IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received July 31, 2018, accepted September 5, 2018, date of publication September 17, 2018, date of current version October 12, 2018.

Digital Object Identifier 10.1109/ACCESS.2018.2869414

Models for Safety Assessment of Construction Project With Some 2-Tuple Linguistic Pythagorean Fuzzy Bonferroni Mean Operators

XIUMEI DENG[1], GUIWU WEI[2], HUI GAO[2], AND JIE WANG[2]
[1]College of Mathematics and Software Science, Sichuan Normal University, Chengdu 610066, China
[2]School of Business, Sichuan Normal University, Chengdu 610101, China

Corresponding author: Jie Wang (jw970326@163.com)

This work was supported in part by the National Natural Science Foundation of China under Grant 71571128, in part by the Humanities and Social Sciences Foundation of Ministry of Education of China under Grant 17XJA630003, and in part by the Construction Plan of Scientific Research Innovation Team for Colleges and Universities in Sichuan Province under Grant 15TD0004.

ABSTRACT For this paper, to consider both Bonferroni mean (BM) operator and two-tuple linguistic Pythagorean fuzzy numbers (2TLPFNs), we combine the weighted BM (WBM) operator, the generalized WBM (GWBM) operator, and the dual GWBM operator with 2TLPFNs to propose the two-tuple linguistic Pythagorean WBM (2TLPFWBM) operator, the 2TLPFWGBM operator, the generalized 2TLPFWBM (G2TLPFWBM) operator, the generalized 2TLPFWGBM operator, the dual G2TLPFWBM operator, and the dual G2TLPFWGBM operator. Then, some MADM procedures are developed based on these operators. At last, an applicable example for a safety assessment of construction project is given.

INDEX TERMS MADM, 2TLPFSs, 2TLPFWBM operator, G2TLPFWBM operator, DG2TLPFWBM operator, safety assessment, construction project.

I. INTRODUCTION

Pythagorean fuzzy set (PFS) [1], [2] is the generalization of the intuitionistic fuzzy set(IFS) which the square sum of the membership degree(MD) and the non-membership degree(NMD) is equal or less than 1. The PFS is proved to be more flexible than IFS since it has greater space than that of IFS. Hence, PFS can deal with the situations that can't be dealt with by the IFS. The PFS has received more and more attention, which has been investigated and applied broadly [3]–[12]. Some MADM methods in PFS have been developed. TOPSIS method was extended to solve MADM problems in PFS [13]–[18]. Ren *et al.* [19] developed the Pythagorean fuzzy TODIM approach which consider the DMs' psychological behaviors. Zhang [20] proposed the hierarchical QUALIFLEX approach in PFS. Chen [21] developed the Pythagorean fuzzy VIKOR models. Peng and Dai [22] studied the Pythagorean fuzzy stochastic MADM with prospect theory. Xue *et al.* [23] studied the Pythagorean Fuzzy LINMAP model with entropy theory. Wan *et al.* [24] studied Pythagorean fuzzy mathematical programming mean for MAGDM with Pythagorean fuzzy truth degrees. Garg [25] proposed linear programming for MADM with interval-valued PFS. Liang *et al.* [26] gave the projection method for MAGDM with PFNs based on GBM. Peng and Yang [27] proposed the MABAC Method for MAGDM with PFNs. Some information measures were investigated by many scholars [10], [22], [28]–[32]. Information aggregation operators are of great importance to the application of MADM and decision support [33]–[49]. The PFS had been generalized to accommodate interval values [50], linguistic arguments [51]–[55], hesitant fuzzy value [56], [57], etc.

Although, PFSs theory has been broadly applied to many domians, however, all the above approaches are unsuitable to depict the MD and NMD of an element to a set by 2TLSs. In order to overcome this issue, we develop the definition of 2TLPFSs based on the PFS [1], [2] and 2TLSs [58]. And Bonferroni mean (BM) [59]–[61] is a famous aggregation operator which can depict interrelationships between any two arguments. Thus, the BM can supply a flexible mode to deal with the information fusion problem to solve MADM problems. Because 2-tuple linguistic Pythagorean fuzzy numbers (2TLPFNs) can easily describe the fuzzy information, and the BM can capture interrelationships between any two arguments, it is necessary to extend the BM operator to deal with the 2TLPFNs. The paper's goal is to expend some BM operators with 2TLPFNs, then to study some properties of these operators, and applied them to cope with the MADM with 2TLPFNs.

The structure of our article is organized as follows. Section 2 develops the 2TLPFSs. Section 3 combines

2TLPFNs with WBM operator to develop 2TLPFWBM and 2TLPFWGBM operators. Section 4 combines 2TLPFNs with GWBM operator to develop G2TLPFWBM and G2TLPFWGBM operators. Section 5 combines 2TLPFNs with DGWBM operator to develop DG2TLPFWBM and DG2TLPFWGBM operators. Section 6 briefly introduces an application for safety assessment of construction project. Conclusions are given in Section 7.

II. PRELIMINARIES

In this section, we briefly introduce some fundamental concepts and theories of the 2TLPFSs [16] based on the PFS [1], [2] and 2TLSs [58].

A. 2TLSs

Definition 1 [58]: Let $l_1, l_2, \ldots, l_t$ be a linguistic term set. Any label l_i shows a possible linguistic variable, and l is defined:

$$l = \left\{ \begin{array}{l} l_0 = extremely\ bad, l_1 = very\ bad, l_2 = bad, \\ l_3 = medium, l_4 = good, l_5 = very\ good, \\ l_6 = extremely\ good. \end{array} \right\}$$

B. PFSs

Let X be a space of points (objects)x. A Pythagorean fuzzy set (PFS) is characterized as following [1], [2]:

$$A = \{\langle x, u_A(x), v_A(x)\rangle | x \in X\} \quad (1)$$

where the MD function $u_A(x)$ and NMD $v_A(x)$ are nonnegative real number in [0,1], that is, $u_A(x) : X \to [0,1]$, $v_A(x) : X \to [0,1]$. And the sum of $u_A(x)$ and $v_A(x)$ satisfy the condition $(u_A(x))^2 + (v_A(x))^2 \le 1$.

C. 2TLPFSS

Definition 2: Assume that $P = \{p_0, p_1, \ldots, p_t\}$ is a 2TLSs. If $p = \{(s_\phi, \varphi), (s_\theta, \vartheta)\}$ is defined for $(s_\phi, \varphi), (s_\theta, \vartheta) \in P$ *and* $\varphi, \vartheta \in [0, t]$, where (s_ϕ, φ)*and*(s_θ, ϑ) express independently the MD and NMD by 2TLSs, then 2TLPFSs is:

$$p = \left\{(s_\phi, \varphi), (s_\theta, \vartheta)\right\} \quad (2)$$

where $0 \le \frac{\Delta^{-1}(s_\phi,\varphi)}{t} \le 1$, $0 \le \frac{\Delta^{-1}(s_\theta,\vartheta)}{t} \le 1$, and $0 \le \left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2 + \left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^2 \le 1$.

Definition 3: Assume that $p = \left\{(s_\phi, \varphi), (s_{\theta_1}, \vartheta)\right\}$ is a 2TLPFN, then the score function $s(p)$ and accuracy function $h(p)$ can be defined:

$$s(p) = \Delta\left\{\frac{\left(t^2 + \Delta^{-1}(s_\phi,\varphi)^2 - \Delta^{-1}(s_\theta,\vartheta)^2\right)}{2t}\right\}, \quad 0 \le s(p) \le t \quad (3)$$

$$h(p) = \Delta\left\{\frac{\Delta^{-1}(s_\phi,\varphi)^2 + \Delta^{-1}(s_\theta,\vartheta)^2}{2t}\right\}, \quad 0 \le h(p) \le t \quad (4)$$

Definition 4: Let $p_1 = \left\{(s_{\phi_1}, \varphi_1), (s_{\theta_1}, \vartheta_1)\right\}$ and $p_2 = \left\{(s_{\phi_2}, \varphi_2), (s_{\theta_2}, \vartheta_2)\right\}$ be two 2TLPNs, then

(1) if $s(p_1) \prec s(p_2)$, then $p_1 \prec p_2$;
(2) if $s(p_1) \succ s(p_2)$, then $p_1 \succ p_2$;
(3) **if** $s(p_1) = s(p_2)$,, **if** $h(p_2) \succ h(p_1)$ **then** $p_2 \succ p_1$;
(4) **if** $s(p_1) = s(p_2)$,, **if** $h(p_2) = h(p_1)$ **then** $p_2 = p_1$.

Definition 5: Let $p_1 = \left\{(s_{\phi_1}, \varphi_1), (s_{\theta_1}, \vartheta_1)\right\}$ and $p_2 = \left\{(s_{\phi_2}, \varphi_2), (s_{\theta_2}, \vartheta_2)\right\}$ be two 2TLPFNs, then $p_1 \oplus p_2$, $p_1 \otimes p_2$, λp_1, $(p_1)^\lambda$, as shown at the top of the next page.

D. BM OPERATORS

Definition 6 [59]: Let $\alpha, \beta > 0$ and $(b_1, b_2, \cdots, b_m)$ be a group of crisp values,$b_i > 0$, then the BM can be defined:

$$BM^{\alpha,\beta}(b_1, b_2, \cdots, b_m) = \left(\frac{1}{m(m-1)} \sum_{\substack{i,j=1 \\ i \ne j}}^{m} b_i^\alpha b_j^\beta\right)^{\frac{1}{\alpha+\beta}} \quad (5)$$

III. TLPFWBM AND 2TLPFWGBM OPERATORS

A. 2TLPFWBM OPERATOR

To pay attention to weights, the weighted BM (WBM) operator is:

Definition 7 [59]: Let $\alpha, \beta > 0$, and $(b_1, b_2, \cdots, b_m)$ be a group of crisp values, weight vector is $\omega = (\omega_1, \omega_2, \cdots \omega_m)^T$ which satisfies $0 \le \omega_i \le 1$ and $\sum_{i=1}^m \omega_i = 1$. Then

$$WBM_\omega^{\alpha,\beta}(b_1, b_2, \cdots, b_m) = \left(\frac{1}{m(m-1)} \sum_{\substack{i,j=1 \\ i \ne j}}^{m} \omega_i \omega_j b_i^\alpha b_j^\beta\right)^{1/(\alpha+\beta)} \quad (6)$$

We expand WBM operator to 2TLPFNs.

Definition 8: Assume that $p_i = \left\{(s_{\phi_i}, \varphi_i), (s_{\theta_i}, \vartheta_i)\right\}$ is a group of 2TLPFNs. The 2-tuple linguistic Pythagorean WBM (2TLPFWBM) operator is:

$$2TLPFWBM_\omega^{\alpha,\beta}(p_1, p_2, \ldots, p_m) = \left(\frac{1}{m(m-1)} \mathop{\oplus}_{\substack{i,j=1 \\ i \ne j}}^{m} \left(\omega_i \omega_j \left(p_i^\alpha \otimes p_j^\beta\right)\right)\right)^{1/(\alpha+\beta)} \quad (7)$$

Theorem 1: Assume that $p_i = \left\{(s_{\phi_i}, \varphi_i), (s_{\theta_i}, \vartheta_i)\right\}$ is a group of 2TLPFNs. The fused result by 2TLPFWBM operator is a 2TLPFN, (8), as shown at the top of the next page.

Proof:

$$p_i^\alpha = \left\{\Delta\left(\frac{\Delta^{-1}(s_{\phi_i}, \varphi_i)^\alpha}{t^{\alpha-1}}\right), \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\theta_i}, \vartheta_i)}{t}\right)^2\right)^\alpha}\right)\right\} \quad (9)$$

$$p_j^\beta = \left\{\Delta\left(\frac{\Delta^{-1}(s_{\phi_j}, \varphi_j)^\beta}{t^{\beta-1}}\right), \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\theta_j}, \vartheta_j)}{t}\right)^2\right)^\beta}\right)\right\} \quad (10)$$

$$p_1 \oplus p_2 = \left\{ \begin{array}{l} \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_1},\varphi_1)}{t}\right)^2\right)\left(1-\left(\frac{\Delta^{-1}(s_{\phi_2},\varphi_2)}{t}\right)^2\right)}\right), \\ \Delta\left(t\left(\frac{\Delta^{-1}(s_{\theta_1},\vartheta_1)}{t}\cdot\frac{\Delta^{-1}(s_{\theta_2},\vartheta_2)}{t}\right)\right) \end{array} \right\};$$

$$p_1 \otimes p_2 = \left\{ \begin{array}{l} \Delta\left(t\left(\frac{\Delta^{-1}(s_{\phi_1},\varphi_1)}{t}\cdot\frac{\Delta^{-1}(s_{\phi_2},\varphi_2)}{t}\right)\right), \\ \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\theta_1},\vartheta_1)}{t}\right)^2\right)\left(1-\left(\frac{\Delta^{-1}(s_{\theta_2},\vartheta_2)}{t}\right)^2\right)}\right) \end{array} \right\};$$

$$\lambda p_1 = \left\{ \begin{array}{l} \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_1},\varphi_1)}{t}\right)^2\right)^{\lambda}}\right), \\ \Delta\left(t\left(\frac{\Delta^{-1}(s_{\theta_1},\vartheta_1)}{t}\right)^{\lambda}\right) \end{array} \right\}, \quad \lambda > 0;$$

$$(p_1)^{\lambda} = \left\{ \begin{array}{l} \Delta\left(t\left(\frac{\Delta^{-1}(s_{\phi_1},\varphi_1)}{t}\right)^{\lambda}\right), \\ \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\theta_1},\vartheta_1)}{t}\right)^2\right)^{\lambda}}\right) \end{array} \right\}, \quad \lambda > 0.$$

$$\begin{aligned} &\text{2TLPFWBM}_{\omega}^{\alpha,\beta}(p_1,p_2,\ldots,p_m) \\ &= \left(\frac{1}{m(m-1)} \bigoplus_{\substack{i,j=1\\ i\neq j}}^{m} \left(\omega_i\omega_j\left(p_i^{\alpha}\otimes p_j^{\beta}\right)\right)\right)^{1/(\alpha+\beta)} \\ &= \left\{ \begin{array}{l} \Delta\left(t\left(\sqrt{1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}}\right)^{1/(\alpha+\beta)}\right), \\ \Delta\left(t\sqrt{1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\theta_j},\vartheta_j)}{t}\right)^2\right)^{\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}\right)^{1/(\alpha+\beta)}}\right) \end{array} \right\} \end{aligned} \tag{8}$$

Thus, (11), as shown at the top of the next page. Thereafter, (12), as shown at the top of the next page. Thus, (13), as shown at the top of the next page. Furthermore, (14), as shown at the top of the next page. Therefore, (15), as shown at the top of the page 5. Hence, (8) is satisfied.

Then we give the proving process of that (8) is also a 2TLPFN.

① $0 \le \frac{\Delta^{-1}(s_\phi,\varphi)}{t} \le 1, 0 \le \frac{\Delta^{-1}(s_\theta,\vartheta)}{t} \le 1,$

② $0 \le \left(\frac{\Delta^{-1}(s_\phi,\varphi)}{t}\right)^2 + \left(\frac{\Delta^{-1}(s_\theta,\vartheta)}{t}\right)^2 \le 1$

$$p_i^{\alpha} \otimes p_j^{\beta} = \left\{ \begin{array}{l} \Delta\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)^{\alpha}}{t^{\alpha-1}} \cdot \frac{\Delta^{-1}(s_{\phi_j},\varphi_j)^{\beta}}{t^{\beta-1}}\right), \\ \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^2\right)^{\alpha} \cdot \left(1-\left(\frac{\Delta^{-1}(s_{\theta_j},\vartheta_j)}{t}\right)^2\right)^{\beta}}\right) \end{array} \right\} \tag{11}$$

$$\omega_i\omega_j\left(p_i^{\alpha} \otimes p_j^{\beta}\right) = \left\{ \begin{array}{l} \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^{2\alpha} \cdot \left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}}\right), \\ \Delta\left(t\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^2\right)^{\alpha} \cdot \left(1-\left(\frac{\Delta^{-1}(s_{\theta_j},\vartheta_j)}{t}\right)^2\right)^{\beta}}\right)^{\omega_i\omega_j}\right) \end{array} \right\} \tag{12}$$

$$\mathop{\oplus}_{\substack{i,j=1\\ i\neq j}}^{m}\left(\omega_i\omega_j\left(p_i^{\alpha} \otimes p_j^{\beta}\right)\right) = \left\{ \begin{array}{l} \Delta\left(t\sqrt{1-\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^{2\alpha} \cdot \left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}}\right), \\ \Delta\left(t\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^2\right)^{\alpha} \cdot \left(1-\left(\frac{\Delta^{-1}(s_{\theta_j},\vartheta_j)}{t}\right)^2\right)^{\beta}}\right)^{\omega_i\omega_j}\right) \end{array} \right\} \tag{13}$$

$$\begin{aligned} &\frac{1}{m(m-1)}\mathop{\oplus}_{\substack{i,j=1\\ i\neq j}}^{m}\left(\omega_i\omega_j\left(p_i^{\alpha} \otimes p_j^{\beta}\right)\right) \\ &= \left\{ \begin{array}{l} \Delta\left(t\sqrt{1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^{2\alpha} \cdot \left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}}\right), \\ \Delta\left(t\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^2\right)^{\alpha} \cdot \left(1-\left(\frac{\Delta^{-1}(s_{\theta_j},\vartheta_j)}{t}\right)^2\right)^{\beta}}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}\right) \end{array} \right\} \end{aligned} \tag{14}$$

Let, $\frac{\Delta^{-1}(s_\phi,\varphi)}{t}$, $\frac{\Delta^{-1}(s_\theta,\vartheta)}{t}$, as shown at the top of the next page.

Proof: Since $0 \le \frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t} \le 1, 0 \le \frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t} \le 1$ we get

$$0 \le 1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^{2\alpha} \cdot \left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^{2\beta} \le 1 \tag{16}$$

Then, (17) and (18), as shown at the top of the next page. By the same way, we can get $0 \le \frac{\Delta^{-1}(s_\theta,\vartheta)}{t} \le 1$. so ① is right.

② Since $0 \le \left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2 + \left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^2 \le 1$, we get the following inequality $\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2 + \left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^2$, as shown at the top of the page 6. That means $0 \le \left(\Delta^{-1}(s_\phi,\varphi)\right)^2 + \left(\Delta^{-1}(s_\theta,\vartheta)\right)^2 \le t^2$, so is maintained.

