\mathcal{P}_i| \\ &\quad - \frac{1}{\xi_i}\sum_{j=1}^{n}\Big[\big|b_{ij}(t)\big|\, l_j + \big|c_{ij}(t)\big|\,\ell_j\Big]\xi_j > 0. \end{aligned}$$

By the property of limits, we can find a constant $\varrho > \theta$, such that (10) holds. As previously discussed in Theorem 1, we can deduce that system (9) is globally uniformly fixed-deviation stable, that is, under the linear-type discontinuous control law (8), system (1) is globally uniformly fixed-deviation stabilizable.

B. FIXED-DEVIATION SYNCHRONIZATION

In this subsection, we consider another type of fractional-order neurodynamic system (1) described as

$$\begin{aligned} {}^{C}D_{t_0}^{q}z_i(t) &= -a_i(t)z_i(t) + \sum_{j=1}^{n}b_{ij}(t)f_j(z_j(t)) \\ &\quad + \sum_{j=1}^{n}c_{ij}(t)g_j(z_j(t-\theta_{ij}(t))), \quad i = 1, 2, \cdots, n. \end{aligned} \tag{18}$$

Let (18) be the drive system. When we consider the unidirectional coupled identical synchronization, the response system is given by

$$\begin{aligned} {}^{C}D_{t_0}^{q}\mathcal{Z}_i(t) &= -a_i(t)\mathcal{Z}_i(t) + \sum_{j=1}^{n}b_{ij}(t)f_j(\mathcal{Z}_j(t)) \\ &\quad + \sum_{j=1}^{n}c_{ij}(t)g_j(\mathcal{Z}_j(t-\theta_{ij}(t))) + u_i(t), \\ &\qquad i = 1, 2, \cdots, n, \end{aligned} \tag{19}$$

where $u_i(t)$ $(i = 1, 2, \cdots, n)$ are control inputs to be designed.

The initial values of system (19) are endowed as

$$\mathcal{Z}_i(t_0+s)=\tilde{\phi}_i(s), \quad -\theta\le s\le 0,\ i=1,2,\cdots,n,$$

where $\tilde{\phi}(s)=(\tilde{\phi}_1(s),\tilde{\phi}_2(s),\cdots,\tilde{\phi}_n(s))^T\in\mathcal{C}([-\theta,0],\Re^n)$.

Define $\mathcal{E}_i(t)=\mathcal{Z}_i(t)-z_i(t)$, $i\in\{1,2,\cdots,n\}$. Then, the error dynamics of (18) and (19) can be expressed by

$$\begin{aligned}{}^CD_{t_0}^q\mathcal{E}_i(t)&=-a_i(t)\mathcal{E}_i(t)+\sum_{j=1}^n b_{ij}(t)f_j(\mathcal{E}_j(t))\\&\quad+\sum_{j=1}^n c_{ij}(t)g_j(\mathcal{E}_j(t-\theta_{ij}(t)))+u_i(t),\\&\qquad i=1,2,\cdots,n,\end{aligned}\tag{20}$$

where

$$\begin{aligned}f_j(\mathcal{E}_j(t))&=f_j(\mathcal{Z}_j(t))-f_j(z_j(t)),\\g_j(\mathcal{E}_j(t-\theta_{ij}(t)))&=g_j(\mathcal{Z}_j(t-\theta_{ij}(t)))-g_j(z_j(t-\theta_{ij}(t))).\end{aligned}$$

Obviously, according to the drive-response coupled theory, we write the initial values of system (20) in the form

$$\mathcal{E}_i(t_0+s)=\tilde{\phi}_i(s)-\phi_i(s), \quad -\theta\le s\le 0,\ i=1,2,\cdots,n.$$

Denote

$$\mathcal{E}_i(t_0+s)=\mathscr{E}_i(s), \quad -\theta\le s\le 0,\ i=1,2,\cdots,n,$$

clearly, $\mathscr{E}(s)=(\mathscr{E}_1(s),\mathscr{E}_2(s),\cdots,\mathscr{E}_n(s))^T\in\mathcal{C}([-\theta,0],\Re^n)$.

For the synchronous control scheme of (20), the linear-type discontinuous control is designed as follows

$$u_i(t)=\mathcal{P}_i\mathcal{E}_i(t)+\mathcal{Q}_i\mathrm{sign}\big(\mathcal{E}_i(t)\big), \quad i=1,2,\cdots,n, \tag{21}$$

where $\mathcal{P}_i$ and $\mathcal{Q}_i$ are control gains.

