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Interval Observer Design for Nonlinear Systems: Stability Radii Approach

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ABSTRACT This paper presents a new approach to design preserving order and interval observers for a family of nonlinear systems in absence and in presence of parametric uncertainties and exogenous disturbances. A preserving order observer provides an upper/lower estimation that is always above/below the state trajectory, depending on the partial ordering of the initial conditions, and asymptotically converges to its true values in the nominal case. An interval observer is then constituted by means of an upper and a lower preserving order observer. In the uncertain/disturbed case, the estimations preserve the partial ordering with respect to the state trajectory, and practically converge to the true values, despite of the uncertainties/perturbations. The design approach relies on the cooperativity property and the stability radii mathematical tools, both applied to the estimation error systems. The objective is to exploit the stability radii analysis for the family of linear positive systems under the time-varying nonlinear perturbations in order to guarantee the exponential convergence property of the observers, while the cooperativity condition determines the partial ordering between the trajectories of the state and the estimations. The proposed approach, defined for Lipschitz nonlinearities, depends only on two observer matrix gains. The design is reduced to the solution of linear matrix inequalities, which are given by the cooperative condition and convergence constraints. An illustrative example is presented to show the effectiveness of the theoretical results.

INDEX TERMS Interval observers, preserving order observers, stability radii, positive systems.

I. INTRODUCTION

Preserving order observers for nonlinear systems provide an upper/lower estimation, depending on the initial conditions, which are always above/below the real state trajectory. Moreover, the estimation converges to its real values when there is no presence of parametric uncertainties and exogenous disturbances. The partial ordering between the trajectories of the state and the estimation is preserved even when the bounded uncertain/disturbed signals are affecting the dynamical system, while the estimation error converges to a ball centered at the origin [1]. Combining the so-called upper and lower preserving order observers, an interval observer is then constituted for the same family of nonlinear systems [1]. In the last two decades the interval observers, mainly based on the theory of cooperative systems [2], have received much attention in order to cope with the robust estimation problem for uncertain/disturbed systems. Several approaches have been widely tested and implemented in highly-uncertain biological nonlinear continuous-time systems [1], [3]–[8]. Inspired by the growing computational interest to use sample-data systems subjected to perturbations, some works have been proposed to estimate variables for discrete-time systems [9]–[13]. Furthermore, [11], [14]–[17] consider the use of interval observers in order to control/stabilize some family of uncertain/disturbed systems. For safety critical systems, robust Fault Detection techniques [18]–[25], using the upper/lower estimations of the interval observers to generate adaptive thresholds, have been proposed in order to indicate the occurrence of faults in the systems with uncertainties and/or perturbations.

In [3] the first design of interval observers, appeared applying the cooperativity property in the estimation error dynamics, which represents a pioneer technique to estimate unknown parameters and variables for biotechnological processes; the cooperative condition is reduced to find a matrix gain in order to have a Metzler matrix, without imposing any stability constraint. In fact, the preserving order and interval observers, that do not have any convergence condition, are called *framers*, according to [6], in the design of bundle observers to generate the best interval estimate at every time from the initialization of a set of interval observers. Recently, a new approach for designing preserving order and interval observers have been developed and applied in [1] for a family of continuous-time nonlinear systems in absence and in presence of parametric uncertainties and exogenous disturbances, which generalizes and unifies several interval observer techniques as in [26]; applying the cooperativity and dissipativity systemic properties to the estimation error dynamics, the observer design parameters have to satisfy the convergence condition, in particular simplified, as a Bilinear Matrix Inequality (BMI), and the cooperative constraints determined by a Metzler Jacobian matrix dependent on the estimation error state, which can be specially treated as finite set of Linear Matrix Inequalities (LMI's).

Since the cooperativity property depends on the transformation of coordinates, some interval observer approaches, like in [27]–[30], have relaxed the cooperative condition in the interval observer design, applying similarity transformations to the family of Linear Time-Invariant (LTI) uncertain systems in order to obtain a Hurwitz and Metzler matrix. Moreover, the interval observer design for exact linearization approximations of some nonlinear systems has been presented in [8], requiring that the observer gain and the transformation matrix fulfill the Sylvester equation. The case for time-varying nonlinear systems is studied in [31], where a static linear transformation of coordinates is used to satisfy the cooperativity conditions. Furthermore, [32] proposed a methodology to design interval observers for Linear Time-Varying (LTV) systems taking into account a time-varying transformation of coordinates.

The purpose of the present paper is to extend the research of the preserving order and interval observers, initially proposed by Avilés and Moreno [1], for a family of nonlinear systems. The proposed approach uses the stability radii mathematical tool instead of the dissipative method in [1], and it is combined with the cooperativity property in order to design the preserving order and interval observers for the same family of continuous-time nonlinear systems, in the absence and in the presence of parametric uncertainties and exogenous inputs. The work is motivated by the analysis of the stability radii for linear positive systems subjected to the linear and nonlinear perturbations [33]–[35]. The stability radii notion, introduced for linear systems under linear structured perturbations in [36], is defined as the smallest bound of the perturbation that destabilizes the perturbed system. There exist three stability radii according to the perturbation spaces: complex, real and positive real. In the particular case, if the perturbed system is positive, then the three stability radii equivalently coincide, and are determined by a simple formula [34]. Furthermore, if the complex stability radius is greater than the bound of perturbation, then the perturbed system is asymptotically stable. This analysis is extended to the family of linear systems under time-varying nonlinear perturbations, which are considered globally Lipschitz [34]. The mathematical framework of the stability radii exactly corresponds with the estimation error systems of the proposed observers, composed by a LTI subsystem in the forward loop connected to a time-varying static nonlinearity in the feedback loop. The method is only valid for Lipschitz nonlinearities, but it is not restrictive for the family of positive systems. The design approach has been extended to treat additive exogenous inputs in the dynamics of the systems, allowing the incorporation of the Input-to-State Stability notion. The designed method requires finding two matrices that fulfill the convergence condition, determined by the stability radii constraints, and the cooperative condition, given by a finite set of Matrix Inequalities. These conditions are lead to LMI's. Particularly, the proposed approach can be seen as a special method of the dissipativity methodology in [1], since the Lipschitz nonlinearities belong to a symmetric sector $[-K, K]$ with $K = \gamma I$ or a matrix triplet $(Q, S, R) = (-I, 0, \gamma^2 I)$. The main advantage of the proposed approach is focused in reducing the number of design parameters, to two observer matrix gains, instead of five variables to design the same preserving and interval observers in [1].