Example 1: Let $\{(s_3, 0.4), (s_2, -0.3)\}$, $\{(s_2, 0.3), (s_1, 0.2)\}$ be two 2TLPFNs, (α,β) =(2,3),ω=(0.4,0.6) according to (8), we have $2TLPFWBM_{(0.4,0.6)}^{(2,3)}(\{(s_3, 0.4), (s_2, -0.3)\}, \{(s_2, 0.3), (s_1, 0.2)\})$, as shown at the top of the page 6. The 2TLPFWBM has three properties.

$$
\begin{aligned}
&2TLPFWBM_{\omega}^{\alpha,\beta}(p_1,p_2,\ldots,p_m)\\
&=\left(\frac{1}{m(m-1)}\mathop{\oplus}_{\substack{i,j=1\\ i\neq j}}^{m}\left(\omega_i\omega_j\left(p_i^{\alpha}\otimes p_j^{\beta}\right)\right)\right)^{1/(\alpha+\beta)}\\
&=\left\{\begin{array}{l}\Delta\left(t\left(\sqrt{1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}}\right)^{1/(\alpha+\beta)}\right),\\
\Delta\left(t\sqrt{1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^{2}\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^{2}\right)^{\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}\right)^{1/(\alpha+\beta)}}\right)\end{array}\right\}
\end{aligned}\tag{15}
$$

$$
\frac{\Delta^{-1}\left(s_{\phi},\varphi\right)}{t}=\left(\sqrt{1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}}\right)^{1/(\alpha+\beta)}
$$

$$
\frac{\Delta^{-1}\left(s_{\theta},\vartheta\right)}{t}=\sqrt{1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^{2}\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^{2}\right)^{\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}\right)^{1/(\alpha+\beta)}}
$$

$$
0\leq 1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}\leq 1 \tag{17}
$$

$$
0\leq\left(\sqrt{1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}}\right)^{1/(\alpha+\beta)}\leq 1 \tag{18}
$$

Property 1 (Idempotency): If $p_i=\left\{(s_{\phi_i},\varphi_i),\left(s_{\theta_i},\vartheta_i\right)\right\}$ $(i=1,2,\ldots,m)$ are equal, then

$$2TLPFWBM_{\omega}^{\alpha,\beta}(p_1,p_2,\ldots,p_m)=p \tag{19}$$

Proof: Since $p_i=\left\{(s_{\phi_i},\varphi_i),\left(s_{\theta_i},\vartheta_i\right)\right\}$, then $2TLPFWBM_{\omega}^{\alpha,\beta}(p_1,p_2,\ldots,p_m)$, as shown at the top of the page 7.

Property 2 (Monotonicity): Let $p_{x_i}=\{(s_{\phi_{x_i}},\varphi_{x_i}),(s_{\theta_{x_i}},\vartheta_{x_i})\}$ and $p_{y_i}=\{(s_{\phi_{y_i}},\varphi_{y_i}),(s_{\theta_{y_i}},\vartheta_{y_i})\}$ be two lists of 2TLPFNs. If $\Delta^{-1}(s_{\phi_{y_i}},\varphi_{y_i})\geq\Delta^{-1}(s_{\phi_{x_i}},\varphi_{x_i})$ and $\Delta^{-1}(s_{\theta_{y_i}},\vartheta_{y_i})\leq\Delta^{-1}(s_{\theta_{x_i}},\vartheta_{x_i})$, $i=1,2,\ldots,m$, then

$$
\begin{aligned}
&2TLPFWBM_{\omega}^{\alpha,\beta}(p_{x_1},p_{x_2},\ldots,p_{x_m})\\
&\quad\leq 2TLPFWBM_{\omega}^{\alpha,\beta}\left(p_{y_1},p_{y_2},\ldots,p_{y_m}\right)
\end{aligned}\tag{20}
$$

Proof: Let $2TLPFWBM_{\omega}^{\alpha,\beta}(p_{x_1},p_{x_2},\ldots,p_{x_m})=\{(s_{\phi_{x_i}},\varphi_{x_i}),(s_{\theta_{x_i}},\vartheta_{x_i})\}$ and $2TLPFWBM_{\omega}^{\alpha,\beta}(p_{y_1},p_{y_2},\ldots,p_{y_m})=\{(s_{\phi_{y_i}},\varphi_{y_i}),(s_{\theta_{y_i}},\vartheta_{y_i})\}$, $i=1,2,\ldots,m$,

$$
\begin{aligned}
&\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2+\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^2\\
&=\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}\right)^{1/(\alpha+\beta)}\\
&\quad+\left(1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^2\right)^{\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}\right)^{1/(\alpha+\beta)}\right)\\
&\leq\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^2\right)^{\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}\right)^{1/(\alpha+\beta)}\\
&\quad+\left(1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^2\right)^{\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}\right)^{1/(\alpha+\beta)}\right)\\
&=1
\end{aligned}
$$

$$
\begin{aligned}
&2TLPFWBM^{(2,3)}_{(0.4,0.6)}\left(\{(s_3,0.4),(s_2,-0.3)\},\{(s_2,0.3),(s_1,0.2)\}\right)\\
&=\left\{\begin{array}{l}
\Delta\left(6\times\left(\sqrt{1-\left(\left(1-\left(\frac{3.4}{6}\right)^4\times\left(\frac{2.3}{6}\right)^6\right)^{0.4\times0.6}\times\left(1-\left(\frac{2.3}{6}\right)^4\times\left(\frac{3.4}{6}\right)^6\right)^{0.6\times0.4}\right)}^{\frac{1}{2}}\right)^{\frac{1}{2+3}}\right),\\
\Delta\left(6\times\sqrt{1-\left(1-\left(\begin{array}{l}\left(1-\left(1-\left(\frac{1.7}{6}\right)^2\right)^2\times\left(1-\left(\frac{1.2}{6}\right)^2\right)^3\right)^{0.4\times0.6}\\ \times\left(1-\left(1-\left(\frac{1.2}{6}\right)^2\right)^2\times\left(1-\left(\frac{1.7}{6}\right)^2\right)^3\right)^{0.6\times0.4}\end{array}\right)^{\frac{1}{2}}\right)^{\frac{1}{2+3}}}\right),
\end{array}\right\}\\
&=\{(s_2,0.4427),(s_2,0.8738)\}
\end{aligned}
$$

if $\Delta^{-1}(s_{\phi_{x_i}},\varphi_{x_i})\leq\Delta^{-1}(s_{\phi_{y_i}},\varphi_{y_i})$, we have

$$
\begin{aligned}
&\left(\frac{\Delta^{-1}\left(s_{\phi_{x_i}},\varphi_{x_i}\right)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}\left(s_{\phi_{x_i}},\varphi_{x_i}\right)}{t}\right)^{2\beta}\\
&\leq\left(\frac{\Delta^{-1}\left(s_{\phi_{y_i}},\varphi_{y_i}\right)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}\left(s_{\phi_{y_i}},\varphi_{y_i}\right)}{t}\right)^{2\beta}
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
&\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{x_i}},\varphi_{x_i}\right)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}\left(s_{\phi_{x_i}},\varphi_{x_i}\right)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}\\
&\geq\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{y_i}},\varphi_{y_i}\right)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}\left(s_{\phi_{y_i}},\varphi_{y_i}\right)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}
\end{aligned}
\tag{22}
$$

Thereafter, (23), as shown at the next page. Furthermore, (24), as shown at the next page.

$$
\begin{aligned}
&\mathrm{2TLPFWBM}_{\omega}^{\alpha,\beta}(p_1,p_2,\ldots,p_m)\\
&=\left(\frac{1}{m(m-1)}\underset{\substack{i,j=1\\ i\neq j}}{\overset{m}{\oplus}}\left(\omega_i\omega_j\left(p_i^{\alpha}\otimes p_j^{\beta}\right)\right)\right)^{1/(\alpha+\beta)}\\
&=\left\{\begin{array}{l}
\Delta\left(t\left(\sqrt{1-\left(\prod\limits_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}}\right)^{1/(\alpha+\beta)}\right),\\
\Delta\left(t\sqrt{1-\left(1-\left(\prod\limits_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^{2}\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^{2}\right)^{\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}\right)^{1/(\alpha+\beta)}}\right)
\end{array}\right\}\\
&=\left\{\begin{array}{l}
\Delta\left(t\left(\sqrt{1-\left(\prod\limits_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi},\varphi\right)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}\left(s_{\phi},\varphi\right)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}}\right)^{1/(\alpha+\beta)}\right),\\
\Delta\left(t\sqrt{1-\left(1-\left(\prod\limits_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta},\vartheta\right)}{t}\right)^{2}\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta},\vartheta\right)}{t}\right)^{2}\right)^{\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}\right)^{1/(\alpha+\beta)}}\right)
\end{array}\right\}\\
&=\left\{\left(s_{\phi},\varphi\right),\left(s_{\theta},\vartheta\right)\right\}=p
\end{aligned}
$$

$$
\begin{aligned}
&1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{x_i}},\varphi_{x_i}\right)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}\left(s_{\phi_{x_i}},\varphi_{x_i}\right)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}\\
&\quad\leq 1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{y_i}},\varphi_{y_i}\right)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}\left(s_{\phi_{y_i}},\varphi_{y_i}\right)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
&\left(\sqrt{1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{x_i}},\varphi_{x_i}\right)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}\left(s_{\phi_{x_i}},\varphi_{x_i}\right)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}}\right)^{1/(\alpha+\beta)}\\
&\quad\leq\left(\sqrt{1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{y_i}},\varphi_{y_i}\right)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}\left(s_{\phi_{y_i}},\varphi_{y_i}\right)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}}\right)^{1/(\alpha+\beta)}
\end{aligned}
\tag{24}
$$

By the same way, we have $\Delta^{-1}\left(s_{\theta_y},\vartheta_y\right) \le \Delta^{-1}\left(s_{\theta_x},\vartheta_x\right)$. If $\Delta^{-1}\left(s_{\phi_x},\varphi_x\right) < \Delta^{-1}\left(s_{\phi_y},\varphi_y\right)$ and $\Delta^{-1}\left(s_{\theta_x},\vartheta_x\right) \ge \Delta^{-1}\left(s_{\theta_y},\vartheta_y\right)$

$$2TLPFWBM_{\omega}^{\alpha,\beta}\left(p_{x_1},p_{x_2},\cdots,p_{x_m}\right) < 2TLPFWBM_{\omega}^{\alpha,\beta}\left(p_{y_1},p_{y_2},\cdots,p_{y_m}\right)$$

If $\Delta^{-1}\left(s_{\phi_x},\varphi_x\right) = \Delta^{-1}\left(s_{\phi_y},\varphi_y\right)$ and $\Delta^{-1}\left(s_{\theta_x},\vartheta_x\right) = \Delta^{-1}\left(s_{\theta_y},\vartheta_y\right)$

$$2TLPFWBM_{\omega}^{\alpha,\beta}\left(p_{x_1},p_{x_2},\cdots,p_{x_m}\right) = 2TLPFWBM_{\omega}^{\alpha,\beta}\left(p_{y_1},p_{y_2},\cdots,p_{y_m}\right)$$

So property 2 is right.

Property 3(Boundedness): Let $p_i = \{(s_{\phi_i},\varphi_i),(s_{\theta_i},\vartheta_i)\}$ $(i = 1,2,\ldots,m)$ be a set of 2TLPFNs. If $p^+ = (\max_i(S_{\phi_i},\varphi_i), \min_i(S_{\theta_i},\vartheta_i))$ and $p^- = (\min_i(S_{\phi_i},\varphi_i), \max_i(S_{\theta_i},\vartheta_i))$ then

$$p^- \le 2TLPFWBM_{\omega}^{\alpha,\beta}(p_1,p_2,\cdots,p_m) \le p^+ \tag{25}$$

By property 1,

$$2TLPFWBM_{\omega}^{\alpha,\beta}\left(p_1^-,p_2^-,\cdots,p_m^-\right) = p^-$$
$$2TLPFWBM_{\omega}^{\alpha,\beta}\left(p_1^+,p_2^+,\cdots,p_m^+\right) = p^+$$

By property 2,

$$p^- \le 2TLPFWBM_{\omega}^{\alpha,\beta}(p_1,p_2,\cdots,p_m) \le p^+$$

B. TLPFWGBM OPERATOR

Similar to WBM operator, WGBM operator can be defined:

Definition 9 [62]: Assume that $\alpha,\beta > 0$ and $(b_1,b_2,\cdots,b_m)$ be a group of non-negative real values with the weights $\omega = (\omega_1,\omega_2,\cdots\omega_m)^T$, thereby satisfying $0 \le \omega_i \le 1$ and $\sum_{i=1}^m \omega_i = 1$.then

$$WGBM_{\omega}^{\alpha,\beta}(b_1,b_2,\cdots,b_m) = \frac{1}{\alpha+\beta}\prod_{i,j=1}^{m}\left(\alpha b_i+\beta b_j\right)^{\omega_i\omega_j} \tag{26}$$

We expand WGBM to 2TLPFNs and develop 2TLPFWGBM operator.

Definition 10: Let $p_i = \{(s_{\phi_i},\varphi_i),(s_{\theta_i},\vartheta_i)\}$ $(i = 1,2,\ldots,m)$ be a group of 2TLPFNs with their weight be $w_i = (w_1,w_2,\ldots,w_n)^T$, thereby satisfying $w_i \in [0,1]$ and $\sum_{i=1}^n w_i = 1$. If

$$2TLPFWGBM_{\omega}^{\alpha,\beta}(p_1,p_2,\ldots,p_m) = \frac{1}{\alpha+\beta}\left(\underset{\substack{i,j=1\\ i\ne j}}{\overset{m}{\otimes}}\left(\alpha p_i \oplus \beta p_j\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}} \tag{27}$$

Theorem 2: Let $p_i = \left\{(s_{\phi_i},\varphi_i),\left(s_{\theta_i},\vartheta_i\right)\right\}$ $(i = 1,2,\ldots,m)$ be a group of 2TLPFNs. The fused result by 2TLPFWGBM operator is also a 2TLPFN where, (28), as shown at the top of the next page.

Proof:

$$\alpha p_i = \left\{\begin{array}{l}\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha}}\right),\\ \Delta\left(t\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^{\alpha}\right)\end{array}\right\} \tag{29}$$

$$\beta p_j = \left\{\begin{array}{l}\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta}}\right),\\ \Delta\left(t\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^{\beta}\right)\end{array}\right\} \tag{30}$$

Thus, (31), as shown at the top of the next page. Therefore, (32), as shown at the top of the next page. Then, (33), as shown at the top of the next page. Thereafter, (34), as shown at the top of the page 10. Furthermore, (35), as shown at the top of the page 10. Hence, (28) is proven.

Then we give the proving process of that (28) is also a 2TLPFN.

① $0 \le \frac{\Delta^{-1}(s_\phi,\varphi)}{t} \le 1, 0 \le \frac{\Delta^{-1}(s_\theta,\vartheta)}{t} \le 1,$

② $0 \le \left(\frac{\Delta^{-1}(s_\phi,\varphi)}{t}\right)^2 + \left(\frac{\Delta^{-1}(s_\theta,\vartheta)}{t}\right)^2 \le 1$

Let, $\frac{\Delta^{-1}(s_\phi,\varphi)}{t}$, $\frac{\Delta^{-1}(s_\theta,\vartheta)}{t}$, as shown at the top of the page 10.

Proof: Since $0 \le \frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t} \le 1, 0 \le \frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t} \le 1$ we get

$$0 \le 1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha} \cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta} \le 1 \tag{36}$$

Then, (37) and (38), as shown at the top of the page 11. That is $0 \le \frac{\Delta^{-1}(s_\phi,\varphi)}{t} \le 1$, so ① is satisfied, similarly, we have $0 \le \frac{\Delta^{-1}(s_\theta,\vartheta)}{t} \le 1$.

② Since $0 \le \left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2 + \left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^2 \le 1$, we get the following inequality $\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2 + \left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^2$, as shown at the top of the page 11. That means $0 \le \left(\frac{\Delta^{-1}(s_\phi,\varphi)}{t}\right)^2 + \left(\frac{\Delta^{-1}(s_\theta,\vartheta)}{t}\right)^2 \le 1$, so is maintained.

Example 2: Let $\{(s_3,0.4),(s_2,-0.3),(s_2,0.3),(s_1,0.2)\}$ be two 2TLPFNs, (α,β) =(2,3),ω=(0.4,0.6) according to (28), we have $2TLPFWGBM_{(0.4,0.6)}^{(2,3)}(\{(s_3,0.4),(s_2,-0.3)\},\{(s_2,0.3),(s_1,0.2)\})$, as shown at the top of the page 12. The 2TLPFWGBM has three properties.

Property 4 (Idempotency): If $p_i = \left\{(s_{\phi_i},\varphi_i),\left(s_{\theta_i},\vartheta_i\right)\right\}$ $(i = 1,2,\ldots,m)$ are equal, then

$$2TLPFWGBM_{\omega}^{\alpha,\beta}(p_1,p_2,\cdots,p_m) = p \tag{39}$$

$$\begin{aligned}
&2TLPFWGBM_{\omega}^{\alpha,\beta}(p_1,p_2,\ldots,p_m)\\
&=\frac{1}{\alpha+\beta}\left(\underset{\substack{i,j=1\\ i\neq j}}{\overset{m}{\otimes}}\left(\alpha p_i\oplus\beta p_j\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}\\
&=\left\{\begin{array}{l}
\Delta\left(t\sqrt{1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}\right)^{\frac{1}{\alpha+\beta}}}\right),\\
\Delta\left(t\left(\sqrt{1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}}\right)^{\frac{1}{\alpha+\beta}}\right)
\end{array}\right\}
\end{aligned}\tag{28}$$

$$\alpha p_i\oplus\beta p_j=\left\{\begin{array}{l}
\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta}}\right),\\
\Delta\left(t\left(\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^{\alpha}\cdot\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^{\beta}\right)\right)
\end{array}\right\}\tag{31}$$

$$\left(\alpha p_i\oplus\beta p_j\right)^{\omega_i\omega_j}=\left\{\begin{array}{l}
\Delta\left(t\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta}}\right)^{\omega_i\omega_j}\right),\\
\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}}\right)
\end{array}\right\}\tag{32}$$

$$\underset{\substack{i,j=1\\ i\neq j}}{\overset{m}{\otimes}}\left(\alpha p_i\oplus\beta p_j\right)^{\omega_i\omega_j}=\left\{\begin{array}{l}
\Delta\left(t\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta}}\right)^{\omega_i\omega_j}\right),\\
\Delta\left(t\sqrt{1-\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}}\right)
\end{array}\right\}\tag{33}$$

Property 5 (Monotonicity): Let $p_{x_i}=\{(s_{\phi_{x_i}},\varphi_{x_i}),(s_{\theta_{x_i}},\vartheta_{x_i})\}$ $(i=1,2,\ldots,m)$ and $p_{y_i}=\{(s_{\phi_{y_i}},\varphi_{y_i}),(s_{\theta_{y_i}},\vartheta_{y_i})\}(i=1,2,\ldots,m)$ be two lists of 2TLPFNs. If $\Delta^{-1}(s_{\phi_{x_i}},\varphi_{x_i})\leq\Delta^{-1}(s_{\phi_{y_i}},\varphi_{y_i})$ and $\Delta^{-1}(s_{\theta_{x_i}},\vartheta_{x_i})\geq\Delta^{-1}(s_{\theta_{y_i}},\vartheta_{y_i})$, $i=1,2,\ldots,m$, then

$$\begin{aligned}&2TLPFWGBM_{\omega}^{\alpha,\beta}\left(p_{x_1},p_{x_2},\cdots,p_{x_m}\right)\\
&\quad\leq 2TLPFWGBM_{\omega}^{\alpha,\beta}\left(p_{y_1},p_{y_2},\cdots,p_{y_m}\right)\end{aligned}\tag{40}$$

$$\left(\underset{\substack{i,j=1\\i\neq j}}{\overset{m}{\otimes}}(\alpha p_i\oplus\beta p_j)^{\omega_i\omega_j}\right)^{\frac{1}{n(n-1)}}=\left\{\begin{array}{l}\Delta\left(t\left(\prod_{\substack{i,j=1\\i\neq j}}^{m}\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta}}\right)^{\omega_i\omega_j}\right)^{\frac{1}{n(n-1)}}\right),\\ \Delta\left(t\sqrt{1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\theta_j},\vartheta_j)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{n(n-1)}}}\right)\end{array}\right\} \tag{34}$$

$$\begin{aligned}&\text{2TLPFWGBM}_{\omega}^{\alpha,\beta}(p_1,p_2,\ldots,p_m)\\&=\frac{1}{\alpha+\beta}\left(\underset{\substack{i,j=1\\i\neq j}}{\overset{m}{\otimes}}(\alpha p_i\oplus\beta p_j)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}\\&=\left\{\begin{array}{l}\Delta\left(t\sqrt{1-\left(1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}\right)^{\frac{1}{\alpha+\beta}}}\right),\\ \Delta\left(t\left(\sqrt{1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\theta_j},\vartheta_j)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}}\right)^{\frac{1}{\alpha+\beta}}\right)\end{array}\right\}\end{aligned} \tag{35}$$

$$\frac{\Delta^{-1}(s_\phi,\varphi)}{t}=\sqrt{1-\left(1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}\right)^{\frac{1}{\alpha+\beta}}}$$

$$\frac{\Delta^{-1}(s_\theta,\vartheta)}{t}=\left(\sqrt{1-\left(\prod_{\substack{i,j=1\\i\neq j}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\theta_j},\vartheta_j)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}}\right)^{\frac{1}{\alpha+\beta}}$$