Under the controller (21), system (20) is transformed into

$$\begin{aligned}{}^CD_{t_0}^q\mathcal{E}_i(t)&=-a_i(t)\mathcal{E}_i(t)+\sum_{j=1}^n b_{ij}(t)f_j(\mathcal{E}_j(t))\\&\quad+\sum_{j=1}^n c_{ij}(t)g_j(\mathcal{E}_j(t-\theta_{ij}(t)))\\&\quad+\mathcal{P}_i\mathcal{E}_i(t)+\mathcal{Q}_i\mathrm{sign}\big(\mathcal{E}_i(t)\big),\\&\qquad i=1,2,\cdots,n.\end{aligned}\tag{22}$$

Theorem 2: If there exist positive constants $\xi_i>0$ ($i\in\{1,2,\cdots,n\}$) and $\varrho>\theta\ge 0$, and if for $i=1,2,\cdots,n$, $t\ge t_0$, and the following conditions

$$\begin{aligned}&a_i(t)-|\mathcal{P}_i|-\frac{1+q-q^2}{\varrho^q\,\Gamma(2-q)}\\&\quad-\frac{1}{\xi_i}\sum_{j=1}^n\left[\big|b_{ij}(t)\big|\,l_j+\big|c_{ij}(t)\big|\,\ell_j\left(\frac{\varrho}{\varrho-\theta}\right)^q\right]\xi_j>0\end{aligned}\tag{23}$$

hold, then the error system (22) is globally uniformly fixed-deviation stable, i.e., the drive-response systems (18) and (19) are globally uniformly fixed-deviation synchronized under the linear-type discontinuous control law (21), where the fixed-deviation degree is

$$\omega=\frac{\Xi\,\|\xi\|}{\Theta}\tag{24}$$

with

$$\begin{aligned}\Theta=\min_{1\le i\le n}\Bigg(&a_i(t)-|\mathcal{P}_i|-\frac{1+q-q^2}{\varrho^q\,\Gamma(2-q)}\\&-\frac{1}{\xi_i}\sum_{j=1}^n\left[\big|b_{ij}(t)\big|\,l_j+\big|c_{ij}(t)\big|\,\ell_j\left(\frac{\varrho}{\varrho-\theta}\right)^q\right]\xi_j\Bigg)\end{aligned}$$

and

$$\Xi=\max_{1\le i\le n}\left(\frac{|\mathcal{Q}_i|}{\xi_i}\right).$$

Proof: To construct the auxiliary function

$$\mathcal{M}(t)=\max\left\{\frac{|\mathcal{E}_i(t)|}{\xi_i},\ i=1,2,\cdots,n\right\}.$$

Let

$$\mathcal{W}(t)=(t-t_0+\varrho)^q\mathcal{M}(t) \quad\text{and}\quad \overline{\mathcal{W}}(t)=\sup_{t_0-\varrho\le s\le t}\mathcal{W}(s).$$

Obviously, it follows that both of $\mathcal{M}(t)$ and $\mathcal{W}(t)$ are positive definite, $\overline{\mathcal{W}}(t)$ is monotone increasing in regard to t.

For any given $t\ge t_0$, there must be a $\nu\in\{1,2,\cdots,n\}$ such that

$$\mathcal{M}(t)=\frac{|\mathcal{E}_\nu(t)|}{\xi_\nu}.$$

Together with (22), using Lemma 1, we calculate the Caputo fractional derivative of $\mathcal{M}(t)$, then

$$\begin{aligned}&{}^CD_{t_0}^q\mathcal{M}(t)\\&=\frac{1}{\xi_\nu}{}^CD_{t_0}^q\,|\mathcal{E}_\nu(t)|\\&\le\frac{\mathrm{sign}\big(\mathcal{E}_\nu(t)\big)}{\xi_\nu}{}^CD_{t_0}^q\mathcal{E}_\nu(t)\\&\le\frac{-(a_\nu(t)-|\mathcal{P}_\nu|)}{\xi_\nu}\mathcal{E}_\nu(t)+\frac{1}{\xi_\nu}\sum_{j=1}^n\big|b_{\nu j}(t)\big|\,l_j\,\big|\mathcal{E}_j(t)\big|\\&\quad+\frac{1}{\xi_\nu}\sum_{j=1}^n\big|c_{\nu j}(t)\big|\,\ell_j\,\big|\mathcal{E}_j(t-\theta_{\nu j}(t))\big|+\frac{|\mathcal{Q}_\nu|}{\xi_\nu}\\&\le-(a_\nu(t)-|\mathcal{P}_\nu|)\mathcal{M}(t)+\frac{1}{\xi_\nu}\sum_{j=1}^n\big|b_{\nu j}(t)\big|\,l_j\xi_j\mathcal{M}(t)\\&\quad+\frac{1}{\xi_\nu}\sum_{j=1}^n\big|c_{\nu j}(t)\big|\,\ell_j\xi_j\mathcal{M}(t-\theta_{\nu j}(t))+\frac{|\mathcal{Q}_\nu|}{\xi_\nu}.\end{aligned}\tag{25}$$