The paper is organized as follows. In Section II, the preliminaries of the preserving and interval observers for nonlinear systems are presented. Section III is devoted to the analysis of the convergence property of observers making use of the stability radius mathematical tool. The results given in Section III is used to define the design conditions of the preserving order and interval observer for nonlinear systems in Section IV. Section V presents some practical issues for the design of the observers. In section VI, an illustrative example demonstrates the efficiency of the approach through numerical simulations. Finally, conclusions are included in Section VII.

Notations: We begin by introducing some notations that will be used throughout this paper. We define especially the partial ordering between a pair of vectors $x,\ y \in \mathbb{R}^n$ by means of the symbol $\succeq$: if $x_i - y_i \geq 0$ for all $i = 1, \ldots, n$, then $x \succeq y$. For matrices $A, B \in \mathbb{R}^{n\times n}$, this definition is also valid, that is $a_{ij} - b_{ij} \geq 0$ then $A \succeq B$. We particularly define the non-negativity for vectors and matrices: x is a nonnegative vector, denoted as $x \succeq 0$, if and only if $x_i \geq 0, \forall i = 1, \ldots, n$; $A \succeq 0$ indicates that A is a nonnegative matrix, if and only if $a_{ij} \geq 0$. It is worth to note that for any symmetric matrix, $A \geq 0$ $(A > 0)$ denotes a positive semidefinite matrix (positive definite matrix, respectively); $A \leq 0$ $(A < 0)$ indicates a negative semidefinite matrix (negative definite matrix, respectively). In addition, we write $M \overset{M}{\succeq} 0$ to denote that M is a Metzler matrix, if and only if $M_{ij} \geq 0,\ \forall i \neq j, \forall i, j \in 1, \ldots, n$. Finally, $D_x f(t, x, u) = \partial f(t, x, u)/\partial x$ denotes the Jacobian matrix of the nonlinear function $f(t, x, u)$ w.r.t. x, while $D_u f(t, x, u) = \partial f(t, x, u)/\partial u$ denotes the derivate w.r.t. u.

II. PRELIMINARIES

We present two important concepts for designing interval and preserving order observers for nonlinear systems: (i) Cooperativity, essential property to preserve the partial ordering of the trajectories of the state and the output [2], [37], and (ii) Stability radii for positive systems [34], provides the stability conditions for a family of nonlinear systems through a simple formula. Several results to these topics will be mentioned here.

A. COOPERATIVE SYSTEMS

Cooperative systems, represent an important subfamily of monotone systems, establish a partial ordering on the trajectories of the state and the output at every time, from a partial ordering on the input signals and the initial states [2], [37]. The definition and characterization of the nonlinear and linear systems are given in the succeeding paragraphs.

Definition 1: Let us consider the nonlinear system of the following form

$$\Sigma_{\mathrm{NL}}:\begin{cases}\dot{x}=f\left(t,x,u\right), & x(0)=x_0,\\ y=h\left(t,x\right),\end{cases}\tag{1}$$

where $x\in\mathbb{R}^n$ *is the* state*,* $u\in\mathbb{R}^m$ *is the control input,* and $y\in\mathbb{R}^p$ is the measurement. Σ_{NL} in (1) is a *cooperative system* if given a partial ordering on initial states and inputs, that is, $x_0^1\preceq x_0^2,\ \ u^1\left(t\right)\preceq u^2\left(t\right),\ \ \forall t\geq t_0$, then, the partial ordering is preserved in the state and output trajectories for all future times $t\geq t_0$, that is,

$$x\left(t,t_0,x_0^1,u^1\left(t\right)\right)\preceq x\left(t,t_0,x_0^2,u^2\left(t\right)\right),$$
$$y\left(t,t_0,x_0^1,u^1\left(t\right)\right)\preceq y\left(t,t_0,x_0^2,u^2\left(t\right)\right).$$

◊

The next proposition establishes the characterization on the continuous-time cooperative systems.

Proposition 2 [2]: Σ_{NL} *in* (1) *is a cooperative system if and only if the next conditions,*

$$\textbf{(a)}.\ D_xf(x,u)\overset{M}{\succeq}0,\quad \textbf{(b)}.\ D_uf(x,u)\succeq 0,\ \ \textbf{(c)}.\ D_xh(x,u)\succeq 0,$$

are fulfilled. ◊

This notion is now presented for the class of the continuous-time Linear Time-Invariant (LTI) systems.

Proposition 3 [2]: Consider the LTI system described by the equations,

$$\Sigma_{\mathrm{L}}:\begin{cases}\dot{x}=Ax+Bu, & x\left(0\right)=x_0,\\ y=Cx,\end{cases}\tag{2}$$

where $x(t)\in\mathbb{R}^n$ *is the state vector,* $u(t)\in\mathbb{R}^m$ *is the control vector, and* $y(t)\in\mathbb{R}^p$ *is the output vector. The known matrices* A*,* B *and* C *have appropriate dimensions.* Σ_{L} *is a cooperative system if and only if the next conditions hold,*

$$\textbf{(a)}.\ A\overset{M}{\succeq}0,\quad \textbf{(b)}.\ B\succeq 0,\ \ \textbf{(c)}.\ C\succeq 0.$$

◊

Remark 4: The positivity property states that if $x_0\succeq 0$ *and* $u\left(t\right)\succeq 0$*, it turns out that all trajectories of the state and the output are nonnegative, that is,* $x(t,t_0,x_0,u(t))\succeq 0$ *and* $y\left(t,t_0,x_0,u\left(t\right)\right)\succeq 0$*. The sufficient and necessary conditions for obtaining linear positive systems equivalently coincide to the characterization on the LTI cooperative systems (see Proposition 3).*

The following Lemma shows a property of the Metzler matrices which will be used in the subsequent sections.