Property 6 (Boundedness): Let $p_i=\{(s_{\phi_i},\varphi_i),(s_{\theta_i},\vartheta_i)\}(i=1,2,\ldots,m)$ be a set of 2TLPFNs. If $p^+=(\max_i(S_{\phi_i},\varphi_i),\min_i(S_{\theta_i},\vartheta_i))$ and $p^-=(\min_i(S_{\phi_i},\varphi_i),\max_i(S_{\theta_i},\vartheta_i))$ then

$$p^-\leq \text{2TLPFWGBM}_{\omega}^{\alpha,\beta}(p_1,p_2,\cdots,p_m)\leq p^+ \tag{41}$$

IV. THE G2TLPFWBM AND G2TLPFWGBM OPERATORS

A. G2TLPFWBM OPERATOR

This section fuses the GBM operator proposed by Beliakov *et al.* [60] and the GWBM operator proposed by Zhu *et al.* [62] with the 2TLPFNs.

$$0 \le \prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta}\right)^{\omega_i\omega_j} \le 1 \tag{37}$$

$$0 \le \sqrt{1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{\alpha+\beta}}} \le 1 \tag{38}$$

$$\begin{aligned}
&\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2+\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^2\\
&=\left(1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}\right)^{\frac{1}{\alpha+\beta}}\right)\\
&\quad+\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}\right)^{\frac{1}{\alpha+\beta}}\\
&\le\left(1-\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^2\right)\right)^{\alpha}\cdot\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^2\right)\right)^{\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}\right)^{\frac{1}{\alpha+\beta}}\right)\\
&\quad+\left(1-\left(\prod_{\substack{i,j=1\\ i\neq j}}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^{2\beta}\right)^{\omega_i\omega_j}\right)^{\frac{1}{m(m-1)}}\right)^{\frac{1}{\alpha+\beta}}=1
\end{aligned}$$

Definition 11 [62]: Let $\alpha, \beta, \gamma > 0$ and $(b_1, b_2, \cdots, b_m)$ be a group of non-negative real values, the weights being $\omega = (\omega_1, \omega_2, \cdots \omega_m)^T$, thereby satisfying $0 \le \omega_i \le 1$ and $\sum_{i=1}^{m}\omega_i = 1$. The GWBM is:

$$\mathrm{GWBM}_{\omega}^{\alpha,\beta,\gamma}(b_1, b_2, \cdots, b_m) = \left(\sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\omega_i\omega_j\omega_k b_i^{\alpha} b_j^{\beta} b_k^{\gamma}\right)^{1/(\alpha+\beta+\gamma)} \tag{42}$$

We expand GWBM to 2TLPFNs and propose generalized 2TLPFWBM (G2TLPFWBM) operator.

Definition 12: Let $p_i = \left\{(s_{\phi_i}, \varphi_i), \left(s_{\theta_i}, \vartheta_i\right)\right\}$ be a group of 2TLPFNs. The G2TLPFWBM operator is, (43), as shown at the next page.

Theorem 3: Let $p_i = \left\{(s_{\phi_i}, \varphi_i), \left(s_{\theta_i}, \vartheta_i\right)\right\}$ be a group of 2TLPFNs. The fused result by G2TLPFWBM operator is also a 2TLPFN, (44), as shown at the next page.

Proof: Equations (45)–(47), as shown at the next page.

Thus, (48), as shown at the top of the page 13. Thereafter, (49), as shown at the top of the page 13. Furthermore, (50), as shown at the top of the page 13. Then, (51), as shown at the top of the page 13. Therefore, (52), as shown at the top of the page 14. Hence, (44) is proven.

$$
\begin{aligned}
&\mathrm{2TLPFWGBM}^{(2,3)}_{(0.4,0.6)}\left(\{(s_3,0.4),(s_2,-0.3)\},\{(s_2,0.3),(s_1,0.2)\}\right)\\
&=\left\{\begin{array}{l}
\Delta\left(6\times\sqrt{1-\left(1-\left(\begin{array}{l}\left(\left(1-\left(1-\left(\frac{3.4}{6}\right)^2\right)^2\times\left(1-\left(\frac{2.3}{6}\right)^2\right)^3\right)^{0.4\times0.6}\\ \times\left(1-\left(1-\left(\frac{2.3}{6}\right)^2\right)^2\times\left(1-\left(\frac{3.4}{6}\right)^2\right)^3\right)^{0.6\times0.4}\end{array}\right)^{\frac{1}{2}}\right)^{\frac{1}{2+3}}}\right),\\
\Delta\left(6\times\left(\sqrt{1-\left(\left(1-\left(\frac{1.7}{6}\right)^4\times\left(\frac{1.2}{6}\right)^6\right)^{0.4\times0.6}\times\left(1-\left(\frac{1.2}{6}\right)^4\times\left(\frac{1.7}{6}\right)^6\right)^{0.6\times0.4}\right)^{\frac{1}{2}}}\right)^{\frac{1}{2+3}}\right),
\end{array}\right\}\\
&=\{(s_5,0.4803),(s_5,0.8212)\}
\end{aligned}
$$

$$
\mathrm{G2TLPFWBM}^{\alpha,\beta,\gamma}_{\omega}=\left(\frac{1}{m(m-1)(m-2)}\bigoplus_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(\omega_i\omega_j\omega_k\left(p_i^{\alpha}\otimes p_j^{\beta}\otimes p_k^{\gamma}\right)\right)\right)^{1/(\alpha+\beta+\gamma)} \tag{43}
$$

$$
\begin{aligned}
&\mathrm{G2TLPFWBM}^{\alpha,\beta,\gamma}_{\omega}(p_1,p_2,\ldots,p_m)\\
&=\left(\frac{1}{m(m-1)(m-2)}\bigoplus_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(\omega_i\omega_j\omega_k\left(p_i^{\alpha}\otimes p_j^{\beta}\otimes p_k^{\gamma}\right)\right)\right)^{1/(\alpha+\beta+\gamma)}\\
&=\left\{\begin{array}{l}
\Delta\left(t\left(\sqrt{1-\left(\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^{2\beta}\cdot\left(\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\right)^{2\gamma}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}}\right)^{1/(\alpha+\beta+\gamma)}\right),\\
\Delta\left(t\sqrt{1-\left(1-\left(\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\theta_j},\vartheta_j)}{t}\right)^2\right)^{\beta}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\theta_k},\vartheta_k)}{t}\right)^2\right)^{\gamma}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}\right)^{1/(\alpha+\beta+\gamma)}}\right)
\end{array}\right\}
\end{aligned} \tag{44}
$$

$$
p_i^{\alpha}=\left\{\Delta\left(t\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^{\alpha}\right),\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^2\right)^{\alpha}}\right)\right\} \tag{45}
$$

$$
p_j^{\beta}=\left\{\Delta\left(t\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^{\beta}\right),\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\theta_j},\vartheta_j)}{t}\right)^2\right)^{\beta}}\right)\right\} \tag{46}
$$

$$
p_k^{\gamma}=\left\{\Delta\left(t\left(\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\right)^{\gamma}\right),\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\theta_k},\vartheta_k)}{t}\right)^2\right)^{\gamma}}\right)\right\} \tag{47}
$$

$$p_i^{\alpha}\otimes p_j^{\beta}\otimes p_k^{\gamma}=\left\{\begin{array}{l}\Delta\left(t\left(\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^{\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^{\beta}\cdot\left(\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\right)^{\gamma}\right)\right),\\ \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^2\right)^{\beta}\cdot\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_k},\vartheta_k\right)}{t}\right)^2\right)^{\gamma}}\right)\end{array}\right\}\tag{48}$$

$$\begin{aligned}&\omega_i\omega_j\omega_k\left(p_i^{\alpha}\otimes p_j^{\beta}\otimes p_k^{\gamma}\right)\\&=\left\{\begin{array}{l}\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^{2\beta}\cdot\left(\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\right)^{2\gamma}\right)^{\omega_i\omega_j\omega_k}}\right),\\ \Delta\left(t\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^2\right)^{\beta}\cdot\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_k},\vartheta_k\right)}{t}\right)^2\right)^{\gamma}}\right)^{\omega_i\omega_j\omega_k}\right)\end{array}\right\}\end{aligned}\tag{49}$$

$$\begin{aligned}&\mathop{\oplus}_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(\omega_i\omega_j\omega_k\left(p_i^{\alpha}\otimes p_j^{\beta}\otimes p_k^{\gamma}\right)\right)\\&=\left\{\begin{array}{l}\Delta\left(t\sqrt{1-\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^{2\beta}\cdot\left(\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\right)^{2\gamma}\right)^{\omega_i\omega_j\omega_k}}\right),\\ \Delta\left(t\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^2\right)^{\beta}\cdot\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_k},\vartheta_k\right)}{t}\right)^2\right)^{\gamma}}\right)^{\omega_i\omega_j\omega_k}\right)\end{array}\right\}\end{aligned}\tag{50}$$

$$\begin{aligned}&\frac{1}{m(m-1)(m-2)}\mathop{\oplus}_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(\omega_i\omega_j\omega_k\left(p_i^{\alpha}\otimes p_j^{\beta}\otimes p_k^{\gamma}\right)\right)\\&=\left\{\begin{array}{l}\Delta\left(t\sqrt{1-\left(\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^{2\beta}\cdot\left(\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\right)^{2\gamma}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}}\right),\\ \Delta\left(t\left(\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^2\right)^{\beta}\cdot\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_k},\vartheta_k\right)}{t}\right)^2\right)^{\gamma}}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}\right)\end{array}\right\}\end{aligned}\tag{51}$$

Then we give the proving process of that (44) is also a 2TLPFN.

① $0\leq\frac{\Delta^{-1}(s_{\phi},\varphi)}{t}\leq 1,\ 0\leq\frac{\Delta^{-1}(s_{\theta},\vartheta)}{t}\leq 1$,

② $0\leq\left(\frac{\Delta^{-1}(s_{\phi},\varphi)}{t}\right)^2+\left(\frac{\Delta^{-1}(s_{\theta},\vartheta)}{t}\right)^2\leq 1$

Let $\frac{\Delta^{-1}(s_{\phi},\varphi)}{t}$, $\frac{\Delta^{-1}(s_{\theta},\vartheta)}{t}$, as shown at the top of the next page.

Proof: Since $0\leq\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\leq 1$, $0\leq\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\leq 1$, $0\leq\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\leq 1$ we get

$$0\leq 1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^{2\beta}\cdot\left(\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\right)^{2\gamma}\leq 1\tag{53}$$

$$\begin{aligned}
&\text{G2TLPFWBM}_{\omega}^{\alpha,\beta,\gamma}(p_1,p_2,\ldots,p_m)\\
&=\left(\frac{1}{m(m-1)(m-2)}\bigoplus_{\substack{i,j,k=1\\i\neq j\neq k}}^{m}\left(\omega_i\omega_j\omega_k\left(p_i^{\alpha}\otimes p_j^{\beta}\otimes p_k^{\gamma}\right)\right)\right)^{1/(\alpha+\beta+\gamma)}\\
&=\left\{\begin{array}{l}
\Delta\left(t\left(\sqrt{1-\left(\prod_{\substack{i,j,k=1\\i\neq j\neq k}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^{2\beta}\cdot\left(\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\right)^{2\gamma}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}}\right)^{1/(\alpha+\beta+\gamma)}\right),\\
\Delta\left(t\sqrt{1-\left(1-\left(\prod_{\substack{i,j,k=1\\i\neq j\neq k}}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\theta_j},\vartheta_j)}{t}\right)^2\right)^{\beta}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\theta_k},\vartheta_k)}{t}\right)^2\right)^{\gamma}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}\right)^{1/(\alpha+\beta+\gamma)}}\right)
\end{array}\right\}
\end{aligned}\tag{52}$$

$$\frac{\Delta^{-1}(s_\phi,\varphi)}{t}=\left(\sqrt{1-\left(\prod_{\substack{i,j,k=1\\i\neq j\neq k}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^{2\beta}\cdot\left(\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\right)^{2\gamma}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}}\right)^{1/(\alpha+\beta+\gamma)}$$

$$\frac{\Delta^{-1}(s_\theta,\vartheta)}{t}=\sqrt{1-\left(1-\left(\prod_{\substack{i,j,k=1\\i\neq j\neq k}}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\theta_j},\vartheta_j)}{t}\right)^2\right)^{\beta}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\theta_k},\vartheta_k)}{t}\right)^2\right)^{\gamma}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}\right)^{1/(\alpha+\beta+\gamma)}}$$

Then, (54) and (55), as shown at the top of the next page. That is $0\leq\frac{\Delta^{-1}(s_\phi,\varphi)}{t}\leq 1$, by the same way, we have $0\leq\frac{\Delta^{-1}(s_\theta,\vartheta)}{t}\leq 1$.

① Since $0\leq\left(\frac{\Delta^{-1}(s_\phi,\varphi)}{t}\right)^2+\left(\frac{\Delta^{-1}(s_\theta,\vartheta)}{t}\right)^2\leq 1$, we get the following inequality $\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2+\left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^2$, as shown at the top of the next page. That means is maintained.

The G2TLPFWBM operator has three properties.

Property 7 (Idempotency): If $p_i=\left\{(s_{\phi_i},\varphi_i),(s_{\theta_i},\vartheta_i)\right\}$ $(i=1,2,\ldots,m)$ are equal, then

$$\text{G2TLPFWBM}_{\omega}^{\alpha,\beta,\gamma}(p_1,p_2,\cdots,p_m)=p\tag{56}$$

Proof: Since $p_i=p=\left\{(s_\phi,\varphi),(s_\theta,\vartheta)\right\}$, then $\text{G2TLPFWBM}_{\omega}^{\alpha,\beta,\gamma}(p_1,p_2,\ldots,p_m)$, as shown at the page 16.

Property 8 (Monotonicity): Let $p_{x_i}=\{(s_{\phi_{x_i}},\varphi_{x_i}),(s_{\theta_{x_i}},\vartheta_{x_i})\}$ and $p_{y_i}=\{(s_{\phi_{y_i}},\varphi_{y_i}),(s_{\theta_{y_i}},\vartheta_{y_i})\}$ be two lists of 2TLPFNs. If $\Delta^{-1}(s_{\phi_{y_i}},\varphi_{y_i})\geq\Delta^{-1}(s_{\phi_{x_i}},\varphi_{x_i})$ and $\Delta^{-1}(s_{\theta_{x_i}},\vartheta_{x_i})\geq\Delta^{-1}(s_{\theta_{y_i}},\vartheta_{y_i})$, $i=1,2,\ldots,m$, then

$$\text{G2TLPFWBM}_{\omega}^{\alpha,\beta,\gamma}(p_{x_1},p_{x_2},\cdots,p_{x_m})\leq\text{G2TLPFWBM}_{\omega}^{\alpha,\beta,\gamma}(p_{y_1},p_{y_2},\cdots,p_{y_m})\tag{57}$$

Proof: Let $\text{G2TLPFWBM}_{\omega}^{\alpha,\beta,\gamma}(p_{x_1},p_{x_2},\cdots,p_{x_m})=\{(s_{\phi_{x_i}},\varphi_{x_i}),(s_{\theta_{x_i}},\vartheta_{x_i})\}$ and $\text{G2TLPFWBM}_{\omega}^{\alpha,\beta,\gamma}(p_{y_1},p_{y_2},\cdots,p_{y_m})=\{(s_{\phi_{y_i}},\varphi_{y_i}),(s_{\theta_{y_i}},\vartheta_{y_i})\}$, if $\Delta^{-1}(s_{\phi_{x_i}},\varphi_{x_i})\leq\Delta^{-1}(s_{\phi_{y_i}},\varphi_{y_i})$, then

$$\left(\frac{\Delta^{-1}\left(s_{\phi_{x_i}},\varphi_{x_i}\right)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}\left(s_{\phi_{x_j}},\varphi_{x_j}\right)}{t}\right)^{2\beta}\cdot\left(\frac{\Delta^{-1}\left(s_{\phi_{x_k}},\varphi_{x_k}\right)}{t}\right)^{2\gamma}$$

$$0 \leq 1 - \left(\prod_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{m} \left(1 - \left(\frac{\Delta^{-1}(s_{\phi_i}, \varphi_i)}{t} \right)^{2\alpha} \cdot \left(\frac{\Delta^{-1}(s_{\phi_j}, \varphi_j)}{t} \right)^{2\beta} \cdot \left(\frac{\Delta^{-1}(s_{\phi_k}, \varphi_k)}{t} \right)^{2\gamma} \right)^{\omega_i \omega_j \omega_k} \right)^{\frac{1}{n(n-1)(n-2)}} \leq 1 \tag{54}$$

$$0 \leq \left(\sqrt{1 - \left(\prod_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{m} \left(1 - \left(\frac{\Delta^{-1}(s_{\phi_i}, \varphi_i)}{t} \right)^{2\alpha} \cdot \left(\frac{\Delta^{-1}(s_{\phi_j}, \varphi_j)}{t} \right)^{2\beta} \cdot \left(\frac{\Delta^{-1}(s_{\phi_k}, \varphi_k)}{t} \right)^{2\gamma} \right)^{\omega_i \omega_j \omega_k} \right)^{\frac{1}{m(m-1)(m-2)}}} \right)^{1/(\alpha+\beta+\gamma)} \leq 1 \tag{55}$$