Using Lemma 2, from (25), then

$$\begin{aligned}&{}^CD_{t_0}^q\mathcal{W}(t)\\&\le(t-t_0+\varrho)^q\,{}^CD_{t_0}^q\mathcal{M}(t)+\frac{1+q-q^2}{\varrho^q\,\Gamma(2-q)}\overline{\mathcal{W}}(t)\\&\le\left[-(a_\nu(t)-|\mathcal{P}_\nu|)+\frac{1}{\xi_\nu}\sum_{j=1}^n\big|b_{\nu j}(t)\big|\,l_j\xi_j\right]\mathcal{W}(t)\\&\quad+\frac{1}{\xi_\nu}\sum_{j=1}^n\big|c_{\nu j}(t)\big|\,\ell_j\xi_j\left(\frac{t-t_0+\varrho}{t-\theta_{\nu j}(t)-t_0+\varrho}\right)^q\mathcal{W}(t-\theta_{\nu j}(t))\\&\quad+(t-t_0+\varrho)^q\frac{|\mathcal{Q}_\nu|}{\xi_\nu}+\frac{1+q-q^2}{\varrho^q\,\Gamma(2-q)}\overline{\mathcal{W}}(t)\end{aligned}$$

$$\leq \left[-(a_\nu(t)-|\mathcal{P}_\nu|)+\frac{1}{\xi_\nu}\sum_{j=1}^{n}\left|b_{\nu j}(t)\right| l_j\xi_j\right]\mathcal{W}(t)$$
$$+\frac{1}{\xi_\nu}\sum_{j=1}^{n}\left|c_{\nu j}(t)\right|\ell_j\xi_j\left(\frac{t-t_0+\varrho}{t-\theta_{\nu j}(t)-t_0+\varrho}\right)^q\sup_{t_0-\varrho\leq s\leq t}\mathcal{W}(s)$$
$$+(t-t_0+\varrho)^q\frac{|\mathcal{Q}_\nu|}{\xi_\nu}+\frac{1+q-q^2}{\varrho^q\,\Gamma(2-q)}\overline{\mathcal{W}}(t)$$
$$=\left[-(a_\nu(t)-|\mathcal{P}_\nu|)+\frac{1}{\xi_\nu}\sum_{j=1}^{n}\left|b_{\nu j}(t)\right| l_j\xi_j\right]\mathcal{W}(t)$$
$$+\frac{1}{\xi_\nu}\sum_{j=1}^{n}\left|c_{\nu j}(t)\right|\ell_j\xi_j\left(\frac{t-t_0+\varrho}{t-\theta_{\nu j}(t)-t_0+\varrho}\right)^q\overline{\mathcal{W}}(t)$$
$$+(t-t_0+\varrho)^q\frac{|\mathcal{Q}_\nu|}{\xi_\nu}+\frac{1+q-q^2}{\varrho^q\,\Gamma(2-q)}\overline{\mathcal{W}}(t).$$

Note that $\dfrac{\varrho+\mathscr{X}}{-\theta_{\nu j}(t)+\varrho+\mathscr{X}}$ is monotone non-increasing with respect to $\mathscr{X}\geq 0$, hence

$$\frac{t-t_0+\varrho}{t-\theta_{\nu j}(t)-t_0+\varrho}\leq\frac{\varrho}{-\theta_{\nu j}(t)+\varrho},$$

on the other hand,

$$\frac{\varrho}{-\theta_{\nu j}(t)+\varrho}\leq\frac{\varrho}{-\theta+\varrho},$$

whereupon,

$$\frac{t-t_0+\varrho}{t-\theta_{\nu j}(t)-t_0+\varrho}\leq\frac{\varrho}{-\theta+\varrho},$$

and then

$$\begin{aligned}
&{}^{C}D_{t_0}^{q}\mathcal{W}(t)\\
&\quad\leq\left[-(a_\nu(t)-|\mathcal{P}_\nu|)+\frac{1+q-q^2}{\varrho^q\,\Gamma(2-q)}\right]\mathcal{W}(t)\\
&\qquad+\frac{1}{\xi_\nu}\sum_{j=1}^{n}\left[\left|b_{\nu j}(t)\right| l_j+\left|c_{\nu j}(t)\right|\ell_j\left(\frac{\varrho}{\varrho-\theta}\right)^q\right]\xi_j\mathcal{W}(t)\\
&\qquad+(t-t_0+\varrho)^q\frac{|\mathcal{Q}_\nu|}{\xi_\nu}\\
&\quad\leq-\Theta\mathcal{W}(t)+(t-t_0+\varrho)^q\,\Xi,
\end{aligned}\tag{26}$$

when $\overline{\mathcal{W}}(t)=\mathcal{W}(t)$ for $t\geq t_0$, where

$$\Theta\triangleq\min_{1\leq i\leq n}\Bigg(a_i(t)-|\mathcal{P}_i|-\frac{1+q-q^2}{\varrho^q\,\Gamma(2-q)}-\frac{1}{\xi_i}\sum_{j=1}^{n}\left[\left|b_{ij}(t)\right| l_j+\left|c_{ij}(t)\right|\ell_j\left(\frac{\varrho}{\varrho-\theta}\right)^q\right]\xi_j\Bigg)$$

and

$$\Xi\triangleq\max_{1\leq i\leq n}\left(\frac{|\mathcal{Q}_i|}{\xi_i}\right).$$

In the following, for any $t\geq t_0$, we are going to consider three cases to prove global uniform fixed-deviation synchronization.