Lemma 5 [34]: Suppose that $A\in\mathbb{R}^{n\times n}$ *is a Metzler matrix, and* $B\succeq 0$ *is an nonnegative matrix.* $\mu(A)$ *is the spectral abscissa of* A*, that is, it is the maximal real eigenvalue of* A *such that* $Ax=\mu(A)x$ *for* $x\neq 0$ *with* $x\succeq 0$*. Then the following inequality holds*

$$\mu(A)\ \leq\ \mu(A+B).\tag{3}$$

B. STABILITY RADII FOR POSITIVE SYSTEMS

We here recall the robust stability conditions for LTI positive systems subjected to the presence of (linear or nonlinear) perturbations [34]. Moreover, the stability radii method is extended in order to consider a family of nonlinear systems with exogenous inputs.

1) SYSTEMS WITH LINEAR PERTURBATIONS

We consider the class of the linear positive systems

$$\Gamma_{\mathrm{L}}:\{\ \dot{x}=Ax,\quad x(0)=x_0,\tag{4}$$

where $x\in\mathbb{R}^n$ is the state vector. The system Γ_{L} in (4) is asymptotically stable if and only if A is Hurwitz. If Γ_{L} is subjected to structured perturbations of output feedback type $A\rightarrow A+B\Delta C$, the perturbed system is then given by

$$\Gamma_{\mathrm{LP}}:\{\ \ \dot{x}=(A+B\Delta C)x,\quad \|\Delta\|<\gamma,\tag{5}$$

where Δ is a (complex, real or nonnegative) matrix of unknown perturbations ($\Delta\in\mathbb{C}^{m\times p}$, $\Delta\in\mathbb{R}^{m\times p}$, $\Delta\in\mathbb{R}_+^{m\times p}$). Moreover, $\|\Delta\|$ is the size of the linear perturbation, and $\gamma>0$ is a known positive scalar that stands for the bound of the disturbance magnitude. We now formally define the complex, real and nonnegative stability radii according to [35] and [36].

Definition 6 (Complex, Real and Nonnegative Stability Radii): Consider the disturbed and uncertain system in Γ_{LP}*. The stability radii are defined as*

$$r_{\mathbb{K}}\left(A;B,C\right)=\inf\left\{\|\Delta\|:\Delta\in\mathbb{K}^{m\times p},\mu\left(A+B\Delta C\right)\geq 0\right\}\tag{6}$$

where $\mathbb{K}=\mathbb{C},\mathbb{R},\mathbb{R}_+$ *stands for the complex, real or nonnegative matrix space.*

In other words, the stability radii $r_{\mathbb{K}}$ in (6) represents the smallest value (minimal bound) of the perturbation Δ that destabilizes the system Γ_{LP}. The three stability radii satisfy the next property,

$$0\leq r_{\mathbb{C}}\left(A;B,C\right)\leq r_{\mathbb{R}}\left(A;B,C\right)\leq r_{\mathbb{R}_+}\left(A;B,C\right).\tag{7}$$

From the transfer matrix $G(s) = C\,(sI - A)^{-1} B$, the complex stability radius is determined by [36] and [38]

$$r_{\mathbb{C}}\,(A; B, C) = \frac{1}{\max_{\omega\in\mathbb{R}} \left\| C\,(j\omega I - A)^{-1} B \right\|}.$$

The problem of computing the stability radii for positive linear systems has been solved in [34], when the function $\omega \rightarrow \|G(\omega)\|$ attains its maximum value at $\omega = 0$. For this family of systems, the complex stability radius equivalently coincides with the nonnegative and real expressions. Moreover, the next theorem provides a formula for computing exactly the stability radii.

Theorem 7 [34]: Consider the system Γ_{LP} in (5). *If Γ_{LP} is a positive system, that is, $A \in \mathbb{R}^{n\times n}$ is a Metzler and Hurwitz matrix, $B \succeq 0$ and $C \succeq 0$ are nonnegative matrices, then stability radii are given by:*

$$r_{\mathbb{C}} = \left\| CA^{-1}B \right\|^{-1}, \quad r_{\mathbb{C}} = r_{\mathbb{R}} = r_{\mathbb{R}^+}. \tag{8}$$

◊

2) SYSTEMS WITH NONLINEAR PERTURBATIONS

The system Γ_L in (4) can be also subjected to nonlinear and time-varying perturbations [34]. The perturbed nonlinear system is now represented by

$$\dot{x} = Ax + Bf(Cx, t), \tag{9}$$

and it is possible to write it as a continuous-time LTI subsystem in the forward loop and connected to a time-varying static nonlinearity in the feedback loop [39] (see Figure 1), described by the following equations

$$\Gamma_{NP}: \begin{cases} \dot{x} = Ax + Bu, \quad x(0) = x_0, \\ y = Cx, \\ u = f\,(y, t)\,, \end{cases} \tag{10}$$

where the time-varying memoryless nonlinearity $f\,(y, t) : \mathbb{R}^q \times [0, \infty) \rightarrow \mathbb{R}^m$ is globally Lipschitz in $y(t)$ and piecewise continuous in t, for the positive constant $\gamma > 0$, such that the following inequality holds,

$$\|f\,(y, t)\| \leq \gamma\,\|y\|\,, \quad \forall y \in \mathbb{R}^p. \tag{11}$$

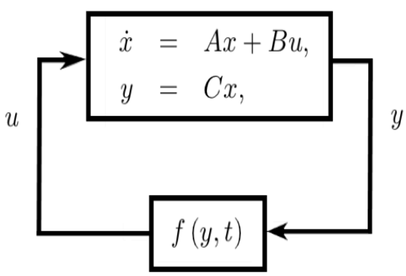

FIGURE 1. **Representation of the stability radius system.**

This inequality expresses, in an equivalent form, that $f\,(y, t)$ belongs to the symmetric sector $[-K,\, K]$, where $K \in \mathbb{R}^{m\times m}$ is a square matrix with the value $K = \gamma I$, and satisfies

$$\gamma^2 y^T y \; - \; f^T(y, t) f(y, t) \;\geq\; 0. \tag{12}$$

Additionally, the size of the nonlinear function is defined as

$$\|f\| = \inf\left\{\gamma \in \mathbb{R}_+; \quad \forall y \in \mathbb{R}^p, t \geq 0 : \|f\,(y, t)\| \leq \gamma\,\|y\|\right\}$$

The following lemma establishes the conditions for assuring the exponential stability for Γ_{NP} using the notion of the stability radii for positive systems [34].