$$\begin{aligned}
&\left(\frac{\Delta^{-1}(s_{\phi_i}, \varphi_i)}{t} \right)^2 + \left(\frac{\Delta^{-1}(s_{\theta_i}, \vartheta_i)}{t} \right)^2 \\
&= \left(1 - \left(\prod_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{m} \left(1 - \left(\frac{\Delta^{-1}(s_{\phi_i}, \varphi_i)}{t} \right)^{2\alpha} \cdot \left(\frac{\Delta^{-1}(s_{\phi_j}, \varphi_j)}{t} \right)^{2\beta} \cdot \left(\frac{\Delta^{-1}(s_{\phi_k}, \varphi_k)}{t} \right)^{2\gamma} \right)^{\omega_i \omega_j \omega_k} \right)^{\frac{1}{m(m-1)(m-2)}} \right)^{1/(\alpha+\beta+\gamma)} \\
&\quad + \left(1 - \left(1 - \left(\prod_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{m} \left(1 - \left(1 - \left(\frac{\Delta^{-1}(s_{\theta_i}, \vartheta_i)}{t} \right)^2 \right)^{\alpha} \cdot \left(1 - \left(\frac{\Delta^{-1}(s_{\theta_j}, \vartheta_j)}{t} \right)^2 \right)^{\beta} \cdot \left(1 - \left(\frac{\Delta^{-1}(s_{\theta_k}, \vartheta_k)}{t} \right)^2 \right)^{\gamma} \right)^{\omega_i \omega_j \omega_k} \right)^{\frac{1}{m(m-1)(m-2)}} \right)^{1/(\alpha+\beta+\gamma)} \right) \\
&\leq \left(1 - \left(\prod_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{m} \left(1 - \left(1 - \left(\frac{\Delta^{-1}(s_{\theta_i}, \vartheta_i)}{t} \right)^2 \right)^{\alpha} \cdot \left(1 - \left(\frac{\Delta^{-1}(s_{\theta_j}, \vartheta_j)}{t} \right)^2 \right)^{\beta} \cdot \left(1 - \left(\frac{\Delta^{-1}(s_{\theta_k}, \vartheta_k)}{t} \right)^2 \right)^{\gamma} \right)^{\omega_i \omega_j \omega_k} \right)^{\frac{1}{m(m-1)(m-2)}} \right)^{1/(\alpha+\beta+\gamma)} \\
&\quad + \left(1 - \left(1 - \left(\prod_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{m} \left(1 - \left(1 - \left(\frac{\Delta^{-1}(s_{\theta_i}, \vartheta_i)}{t} \right)^2 \right)^{\alpha} \cdot \left(1 - \left(\frac{\Delta^{-1}(s_{\theta_j}, \vartheta_j)}{t} \right)^2 \right)^{\beta} \cdot \left(1 - \left(\frac{\Delta^{-1}(s_{\theta_k}, \vartheta_k)}{t} \right)^2 \right)^{\gamma} \right)^{\omega_i \omega_j \omega_k} \right)^{\frac{1}{m(m-1)(m-2)}} \right)^{1/(\alpha+\beta+\gamma)} \right) \\
&= 1
\end{aligned}$$

$$\leq \left(\frac{\Delta^{-1}\left(s_{\phi_{y_i}}, \varphi_{y_i}\right)}{t} \right)^{2\alpha} \cdot \left(\frac{\Delta^{-1}\left(s_{\phi_{y_j}}, \varphi_{y_j}\right)}{t} \right)^{2\beta} \cdot \left(\frac{\Delta^{-1}\left(s_{\phi_{y_k}}, \varphi_{y_k}\right)}{t} \right)^{2\gamma} \tag{58}$$

$$\left(1 - \left(\frac{\Delta^{-1}\left(s_{\phi_{x_i}}, \varphi_{x_i}\right)}{t} \right)^{2\alpha} \cdot \left(\frac{\Delta^{-1}\left(s_{\phi_{x_j}}, \varphi_{x_j}\right)}{t} \right)^{2\beta} \cdot \left(\frac{\Delta^{-1}\left(s_{\phi_{x_k}}, \varphi_{x_k}\right)}{t} \right)^{2\gamma} \right)^{\omega_i \omega_j \omega_k}$$

$$\geq \left(1 - \left(\frac{\Delta^{-1}\left(s_{\phi_{y_i}}, \varphi_{y_i}\right)}{t} \right)^{2\alpha} \cdot \left(\frac{\Delta^{-1}\left(s_{\phi_{y_j}}, \varphi_{y_j}\right)}{t} \right)^{2\beta} \cdot \left(\frac{\Delta^{-1}\left(s_{\phi_{y_k}}, \varphi_{y_k}\right)}{t} \right)^{2\gamma} \right)^{\omega_i \omega_j \omega_k} \tag{59}$$

Thereafter, (60), as shown at the next page. Furthermore, (61), as shown at the next page. That is $\Delta^{-1}\left(s_{\phi_y}, \varphi_y\right) \geq \Delta^{-1}\left(s_{\phi_x}, \varphi_x\right)$. By the same, we have $\Delta^{-1}\left(s_{\theta_y}, \vartheta_y\right) \leq \Delta^{-1}\left(s_{\theta_x}, \vartheta_x\right)$.

If $\Delta^{-1}\left(s_{\phi_y}, \varphi_y\right) < \Delta^{-1}\left(s_{\phi_x}, \varphi_x\right)$ and $\Delta^{-1}\left(s_{\theta_y}, \vartheta_y\right) \geq \Delta^{-1}\left(s_{\theta_x}, \vartheta_x\right)$

$$\begin{aligned}
&G2TLPFWBM_{\omega}^{\alpha,\beta,\gamma}\left(p_{y_1}, p_{y_2}, \cdots, p_{y_m}\right) \\
&\quad < G2TLPFWBM_{\omega}^{\alpha,\beta,\gamma}\left(p_{x_1}, p_{x_2}, \cdots, p_{x_m}\right)
\end{aligned}$$

$$\begin{aligned}
&\mathrm{G2TLPFWBM}_{\omega}^{\alpha,\beta,\gamma}(p_1,p_2,\ldots,p_m)\\
&=\left(\frac{1}{m(m-1)(m-2)}\bigoplus_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(\omega_i\omega_j\omega_k\left(p_i^{\alpha}\otimes p_j^{\beta}\otimes p_k^{\gamma}\right)\right)\right)^{1/(\alpha+\beta+\gamma)}\\
&=\left\{\begin{array}{l}
\Delta\left(t\left(\sqrt{1-\left(\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^{2\beta}\cdot\left(\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\right)^{2\gamma}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}}\right)^{1/(\alpha+\beta+\gamma)}\right),\\
\Delta\left(t\sqrt{1-\left(1-\left(\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\theta_j},\vartheta_j)}{t}\right)^2\right)^{\beta}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\theta_k},\vartheta_k)}{t}\right)^2\right)^{\gamma}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}\right)^{1/(\alpha+\beta+\gamma)}}\right)
\end{array}\right\}\\
&=\left\{\begin{array}{l}
\Delta\left(t\left(\sqrt{1-\left(\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\phi},\varphi)}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\phi},\varphi)}{t}\right)^{2\beta}\cdot\left(\frac{\Delta^{-1}(s_{\phi},\varphi)}{t}\right)^{2\gamma}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}}\right)^{1/(\alpha+\beta+\gamma)}\right),\\
\Delta\left(t\sqrt{1-\left(1-\left(\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}(s_{\theta},\vartheta)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\theta},\vartheta)}{t}\right)^2\right)^{\beta}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\theta},\vartheta)}{t}\right)^2\right)^{\gamma}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}\right)^{1/(\alpha+\beta+\gamma)}}\right)
\end{array}\right\}\\
&=\left\{(s_{\phi},\varphi),(s_{\theta},\vartheta)\right\}=p
\end{aligned}$$

$$\begin{aligned}
&1-\left(\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\phi_{x_i}},\varphi_{x_i})}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\phi_{x_j}},\varphi_{x_j})}{t}\right)^{2\beta}\cdot\left(\frac{\Delta^{-1}(s_{\phi_{x_k}},\varphi_{x_k})}{t}\right)^{2\gamma}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}\\
&\quad\leq 1-\left(\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\phi_{y_i}},\varphi_{y_i})}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\phi_{y_j}},\varphi_{y_j})}{t}\right)^{2\beta}\cdot\left(\frac{\Delta^{-1}(s_{\phi_{y_k}},\varphi_{y_k})}{t}\right)^{2\gamma}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}
\end{aligned}\tag{60}$$

$$\begin{aligned}
&\left(\sqrt{1-\left(\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\phi_{x_i}},\varphi_{x_i})}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\phi_{x_j}},\varphi_{x_j})}{t}\right)^{2\beta}\cdot\left(\frac{\Delta^{-1}(s_{\phi_{x_k}},\varphi_{x_k})}{t}\right)^{2\gamma}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}}\right)^{1/(\alpha+\beta+\gamma)}\\
&\leq\left(\sqrt{1-\left(\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(1-\left(\frac{\Delta^{-1}(s_{\phi_{y_i}},\varphi_{y_i})}{t}\right)^{2\alpha}\cdot\left(\frac{\Delta^{-1}(s_{\phi_{y_j}},\varphi_{y_j})}{t}\right)^{2\beta}\cdot\left(\frac{\Delta^{-1}(s_{\phi_{y_k}},\varphi_{y_k})}{t}\right)^{2\gamma}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}}\right)^{1/(\alpha+\beta+\gamma)}
\end{aligned}\tag{61}$$

If $\Delta^{-1}\left(s_{\phi_x}, \varphi_x\right) = \Delta^{-1}\left(s_{\phi_y}, \varphi_y\right)$ and $\Delta^{-1}\left(s_{\theta_x}, \vartheta_x\right) = \Delta^{-1}\left(s_{\theta_y}, \vartheta_y\right)$

$$\begin{aligned} &\mathrm{G2TLPFWBM}_{\omega}^{\alpha,\beta,\gamma}\left(p_{x_1}, p_{x_2}, \cdots, p_{x_m}\right) \\ &\quad = \mathrm{G2TLPFWBM}_{\omega}^{\alpha,\beta,\gamma}\left(p_{y_1}, p_{y_2}, \cdots, p_{y_m}\right) \end{aligned}$$

So property 8 is correct.

Property 9 (Boundedness): Let $p_i = \left\{\left(s_{\phi_i}, \varphi_i\right), \left(s_{\theta_i}, \vartheta_i\right)\right\}$ be a set of 2TLPFNs. If $p^+ = \left(\max_i\left(S_{\phi_i}, \varphi_i\right), \min_i\left(S_{\theta_i}, \vartheta_i\right)\right)$ and $p^- = \left(\min_i\left(S_{\phi_i}, \varphi_i\right), \max_i\left(S_{\theta_i}, \vartheta_i\right)\right)$ then

$$p^- \leq \mathrm{G2TLPFWBM}_{\omega}^{\alpha,\beta,\gamma}\left(p_1, p_2, \cdots, p_m\right) \leq p^+ \quad (62)$$

From property 7,

$$\begin{aligned} &\mathrm{G2TLPFWBM}_{\omega}^{\alpha,\beta,\gamma}\left(p_1^-, p_2^-, \cdots, p_m^-\right) = p^- \\ &\mathrm{G2TLPFWBM}_{\omega}^{\alpha,\beta,\gamma}\left(p_1^+, p_2^+, \cdots, p_m^+\right) = p^+ \end{aligned}$$

From property 8,

$$p^- \leq \mathrm{G2TLPFWBM}_{\omega}^{\alpha,\beta,\gamma}\left(p_1, p_2, \cdots, p_m\right) \leq p^+$$

B. G2TLPFWGBM OPERATOR

To pay attention to the attribute weights, the generalized WGBM (GWGBM) operator is:

Definition 13 [62]: Let $\alpha, \beta, \gamma > 0$ and $(b_1, b_2, \cdots, b_m)$ be a group of non-negative real values, the weights vector is $\omega = (\omega_1, \omega_2, \cdots \omega_m)^T$, thereby satisfying $0 \leq \omega_i \leq 1$, $\sum_{i=1}^m \omega_i = 1$.if

$$\begin{aligned} &\mathrm{GWGBM}_{\omega}^{\alpha,\beta,\gamma}\left(b_1, b_2, \cdots, b_m\right) \\ &\quad = \frac{1}{\alpha+\beta+\gamma} \prod_{i,j,k=1}^{m} \left(\alpha b_i + \beta b_j + \gamma b_k\right)^{\omega_i \omega_j \omega_k} \quad (63) \end{aligned}$$

Then we expand GWGBM to 2TLPFNs and propose generalized 2TLPFWGBM (G2TLPFWGBM) operator.

Definition 14: Assume that $p_i = \left\{\left(s_{\phi_i}, \varphi_i\right), \left(s_{\theta_i}, \vartheta_i\right)\right\}$ is a group of 2TLPFNs, and their weight vector is $w_i = (w_1, w_2, \ldots, w_m)^T$, thereby satisfying $0 \leq \omega_i \leq 1$, $\sum_{i=1}^m \omega_i = 1$. If

$$\begin{aligned} &\mathrm{G2TLPFWGBM}_{\omega}^{\alpha,\beta,\gamma}(p_1, p_2, \ldots, p_m) \\ &= \frac{1}{\alpha+\beta+\gamma} \left(\bigotimes_{\substack{i,j,k=1 \\ i\neq j\neq k}}^{m} \left(\alpha p_i \oplus \beta p_j \oplus \gamma p_k\right)^{\omega_i \omega_j \omega_k} \right)^{\frac{1}{m(m-1)(m-2)}} \end{aligned} \quad (64)$$

Theorem 4: Assume that $p_i = \left\{\left(s_{\phi_i}, \varphi_i\right), \left(s_{\theta_i}, \vartheta_i\right)\right\}$ is a group of 2TLPFNs. The fused value by G2TLPFWGBM operators is a 2TLPFN where, (65), as shown at the next page.

Proof: Equations (66)–(68), as shown at the next page.

Thus, (69), as shown at the next page. Therefore, (70), as shown at the next page. Thereafter, (71), as shown at the top of the page 19. Then, (72), as shown at the top of the page 19. Furthermore, (73), as shown at the top of the page 19. Hence, (65) is proven.

Then we give the proving process of that (65) is also a 2TLPFN.

① $0 \leq \frac{\Delta^{-1}(s_\phi,\varphi)}{t} \leq 1, 0 \leq \frac{\Delta^{-1}(s_\theta,\vartheta)}{t} \leq 1$,

② $0 \leq \left(\frac{\Delta^{-1}(s_\phi,\varphi)}{t}\right)^2 + \left(\frac{\Delta^{-1}(s_\theta,\vartheta)}{t}\right)^2 \leq 1$

Let, $\frac{\Delta^{-1}(s_\phi,\varphi)}{t}$, $\frac{\Delta^{-1}(s_\theta,\vartheta)}{t}$, as shown at the page 20.

Proof: if $0 \leq \frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t} \leq 1, 0 \leq \frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t} \leq 1, 0 \leq \frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t} \leq 1$, then

$$\begin{aligned} 0 \leq &\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha} \\ &\cdot \left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta} \\ &\cdot \left(1-\left(\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\right)^2\right)^{\gamma} \leq 1 \end{aligned} \quad (74)$$

Then, (75) and (76), as shown at the page 20. That means $0 \leq \frac{\Delta^{-1}(s_\phi,\varphi)}{t} \leq 1$, so ① is maintained, similarly, we have $0 \leq \frac{\Delta^{-1}(s_\theta,\vartheta)}{t} \leq 1$.

② Since $0 \leq \left(\frac{\Delta^{-1}(s_\phi,\varphi)}{t}\right)^2 + \left(\frac{\Delta^{-1}(s_\theta,\vartheta)}{t}\right)^2 \leq 1$, we get the following inequality, $\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2 + \left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^2$, as shown at the page 20. That means is maintained.

The G2TLPFWGBM has three properties.

Property 10 (Idempotency): If $p_i = \left\{\left(s_{\phi_i}, \varphi_i\right), \left(s_{\theta_i}, \vartheta_i\right)\right\}$ are equal, then

$$\mathrm{G2TLPFWGBM}_{\omega}^{\alpha,\beta,\gamma}\left(p_1, p_2, \cdots, p_m\right) = p \quad (77)$$

Property 11 (Monotonicity): Let $p_{x_i} = \{(s_{\phi_{x_i}}, \varphi_{x_i}), (s_{\theta_{x_i}}, \vartheta_{x_i})\}$ and $p_{y_i} = \{(s_{\phi_{y_i}}, \varphi_{y_i}), (s_{\theta_{y_i}}, \vartheta_{y_i})\}$ be two sets of 2TLPFNs. If $\Delta^{-1}(s_{\phi_{y_i}}, \varphi_{y_i}) \leq \Delta^{-1}(s_{\phi_{x_i}}, \varphi_{x_i})$ and $\Delta^{-1}(s_{\theta_{y_i}}, \vartheta_{y_i}) \geq \Delta^{-1}(s_{\theta_{x_i}}, \vartheta_{x_i}), (i = 1, 2, \ldots, m)$, then

$$\begin{aligned} &\mathrm{G2TLPFWGBM}_{\omega}^{\alpha,\beta,\gamma}\left(p_{y_1}, p_{y_2}, \cdots, p_{y_m}\right) \\ &\quad \leq \mathrm{G2TLPFWGBM}_{\omega}^{\alpha,\beta,\gamma}\left(p_{x_1}, p_{x_2}, \cdots, p_{x_m}\right) \quad (78) \end{aligned}$$

Property 12 (Boundedness): Let $p_i = \left\{\left(s_{\phi_i}, \varphi_i\right), \left(s_{\theta_i}, \vartheta_i\right)\right\}$ be a set of 2TLPFNs. If $p^+ = \left(\max_i\left(S_{\phi_i}, \varphi_i\right), \min_i\left(S_{\theta_i}, \vartheta_i\right)\right)$ and $p^- = \left(\min_i\left(S_{\phi_i}, \varphi_i\right), \max_i\left(S_{\theta_i}, \vartheta_i\right)\right)$ then

$$p^- \leq \mathrm{G2TLPFWGBM}_{\omega}^{\alpha,\beta,\gamma}\left(p_1, p_2, \cdots, p_m\right) \leq p^+ \quad (79)$$

V. DG2TLPFWBM AND DG2TLPFWGBM OPERATORS

A. DG2TLPFWBM OPERATOR

Zhang *et al.* [63] proposed the dual GWBM (DGBM).