According to the definition $\overline{\mathcal{W}}(t)=\sup_{t_0-\varrho\leq s\leq t}\mathcal{W}(s)$, that is, $\overline{\mathcal{W}}(t)$ is the maximum value of $\mathcal{W}(s)$ in $[t_0-\varrho,t]$, then three situations may be happen: (1) The maximum value is just at time t_0; (2) The maximum value is mutative, which changes with time; (3) The maximum value is just at one moment in (t_0,t).

Case 1: $\overline{\mathcal{W}}(s)>\mathcal{W}(s)$ for any $t_0<s\leq t$. Then, it follows that

$$\overline{\mathcal{W}}(t)=\overline{\mathcal{W}}(t_0),\quad\forall t\geq t_0.$$

Consequently,

$$\begin{aligned}
\|\mathcal{E}(t)\|&\leq\|\xi\|\,\mathcal{M}(t)=\frac{\|\xi\|}{(t-t_0+\varrho)^q}\mathcal{W}(t)\\
&\leq\frac{\|\xi\|}{(t-t_0+\varrho)^q}\overline{\mathcal{W}}(t)\\
&=\frac{\|\xi\|}{(t-t_0+\varrho)^q}\overline{\mathcal{W}}(t_0)\\
&\leq\frac{\|\xi\|\,\varrho^q}{(t-t_0+\varrho)^q\xi_{min}}\|\mathscr{E}\|_{\mathcal{C}}\\
&\leq\frac{\|\xi\|\,\varrho^q\varsigma}{(t-t_0+\varrho)^q\xi_{min}},
\end{aligned}$$

when $\|\mathscr{E}\|_{\mathcal{C}}\leq\varsigma$, where $\xi_{min}=\min_{1\leq i\leq n}\{\xi_i\}$.

Case 2: $\overline{\mathcal{W}}(t)=\mathcal{W}(t)$. One can conclude that for $t\geq t_0$,

$$\begin{aligned}
&\int_{t_0}^{t}\frac{\overline{\mathcal{W}}'(s)-\mathcal{W}'(s)}{(t-s)^q}\mathrm{d}s\\
&\quad=\lim_{s\to t_-}\frac{\overline{\mathcal{W}}(s)-\mathcal{W}(s)}{(t-s)^q}\\
&\qquad-\frac{\overline{\mathcal{W}}(t_0)-\mathcal{W}(t_0)}{(t-t_0)^q}-q\int_{t_0}^{t}\frac{\overline{\mathcal{W}}(s)-\mathcal{W}(s)}{(t-s)^{q+1}}\mathrm{d}s\\
&\quad=\lim_{s\to t_-}\frac{1}{q}\left[\overline{\mathcal{W}}'(s)-\mathcal{W}'(s)\right](t-s)^{1-q}(-1)\\
&\qquad-\frac{\overline{\mathcal{W}}(t_0)-\mathcal{W}(t_0)}{(t-t_0)^q}-q\int_{t_0}^{t}\frac{\overline{\mathcal{W}}(s)-\mathcal{W}(s)}{(t-s)^{q+1}}\mathrm{d}s\\
&\quad=-\frac{\overline{\mathcal{W}}(t_0)-\mathcal{W}(t_0)}{(t-t_0)^q}-q\int_{t_0}^{t}\frac{\overline{\mathcal{W}}(s)-\mathcal{W}(s)}{(t-s)^{q+1}}\mathrm{d}s\\
&\quad\leq 0,
\end{aligned}$$

which implies that

$${}^{C}D_{t_0}^{q}\overline{\mathcal{W}}(t)\leq{}^{C}D_{t_0}^{q}\mathcal{W}(t),\quad t\geq t_0.\tag{27}$$

Next, we show that

$$\mathcal{M}(t)\leq\frac{\Xi}{\Theta},\quad t\geq t_0.\tag{28}$$

Otherwise, by virtue of (26) and (27), for $t\geq t_0$,

$$\begin{aligned}
{}^{C}D_{t_0}^{q}\overline{\mathcal{W}}(t)&\leq{}^{C}D_{t_0}^{q}\mathcal{W}(t)\\
&\leq-\Theta\mathcal{W}(t)+(t-t_0+\varrho)^q\,\Xi\\
&=-\Theta(t-t_0+\varrho)^q\,\mathcal{M}(t)+(t-t_0+\varrho)^q\,\Xi\\
&<\ 0.
\end{aligned}$$