Lemma 8: Consider that Γ_{NP} in (9) *is a positive system, that is $A \overset{M}{\succeq} 0$, $B \succeq 0$, $C \succeq 0$, and moreover A is a Hurwitz matrix. Suppose that $f(y, t)$ satisfies* (11). *If the stability radius condition*

$$\left\| CA^{-1}B \right\|^{-1} \;>\; \gamma \tag{13}$$

is satisfied with $\|f\| \leq \gamma$, then Γ_{NP} is a globally and exponentially stable system, that is, there exists positive constants $\alpha > 0$, $\beta > 0$ such that

$$\|x\,(t)\| \leq \alpha\,\|x_0\|\,\exp(-\beta t), \quad t > 0. \tag{14}$$

◊

Proof: We consider the Lyapunov function candidate $V = x^T Px$, where P is a definite-positive symmetric matrix. Assume that $f\,(t, y)$ is globally Lipschitz in y and piece-wise continuous in t. Suppose $\left\|CA^{-1}B\right\|^{-1} > \gamma$ then there exists $\epsilon > 0$ and a stabilizing positive solution $P = P^T > 0$ such that

$$A^T P + PA + \epsilon P + \gamma^2 C^T C + PBB^T P = 0. \tag{15}$$

This is due to the stability radii that is characterized by the algebraic Riccati equation [33]. Hence, there exists a solution set P and ϵ such that

$$A^T P + PA + \epsilon P + \gamma^2 C^T C + PBB^T P \leq 0. \tag{16}$$

Applying Schur's lemma, we obtain the next matrix inequality

$$\begin{bmatrix} A^T P + PA + \epsilon P + \gamma^2 C^T C & PB \\ B^T P & -I \end{bmatrix} \leq 0 \tag{17}$$

which is equivalent to that $\dot{V}\,(x) \leq -\epsilon V\,(x)$. Making use of the comparison Lemma [40], we get

$$V\,(x\,(t)) \leq V\,(x(0))\exp(-\epsilon t)$$

then, it is established that

$$\begin{aligned} \lambda_{\min}\,(P)\,\|x(t)\|_2^2 &\leq x^T(t)Px(t) \\ &\leq x^T(0)Px(0)\exp(-\epsilon t) \\ &\leq \lambda_{\max}\,(P)\,\|x(0)\|_2^2\exp(-\epsilon t), \end{aligned}$$

where $\lambda_{\min,\max}(P)$ are the smallest and the greatest eigenvalues of P, respectively. Therefore,

$$\|x(t)\|_2 \le \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\|x(0)\|_2 \exp(-\frac{\epsilon}{2}t),$$

which implies that the equilibrium point $x = 0$ of Γ_{NP} is globally and exponentially stable. □

Remark 9: The exponential stability property of the system Γ_{NP} can be also achieved by means of the dissipativity theory in [39] and [41], taking into account a storage function defined as $V(x) = x^T Px$ with $P = P^T \ge 0$ and quadratic supply rate $\omega(f(y,t), y) = y^T Qy + 2y^T Sf(y,t) + f^T(y,t)Rf(y,t)$ with $Q = Q^T \ge 0$ and $R = R^T \ge 0$. According to [39], the following dissipative inequality holds

$$\dot{V}(x) \le -\epsilon V(x) + \omega(y,u),$$

selecting the matrix triplet $(Q, S, R) = (-R, S^T, -Q)$. In particular, if the nonlinearity $f(y,t)$ satisfies the Lipschitz condition in (11), which is equivalently described by the matrix triplet $(Q, S, R) = (-I, 0, \gamma^2 I)$, then we have

$$\begin{aligned}\dot{V}(x) &= \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \\ &\le -\epsilon V(x) + \begin{bmatrix} x \\ f \end{bmatrix}^T \begin{bmatrix} -\gamma^2 C^T C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ f \end{bmatrix} \\ &\le -\epsilon V(x).\end{aligned}$$

Therefore, the resulting dissipative matrix inequality,

$$\begin{bmatrix} A^T P + PA + \epsilon P & PB \\ B^T P & 0 \end{bmatrix} - \begin{bmatrix} -\gamma^2 C^T C & 0 \\ 0 & I \end{bmatrix} \le 0,$$

exactly coincides with the matrix inequality provided by the stability radii (17).

Remark 10: Note that the stability radii conditions of the Lemma 8, defined for Lipschitz nonlinearities, can guarantee the exponential stability property of the system Γ_{NP}, using the inequality in (13) without finding any matrix and/or scalar. The dissipative method in [39] ensures the same stability property of Γ_{NP} for a wide variety of nonlinearities, requiring to found, in general, a matrix P, a vector θ and the scalar ϵ. In fact, the dissipative method can be considered as a generalization of the stability radii technique. The main advantage of the proposed technique is that it reduces the number of variables to examine the stability property.

It is possible to consider that Γ_{NP} is affected by additive exogenous signals, described by the following equations,

$$\Gamma_{NE}: \begin{cases} \dot{x} = Ax + Bu + b, \quad x(0) = x_0, \\ y = Cx, \\ u = f(y,t), \end{cases} \tag{18}$$

where $b(t)$ stands for the external signals to the system Γ_{NE}. In this case, the state trajectories of Γ_{NE} asymptotically converge to a centered vicinity at zero, if $b(t)$ is bounded. This notion is addressed by the *Input-to-State-Stability*.