Definition 15 [63]: Assume that $(b_1, b_2, \cdots, b_m)$ be a group of non-negative real values, the weights vector is $\omega = (\omega_1, \omega_2, \cdots \omega_m)^T$, thereby satisfying $0 \leq \omega_i \leq 1$, $\sum_{i=1}^m \omega_i = 1$. Suppose that $R = (r_1, r_2, \ldots, r_m)^T$ and $r_i \geq 0$ $(i = 1, 2, \ldots, m)$. then

$$\begin{aligned} &\mathrm{DGWBM}_{w}^{R}(b_1, b_2, \ldots, b_m) \\ &\quad = \left(\sum_{\substack{i_1, i_2, \ldots, i_m = 1 \\ i_1 \neq i_2 \neq \ldots \neq i_m}}^{m} \left(\prod_{j=1}^{m} w_{i_j} b_{i_j}^{r_j} \right) \right)^{1/\sum_{j=1}^m r_j} \quad (80) \end{aligned}$$

$$\begin{aligned}
&\text{G2TLPFWGBM}_{\omega}^{\alpha,\beta,\gamma}(p_1,p_2,\ldots,p_m)\\
&=\frac{1}{\alpha+\beta+\gamma}\left(\underset{\substack{i,j,k=1\\ i\neq j\neq k}}{\overset{m}{\otimes}}\left(\alpha p_i\oplus\beta p_j\oplus\gamma p_k\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}\\
&=\left\{\begin{array}{l}
\Delta\left(t\sqrt{1-\left(1-\left(\prod\limits_{\substack{i,j,k=1\\ i\neq j\neq k}}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\right)^2\right)^{\gamma}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{n(n-1)(n-2)}}\right)^{\frac{1}{\alpha+\beta+\gamma}}}\right),\\
\Delta\left(t\left(\sqrt{1-\left(\prod\limits_{\substack{i,j,k=1\\ i\neq j\neq k}}^{n}\left(1-\left(\frac{\Delta^{-1}(s_{\theta_k},\vartheta_k)}{t}\right)^{2\gamma}\cdot\left(\frac{\Delta^{-1}(s_{\theta_j},\vartheta_j)}{t}\right)^{2\beta}\cdot\left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^{2\alpha}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{n(n-1)(n-2)}}}\right)^{\frac{1}{\alpha+\beta+\gamma}}\right)
\end{array}\right\}
\end{aligned}\tag{65}$$

$$\alpha p_i=\left\{\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha}}\right),\Delta\left(t\left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^{\alpha}\right)\right\}\tag{66}$$

$$\beta p_j=\left\{\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta}}\right),\Delta\left(t\left(\frac{\Delta^{-1}(s_{\theta_j},\vartheta_j)}{t}\right)^{\beta}\right)\right\}\tag{67}$$

$$\gamma p_k=\left\{\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\right)^2\right)^{\gamma}}\right),\Delta\left(t\left(\frac{\Delta^{-1}(s_{\theta_k},\vartheta_k)}{t}\right)^{\gamma}\right)\right\}\tag{68}$$

$$\alpha p_i\oplus\beta p_j\oplus\gamma p_k=\left\{\begin{array}{l}
\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\right)^2\right)^{\gamma}}\right),\\
\Delta\left(t\left(\frac{\Delta^{-1}(s_{\theta_k},\vartheta_k)}{t}\right)^{\gamma}\cdot\left(\frac{\Delta^{-1}(s_{\theta_j},\vartheta_j)}{t}\right)^{\beta}\cdot\left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^{\alpha}\right)
\end{array}\right\}\tag{69}$$

$$\begin{aligned}
&\left(\alpha p_i\oplus\beta p_j\oplus\gamma p_k\right)^{\omega_i\omega_j\omega_k}\\
&=\left\{\begin{array}{l}
\Delta\left(t\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\right)^2\right)^{\gamma}}\right)^{\omega_i\omega_j\omega_k}\right),\\
\Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\theta_k},\vartheta_k)}{t}\right)^{2\gamma}\cdot\left(\frac{\Delta^{-1}(s_{\theta_j},\vartheta_j)}{t}\right)^{2\beta}\cdot\left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^{2\alpha}\right)^{\omega_i\omega_j\omega_k}}\right)
\end{array}\right\}
\end{aligned}\tag{70}$$

$$\underset{\substack{i,j,k=1\\i\neq j\neq k}}{\overset{m}{\otimes}}\left(\alpha p_i\oplus\beta p_j\oplus\gamma p_k\right)^{\omega_i\omega_j\omega_k}=\left\{\begin{array}{l}\Delta\left(t\prod\limits_{\substack{i,j,k=1\\i\neq j\neq k}}^{m}\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\right)^2\right)^{\gamma}}\right)^{\omega_i\omega_j\omega_k}\right),\\ \Delta\left(t\sqrt{1-\prod\limits_{\substack{i,j,k=1\\i\neq j\neq k}}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_k},\vartheta_k\right)}{t}\right)^{2\gamma}\cdot\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^{2\beta}\cdot\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^{2\alpha}\right)^{\omega_i\omega_j\omega_k}}\right)\end{array}\right\} \tag{71}$$

$$\left(\underset{\substack{i,j,k=1\\i\neq j\neq k}}{\overset{m}{\otimes}}\left(\alpha p_i\oplus\beta p_j\oplus\gamma p_k\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}=\left\{\begin{array}{l}\Delta\left(t\left(\prod\limits_{\substack{i,j,k=1\\i\neq j\neq k}}^{m}\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\right)^2\right)^{\gamma}}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}\right),\\ \Delta\left(t\sqrt{1-\left(\prod\limits_{\substack{i,j,k=1\\i\neq j\neq k}}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_k},\vartheta_k\right)}{t}\right)^{2\gamma}\cdot\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^{2\beta}\cdot\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^{2\alpha}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}}\right)\end{array}\right\} \tag{72}$$

$$\begin{aligned}&\text{G2TLPFWGBM}_{\omega}^{\alpha,\beta,\gamma}(p_1,p_2,\ldots,p_m)\\&=\frac{1}{\alpha+\beta+\gamma}\left(\underset{\substack{i,j,k=1\\i\neq j\neq k}}{\overset{m}{\otimes}}\left(\alpha p_i\oplus\beta p_j\oplus\gamma p_k\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}\\&=\left\{\begin{array}{l}\Delta\left(t\sqrt{1-\left(1-\left(\prod\limits_{\substack{i,j,k=1\\i\neq j\neq k}}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\right)^2\right)^{\gamma}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}\right)^{\frac{1}{\alpha+\beta+\gamma}}}\right),\\ \Delta\left(t\left(\sqrt{1-\left(\prod\limits_{\substack{i,j,k=1\\i\neq j\neq k}}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_k},\vartheta_k\right)}{t}\right)^{2\gamma}\cdot\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^{2\beta}\cdot\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^{2\alpha}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}}\right)^{\frac{1}{\alpha+\beta+\gamma}}\right)\end{array}\right\}\end{aligned} \tag{73}$$

Then we expand DGWBM_{w}^{R} to 2TLPFNs and propose dual G2TLPFWBM (DG2TLPFWBM) operator.

Definition 16: Let $p_i = \left\{(s_{\phi_i},\varphi_i), \left(s_{\theta_i},\vartheta_i\right)\right\}$ be a group of 2TLPFNs. The DG2TLPFWBM operator is:

$$\text{DG2TLPFWBM}_{w}^{R}(p_1,p_2,\ldots,p_n)$$

$$=\left(\frac{1}{m!}\underset{\substack{i_1,i_2,\ldots,i_m=1\\i_1\neq i_2\neq\ldots\neq i_m}}{\overset{m}{\oplus}}\left(\underset{j=1}{\overset{m}{\otimes}}w_{i_j}p_{i_j}^{r_j}\right)\right)^{1/\sum_{i=1}^{m}r_j} \tag{81}$$

Theorem 5: Let $p_i = \left\{(s_{\phi_i},\varphi_i), \left(s_{\theta_i},\vartheta_i\right)\right\}$ be a group of 2TLPFNs. The fused value by g DG2TLPFWBM

$$\frac{\Delta^{-1}\left(s_{\phi},\varphi\right)}{t} = \sqrt{1-\left(1-\left(\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\right)^2\right)^{\gamma}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}\right)^{\frac{1}{\alpha+\beta+\gamma}}}$$

$$\frac{\Delta^{-1}\left(s_{\theta},\vartheta\right)}{t} = \left(\sqrt{1-\left(\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_k},\vartheta_k\right)}{t}\right)^{2\gamma}\cdot\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^{2\beta}\cdot\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^{2\alpha}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}}\right)^{\frac{1}{\alpha+\beta+\gamma}}$$

$$0 \leq \prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\right)^2\right)^{\gamma}\right)^{\omega_i\omega_j\omega_k} \leq 1 \tag{75}$$

$$0 \leq \sqrt{1-\left(1-\left(\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\right)^2\right)^{\gamma}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{n(n-1)(n-2)}}\right)^{\frac{1}{\alpha+\beta+\gamma}}} \leq 1 \tag{76}$$

$$\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2+\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^2$$

$$= \left(1-\left(1-\left(\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2\right)^{\alpha}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^2\right)^{\beta}\cdot\left(1-\left(\frac{\Delta^{-1}(s_{\phi_k},\varphi_k)}{t}\right)^2\right)^{\gamma}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}\right)^{\frac{1}{\alpha+\beta+\gamma}}\right)$$

$$+\left(1-\left(\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_k},\vartheta_k\right)}{t}\right)^{2\gamma}\cdot\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^{2\beta}\cdot\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^{2\alpha}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}\right)^{\frac{1}{\alpha+\beta+\gamma}}$$

$$\leq \left(1-\left(1-\left(\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(1-\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_k},\vartheta_k\right)}{t}\right)^2\right)\right)^{\alpha}\cdot\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^2\right)\right)^{\beta}\cdot\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^2\right)\right)^{\gamma}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}\right)^{\frac{1}{\alpha+\beta+\gamma}}\right)$$

$$+\left(1-\left(\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_k},\vartheta_k\right)}{t}\right)^{2\gamma}\cdot\left(\frac{\Delta^{-1}\left(s_{\theta_j},\vartheta_j\right)}{t}\right)^{2\beta}\cdot\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^{2\alpha}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{m(m-1)(m-2)}}\right)^{\frac{1}{\alpha+\beta+\gamma}} = 1$$

$$\mathrm{DG2TLPFWBM}_w^R(p_1,p_2,\ldots,p_m) = \left(\frac{1}{n!}\bigoplus_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(\bigotimes_{j=1}^{m} w_{i_j}p_{i_j}^{r_j}\right)\right)^{1/\sum_{i=1}^{m} r_j}$$

$$= \left\{\begin{array}{l} \Delta\left(t\left(\sqrt{1-\prod\limits_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(1-\prod\limits_{j=1}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^{2r_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{m!}}}\right)^{1/\sum_{i=1}^{m} r_j}\right), \\ \Delta\left(t\sqrt{1-\left(1-\prod\limits_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(1-\prod\limits_{j=1}^{m}\left(1-\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{i_j}},\vartheta_{i_j}\right)}{t}\right)^{2}\right)^{r_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{m!}}\right)^{1/\sum_{i=1}^{m} r_j}}\right) \end{array}\right\} \tag{82}$$

$$p_{i_j}^{r_j} = \left\{\Delta\left(t\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^{r_j}\right), \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{i_j}},\vartheta_{i_j}\right)}{t}\right)^{2}\right)^{r_j}}\right)\right\} \tag{83}$$

operators is o a 2TLPFN where, (82), as shown at the top of this page.

Proof: Equation (83), as shown at the top of this page.

Thus,

$$w_{i_j}p_{i_j}^{r_j} = \left\{\begin{array}{l} \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^{2r_j}\right)^{w_{i_j}}}\right), \\ \Delta\left(t\left(\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{i_j}},\vartheta_{i_j}\right)}{t}\right)^{2}\right)^{r_j}}\right)^{w_{i_j}}\right) \end{array}\right\} \tag{84}$$

Thereafter, (85), as shown at the next page. Furthermore, (86), as shown at the next page. Then, (87), as shown at the next page. Therefore, (88), as shown at the next page. Hence, (82) is proven.

Then we give the proving process of that (82) is also a 2TLPFN.

① $0 \le \frac{\Delta^{-1}(s_\phi,\varphi)}{t} \le 1, 0 \le \frac{\Delta^{-1}(s_\theta,\vartheta)}{t} \le 1,$

② $0 \le \left(\frac{\Delta^{-1}(s_\phi,\varphi)}{t}\right)^2 + \left(\frac{\Delta^{-1}(s_\theta,\vartheta)}{t}\right)^2 \le 1$

Let $\frac{\Delta^{-1}(s_\phi,\varphi)}{t}$ and $\frac{\Delta^{-1}(s_\theta,\vartheta)}{t}$, as shown at the top of the page 23.

Proof: Since $0 \le \frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t} \le 1$, we get

$$0 \le 1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^{2r_j} \le 1 \tag{89}$$

Then,

$$0 \le 1-\prod_{j=1}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^{2r_j}\right)^{w_{i_j}}\right) \le 1 \tag{90}$$

Furthermore,

$$0 \le 1-\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m} \times\left(1-\prod_{j=1}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^{2r_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{m!}} \le 1 \tag{91}$$

Therefore, (92), as shown at the top of the page 23. By the same, we can get $0 \le \frac{\Delta^{-1}(s_\theta,\vartheta)}{t} \le 1$.

② Since $0 \le \left(\frac{\Delta^{-1}(s_\phi,\varphi)}{t}\right)^2 + \left(\frac{\Delta^{-1}(s_\theta,\vartheta)}{t}\right)^2 \le 1$, we get the following inequality $\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2 + \left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^2$, as shown at the top of the page 23. That means is maintained.

The DG2TLPFWBM operator has two properties.

Property 13 (Monotonicity): Let $p_{x_i} = \{(s_{\phi_{x_i}},\varphi_{x_i}), (s_{\theta_{x_i}},\vartheta_{x_i})\}$ and $p_{y_i} = \{(s_{\phi_{y_i}},\varphi_{y_i}), (s_{\theta_{y_i}},\vartheta_{y_i})\}$ be two sets of 2TLPFNs. If $\Delta^{-1}(s_{\phi_{y_i}},\varphi_{y_i}) \le \Delta^{-1}(s_{\phi_{x_i}},\varphi_{x_i})$ and $\Delta^{-1}(s_{\theta_{y_i}},\vartheta_{y_i}) \ge \Delta^{-1}(s_{\theta_{x_i}},\vartheta_{x_i})$, $i = 1,2,\ldots,m$, then

$$\mathrm{DG2TLPFWBM}_w^R\left(p_{y_1},p_{y_2},\cdots,p_{y_m}\right) \le \mathrm{DG2TLPFWBM}_w^R\left(p_{x_1},p_{x_2},\cdots,p_{x_m}\right) \tag{93}$$

$$\underset{j=1}{\overset{m}{\otimes}} w_{i_j} p_{i_j}^{r_j} = \left\{ \begin{array}{l} \Delta\left(t \prod_{j=1}^{m} \sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}, \varphi_{i_j}\right)}{t}\right)^{2r_j}\right)^{w_{i_j}}} \right), \\ \Delta\left(t \sqrt{1-\prod_{j=1}^{m}\left(1-\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{i_j}}, \vartheta_{i_j}\right)}{t}\right)^{2}\right)^{r_j}\right)^{w_{i_j}}\right)} \right) \end{array} \right\} \tag{85}$$

$$\underset{\substack{i_1,i_2,\ldots,i_m=1 \\ i_1\neq i_2\neq\ldots\neq i_m}}{\overset{m}{\oplus}} \left(\underset{j=1}{\overset{m}{\otimes}} w_{i_j} p_{i_j}^{r_j} \right) = \left\{ \begin{array}{l} \Delta\left(t \sqrt{1-\prod_{\substack{i_1,i_2,\ldots,i_m=1 \\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}, \varphi_{i_j}\right)}{t}\right)^{2r_j}\right)^{w_{i_j}}\right)\right)} \right), \\ \Delta\left(t \prod_{\substack{i_1,i_2,\ldots,i_m=1 \\ i_1\neq i_2\neq\ldots\neq i_m}}^{m} \sqrt{1-\prod_{j=1}^{m}\left(1-\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{i_j}}, \vartheta_{i_j}\right)}{t}\right)^{2}\right)^{r_j}\right)^{w_{i_j}}\right)} \right) \end{array} \right\} \tag{86}$$

$$\frac{1}{m!} \underset{\substack{i_1,i_2,\ldots,i_m=1 \\ i_1\neq i_2\neq\ldots\neq i_m}}{\overset{m}{\oplus}} \left(\underset{j=1}{\overset{m}{\otimes}} w_{i_j} p_{i_j}^{r_j} \right) = \left\{ \begin{array}{l} \Delta\left(t \sqrt{1-\prod_{\substack{i_1,i_2,\ldots,i_m=1 \\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}, \varphi_{i_j}\right)}{t}\right)^{2r_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{m!}}} \right), \\ \Delta\left(t \prod_{\substack{i_1,i_2,\ldots,i_m=1 \\ i_1\neq i_2\neq\ldots\neq i_m}}^{m} \left(\sqrt{1-\prod_{j=1}^{m}\left(1-\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{i_j}}, \vartheta_{i_j}\right)}{t}\right)^{2}\right)^{r_j}\right)^{w_{i_j}}\right)} \right)^{\frac{1}{m!}} \right) \end{array} \right\} \tag{87}$$

$$\begin{aligned} & \text{DG2TLPFWBM}_w^R(p_1, p_2, \ldots, p_m) \\ & = \left(\frac{1}{n!} \underset{\substack{i_1,i_2,\ldots,i_m=1 \\ i_1\neq i_2\neq\ldots\neq i_m}}{\overset{m}{\oplus}} \left(\underset{j=1}{\overset{m}{\otimes}} w_{i_j} p_{i_j}^{r_j} \right) \right)^{1/\sum_{i=1}^{m} r_j} \\ & = \left\{ \begin{array}{l} \Delta\left(t \left(\sqrt{1-\prod_{\substack{i_1,i_2,\ldots,i_m=1 \\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}}, \varphi_{i_j}\right)}{t}\right)^{2r_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{m!}}} \right)^{1/\sum_{i=1}^{m} r_j} \right), \\ \Delta\left(t \sqrt{1-\left(1-\prod_{\substack{i_1,i_2,\ldots,i_m=1 \\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{i_j}}, \vartheta_{i_j}\right)}{t}\right)^{2}\right)^{r_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{m!}}\right)^{1/\sum_{i=1}^{m} r_j}} \right) \end{array} \right\} \end{aligned} \tag{88}$$

$$\frac{\Delta^{-1}\left(s_{\phi},\varphi\right)}{t}=\left(\sqrt{1-\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^{2r_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{m!}}}\right)^{1/\sum_{i=1}^{m}r_j}$$

$$\frac{\Delta^{-1}\left(s_{\theta},\vartheta\right)}{t}=\sqrt{1-\left(1-\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{i_j}},\vartheta_{i_j}\right)}{t}\right)^{2}\right)^{r_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{m!}}\right)^{1/\sum_{i=1}^{m}r_j}}$$

$$0\leq\left(\sqrt{1-\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^{2r_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{m!}}}\right)^{1/\sum_{i=1}^{m}r_j}\leq 1 \tag{92}$$

$$\begin{aligned}
&\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2+\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^2\\
&=\left(1-\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^{2r_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{m!}}\right)^{1/\sum_{i=1}^{m}r_j}\\
&\quad+\left(1-\left(1-\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{i_j}},\vartheta_{i_j}\right)}{t}\right)^{2}\right)^{r_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{m!}}\right)^{1/\sum_{i=1}^{m}r_j}\right)\\
&\leq\left(1-\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{i_j}},\vartheta_{i_j}\right)}{t}\right)^{2}\right)^{r_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{m!}}\right)^{1/\sum_{i=1}^{m}r_j}\\
&\quad+\left(1-\left(1-\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{i_j}},\vartheta_{i_j}\right)}{t}\right)^{2}\right)^{r_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{m!}}\right)^{1/\sum_{i=1}^{m}r_j}\right)\\
&=1
\end{aligned}$$

Proof: Let $\mathrm{DG2TLPFWBM}_w^R(p_{x_1},p_{x_2},\cdots,p_{x_m})=\{(s_{\phi_{x_i}},\varphi_{x_i}),(s_{\theta_{x_i}},\vartheta_{x_i})\}$ and $\mathrm{DG2TLPFWBM}_w^R(p_{y_1},p_{y_2},\cdots,p_{y_m})=\{(s_{\phi_{y_i}},\varphi_{y_i}),\ (s_{\theta_{y_i}},\vartheta_{y_i})\}$, if $\Delta^{-1}(s_{\phi_{x_i}},\varphi_{x_i})\leq\Delta^{-1}(s_{\phi_{y_i}},\varphi_{y_i})$, then

$$\left(\frac{\Delta^{-1}(s_{\phi_{x_{i_j}}},\varphi_{x_{i_j}})}{t}\right)^{2r_j}\leq\left(\frac{\Delta^{-1}(s_{\phi_{y_{i_j}}},\varphi_{y_{i_j}})}{t}\right)^{2r_j} \tag{94}$$

$$\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{x_{i_j}}},\varphi_{x_{i_j}}\right)}{t}\right)^{2r_j}\right)^{w_{i_j}}\geq\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{y_{i_j}}},\varphi_{y_{i_j}}\right)}{t}\right)^{2r_j}\right)^{w_{i_j}} \tag{95}$$

$$\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{x_{i_j}}},\varphi_{x_{i_j}}\right)}{t}\right)^{2r_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{n!}}$$

$$\geq\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{y_{i_j}}},\varphi_{y_{i_j}}\right)}{t}\right)^{2r_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{m!}} \quad (97)$$

$$\left(\sqrt{1-\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{x_{i_j}}},\varphi_{x_{i_j}}\right)}{t}\right)^{2r_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{m!}}}\right)^{1/\sum_{i=1}^{m}r_j}$$

$$\leq\left(\sqrt{1-\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{y_{i_j}}},\varphi_{y_{i_j}}\right)}{t}\right)^{2r_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{m!}}}\right)^{1/\sum_{i=1}^{m}r_j} \quad (98)$$

Thereafter,

$$\prod_{j=1}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{x_{i_j}}},\varphi_{x_{i_j}}\right)}{t}\right)^{2r_j}\right)^{w_{i_j}}\right)$$

$$\leq\prod_{j=1}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{y_{i_j}}},\varphi_{y_{i_j}}\right)}{t}\right)^{2r_j}\right)^{w_{i_j}}\right) \quad (96)$$

Furthermore, (97) and (98), as shown at the top of this page. By the same, we have $\Delta^{-1}\left(s_{\theta_y},\vartheta_y\right)\leq\Delta^{-1}\left(s_{\theta_x},\vartheta_x\right)$.