Recall that $\overline{\mathcal{W}}(t)$ is monotone increasing in regard to t, so $\overline{\mathcal{W}}'(t) \geq 0$, then

$$^{C}D_{t_0}^{q}\overline{\mathcal{W}}(t) = \frac{1}{\Gamma(1-q)}\int_{t_0}^{t}\frac{\overline{\mathcal{W}}'(s)}{(t-s)^q}\mathrm{d}s \geq 0, \quad t \geq t_0,$$

which is a contradiction. Thus, it follows that (28) is true. Then we get

$$\|\mathcal{E}(t)\| \leq \|\xi\|\,\mathcal{M}(t) \leq \frac{\Xi\,\|\xi\|}{\Theta}, \quad t \geq t_0.$$

Case 3: $\overline{\mathcal{W}}(\breve{t}) = \mathcal{W}(\breve{t})$, $t_0 \leq \breve{t} < t$, and $\overline{\mathcal{W}}(s) > \mathcal{W}(s)$, $\forall\, s \in (\breve{t}, t]$.

Applying a similar argument to Case 1, we finally obtain that

$$\mathcal{M}(\breve{t}) \leq \frac{\Xi}{\Theta}$$

and

$$\begin{aligned}\mathcal{W}(t) &< \overline{\mathcal{W}}(t) = \overline{\mathcal{W}}(\breve{t}) = \mathcal{W}(\breve{t}) \\ &= (\breve{t} - t_0 + \varrho)^q \mathcal{M}(\breve{t}) \leq (\breve{t} - t_0 + \varrho)^q \frac{\Xi}{\Theta}.\end{aligned}$$

Hence, for $t \geq t_0$, one has

$$\begin{aligned}\|\mathcal{E}(t)\| &\leq \|\xi\|\,\mathcal{M}(t) = \frac{\|\xi\|}{(t-t_0+\varrho)^q}\mathcal{W}(t) \\ &\leq \frac{\|\xi\|}{(t-t_0+\varrho)^q}(\breve{t}-t_0+\varrho)^q\frac{\Xi}{\Theta} \\ &\leq \frac{\Xi\,\|\xi\|}{\Theta}.\end{aligned}$$

In summary, let

$$\mathcal{T}(\varsigma) = \max\left\{\left[\left(\frac{\Theta\,\varsigma}{\Xi\,\xi_{min}}\right)^{\frac{1}{q}} - 1\right]\varrho,\ 0\right\},$$

then

$$\|\mathcal{E}(t)\| \leq \frac{\Xi\,\|\xi\|}{\Theta} \triangleq \omega$$

for all $t \geq t_0 + \mathcal{T}(\varsigma)$, when $\|\mathscr{E}\|_{\mathcal{C}} \leq \varsigma$. Therefore, it can be deduced that error system (22) is globally uniformly fixed-deviation stable, namely, the drive-response systems (18) and (19) are globally uniformly fixed-deviation synchronized under the linear-type discontinuous control law (21).

Corollary 2: If there are $\xi_i > 0$ ($i \in \{1, 2, \cdots, n\}$), for $i = 1, 2, \cdots, n$, $t \geq t_0$, and the following conditions

$$a_i(t) - |\mathcal{P}_i| - \frac{1}{\xi_i}\sum_{j=1}^{n}\left[\left|b_{ij}(t)\right| l_j + \left|c_{ij}(t)\right| \ell_j\right]\xi_j > 0 \tag{29}$$

hold, then error system (22) is globally uniformly fixed-deviation stable, i.e., the drive-response systems (18) and (19) are globally uniformly fixed-deviation synchronized under the linear-type discontinuous control law (21).

Proof: Let

$$\begin{aligned}\mathscr{H}_i(\chi) = a_i(t) - |\mathcal{P}_i| - \frac{1+q-q^2}{\chi^q\,\Gamma(2-q)} \\ - \frac{1}{\xi_i}\sum_{j=1}^{n}\left[\left|b_{ij}(t)\right| l_j + \left|c_{ij}(t)\right| \ell_j\left(\frac{\chi}{\chi-\theta}\right)^q\right]\xi_j\end{aligned} \tag{30}$$

for $i = 1, 2, \cdots, n$, $t \geq t_0$, $\chi > \theta$, then

$$\lim_{\chi\to+\infty}\mathscr{H}_i(\chi) = a_i(t) - |\mathcal{P}_i| - \frac{1}{\xi_i}\sum_{j=1}^{n}\left[\left|b_{ij}(t)\right| l_j + \left|c_{ij}(t)\right| \ell_j\right]\xi_j > 0.$$

By the property of limits, we can find a constant $\varrho > \theta$, such that (23) holds. As previously discussed in Theorem 2, we can then deduce that error system (22) is globally uniformly fixed-deviation stable, i.e., the drive-response systems (18) and (19) are globally uniformly fixed-deviation synchronized under the linear-type discontinuous control law (21).