Definition 11 [1], [40], [42]: Γ_{NE} in (18) is said to be (globally) Input-to-State Stable (ISS) w.r.t. $b(t)$ if there exists a $\mathcal{KL}$ function β, a $\mathcal{K}$ function γ such that it satisfies

$$\|x(t)\| \le \beta(\|x(t_0)\|, t) + \gamma\left(\sup_{t\ge 0}\|b(t)\|\right), \quad \forall t \ge 0$$

for any initial state x_0 and any bounded input $b(t)$.

The following result provides the conditions of the practical stability for Γ_{NE} in (18) in terms of the stability radii for positive systems.

Lemma 12: It is supposed that the nonlinearity satisfies (11). It is assumed that the stability radii conditions of Lemma 8 are fulfilled. Under these conditions, the system Γ_{NE} in (18) is ISS w.r.t. $b(t)$. ⋄

Proof: We consider the same Lyapunov function candidate $V = x^T Px$ for the system Γ_{NE}, when b is different to zero. Hence, the inequality $\dot{V} \le -\epsilon V + 2x^T Pb$ is fulfilled, for any $\theta \in (0,1)$. This expression can be rewritten as

$$\begin{aligned}\dot{V}(x) &\le -(1-\theta)\epsilon V - \theta\epsilon x^T Px + 2x^T Pb \\ &\le -(1-\theta)\epsilon V - \theta\epsilon\lambda_{\max}(P)\|x\|_2^2 + 2\lambda_{\max}(P)\|b\|_2\|x\|_2 \\ &\le -(1-\theta)\epsilon V + \lambda_{\max}(P)\|x\|_2(2\|b\|_2 - \theta\epsilon\|x\|_2) \\ &\le -(1-\theta)\epsilon V, \quad \forall \|x\|_2 \ge \frac{2}{\theta\epsilon}\|b\|_2.\end{aligned}$$

According to [40, Th. 4.19], Γ_{NE} is ISS w.r.t. $b(t)$. □

III. OBSERVERS FOR NONLINEAR SYSTEMS BASED ON STABILITY RADII APPROACH

In this section, we study the convergence property of the observers for a family of nonlinear systems in the absence and in the presence of bounded unknown signals, applying the stability radii notions in the estimation error dynamics.

A. NOMINAL SYSTEMS

Let us consider the nonlinear system of the following form

$$\Xi_S: \begin{cases} \dot{x} = Ax + G\xi(\sigma; t, y, u) + \varphi(t, y, u), \\ \sigma = Hx, \quad x(0) = x_0, \\ y = Cx, \end{cases} \tag{19}$$

where $(x, u, y) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ are the state, the input and the output vectors, respectively. Moreover, $\sigma \in \mathbb{R}^r$ stands for a linear function of the state. Additionally, $\xi(\cdot) \in \mathbb{R}^m$ is a nonlinear function locally Lipschitz in σ, which depends on the known variables (u, y). Finally, the known nonlinear function $\varphi(\cdot)$ is locally Lipschitz in (u, y) and piece-wise continuous in t.

We use the following state observer for Ξ_S, which has been introduced in [39] and [43],

$$\Xi_O: \begin{cases} \dot{\hat{x}} = A\hat{x} + L(\hat{y} - y) \\ \qquad + G\xi(\hat{\sigma} + N(\hat{y} - y); t, y, u) \\ \qquad + \varphi(t, y, u), \quad \hat{x}(0) = \hat{x}_0, \\ \hat{\sigma} = H\hat{x}, \\ \hat{y} = C\hat{x}, \end{cases} \tag{20}$$

where $\widehat{x} \in \mathbb{R}^n$ is the estimate of x. The aim consists on finding the matrices $L \in \mathbb{R}^{n\times q}$ and $N \in \mathbb{R}^{r\times q}$ such that the state estimation error $e \triangleq \widehat{x} - x$ exponentially converges towards zero. This is achieved because the observer Ξ_{O} contains a pair of correction terms in its dynamics: an output injection linear term $L(\widehat{y} - y)$ and an injection linear term in the nonlinearity $\xi(\cdot)$ described by $N(\widehat{y} - y)$. Using the linear expression $\sigma + N(\widehat{y} - y) = Hx + (H + NC)(\widehat{x} - x) = \sigma + H_N e$, we obtain the following estimation error dynamics

$$\Xi_{\text{E}}: \begin{cases} \dot{e} = A_L e + G\upsilon, \quad e(0) = e_0, \\ z = H_N e, \\ \upsilon = -\psi(\sigma, z), \end{cases} \tag{21}$$

where $A_L \triangleq A + LC$ and $H_N \triangleq H + NC$. We consider the incremental nonlinearity described by,

$$\psi(z, \sigma; t, y, u) \triangleq \xi(\sigma;\ t, y, u) - \xi(\sigma + z;\ t, y, u). \tag{22}$$

The following Theorem is quite useful in deriving the main results of this paper. It states the exponential stability result in the estimation error dynamics Ξ_{E} from applying Lemma 8.

Theorem 13: For the nonlinear system Ξ_{S} in (19)*, it is assumed that $G \succeq 0$. It is supposed that the incremental nonlinearity $\psi(\cdot)$ in* (22) *satisfies the Lipschitz condition in* (11)*. If the matrices L and N, can be found such that the following conditions*

(i). *A_L is a Hurwitz Matrix,*
(ii). $A_L \overset{M}{\succeq} 0$,
(iii). $H_N \succeq 0$,
(iv). $\left\| H_N (A_L)^{-1} G \right\|^{-1} > \gamma$,

are fulfilled. Then Ξ_{O} in (20) *is a globally exponentially convergent observer for Ξ_S.*

Proof: Applying Lemma 8 in the estimation error system Ξ_{E} in (21), it is ensured the exponential convergence property in Ξ_{O}. □

Remark 14: Note that the conditions of the Theorem 13 depends only on the variables L and N. It is worth to mention that the convergence property of the observer can be equivalently achieved by means of the dissipative method in [1], [39], and [41], solving in general a feasibility problem of Nonlinear Matrix Inequality in the variables $(P, L, N, \epsilon, \theta)$. Therefore, the proposed method reduces the number of variables for designing convergent observers for the class of the systems Ξ_S. However, it is more restrictive.