If $\Delta^{-1}\left(s_{\phi_y},\varphi_y\right)>\Delta^{-1}\left(s_{\phi_x},\varphi_x\right)$ and $\Delta^{-1}\left(s_{\theta_y},\vartheta_y\right)\leq\Delta^{-1}\left(s_{\theta_x},\vartheta_x\right)$

$$\mathrm{DG2TLPFWBM}_w^R\left(p_{x_1},p_{x_2},\cdots,p_{x_m}\right)<\mathrm{DG2TLPFWBM}_w^R\left(p_{y_1},p_{y_2},\cdots,p_{y_m}\right)$$

If $\Delta^{-1}\left(s_{\phi_x},\varphi_x\right)=\Delta^{-1}\left(s_{\phi_y},\varphi_y\right)$ and $\Delta^{-1}\left(s_{\theta_x},\vartheta_x\right)=\Delta^{-1}\left(s_{\theta_y},\vartheta_y\right)$

$$\mathrm{DG2TLPFWBM}_w^R\left(p_{x_1},p_{x_2},\cdots,p_{x_m}\right)=\mathrm{DG2TLPFWBM}_w^R\left(p_{y_1},p_{y_2},\cdots,p_{y_m}\right)$$

So property 13 is correct.

Property 14 (Boundedness): Let $p_i=\left\{\left(s_{\phi_i},\varphi_i\right),\left(s_{\theta_i},\vartheta_i\right)\right\}$ be a group of 2TLPFNs. If $p^-=(\min_i(S_{\phi_i},\varphi_i),\max_i(S_{\theta_i},\vartheta_i))$ and $p^+=\left(\max_i\left(S_{\phi_i},\varphi_i\right),\min_i\left(S_{\theta_i},\vartheta_i\right)\right)$ then

$$p^-\leq\mathrm{DG2TLPFWBM}_\omega^R\left(p_1,p_2,\cdots,p_m\right)\leq p^+ \quad (99)$$

From theorem 5, we can obtain $\mathrm{DG2TLPFWBM}_w^R\left(p_1^-,p_2^-,\cdots,p_m^-\right)$ and $\mathrm{DG2TLPFWBM}_w^R(p_1^+,p_2^+,\cdots,p_m^+)$, as shown at the next page. From property 13,

$$p^-\leq\mathrm{DG2TLPFWBM}_w^R\left(p_1,p_2,\cdots,p_m\right)\leq p^+$$

B. DG2TLPFWGBM OPERATOR

In order to pay attention to the attribute weights, the dual GWGBM (DGWGBM) is:

Definition 17 [63]: Assume that $(b_1,b_2,\cdots,b_m)$ be a group of non-negative real values, the weights vector is $\omega=(\omega_1,\omega_2,\cdots\omega_m)^T$, thereby satisfying $0\leq\omega_i\leq1$, $\sum_{i=1}^{m}\omega_i=1$. Suppose that $R=(r_1,r_2,\ldots,r_m)^T$ and $r_i\geq0$ $(i=1,2,\ldots,m)$. then

$$\mathrm{DGWGBM}_w^R\left(b_1,b_2,\cdots,b_m\right)=\frac{1}{\sum_{j=1}^{m}r_j}\left(\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(\sum_{j=1}^{m}\left(r_jb_{i_j}\right)\right)\right)^{\prod_{j=1}^{m}w_{i_j}} \quad (100)$$

Then we expand DGWGBM to 2TLPFNs and propose dual G2TLPFWGBM (DG2TLPFWGBM) operator.

Definition 18: Let $p_i=\left\{\left(s_{\phi_i},\varphi_i\right),\left(s_{\theta_i},\vartheta_i\right)\right\}$ be a group of 2TLPFNs, their weight vector is $w_i=(w_1,w_2,\ldots,w_m)^T$, thereby satisfying g $0\leq\omega_i\leq1$, $\sum_{i=1}^{m}\omega_i=1$. If

$$\mathrm{DG2TLPFWGBM}_\omega^R(p_1,p_2,\ldots,p_m)=\frac{1}{\sum_{j=1}^{m}r_j}\left(\bigotimes_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(\bigoplus_{j=1}^{m}\left(r_jp_{i_j}\right)\right)^{\prod_{j=1}^{m}w_{i_j}}\right)^{\frac{1}{m!}} \quad (101)$$

Theorem 6: Let $p_i=\left\{\left(s_{\phi_i},\varphi_i\right),\left(s_{\theta_i},\vartheta_i\right)\right\}$ be a group of 2TLPFNs. The fused result by DG2TLPFWGBM operator is, (102), as shown at the next page.

$$
\begin{aligned}
&\mathrm{DG2TLPFWBM}_w^R\left(p_1^-, p_2^-, \cdots, p_m^-\right)\\
&=\left\{\begin{array}{l}
\Delta\left(t\left(\sqrt{1-\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\left(\frac{\min\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^{2r_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{m!}}}\right)^{1/\sum_{i=1}^{m} r_j}\right),\\
\Delta\left(t\sqrt{1-\left(1-\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\left(1-\left(\frac{\max\Delta^{-1}\left(s_{\theta_{i_j}},\vartheta_{i_j}\right)}{t}\right)^{2}\right)^{r_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{m!}}\right)^{1/\sum_{i=1}^{m} r_j}}\right)
\end{array}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{DG2TLPFWBM}_w^R\left(p_1^+, p_2^+, \cdots, p_m^+\right)\\
&=\left\{\begin{array}{l}
\Delta\left(t\left(\sqrt{1-\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\left(\frac{\max\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^{2r_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{m!}}}\right)^{1/\sum_{i=1}^{m} r_j}\right),\\
\Delta\left(t\sqrt{1-\left(1-\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\left(1-\left(\frac{\min\Delta^{-1}\left(s_{\theta_{i_j}},\vartheta_{i_j}\right)}{t}\right)^{2}\right)^{r_j}\right)^{w_{i_j}}\right)\right)^{\frac{1}{m!}}\right)^{1/\sum_{i=1}^{m} r_j}}\right)
\end{array}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{DG2TLPFWGBM}_w^R(p_1, p_2, \ldots, p_m)\\
&=\frac{1}{\sum_{j=1}^{m} r_j}\left(\bigotimes_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(\bigoplus_{j=1}^{m}\left(r_j p_{i_j}\right)\right)^{\prod_{j=1}^{m} w_{i_j}}\right)^{\frac{1}{m!}}\\
&=\left\{\begin{array}{l}
\Delta\left(t\sqrt{1-\left(1-\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(\left(\left(1-\prod_{j=1}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^{2}\right)\right)^{r_j}\right)^{\prod_{j=1}^{m} w_{i_j}}\right)^{\frac{1}{m!}}\right)^{\frac{1}{\sum_{j=1}^{m} r_j}}}\right),\\
\Delta\left(t\left(\sqrt{1-\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(\left(1-\prod_{j=1}^{m}\left(\frac{\Delta^{-1}\left(s_{\theta_{i_j}},\vartheta_{i_j}\right)}{t}\right)^{2r_j}\right)^{\prod_{j=1}^{m} w_{i_j}}\right)^{\frac{1}{m!}}}\right)^{\frac{1}{\sum_{j=1}^{m} r_j}}\right)
\end{array}\right\}
\end{aligned}
\tag{102}
$$

Proof: Equation (103), as shown at the top of the next page.

Thus, (104), as shown at the top of the next page. Therefore, (105), as shown at the top of the next page. Thereafter, (106), as shown at the top of the next page. Then, (107), as shown at the top of the next page. Furthermore, (108), as shown at the top of the page 27. Hence, (102) is proven.

Then we give the proving process of that (102) is also a 2TLPFN.

① $0 \leq \frac{\Delta^{-1}(s_\phi,\varphi)}{t} \leq 1, 0 \leq \frac{\Delta^{-1}(s_\theta,\vartheta)}{t} \leq 1,$

② $0 \leq \left(\frac{\Delta^{-1}(s_\phi,\varphi)}{t}\right)^2 + \left(\frac{\Delta^{-1}(s_\theta,\vartheta)}{t}\right)^2 \leq 1$

Let $\frac{\Delta^{-1}(s_\phi,\varphi)}{t}$ and $\frac{\Delta^{-1}(s_\theta,\vartheta)}{t}$, as shown at the top of the page 27.

$$r_j p_{i_j} = \left\{ \Delta\left(t\sqrt{1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^2\right)^{r_j}} \right), \Delta\left(t\left(\frac{\Delta^{-1}\left(s_{\theta_{i_j}},\vartheta_{i_j}\right)}{t}\right)^{r_j} \right) \right\} \tag{103}$$

$$\overset{m}{\underset{j=1}{\oplus}}\left(r_j p_{i_j}\right) = \left\{ \Delta\left(t\sqrt{1-\prod_{j=1}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^2\right)^{r_j}} \right), \Delta\left(t\prod_{j=1}^{m}\left(\frac{\Delta^{-1}\left(s_{\theta_{i_j}},\vartheta_{i_j}\right)}{t}\right)^{r_j} \right) \right\} \tag{104}$$

$$\left(\overset{m}{\underset{j=1}{\oplus}}\left(r_j p_{i_j}\right)\right)^{\prod_{j=1}^{m} w_{i_j}} = \left\{ \Delta\left(t\left(\sqrt{1-\prod_{j=1}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^2\right)^{r_j}}\right)^{\prod_{j=1}^{m} w_{i_j}} \right), \Delta\left(t\sqrt{1-\left(1-\prod_{j=1}^{m}\left(\frac{\Delta^{-1}\left(s_{\theta_{i_j}},\vartheta_{i_j}\right)}{t}\right)^{2r_j}\right)^{\prod_{j=1}^{m} w_{i_j}}} \right) \right\} \tag{105}$$

$$\underset{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}{\overset{m}{\otimes}}\left(\overset{m}{\underset{j=1}{\oplus}}\left(r_j p_{i_j}\right)\right)^{\prod_{j=1}^{m} w_{i_j}} = \left\{ \begin{array}{l} \Delta\left(t\prod\limits_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(\sqrt{1-\prod\limits_{j=1}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^2\right)^{r_j}}\right)^{\prod_{j=1}^{m} w_{i_j}} \right), \\ \Delta\left(t\sqrt{1-\prod\limits_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(1-\left(1-\prod\limits_{j=1}^{m}\left(\frac{\Delta^{-1}\left(s_{\theta_{i_j}},\vartheta_{i_j}\right)}{t}\right)^{2r_j}\right)^{\prod_{j=1}^{m} w_{i_j}}\right)} \right) \end{array} \right\} \tag{106}$$

$$\left(\underset{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}{\overset{m}{\otimes}}\left(\overset{m}{\underset{j=1}{\oplus}}\left(r_j p_{i_j}\right)\right)^{\prod_{j=1}^{m} w_{i_j}}\right)^{\frac{1}{m!}} = \left\{ \begin{array}{l} \Delta\left(t\left(\prod\limits_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(\left(\sqrt{1-\prod\limits_{j=1}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^2\right)^{r_j}}\right)^{\prod_{j=1}^{m} w_{i_j}}\right)\right)^{\frac{1}{m!}} \right), \\ \Delta\left(t\sqrt{1-\left(\prod\limits_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(\left(1-\prod\limits_{j=1}^{m}\left(\frac{\Delta^{-1}\left(s_{\theta_{i_j}},\vartheta_{i_j}\right)}{t}\right)^{2r_j}\right)^{\prod_{j=1}^{m} w_{i_j}}\right)\right)^{\frac{1}{m!}}} \right) \end{array} \right\} \tag{107}$$

Proof: if $0 \leq \frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t} \leq 1$, then

$$0 \leq 1-\prod_{j=1}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^2\right)^{r_j} \leq 1 \tag{109}$$

Then, (110) and (111), as shown at the top of the next page. That means $0 \leq \frac{\Delta^{-1}(s_\phi,\varphi)}{t} \leq 1$, by the same, we have $0 \leq \frac{\Delta^{-1}(s_\theta,\vartheta)}{t} \leq 1$.

② Since $0 \leq \left(\frac{\Delta^{-1}(s_\phi,\varphi)}{t}\right)^2 + \left(\frac{\Delta^{-1}(s_\theta,\vartheta)}{t}\right)^2 \leq 1$, we get the following inequality $\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2 + \left(\frac{\Delta^{-1}(s_{\theta_i},\vartheta_i)}{t}\right)^2$, as shown at the bottom of the page 28. That means is maintained.

The DG2TLPFWGBM has two properties.

Property 15 (Monotonicity): Let $p_{x_i} = \{(s_{\phi_{x_i}},\varphi_{x_i}), (s_{\theta_{x_i}},\vartheta_{x_i})\}$ and $p_{y_i} = \{(s_{\phi_{y_i}},\varphi_{y_i}), (s_{\theta_{y_i}},\vartheta_{y_i})\}$ be two sets of 2TLPFNs. If $\Delta^{-1}(s_{\phi_{x_i}},\varphi_{x_i}) \leq \Delta^{-1}(s_{\phi_{y_i}},\varphi_{y_i})$ and

$$\begin{aligned}
&\mathrm{DG2TLPFWGBM}_w^R(p_1, p_2, \ldots, p_m) \\
&= \frac{1}{\sum_{j=1}^m r_j}\left(\underset{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}{\overset{m}{\otimes}}\left(\overset{m}{\underset{j=1}{\oplus}}(r_j p_{i_j})\right)^{\prod_{j=1}^m w_{i_j}}\right)^{\frac{1}{m!}} \\
&= \left\{\begin{array}{l}
\Delta\left(t\left(\sqrt{1-\left(1-\prod\limits_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(\left(1-\prod\limits_{j=1}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^2\right)\right)^{r_j}\right)^{\prod_{j=1}^m w_{i_j}}\right)^{\frac{1}{m!}}}\right)^{\frac{1}{\sum_{j=1}^m r_j}}\right), \\
\Delta\left(t\left(\sqrt{1-\prod\limits_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(\left(1-\prod\limits_{j=1}^{m}\left(\frac{\Delta^{-1}\left(s_{\theta_{i_j}},\vartheta_{i_j}\right)}{t}\right)^{2r_j}\right)^{\prod_{j=1}^m w_{i_j}}\right)^{\frac{1}{m!}}}\right)^{\frac{1}{\sum_{j=1}^m r_j}}\right)
\end{array}\right\}
\end{aligned} \tag{108}$$

$$\frac{\Delta^{-1}\left(s_\phi,\varphi\right)}{t} = \sqrt{1-\left(1-\prod\limits_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(\left(1-\prod\limits_{j=1}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^2\right)\right)^{r_j}\right)^{\prod_{j=1}^m w_{i_j}}\right)^{\frac{1}{m!}}}^{\frac{1}{\sum_{j=1}^m r_j}}$$

$$\frac{\Delta^{-1}\left(s_\theta,\vartheta\right)}{t} = \left(\sqrt{1-\prod\limits_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(\left(1-\prod\limits_{j=1}^{m}\left(\frac{\Delta^{-1}\left(s_{\theta_{i_j}},\vartheta_{i_j}\right)}{t}\right)^{2r_j}\right)^{\prod_{j=1}^m w_{i_j}}\right)^{\frac{1}{m!}}}\right)^{\frac{1}{\sum_{j=1}^m r_j}}$$

$$0 \leq \prod\limits_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(\left(1-\prod\limits_{j=1}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^2\right)\right)^{r_j}\right)^{\prod_{j=1}^m w_{i_j}\cdot\frac{1}{m!}} \leq 1 \tag{110}$$

$$0 \leq \sqrt{1-\left(1-\prod\limits_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(\left(1-\prod\limits_{j=1}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^2\right)\right)^{r_j}\right)^{\prod_{j=1}^m w_{i_j}}\right)^{\frac{1}{m!}}}^{\frac{1}{\sum_{j=1}^m r_j}} \leq 1 \tag{111}$$

$\Delta^{-1}(s_{\theta_{x_i}}, \vartheta_{x_i}) \geq \Delta^{-1}(s_{\theta_{y_i}}, \vartheta_{y_i}), i = 1, 2, \ldots, m$, then

$$\mathrm{DG2TLPFWGBM}_w^R\left(p_{x_1}, p_{x_2}, \cdots, p_{x_m}\right) \leq \mathrm{DG2TLPFWGBM}_w^R\left(p_{y_1}, p_{y_2}, \cdots, p_{y_m}\right) \tag{112}$$

Property 16 (Boundedness): Let $p_i = \left\{\left(s_{\phi_i}, \varphi_i\right), \left(s_{\theta_i}, \vartheta_i\right)\right\}$ be a set of 2TLPFNs. If $p^- = \left(\min_i\left(S_{\phi_i}, \varphi_i\right), \max_i\left(S_{\theta_i}, \vartheta_i\right)\right)$ and $p^+ = \left(\max_i\left(S_{\phi_i}, \varphi_i\right), \min_i\left(S_{\theta_i}, \vartheta_i\right)\right)$ then

$$p^- \leq \mathrm{DG2TLPFWGBM}_\omega^R\left(p_1, p_2, \cdots, p_m\right) \leq p^+ \tag{113}$$

TABLE 1. 2TLPFN decision matrix (R_1).

	G_1	G_2	G_3	G_4
A_1	$<(s_3,0), (s_2,0)>$	$<(s_4,0), (s_1,0)>$	$<(s_2,0), (s_4,0)>$	$<(s_2,0), (s_2,0)>$
A_2	$<(s_4,0), (s_1,0)>$	$<(s_4,0), (s_2,0)>$	$<(s_4,0), (s_1,0)>$	$<(s_4,0), (s_2,0)>$
A_3	$<(s_3,0), (s_3,0)>$	$<(s_1,0), (s_4,0)>$	$<(s_3,0), (s_3,0)>$	$<(s_4,0), (s_2,0)>$
A_4	$<(s_1,0), (s_4,0)>$	$<(s_1,0), (s_5,0)>$	$<(s_2,0), (s_4,0)>$	$<(s_1,0), (s_2,0)>$
A_5	$<(s_1,0), (s_4,0)>$	$<(s_3,0), (s_3,0)>$	$<(s_2,0), (s_1,0)>$	$<(s_2,0), (s_2,0)>$

TABLE 2. 2TLPFN decision matrix (R_2).