Remark 5: Theorems 1 and 2 offer some sufficient conditions for fixed-deviation stabilization and synchronization in fractional-order neurodynamic systems with communication delays from the linear-type discontinuous control point of view, which provides a new angle of view and a meaningful reference to analyse and control fractional-order time-delay systems. This result indicates that even for fractional-order continuous systems, fixed-deviation stabilization and synchronization can still be guaranteed via discontinuous control.

Remark 6: Note that condition (10) in Theorem 1 and condition (23) in Theorem 2 contain the free parameter ϱ. Indeed, it is also a crucial step to find the suitable free parameter ϱ to satisfy (10) or (23). In addition, if we use another condition that the free parameter ϱ is abandoned, then condition (10) or (23) can be improved. This way, we derive Corollaries 1 and 2.

Remark 7: Currently, there are many results on the dynamics of fractional-order systems from the nonlinear control point of view, for instance, see [4], [6]. Unfortunately, few studies have been reported about the fixed-deviation dynamics of fractional-order systems [14]. Taking this situation into consideration, we admit the feasibility and suitability of discontinuous control, provided fixed-deviation stabilization and synchronization in fractional-order time-delay systems. In addition, it is worth noting that the developing results might be able to be applied to the fixed-deviation dynamics of more general fractional-order systems.

IV. TWO NUMERICAL EXAMPLES

In this section, the superiority of the derived theoretical results is analysed by two illustrative examples as well as the relevant simulation results.

Example 1: We discuss a fractional-order neurodynamic system as follows

$$\begin{aligned}^{C}D_{t_0}^{q}z_1(t) &= -5z_1(t) + \tanh(z_1(t)) + \tanh(z_2(t)) \\ &\quad + \tanh(z_1(t-1)) + \tanh(z_2(t-1)) + u_1(t), \\ ^{C}D_{t_0}^{q}z_2(t) &= -7z_2(t) + 1.5\tanh(z_1(t)) + 1.5\tanh(z_2(t)) \\ &\quad + 1.5\tanh(z_1(t-1)) + \tanh(z_2(t-1)) + u_2(t),\end{aligned} \tag{31}$$

where $q = 0.9$, $t_0 = 0$.

From Theorem 1, it is required that there exist positive constants $\xi_1 > 0$, $\xi_2 > 0$, and $\varrho > 1$ to satisfy

$$5 - |\mathcal{P}_1| - \frac{1.09}{\varrho^{0.9}\,\Gamma(1.1)} - \frac{1}{\xi_1}\left[(\xi_1+\xi_2) + \left(\frac{\varrho}{\varrho-1}\right)^{0.9}(\xi_1+\xi_2)\right] > 0, \tag{32}$$

$$7 - |\mathcal{P}_2| - \frac{1.09}{\varrho^{0.9}\,\Gamma(1.1)} - \frac{1}{\xi_2}\left[1.5(\xi_1+\xi_2) + 1.5\left(\frac{\varrho}{\varrho-1}\right)^{0.9}\xi_1 + \left(\frac{\varrho}{\varrho-1}\right)^{0.9}\xi_2\right] > 0. \tag{33}$$

If $\xi_1 = 1$, $\xi_2 = 1$, $\varrho = 10$, $|\mathcal{P}_1| = 0.6$, and $|\mathcal{P}_2| = 1.1$ are chosen, then (32) and (33) hold. Moreover, if $|\mathcal{Q}_1| = 0.03$ and $|\mathcal{Q}_2| = 0.03$ are selected, it can be deduced that the fixed-deviation degree is

$$\omega = \frac{0.03}{0.0309} = 0.9709.$$

The fixed-deviation stabilization of system (31) under the linear-type discontinuous control law $u_1(t) = 0.6z_1(t) + 0.03\text{sign}(z_1(t))$, and $u_2(t) = 1.1z_2(t) + 0.03\text{sign}(z_2(t))$ is shown in Figure 1.

Remark 8: For Example 1, if the controlled system is asymptotically stabilizable in Lyapunov sense, then all the system trajectories will asymptotically converge to the origin. By contrast to Figure 1, system trajectories always keep around the origin at a distance (≤ 0.9709). This distance is exactly deviation degree.

Example 2: We consider another fractional-order neurodynamic system whose dynamics are described as

$$\begin{aligned} {}^{C}D^{q}_{t_0} z_1(t) &= -4z_1(t) + 2\sin(z_1(t)) - 0.2\sin(z_2(t)) \\ &\quad - 1.5\sin(z_1(t-1)) + 0.2\sin(z_2(t-1)), \\ {}^{C}D^{q}_{t_0} z_2(t) &= -3z_2(t) - 0.9\sin(z_1(t)) + 0.5\sin(z_2(t)) \\ &\quad - 0.3\sin(z_1(t-1)) - \sin(z_2(t-1)), \end{aligned} \tag{34}$$

where $q = 0.95$, $t_0 = 0$.