Remark 15: According to [34, Lemma 2], if we select any nonnegative design matrix $L \succeq 0$, then the convergence of the output injection observer Ξ_O is slower than of a open-loop observer with $L = 0$, since the eigenvalues of the linear part of the estimation error system Ξ_E are larger than of the open-loop observer, that is $\mu(A) \leq \mu(A + LC)$.

B. SYSTEMS IN PRESENCE OF DISTURBANCES/UNCERTAINTIES

Let the class of nonlinear systems in the presence of parametric uncertainties and external disturbances be described by

$$\Psi_{\text{S}}: \begin{cases} \dot{x} = Ax + G\xi(\sigma; t, y, u) \\ \qquad + w(t, x, u) + \varphi(t, y, u), \\ \sigma = Hx, \quad x(0) = x_0, \\ y = Cx, \end{cases} \tag{23}$$

where $w(t, x, u) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m$ stands for the parametric uncertainties and/or external disturbances in the system Ψ_{S}, which are unknown signals but upper and lower bounded, satisfying the following inequality

$$w^-(t,\ y(t),\ u(t)) \preceq w \preceq w^+(t, y(t), u(t)), \tag{24}$$

where $w^+(t, y, u)$ and $w^-(t, y, u)$ are known bounds, respectively. Using the same state observer Ξ_{O} in (20) for the disturbed system Ψ_{S}, we obtain the following estimation error dynamics,

$$\Psi_{\text{E}}: \begin{cases} \dot{e} = A_L e + G\upsilon + b, \quad e(0) = e_0, \\ z = H_N e, \\ \upsilon = -\psi(\sigma, z), \end{cases} \tag{25}$$

where $b(t) = -w(t, x, u)$ represents the exogenous signal to the system Ψ_{E}. The matrices L and N are the design parameters for ensuring the practical convergence property on the observer Ξ_{O} in (20). Hence, the trajectories of the estimation error will exponentially converge to a ball centered at zero, with a radius that depends on the bound of $b(t)$. In the following Theorem we establish the conditions on the practical convergence in terms of stability radii for positive systems.

Theorem 16: For the nonlinear system Ψ_{S} in (23)*, it is assumed that $G \succeq 0$. It is supposed that the incremental nonlinearity $\psi(\cdot)$ satisfies the Lipschitz condition in* (11)*. If the matrices L and N, can be found such that the conditions of the Theorem 13 are fulfilled, then Ξ_O in* (20) *is a globally ISS convergent observer for Ψ_S.*

Proof: Making use of the Lemma 12 in the estimation error system Ξ_{E} in (21), the practical convergence property in Ξ_{O} is guaranteed. □

IV. PRESERVING ORDER AND INTERVAL OBSERVERS

This section is devoted to the design method of preserving order and interval observers for a class of nominal nonlinear systems. This new design is also able to solve the estimation problem for a class of nonlinear systems in the presence of parametric uncertainties and external disturbances.

A. NOMINAL SYSTEMS

The initial condition $x_0 \in \mathbb{R}^n$ of the system Ξ_{S} is here supposed to be bounded in the form

$$x_0^- \preceq x_0 \preceq x_0^+, \tag{26}$$

where x_0^+ and x_0^- are two known bounds. Inspired by [1], we firstly define the conditions of preserving order and interval observers for the class of the nonlinear systems Ξ_S.

Definition 17: The observer Ξ_O is an upper (lower) preserving order observer for the nominal nonlinear system Ξ_S, if the following conditions hold

(i). *the estimate $\widehat{x}(t)$ asymptotically converges to the state trajectory $x(t)$, that is,*

$$\|\widehat{x}(t) - x(t)\| \to 0, \quad \text{as } t \to \infty,$$

(ii). *there exists a partial ordering between the trajectories of the state and the estimate, depending on a proper partial ordering on the initial conditions, that is,*

$$\begin{aligned} &\text{If } x_0 \preceq x_0^+ \preceq \widehat{x}_0 \quad \left(\widehat{x}_0 \preceq x_0^- \preceq x_0\right) \\ &\Rightarrow \quad x(t) \preceq \widehat{x}(t) \quad (\widehat{x}(t) \preceq x(t)), \quad \forall t \geq 0. \end{aligned}$$

Remark 18: In addition, if both upper and lower preserving order observers, denoted as Ξ_{O^+} and Ξ_{O^-}, are run in parallel, then they define an interval observer for a nominal system Ξ_S. Thus, the state trajectory is always bounded by the estimations, satisfying the next inequality

$$\widehat{x}^-(t) \preceq x(t) \preceq \widehat{x}^+(t), \quad \forall t \geq 0.$$

According to the above definition, we have established the characterization of convergence condition of Ξ_O in the Theorem 13. Now, it is required to satisfy the cooperativity conditions of the Proposition 2 in the estimation error dynamics Ξ_E for ensuring the partial ordering between the real trajectory and the estimation. Therefore, Ξ_E in (21) is a cooperative system, if the following Jacobian matrix

$$A_L - GD_z\psi(z, \sigma;\, t, y, u)H_N \overset{M}{\geq} 0, \quad \forall z \in \mathbb{R}^r,\ \forall t, \sigma, y, u. \tag{27}$$

is Metzler. Since $D_z\psi(z, \sigma;\, t, y, u) = -D_\sigma\xi(\sigma + z;\, t, y, u)$, we can rewrite the cooperativity condition in (27) as

$$A_L + GD_\sigma\xi\,(\sigma + z; t, y, u)\, H_N \succeq 0, \quad \forall z \in \mathbb{R}^r,\ \forall t, \sigma, y, u \tag{28}$$

Thus, we can now state the design method of the preserving order observers for a class of nominal nonlinear systems of the form Ξ_O.