	G_1	G_2	G_3	G_4
A_1	$<(s_3,0), (s_1,0)>$	$<(s_2,0), (s_1,0)>$	$<(s_1,0), (s_1,0)>$	$<(s_5,0), (s_1,0)>$
A_2	$<(s_5,0), (s_1,0)>$	$<(s_4,0), (s_1,0)>$	$<(s_4,0), (s_2,0)>$	$<(s_5,0), (s_1,0)>$
A_3	$<(s_3,0), (s_1,0)>$	$<(s_4,0), (s_1,0)>$	$<(s_3,0), (s_1,0)>$	$<(s_2,0), (s_1,0)>$
A_4	$<(s_1,0), (s_3,0)>$	$<(s_2,0), (s_4,0)>$	$<(s_3,0), (s_3,0)>$	$<(s_2,0), (s_2,0)>$
A_5	$<(s_2,0), (s_3,0)>$	$<(s_4,0), (s_2,0)>$	$<(s_1,0), (s_1,0)>$	$<(s_5,0), (s_1,0)>$

VI. PRACTICAL APPLICATION OF THE PROPOSED OPERATORS

A. ILLUSTRATIVE EXAMPLE

The current safety situation of construction in our country is still not optimistic, large-scale construction of the huge investment and large numbers of participants in construction would easily result in more serious accidents. Frequent construction accidents, would not only increase the cost of construction enterprises, result in waste of social resources, but also threaten people's lives. Safety evaluation for construction projects could help construction companies effectively forecast dangerous factors in order to put forward

$$
\begin{aligned}
&\left(\frac{\Delta^{-1}(s_{\phi_i},\varphi_i)}{t}\right)^2+\left(\frac{\Delta^{-1}\left(s_{\theta_i},\vartheta_i\right)}{t}\right)^2\\
&=\left(1-\left(1-\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(\left(1-\prod_{j=1}^{m}\left(1-\left(\frac{\Delta^{-1}\left(s_{\phi_{i_j}},\varphi_{i_j}\right)}{t}\right)^2\right)^{r_j}\right)^{\prod_{j=1}^{m}w_{i_j}}\right)^{\frac{1}{m!}}\right)^{\frac{1}{\sum_{j=1}^{m}r_j}}\right.\\
&\quad\left.+\left(1-\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(\left(1-\prod_{j=1}^{m}\left(\frac{\Delta^{-1}\left(s_{\theta_{i_j}},\vartheta_{i_j}\right)}{t}\right)^{2r_j}\right)^{\prod_{j=1}^{m}w_{i_j}}\right)^{\frac{1}{m!}}\right)^{\frac{1}{\sum_{j=1}^{m}r_j}}\right)\\
&\leq\left(1-\left(1-\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(\left(1-\prod_{j=1}^{m}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{\theta_{i_j}},\vartheta_{i_j}\right)}{t}\right)^2\right)\right)^{r_j}\right)^{\prod_{j=1}^{m}w_{i_j}}\right)^{\frac{1}{m!}}\right)^{\frac{1}{\sum_{j=1}^{m}r_j}}\right.\\
&\quad\left.+\left(1-\prod_{\substack{i_1,i_2,\ldots,i_m=1\\ i_1\neq i_2\neq\ldots\neq i_m}}^{m}\left(\left(1-\prod_{j=1}^{m}\left(\frac{\Delta^{-1}\left(s_{\theta_{i_j}},\vartheta_{i_j}\right)}{t}\right)^{2r_j}\right)^{\prod_{j=1}^{m}w_{i_j}}\right)^{\frac{1}{m!}}\right)^{\frac{1}{\sum_{j=1}^{m}r_j}}\right)=1
\end{aligned}
$$

TABLE 3. 2TLPFN decision matrix (R_3)

	G_1	G_2	G_3	G_4
A_1	$<(s_2,0), (s_2,0)>$	$<(s_3,0), (s_1,0)>$	$<(s_1,0), (s_3,0)>$	$<(s_1,0), (s_3,0)>$
A_2	$<(s_5,0), (s_1,0)>$	$<(s_3,0), (s_2,0)>$	$<(s_4,0), (s_1,0)>$	$<(s_4,0), (s_3,0)>$
A_3	$<(s_2,0), (s_2,0)>$	$<(s_1,0), (s_2,0)>$	$<(s_2,0), (s_1,0)>$	$<(s_1,0), (s_2,0)>$
A_4	$<(s_1,0), (s_4,0)>$	$<(s_1,0), (s_5,0)>$	$<(s_1,0), (s_2,0)>$	$<(s_2,0), (s_2,0)>$
A_5	$<(s_1,0), (s_4,0)>$	$<(s_1,0), (s_2,0)>$	$<(s_2,0), (s_1,0)>$	$<(s_1,0), (s_3,0)>$

TABLE 4. The fused values by 2TLPFWAA operator.

	G_1	G_2	G_3	G_4
A_1	$\{(s_2,0.66),(s_1,0.62)\}$	$\{(s_2,0.75),(s_1,0.00)\}$	$\{(s_1,0.39),(s_2,0.35)\}$	$\{(s_3,0.45),(s_1,0.91)\}$
A_2	$\{(s_5,0.00),(s_1,0.00)\}$	$\{(s_3,0.67),(s_1,0.62)\}$	$\{(s_4,0.00),(s_1,0.23)\}$	$\{(s_4,0.39),(s_1,0.91)\}$
A_3	$\{(s_2,0.66),(s_1,0.83)\}$	$\{(s_2,0.53),(s_2,0.00)\}$	$\{(s_2,0.66),(s_1,0.39)\}$	$\{(s_2,0.68),(s_1,0.62)\}$
A_4	$\{(s_1,0.00),(s_3,0.67)\}$	$\{(s_1,0.39),(s_4,0.68)\}$	$\{(s_2,0.12),(s_2,0.78)\}$	$\{(s_1,0.77),(s_2,0.00)\}$
A_5	$\{(s_1,0.39),(s_3,0.67)\}$	$\{(s_2,0.94),(s_2,0.26)\}$	$\{(s_1,0.77),(s_1,0.00)\}$	$\{(s_3,0.45),(s_1,0.91)\}$

TABLE 5. The fused results by DG2TLPFWBM (DG2TLPFWGBM) operator.

	DG2TLPFWBM	DG2TLPFWGBM
A_1	$\{(s_0,0.17),(s_3,0.91)\}$	$\{(s_4,0.21),(s_0,0.05)\}$
A_2	$\{(s_0,0.68),(s_3,0.81)\}$	$\{(s_4,0.84),(s_0,0.03)\}$
A_3	$\{(s_0,0.17),(s_3,0.89)\}$	$\{(s_4,0.20),(s_0,0.05)\}$
A_4	$\{(s_0,0.04),(s_4,0.53)\}$	$\{(s_3,0.86),(s_0,0.36)\}$
A_5	$\{(s_0,0.14),(s_4,0.08)\}$	$\{(s_4,0.17),(s_0,0.10)\}$

TABLE 6. The $s(A_i)$ of the construction projects.

	DG2TLPFWBM	DG2TLPFWGBM
A_1	$(s_1,0.73)$	$(s_2,0.00)$
A_2	$(s_1,0.83)$	$(s_1,0.00)$
A_3	$(s_1,0.74)$	$(s_3,0.00)$
A_4	$(s_1,0.29)$	$(s_5,0.00)$
A_5	$(s_1,0.62)$	$(s_4,0.00)$

TABLE 7. Order of the construction projects.

	Order
DG2TLPFWBM	$A_2>A_3>A_1>A_5>A_4$
DG2TLPFWGBM	$A_2>A_1>A_3>A_5>A_4$

reasonable measures to prevent accidents occurring. So the research in this paper has important theoretical and practical meaning. Thus, we propose a numerical example to select best construction projects with 2TLPFNs. There are five possible construction projects $A_i\,(i = 1, 2, 3, 4, 5)$ to choose and four attribute to assess these construction projects: ① G_1 is the human factors in construction projects; ② G_2 is the building materials and equipment factors; ③ G_3 is the management factors; ④ G_4 is the environmental factors. The five possible construction projects $A_i\,(i = 1, 2, 3, 4, 5)$ are to be evaluated with 2TLPFNs with the four attributes (attributes weight $\omega = (0.15, 0.35, 0.30, 0.20)$, experts weight $\omega = (0.3, 0.3, 0.4)$.), which are listed in Table 1-3.

Then, we utilize these operators developed to select best construction project.

Definition 19: Let $p_j = \{(s_{\phi_j}, \varphi_j), (s_{\theta_j}, \vartheta_j)\}$ be a group of 2TLPFNs with weight values be$w_i = (w_1, w_2, \ldots, w_n)^T$, thereby satisfying $w_i \in [0, 1]$ and$\sum_{i=1}^{n} w_i = 1$, then we can obtain

$$
\begin{aligned}
&2\text{TLPFWAA}\,(p_1, p_2, \ldots, p_n) \\
&\quad = \sum_{j=1}^{n} w_j p_j \\
&\quad = \left\{ \begin{array}{l} \Delta\left(t\sqrt{1 - \prod_{j=1}^{n}\left(1 - \left(\dfrac{\Delta^{-1}(s_{\phi_j}, \varphi_j)}{t}\right)^2\right)^{w_j}} \right), \\ \Delta\left(t\prod_{j=1}^{n}\left(\dfrac{\Delta^{-1}\left(s_{\theta_j}, \vartheta_j\right)}{t}\right)^{w_j} \right) \end{array} \right\}
\end{aligned}
\tag{114}
$$

TABLE 8. Order by altering parameters of the DG2TLPFWBM operator.

P	A_1	A_2	A_3	A_4	A_5	Ordering
(1,1,1,1)	$(s_1,0.40)$	$(s_1,0.78)$	$(s_1,0.44)$	$(s_0,0.92)$	$(s_1,0.28)$	$A_2>A_3>A_1>A_5>A_4$
(2,2,2,2)	$(s_1,0.73)$	$(s_1,0.83)$	$(s_1,0.74)$	$(s_1,0.29)$	$(s_1,0.62)$	$A_2>A_3>A_1>A_5>A_4$
(3,3,3,3)	$(s_2,0.01)$	$(s_2,0.07)$	$(s_2,0.02)$	$(s_1,0.55)$	$(s_1,0.90)$	$A_2>A_3>A_1>A_5>A_4$
(4,4,4,4)	$(s_2,0.17)$	$(s_2,0.23)$	$(s_2,0.18)$	$(s_1,0.71)$	$(s_2,0.07)$	$A_2>A_3>A_1>A_5>A_4$
(5,5,5,5)	$(s_2,0.28)$	$(s_2,0.34)$	$(s_2,0.29)$	$(s_1,0.82)$	$(s_2,0.18)$	$A_2>A_3>A_1>A_5>A_4$
(6,6,6,6)	$(s_2,0.35)$	$(s_2,0.41)$	$(s_2,0.37)$	$(s_1,0.90)$	$(s_2,0.27)$	$A_2>A_3>A_1>A_5>A_4$
(7,7,7,7)	$(s_2,0.41)$	$(s_2,0.47)$	$(s_2,0.42)$	$(s_1,0.97)$	$(s_2,0.33)$	$A_2>A_3>A_1>A_5>A_4$
(8,8,8,8)	$(s_2,0.46)$	$(s_2,0.51)$	$(s_2,0.47)$	$(s_2,0.02)$	$(s_2,0.38)$	$A_2>A_3>A_1>A_5>A_4$
(9,9,9,9)	$(s_2,0.49)$	$(s_2,0.55)$	$(s_2,0.50)$	$(s_2,0.06)$	$(s_2,0.42)$	$A_2>A_3>A_1>A_5>A_4$
(10,10,10,10)	$(s_2,0.52)$	$(s_2,0.58)$	$(s_2,0.53)$	$(s_2,0.10)$	$(s_2,0.45)$	$A_2>A_3>A_1>A_5>A_4$

TABLE 9. Order by altering parameters of the DG2TLPFWGBM operator.

P	A_1	A_2	A_3	A_4	A_5	Ordering
(1,1,1,1)	$(s_4,0.89)$	$(s_5,0.26)$	$(s_4,0.88)$	$(s_4,0.43)$	$(s_4,0.82)$	$A_2>A_1>A_3>A_5>A_4$
(2,2,2,2)	$(s_4,0.48)$	$(s_4,0.95)$	$(s_4,0.47)$	$(s_4,0.23)$	$(s_4,0.45)$	$A_2>A_1>A_3>A_5>A_4$
(3,3,3,3)	$(s_4,0.20)$	$(s_4,0.74)$	$(s_4,0.20)$	$(s_3,0.96)$	$(s_4,0.17)$	$A_2>A_1>A_3>A_5>A_4$
(4,4,4,4)	$(s_4,0.04)$	$(s_4,0.61)$	$(s_4,0.04)$	$(s_3,0.80)$	$(s_4,0.00)$	$A_2>A_1>A_3>A_5>A_4$
(5,5,5,5)	$(s_3,0.93)$	$(s_4,0.53)$	$(s_3,0.93)$	$(s_3,0.69)$	$(s_3,0.89)$	$A_2>A_3>A_1>A_5>A_4$
(6,6,6,6)	$(s_3,0.85)$	$(s_4,0.47)$	$(s_3,0.86)$	$(s_3,0.62)$	$(s_3,0.81)$	$A_2>A_3>A_1>A_5>A_4$
(7,7,7,7)	$(s_3,0.80)$	$(s_4,0.42)$	$(s_3,0.81)$	$(s_3,0.56)$	$(s_3,0.75)$	$A_2>A_3>A_1>A_5>A_4$
(8,8,8,8)	$(s_3,0.75)$	$(s_4,0.38)$	$(s_3,0.76)$	$(s_3,0.51)$	$(s_3,0.71)$	$A_2>A_3>A_1>A_5>A_4$
(9,9,9,9)	$(s_3,0.71)$	$(s_4,0.36)$	$(s_3,0.73)$	$(s_3,0.48)$	$(s_3,0.67)$	$A_2>A_3>A_1>A_5>A_4$
(10,10,10,10)	$(s_3,0.68)$	$(s_4,0.33)$	$(s_3,0.70)$	$(s_3,0.45)$	$(s_3,0.64)$	$A_2>A_3>A_1>A_5>A_4$

$$\begin{aligned}&2\text{TLPFWGA}(p_1,p_2,\ldots,p_n)\\&=\prod_{j=1}^{n}(p_j)^{w_j}\\&=\left\{\begin{array}{l}\Delta\left(t\prod_{j=1}^{n}\left(\frac{\Delta^{-1}(s_{\phi_j},\varphi_j)}{t}\right)^{w_j}\right),\\ \Delta\left(t\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\frac{\Delta^{-1}(s_{\theta_j},\vartheta_j)}{t}\right)^2\right)^{w_j}}\right)\end{array}\right\}\end{aligned}\tag{115}$$

Step 1: In accordance with 2TLPFNs $r_{ij}(i = 1, 2, 3, 4, 5, j = 1, 2, 3, 4)$, we fuse all 2TLPFNs r_{ij} by 2TLPFWAA (2TLPFWGA) operator to calculate the overall 2TLPFNs A_i $(i = 1, 2, 3, 4, 5)$ of the construction projects A_i. The fused values are listed in Table4.

Step 2: By table 4, we fuse all 2TLPFNs r_{ij} with the DG2TLPFWBM (DG2TLPFWGBM) operator to get the aggregation values of the construction projects A_i. Suppose that $P = (2, 2, 2, 2)$, then the results are in Table5.

Step 3: According to the score function and Table 5, we can get the score values of the construction projects which listed in Table 6.

Step 4: By Table 6 and the order of the construction projects are listed in Table 7.

VII. INFLUENCE ANALYSIS

By altering parameters of P in the DG2TLPFWBM (DG2TLPFWGBM) operators, we can depict the effects on the ordering, the calculating results are shown as follows.

VIII. COMPARATIVE ANALYSIS

In this section, we compare DG2TLPFWBM, DG2TLPFW-GBM method with LPFWAA and LPFWGA operator [64].

From above analysis, we can have the same best construction project. However, the LPFWAA and LPFWGA operator have the shortcoming to consider the interrelationship

TABLE 10. Order of the construction projects.

	Order
LPFWAA[76]	$A_2 > A_1 > A_3 > A_5 > A_4$
LPFWGA[76]	$A_2 > A_3 > A_1 > A_5 > A_4$

between 2TLPFNs. The DG2TLPFWBM and DG2TLPFWGBM operators can overcome the shortcoming to consider the relationship among the 2TLPFNs.

IX. CONCLUSION

Considering the relationship among the 2TLPFNs, we utilize the WBM operator, GWBM operator and DGWBM operator to propose some BM operators with 2TLPFNs: 2TLPFWBM operator, 2TLPFWGBM operator, G2TLPFWBM operator, G2TLPFWGBM operator, DG2TLPFWBM operator, and DG2TLPFWGBM operator. Shortcomings have been overcome by considering relationship of 2TLPFNs. We present the new MADM method based on the new aggregation operators. Numerical example for safety assessment of construction project has been proposed to illustrate the new method and some comparisons are also conducted to further illustrate advantages of the new method. In subsequent studies, the application and methods of 2TLPFNs needs to be investigated in the any other uncertain decision making environments [65]–[82].

REFERENCES

[1] R. R. Yager, "Pythagorean fuzzy subsets," in *Proc. Joint IFSA World Congr. NAFIPS Annu. Meeting*, Edmonton, AB, Canada, Jun. 2013, pp. 57–61.

[2] R. R. Yager, "Pythagorean membership grades in multicriteria decision making," *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 4, pp. 958–965, Aug. 2014.

[3] X. Peng and Y. Yang, "Some results for pythagorean fuzzy sets," *Int. J. Intell. Syst.*, vol. 30, no. 11, pp. 1133–1160, 2015.

[4] G. Beliakov and S. James, "Averaging aggregation functions for preferences expressed as Pythagorean membership grades and fuzzy orthopairs," in *Proc. IEEE Int. Conf. Fuzzy Syst. (FUZZ-IEEE)*, Jul. 2014, pp. 298–305.

[5] X. Peng and Y. Yang, "Some results for pythagorean fuzzy sets," *Int. J. Intell. Syst.*, vol. 30, no. 11, pp. 1133–1160, 2015.

[6] S. Dick, R. R. Yager, and O. Yazdanbakhsh, "On pythagorean and complex fuzzy set operations," *IEEE Trans. Fuzzy Syst.*, vol. 24, no. 5, pp. 1009–1021, Oct. 2016.

[7] J. H. Kong, S. H. Lee, and G. Kim, "Minkowski Pythagorean-hodograph preserving mappings," *J. Comput. Appl. Math.*, vol. 308, no. 2, pp. 166–176, 2016.

[8] C. Zhang, D. Li, and R. Ren, "Pythagorean fuzzy multigranulation rough set over two universes and its applications in merger and acquisition," *Int. J. Intell. Syst.*, vol. 31, no. 9, pp. 921–943, 2016.