System (34) has a chaotic attractor, which is portrayed in Figure 2.

Let (34) be the drive system. When we consider the unidirectional coupled identical synchronization, the response system is given by

$$\begin{aligned} &{}^{C}D^{q}_{t_0} \mathcal{Z}_1(t) \\ &\quad = -4\mathcal{Z}_1(t) + 2\sin(\mathcal{Z}_1(t)) - 0.2\sin(\mathcal{Z}_2(t)) \\ &\qquad - 1.5\sin(\mathcal{Z}_1(t-1)) + 0.2\sin(\mathcal{Z}_2(t-1)) + u_1(t), \\ &{}^{C}D^{q}_{t_0} \mathcal{Z}_2(t) \\ &\quad = -3\mathcal{Z}_2(t) - 0.9\sin(\mathcal{Z}_1(t)) + 0.5\sin(\mathcal{Z}_2(t)) \\ &\qquad - 0.3\sin(\mathcal{Z}_1(t-1)) - \sin(\mathcal{Z}_2(t-1)) + u_2(t), \end{aligned} \tag{35}$$

where $q = 0.95$, $t_0 = 0$.

Denote $\mathcal{E}_1(t) = \mathcal{Z}_1(t) - z_1(t)$ and $\mathcal{E}_2(t) = \mathcal{Z}_2(t) - z_2(t)$ as the synchronization errors.

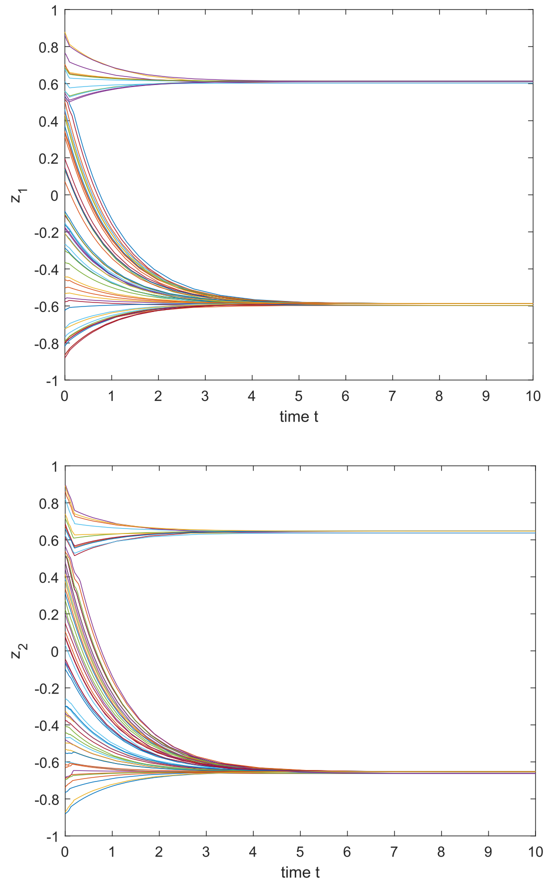

FIGURE 1. The fixed-deviation dynamics of system (31) under the linear-type discontinuous control law $u_1(t) = 0.6z_1(t) + 0.03\text{sign}(z_1(t))$, $u_2(t) = 1.1z_2(t) + 0.03\text{sign}(z_2(t))$.

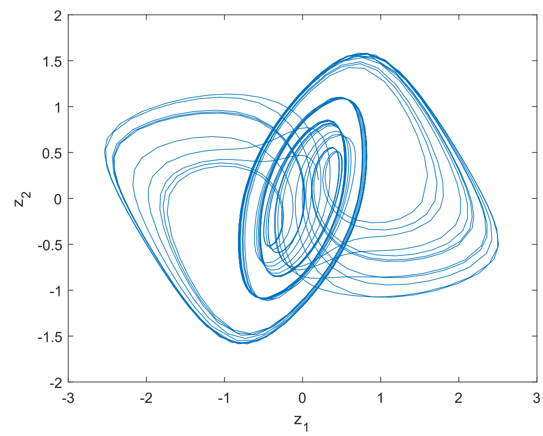

FIGURE 2. Chaotic attractor of system (34).