Theorem 19: For the continuous-time nonlinear nominal system Ξ_S, it is assumed that $G \succeq 0$. Suppose that $\psi(\cdot)$ in (22) *satisfies the Lipschitz condition in* (11). *If the matrices L and N can be found such that*

1) *the conditions of the Theorem 13,*
2) *the cooperativity condition in* (28),

are fulfilled. Then, Ξ_O is a globally exponentially convergent - (upper/lower) preserving order observer for Ξ_S. An upper and a lower preserving order observer form an interval observer for Ξ_S.

B. SYSTEMS IN PRESENCE OF DISTURBANCES/UNCERTAINTIES

We now focus on the design of the preserving order and interval observers for the class of nonlinear systems in presence of parametric uncertainties and exogenous disturbances Ψ_S, when $w(t, x, u)$ is not vanishing.

It is not possible to use the state observer Ξ_O for designing the preserving order and interval observers for a class of disturbed systems Ψ_S, since it does not preserve the partial ordering between the trajectories of the state and the estimate, because the disturbance/uncertain signal, which acts like an exogenous input in the estimation error system Ξ_E in (21), is not partially ordered $\left(b(t) \not\succeq 0\right)$. To cope with this inconvenient, we propose the following pair of observers for Ψ_S,

$$\Psi_{O^+}: \begin{cases} \dot{\widehat{x}}^+ = A\widehat{x}^+ + G\xi\left(\widehat{\sigma}^+ + N^+\left(\widehat{y}^+ - y\right);\, t, y, u\right) \\ \qquad + L^+\left(\widehat{y}^+ - y\right) + w^+(t, y, u) + \varphi(t, y, u), \\ \widehat{\sigma}^+ = H\widehat{x}^+, \quad \widehat{x}^+(0) = \widehat{x}_0^+, \\ \widehat{y}^+ = C\widehat{x}^+, \end{cases} \tag{29}$$

$$\Psi_{O^-}: \begin{cases} \dot{\widehat{x}}^- = A\widehat{x}^- + G\xi\left(\widehat{\sigma}^- + N^-\left(\widehat{y}^- - y\right);\, t, y, u\right) \\ \qquad + L^-\left(\widehat{y}^- - y\right) + w^-(t, y, u) + \varphi(t, y, u), \\ \widehat{\sigma}^- = H\widehat{x}^-, \quad \widehat{x}^-(0) = \widehat{x}_0^-, \\ \widehat{y}^- = C\widehat{x}^-, \end{cases} \tag{30}$$

where $\widehat{x}^+(t)$ $\left(\widehat{x}^-(t)\right)$ is an upper (lower) estimated state and $\widehat{y}^+(t)$ $\left(\widehat{y}^-(t)\right)$ is an upper (lower) estimated output. The design parameters of the observers Ψ_{O^+} and Ψ_{O^-} are $\left(L^+, N^+\right)$ and $\left(L^-, N^-\right)$, respectively. These observers contain the known bounds of the disturbance in order to correct the cooperativity property in the estimation error systems.

We next extend the conditions to ensure the existence of preserving order and interval observers for the family of disturbed/uncertain nonlinear systems. The definition and characterization of this family of observers are given in the succeeding paragraphs.

Definition 20: Ψ_{O^+} $\left(\Psi_{O^-}\right)$ is an upper (lower) preserving order observer for the disturbed system Ψ_S, if the following conditions hold:

(i) *the parametric uncertainties and exogenous disturbances $w(t, x, u)$ satisfy the inequality* (24) $\forall t \geq 0$,

(ii) *there exists a partial ordering between the trajectories of the state and the upper (lower) estimate, from a partial ordering on the initial conditions, that is,*

$$\begin{aligned} &\text{If } x_0 \preceq x_0^+ \preceq \widehat{x}_0^+ \quad \left(\widehat{x}_0^- \preceq x_0^- \preceq x_0\right) \\ &\Rightarrow \quad x(t) \preceq \widehat{x}^+(t) \quad \left(\widehat{x}^-(t) \preceq x(t)\right), \quad \forall t \geq 0, \end{aligned}$$

(iii) *the upper (lower) estimate $\widehat{x}^+(t)$ $\left(\widehat{x}^-(t)\right)$ practically converges to a neighborhood of $x(t)$, that is, when $\|w^\pm\| \leq \omega^\pm$ for constant $\omega^\pm$ then*

$$\|\widehat{x}^\pm(t) - x(t)\| \leq \delta^\pm\left(\omega^\pm\right) \text{ for all } t \geq T,$$

for some $T > 0$ and where $\delta^\pm$ are class $\mathcal{K}$ functions.

Moreover, if the pair of observers $\left(\Psi_{O^+}, \Psi_{O^-}\right)$ *are run in parallel, they constitute an interval observer for the disturbed system,* Ψ_S. *Thus, the state trajectory is always bounded by the upper and lower estimations, that is, it is satisfied the next inequality*

$$\widehat{x}^-(t) \preceq x(t) \preceq \widehat{x}^+(t) \quad \forall t \geq 0,$$

and the ultimate bound of the interval observer, when $\|w^+ - w^-\| \leq \omega$ *for constant* ω, *is given by*

$$\|\widehat{x}^+(t) - \widehat{x}^-(t)\| \leq \delta(\omega) \quad \text{for all } t \geq T,$$

for some $T > 0$ *and where* δ *is a class* $\mathcal{K}$ *function.*

Theorem 21: For the continuous-time nonlinear disturbed/uncertain system Ψ_S, *it is assumed that* $G \succeq 0$. *Suppose that* $\psi^+(\cdot)$ $\left(\psi^-(\cdot)\right)$, *described in similar form to* (22), *satisfies the Lipschitz condition in* (11). *Moreover, suppose that the disturbance* $w(t, x, u)$ *satisfies* (24) *for all* $t \geq 0$. *If the matrix gains* L^+ *and* N^+ (L^-, N^-), *can be found such that*

1) *the* conditions *of the Theorem 16, and*
2) *the* cooperative condition *in* (28),

are fulfilled. Then, Ψ_{O^+} $\left(\Psi_{O^+}\right)$ *is a globally ISS - upper (lower) preserving order observer for* Ψ_S. *The observers* (Ψ_{O^+}, Ψ_{O^-}) *are then an* interval observer *for* Ψ_S.