[9] X. Gou, Z. Xu, and P. Ren, "The properties of continuous Pythagorean fuzzy information," *Int. J. Intell. Syst.*, vol. 31, no. 5, pp. 401–424, 2016.

[10] X. Peng, H. Yuan, and Y. Yang, "Pythagorean fuzzy information measures and their applications," *Int. J. Intell. Syst.*, vol. 32, no. 2, pp. 991–1029, 2017.

[11] P. Mandal and A. S. Ranadive, "Decision-theoretic rough sets under Pythagorean fuzzy information," *Int. J. Intell. Syst*, vol. 33, no. 4, pp. 818–835, 2018.

[12] U. B. Baloglu and Y. Demir, "An agent-based pythagorean fuzzy approach for demand analysis with incomplete information," *Int. J. Intell. Syst.*, vol. 33, no. 5, pp. 983–997, 2018.

[13] X. Zhang and Z. Xu, "Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets," *Int. J. Intell. Syst.*, vol. 29, pp. 1061–1078, Dec. 2014.

[14] Y. Yang, H. Ding, Z.-S. Chen, and Y.-L. Li, "A note on extension of TOPSIS to multiple criteria decision making with pythagorean fuzzy sets," *Int. J. Intell. Syst.*, vol. 31, no. 1, pp. 68–72, 2016.

[15] D. Liang and Z. Xu, "The new extension of TOPSIS method for multiple criteria decision making with hesitant Pythagorean fuzzy sets," *Appl. Soft Comput.*, vol. 60, pp. 167–179, Nov. 2017.

[16] D. Liang, Z. Xu, D. Liu, and Y. Wu, "Method for three-way decisions using ideal TOPSIS solutions at Pythagorean fuzzy information," *Inf. Sci.*, vol. 435, pp. 282–295, Apr. 2018.

[17] M. S. A. Khan, S. Abdullah, M. Y. Ali, I. Hussain, and M. Farooq, "Extension of TOPSIS method base on Choquet integral under interval-valued Pythagorean fuzzy environment," *J. Intell. Fuzzy Syst.*, vol. 34, no. 1, pp. 267–282, 2018.

[18] S. C. Onar, B. Oztaysi, and C. Kahraman, "Multicriteria evaluation of cloud service providers using pythagorean fuzzy TOPSIS," *J. Multiple-Valued Logic Soft Comput.*, vol. 30, nos. 2–3, pp. 263–283, 2018.

[19] P. Ren, Z. Xu, and X. Gou, "Pythagorean fuzzy TODIM approach to multi-criteria decision making," *Appl. Soft Comput.*, vol. 42, pp. 246–259, May 2016.

[20] X. Zhang, "Multicriteria Pythagorean fuzzy decision analysis: A hierarchical QUALIFLEX approach with the closeness index-based ranking methods," *Inf. Sci.*, vol. 330, pp. 104–124, Feb. 2016.

[21] T.-Y. Chen, "Remoteness index-based Pythagorean fuzzy VIKOR methods with a generalized distance measure for multiple criteria decision analysis," *Inf. Fusion*, vol. 41, pp. 129–150, May 2018.

[22] X. Peng and J. Dai, "Approaches to Pythagorean fuzzy stochastic multi-criteria decision making based on prospect theory and regret theory with new distance measure and score function," *Int. J. Intell. Syst.*, vol. 32, no. 11, pp. 1187–1214, 2017.

[23] W. Xue, Z. Xu, X. Zhang, and X. Tian, "Pythagorean fuzzy LINMAP method based on the entropy theory for railway project investment decision making," *Int. J. Intell. Syst.*, vol. 33, no. 1, pp. 93–125, 2018.

[24] S.-P. Wan, Z. Jin, and J.-Y. Dong, "Pythagorean fuzzy mathematical programming method for multi-attribute group decision making with Pythagorean fuzzy truth degrees," *Knowl. Inf. Syst.*, vol. 55, no. 2, pp. 437–466, 2018.

[25] H. Garg, "A linear programming method based on an improved score function for interval-valued pythagorean fuzzy numbers and its application to decision-making," *Int. J. Uncertainty, Fuzziness Knowl.-Based Syst.*, vol. 26, no. 1, pp. 67–80, 2018.

[26] D. Liang, Z. Xu, and A. P. Darko, "Projection model for fusing the information of Pythagorean fuzzy multicriteria group decision making based on geometric bonferroni mean," *Int. J. Intell. Syst.*, vol. 32, no. 2, pp. 966–987, 2017.

[27] X. Peng and Y. Yang, "Pythagorean fuzzy choquet integral based MABAC method for multiple attribute group decision making," *Int. J. Intell. Syst.*, vol. 31, no. 10, pp. 989–1020, 2016.

[28] Y. Qin, Y. Liu, and Z. Hong, "Multicriteria decision making method based on generalized Pythagorean fuzzy ordered weighted distance measures," *J. Intell. Fuzzy Syst.*, vol. 33, no. 6, pp. 3665–3675, 2017.

[29] G. Wei and Y. Wei, "Similarity measures of Pythagorean fuzzy sets based on the cosine function and their applications," *Int. J. Intell. Syst.*, vol. 33, no. 3, pp. 634–652, 2018.

[30] D. Li and W. Zeng, "Distance measure of pythagorean fuzzy sets," *Int. J. Intell. Syst.*, vol. 33, no. 2, pp. 348–361, 2018.

[31] X. Zhang, "A novel approach based on similarity measure for pythagorean fuzzy multiple criteria group decision making," *Int. J. Intell. Syst.*, vol. 31, no. 6, pp. 593–611, 2016.

[32] H. Garg, "A novel correlation coefficients between pythagorean fuzzy sets and its applications to decision-making processes," *Int. J. Intell. Syst.*, vol. 31, no. 12, pp. 1234–1252, 2016.

[33] Z. Ma and Z. Xu, "Symmetric Pythagorean fuzzy weighted geometric/averaging operators and their application in multicriteria decision-making problems," *Int. J. Intell. Syst.*, vol. 31, no. 12, pp. 1198–1219, 2016.

[34] X. Peng and Y. Yang, "Fundamental properties of interval-valued pythagorean fuzzy aggregation operators," *Int. J. Intell. Syst.*, vol. 31, no. 5, pp. 444–487, 2016.

[35] X. Peng and H. Yuan, "Fundamental properties of pythagorean fuzzy aggregation operators," *Fundam. Inform.*, vol. 147, no. 4, pp. 415–446, 2016.

[36] K. Rahman, S. Abdullah, R. Ahmed, and M. Ullah, "Pythagorean fuzzy Einstein weighted geometric aggregation operator and their application to multiple attribute group decision making," *J. Intell. Fuzzy Syst.*, vol. 33, no. 1, pp. 635–647, 2017.

[37] G. Wei, "Pythagorean fuzzy interaction aggregation operators and their application to multiple attribute decision making," *J. Intell. Fuzzy Syst.*, vol. 33, no. 4, pp. 2119–2132, 2017.

[38] G. Wei and M. Lu, "Pythagorean fuzzy Maclaurin symmetric mean operators in multiple attribute decision making," *Int. J. Intell. Syst.*, vol. 33, no. 5, pp. 1043–1070, 2018.

[39] S.-J. Wu and G.-W. Wei, "Pythagorean fuzzy Hamacher aggregation operators and their application to multiple attribute decision making," *Int. J. Knowl.-Based Intell. Eng. Syst.*, vol. 21, no. 3, pp. 189–201, 2017.

[40] S. Zeng, Z. Mu, and T. Baležentis, "A novel aggregation method for Pythagorean fuzzy multiple attribute group decision making," *Int. J. Intell. Syst.*, vol. 33, no. 3, pp. 573–585, 2018.

[41] G. Wei, "Correction to Pythagorean fuzzy interaction aggregation operators and their application to multiple attribute decision making," *J. Intell. Fuzzy Syst.*, vol. 34, no. 4, pp. 2817–2824, 2018.

[42] V. Mohagheghi, S. M. Mousavi, and B. Vahdani, "Enhancing decision-making flexibility by introducing a new last aggregation evaluating approach based on multi-criteria group decision making and Pythagorean fuzzy sets," *Appl. Soft Comput.*, vol. 61, pp. 527–535, Dec. 2017.

[43] H. Garg, "Confidence levels based Pythagorean fuzzy aggregation operators and its application to decision-making process," *Comput. Math. Org. Theory*, vol. 23, no. 4, pp. 546–571, 2017.

[44] H. Garg, "Generalized Pythagorean fuzzy geometric aggregation operators using Einstein t-norm and t-conorm for multicriteria decision-making process," *Int. J. Intell. Syst.*, vol. 32, no. 2, pp. 597–630, 2017.

[45] G. Wei and M. Lu, "Pythagorean fuzzy power aggregation operators in multiple attribute decision making," *Int. J. Intell. Syst.*, vol. 33, no. 1, pp. 169–186, 2018.

[46] D. Liang, Y. Zhang, Z. Xu, and A. P. Darko, "Pythagorean fuzzy Bonferroni mean aggregation operator and its accelerative calculating algorithm with the multithreading," *Int. J. Intell. Syst.*, vol. 33, no. 3, pp. 615–633, 2018.

[47] H. Gao, M. Lu, G. Wei, and Y. Wei, "Some novel Pythagorean fuzzy interaction aggregation operators in multiple attribute decision making," *Fund. Inf.*, vol. 159, no. 2, pp. 385–428, 2018.

[48] H. Garg, "New exponential operational laws and their aggregation operators for interval-valued Pythagorean fuzzy multicriteria decision-making," *Int. J. Intell. Syst.*, vol. 33, no. 3, pp. 653–683, 2018.

[49] H. Garg, "Some methods for strategic decision-making problems with immediate probabilities in Pythagorean fuzzy environment," *Int. J. Intell. Syst.*, vol. 33, no. 4, pp. 687–712, 2018.

[50] H. Garg, "A novel improved accuracy function for interval valued pythagorean fuzzy sets and its applications in the decision-making process," *Int. J. Intell. Syst.*, vol. 32, no. 12, pp. 1247–1260, 2017.

[51] X. Tang, Y. Huang, and G. Wei, "Approaches to multiple-attribute decision-making based on pythagorean 2-tuple linguistic Bonferroni mean operators," *Algorithms*, vol. 11, no. 1, p. 5, 2018, doi: 10.3390/a11010005.

[52] X. Tang and G. Wei, "Models for green supplier selection in green supply chain management with Pythagorean 2-tuple linguistic information," *IEEE Access*, vol. 6, pp. 18042–18060, 2018.

[53] Y.-H. Huang and G.-W. Wei, "TODIM method for pythagorean 2-tuple linguistic multiple attribute decision making," *J. Intell. Fuzzy Syst.*, vol. 35, no. 1, pp. 901–915, 2018.

[54] G. Wei, M. Lu, F. E. Alsaadi, T. Hayat, and A. Alsaedi, "Pythagorean 2-tuple linguistic aggregation operators in multiple attribute decision making," *J. Intell. Fuzzy Syst.*, vol. 33, no. 2, pp. 1129–1142, 2017.

[55] G. Wei, H. Gao, and Y. Wei, "Some q-rung orthopair fuzzy Heronian mean operators in multiple attribute decision making," *Int. J. Intell. Syst.*, vol. 33, no. 7, pp. 1426–1458, 2018, 10.1002/int.21985.

[56] M. Lu, G. Wei, F. E. Alsaadi, T. Hayat, and A. Alsaedi, "Hesitant pythagorean fuzzy Hamacher aggregation operators and their application to multiple attribute decision making," *J. Intell. Fuzzy Syst.*, vol. 33, no. 2, pp. 1105–1117, 2017.

[57] G. Wei, M. Lu, X. Tang, and Y. Wei, "Pythagorean hesitant fuzzy Hamacher aggregation operators and their application to multiple attribute decision making," *Int. J. Intell. Syst.*, vol. 33, no. 6, pp. 1197–1233, 2018.

[58] F. Herrera and L. Martinez, "A 2-tuple fuzzy linguistic representation model for computing with words," *IEEE Trans. Fuzzy Syst.*, vol. 8, no. 6, pp. 746–752, Dec. 2000.

[59] C. Bonferroni, "Sulle medie multiple di potenze," *Bollettino Dell'Unione Matematica Italiana*, vol. 5, nos. 3–4, pp. 267–270, 1950.

[60] G. Beliakov, S. James, J. Mordelová, T. Rückschlossová, and R. R. Yager, "Generalized Bonferroni mean operators in multi-criteria aggregation," *Fuzzy Sets Syst.*, vol. 161, pp. 2227–2242, Sep. 2010.

[61] G. Wei, "Picture uncertain linguistic Bonferroni mean operators and their application to multiple attribute decision making," *Kybernetes*, vol. 46, no. 10, pp. 1777–1800, 2017.

[62] B. Zhu, Z. Xu, and M. Xia, "Hesitant fuzzy geometric Bonferroni means," *Inf. Sci.*, vol. 205, pp. 72–85, Nov. 2012.

[63] R. Zhang, J. Wang, X. Zhu, M. Xia, and M. Yu, "Some generalized pythagorean fuzzy Bonferroni mean aggregation operators with their application to multiattribute group decision-making," *Complexity*, vol. 2017, Aug. 2017, Art. no. 5937376.

[64] H. Garg, "Linguistic Pythagorean fuzzy sets and its applications in multiattribute decision-making process," *Int. J. Intell. Syst.*, vol. 33, no. 6, pp. 1234–1263, 2018.

[65] G. Wei and Y. Wei, "Some single-valued neutrosophic dombi prioritized weighted aggregation operators in multiple attribute decision making," *J. Intell. Fuzzy Syst.*, vol. 35, no. 2, pp. 2001–2013, 2018.

[66] Y. Hao and X. Chen, "Study on the ranking problems in multiple attribute decision making based on interval-valued intuitionistic fuzzy numbers," *Int. J. Intell. Syst.*, vol. 33, no. 3, pp. 560–572, 2018.

[67] G. Wei, "Interval-valued dual hesitant fuzzy uncertain linguistic aggregation operators in multiple attribute decision making," *J. Intell. Fuzzy Syst.*, vol. 33, no. 3, pp. 1881–1893, 2017.

[68] H. Li, J.-X. You, H.-C. Liu, and G. Tian, "Acquiring and sharing tacit knowledge based on interval 2-tuple linguistic assessments and extended fuzzy Petri nets," *Int. J. Uncertainty, Fuzziness Knowl.-Based Syst.*, vol. 26, no. 1, pp. 43–65, 2018.

[69] S. Wu, J. Wang, G. Wei, and Y. Wei, "Research on construction engineering project risk assessment with some 2-tuple linguistic neutrosophic hamy mean operators," *Sustainability*, vol. 10, no. 5, p. 1536, 2018, 10.3390/su10051536.

[70] G. Wei, F. E. Alsaadi, T. Hayat, and A. Alsaedi, "Picture 2-tuple linguistic aggregation operators in multiple attribute decision making," *Soft Comput.*, vol. 22, no. 2, pp. 989–1002, 2018.

[71] H. Gao, G. W. Wei, and Y. H. Huang, "Dual hesitant bipolar fuzzy Hamacher prioritized aggregation operators in multiple attribute decision making," *IEEE Access*, vol. 6, pp. 11508–11522, 2018.

[72] J. Wang, G. Wei, and Y. Wei, "Models for green supplier selection with some 2-tuple linguistic neutrosophic number bonferroni mean operators," *Symmetry*, vol. 10, no. 5, p. 131, 2018, doi: 10.3390/sym10050131.

[73] C.-Y. Wang and S.-M. Chen, "A new multiple attribute decision making method based on linear programming methodology and novel score function and novel accuracy function of interval-valued intuitionistic fuzzy values," *Inf. Sci.*, vol. 438, pp. 145–155, Apr. 2018.

[74] G. Wei, F. E. Alsaadi, T. Hayat, and A. Alsaedi, "Projection models for multiple attribute decision making with picture fuzzy information," *Int. J. Mach. Learn. Cybern.*, vol. 9, no. 2, pp. 713–719, 2018.

[75] D. Ju, Y. Ju, and A. Wang, "Multiple attribute group decision making based on Maclaurin symmetric mean operator under single-valued neutrosophic interval 2-tuple linguistic environment," *J. Intell. Fuzzy Syst.*, vol. 34, no. 4, pp. 2579–2595, 2018.

[76] G. W. Wei and H. Gao, "The generalized Dice similarity measures for picture fuzzy sets and their applications," *Informatica*, vol. 29, no. 2, pp. 1–18, 2018.

[77] H.-B. Yan, X. Zhang, and Y. Li, "Linguistic multi-attribute decision making with multiple priorities," *Comput. Ind. Eng.*, vol. 109, pp. 15–27, Jul. 2017.

[78] G. Wei, F. E. Alsaadi, T. Hayat, and A. Alsaedi, "Bipolar fuzzy Hamacher aggregation operators in multiple attribute decision making," *Int. J. Fuzzy Syst.*, vol. 20, no. 1, pp. 1–12, 2018.

[79] J. Ye, "Some weighted aggregation operators of trapezoidal neutrosophic numbers and their multiple attribute decision making method," *Informatica*, vol. 28, no. 2, pp. 387–402, 2017.

[80] G. W. Wei, "Some similarity measures for picture fuzzy sets and their applications," *Iranian J. Fuzzy Syst.*, vol. 15, no. 1, pp. 77–89, 2018.

[81] G. Wei, H. Gao, J. Wang, and Y. Huang, "Research on risk evaluation of enterprise human capital investment with interval-valued bipolar 2-tuple linguistic information," *IEEE Access*, vol. 6, pp. 35697–35712, 2018.

[82] L. Shi and J. Ye, "Cosine measures of linguistic neutrosophic numbers and their application in multiple attribute group decision-making," *Information*, vol. 8, no. 4, p. 117, 2017.

XIUMEI DENG is currently pursuing the master's degree with the College of Mathematics and Software Science, Sichuan Normal University, Chengdu, China.

GUIWU WEI received the M.Sc. and Ph.D. degrees in applied mathematics from Southwest Petroleum University, the Business Administration degree from the School of Economics and Management, Southwest Jiaotong University, China. From 2010 to 2012, he was a Post-Doctoral Researcher with the School of Economics and Management, Tsinghua University, Beijing, China. He is currently a Professor with the School of Business, Sichuan Normal University. He has published over 100 papers in journals, books, and conference proceedings, including journals such as *Omega*, *Decision Support Systems*, *Expert Systems with Applications*, *Applied Soft Computing*, *Knowledge and Information Systems*, *Computers & Industrial Engineering*, *Knowledge-Based Systems*, the *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, the *International Journal of Computational Intelligence Systems*, and *Information: An International Interdisciplinary Journal*. He has also published one book. He is currently interested in aggregation operators, decision-making, and computing with words. He has participated in several scientific committees and serves as a Reviewer in a wide range of journals, including *Computers & Industrial Engineering*, the *International Journal of Information Technology and Decision Making*, *Knowledge-Based Systems*, *Information Sciences*, the *International Journal of Computational Intelligence Systems*, and the *European Journal of Operational Research*.

HUI GAO received the M.Sc. degree in management sciences and engineering from the School of Economics and Management, University of Electronic Science and Technology of China, China. She is currently an Associate Professor with the School of Business, Sichuan Normal University.

JIE WANG is currently pursuing the master's degree with the School of Business, Sichuan Normal University, Chengdu, China.