From Theorem 2, it is required that there exist positive constants $\xi_1 > 0$, $\xi_2 > 0$, and $\varrho > 1$ to satisfy

$$4 - |\mathcal{P}_1| - \frac{1.0475}{\varrho^{0.95}\,\Gamma(1.05)} - \frac{1}{\xi_1}\left[2\xi_1 + 0.2\xi_2 + 1.5\xi_1\left(\frac{\varrho}{\varrho-1}\right)^{0.95} + 0.2\xi_2\left(\frac{\varrho}{\varrho-1}\right)^{0.95}\right] > 0, \tag{36}$$

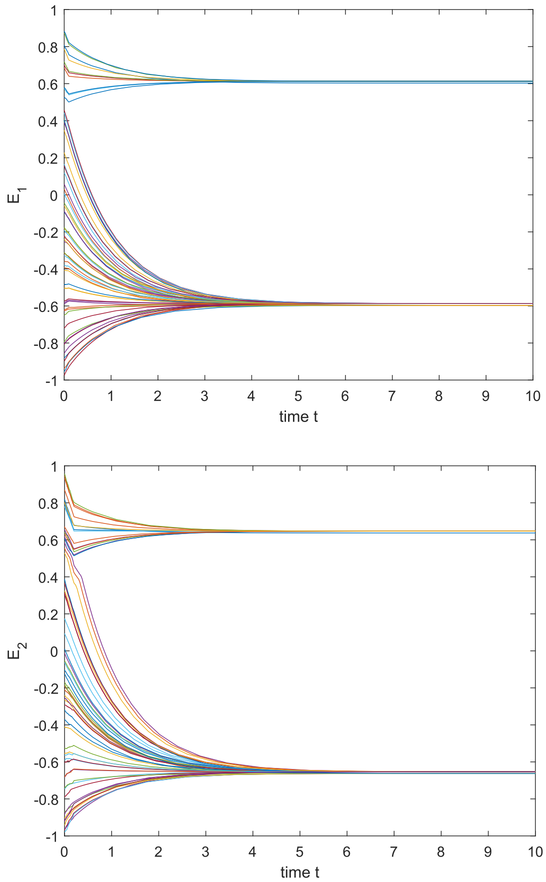

FIGURE 3. **The error dynamics of the drive-response systems (34) and (35) under the linear-type discontinuous control law $u_1(t) = 0.005\mathcal{E}_1(t)+ 0.0006\mathrm{sign}(\mathcal{E}_1(t))$, $u_2(t) = 0.21\mathcal{E}_2(t) + 0.0006\mathrm{sign}(\mathcal{E}_2(t))$.**

$$3 - |\mathcal{P}_2| - \frac{1.0475}{\varrho^{0.95}\,\Gamma(1.05)} - \frac{1}{\xi_2}\left[0.9\xi_1 + 0.5\xi_2 + 0.3\xi_1\left(\frac{\varrho}{\varrho-1}\right)^{0.95} + \xi_2\left(\frac{\varrho}{\varrho-1}\right)^{0.95}\right] > 0. \tag{37}$$

If $\xi_1 = 1$, $\xi_2 = 1$, $\varrho = 30$, $|\mathcal{P}_1| = 0.005$, and $|\mathcal{P}_2| = 0.21$ are chosen, then (36) and (37) hold. Moreover, if $|\mathcal{Q}_1| = 0.0006$ and $|\mathcal{Q}_2| = 0.0006$, it can be deduced that the fixed-deviation degree is

$$\omega = \frac{0.0006}{0.0007} = 0.8571.$$

Obviously, the drive-response systems (34) and (35) are globally uniformly fixed-deviation synchronized under the linear-type discontinuous control law $u_1(t) = 0.005\mathcal{E}_1(t) + 0.0006\mathrm{sign}\big(\mathcal{E}_1(t)\big)$ and $u_2(t) = 0.21\mathcal{E}_2(t) + 0.0006\mathrm{sign}\big(\mathcal{E}_2(t)\big)$. Accordingly, the error dynamics are depicted in Figure 3.

Remark 9: For Example 2, if the drive-response systems are asymptotically synchronized in Lyapunov sense, then all the trajectories in error system will asymptotically converge to the origin. But by contrasting Figure 3, the trajectories for error system always keep around the origin at a distance (≤ 0.8571). This distance is also exactly deviation degree.

Remark 10: Taken together Examples 1 and 2, combining with the numerical simulation results, it is easy to understand the differences between fixed-deviation stability and asymptotical stability in Lyapunov sense. True, fixed-deviation dynamics is a meaningful theme, which could be a sign of deeper problems and should be investigated.

V. CONCLUDING REMARKS

In this paper, we discuss linear-type discontinuous controls for the fixed-deviation stabilization and synchronization of functional fractional-order neurodynamic systems, and we establish some criteria about fixed-deviation stabilization and synchronization from a cybernatic angle. The key concerns are the integration of discontinuous effects to fractional-order control systems and the balance between fixed-deviation dynamics and fractional operators. To overcome these problems, the construction of auxiliary functions and the time-varying Lyapunov method are crucial. The present analysis framework may be extended to address the fixed-deviation dynamics of other functional fractional-order differential systems, which would be an interesting development in future works.

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