Remark 22: It is important to mention that the interval observer can be designed through an only design of a preserving order observer. In this case, the observer gains are given as $L = L^+ = L^-$ *and* $N = N^+ = N^-$. *Additionally, the gains do not depend on the unknown signals.*

Proof: We consider the observers Ψ_{O^+} and Ψ_{O^-} for the disturbed nonlinear system Ψ_S. If $e^+(t) \triangleq \widehat{x}^+(t) - x(t)$ and $e^-(t) \triangleq x(t) - \widehat{x}^-(t)$, are defined as the upper and lower estimation error systems. Then their dynamics can be expressed by the equations,

$$\Psi_{E^+}: \begin{cases} \dot{e}^+ = A_L^+ e^+ + G v^+ + b^+, \quad e^+(0) = e_0^+ \succeq 0, \\ z^+ = H_N^+ e^+, \\ v^+ = \psi^+\left(z^+, \sigma; t, y, u\right), \end{cases} \tag{31}$$

$$\Psi_{E^-}: \begin{cases} \dot{e}^- = A_L^+ e^- + G v^- + b^-, \quad e^-(0) = e_0^- \succeq 0, \\ z^- = H_N^- e^-, \\ v^- = \psi^-\left(z^-, \sigma;\, t,\, y,\, u\right), \end{cases} \tag{32}$$

where the exogenous signals $b^+(t) = w^+(t, y, u) - w(t, x, u) \succeq 0$ and $b^-(t) = w(t, x, u) - w^-(t, y, u) \succeq 0$ are nonnegative vectors. Besides, the incremental nonlinearities are given by $\psi^+\left(z^+, \sigma; t, y, u\right) = \xi(\sigma; t, y, u) - \xi(\sigma + z^+; t, y, u)$ and $\psi^-\left(z^-, \sigma; t, y, u\right) = \xi(\sigma - z^-; t, y, u) - \xi(\sigma; t, y, u)$.

Making use of the Lemma 12 in the systems Ψ_{E^+} and Ψ_{E^-}, it is guaranteed the practical convergence property of the observers Ψ_{O^+} and Ψ_{O^-}. It is assumed that the perturbation $w(t, x, u)$ satisfies (24) for all $t \geq 0$. The partial ordering, between the trajectories of the state and estimations, is preserved using the proposition 2 in the systems Ψ_{E^+} and Ψ_{E^-}. □

V. PRACTICAL ISSUES

Notice that the design of the Preserving Order and Interval Observers, according to the Theorems 19 and 21, depends on two main conditions: (a) Stability radii constraints (i)-(iv) of the Theorem 13, which guarantee the internal stability properties of the observers, and (b) The cooperative matrix inequality in (28) determines the partial ordering between the trajectories of the state and the estimations. In the following paragraphs we discuss some proposed solutions of these conditions.

A. STABILITY RADII CONDITIONS

Theorem 13 and 16 provide a novel design method for ensuring the convergence property of nonlinear observers, using Lipschitz nonlinearities depending on the unmeasured state variables. The difficulty is computationally focused on finding the matrix gains L and N such that the stability radii constraints are satisfied. These conditions are next studied.

1) Using an additional variable P, it is possible to represent the condition (i) of the Theorem 13 through the Lyapunov inequality,

$$A_L^T P + P A_L \leq 0. \tag{33}$$

In general, the matrix inequality in (33) is represented as a feasibility problem of a Bilinear Matrix Inequality (BMI) in the variables (P, L). Particularly, (33) becomes an LMI in (PL, P), the matrix gain L can be then calculated by $L = P^{-1}PL$.

2) A_L is a Metzler matrix can be also represented as a LMI in the variable L.

3) The nonnegative condition (iii), $H_N \succeq 0$, is also expressed in terms of an LMI in N.

4) Finally, the condition (iv) of the Theorem 13 stands for the main stability radii constraint, which is determined as $\left\|H_N (A_L)^{-1} G\right\|^{-1} > \gamma$. This latter constraint is written as

$$\gamma^{-2} I - G^T \left(A_L^{-1}\right)^T H_N^T H_N A_L^{-1} G > 0. \tag{34}$$

Applying Schur's complement, we have the next matrix inequality

$$\begin{bmatrix} \gamma^{-2} I & H_N A_L^{-1} G \\ G^T A_L^{-1} H_N^T & I \end{bmatrix} > 0, \tag{35}$$

which can become an LMI in N, fixing the matrix L.

A solution to the previous constraints, can be iteratively found by means of several efficient computational packages, e.g., the interface Yalmip with the solvers as Sedumi or Penbmi-Tomlab. We now summarize the design procedure for guaranteeing the convergence property of the preserving order and interval observers.

- *Step 1:* We firstly consider the matrix variable L, such that the next conditions are solved,
  [a]. (33) is an LMI in (PL, P).
  [b]. $A_L \overset{M}{\succeq} 0$ becomes an LMI in L.

- *Step 2:* Fixing the matrix L and defining N as the matrix variable, we can simultaneously solve the following inequalities.
  [a]. (35) becomes an LMI in N.
  [b]. $H_N \succeq 0$ is also an LMI in N.

It is worth to mention that the condition (i) of the Theorem 13 is equivalently represented by the matrix inequality in (33), considering the matrices P and L. To solve (33) in the step 1, it is easy to do it with computational packages. The matrix P is not required to ensure the convergence of the observers, but it is considered for purposes of the numerical results of L and N.

B. COOPERATIVE CONDITION

In order to guarantee the cooperativity property in the estimation error systems, the matrices L and N are required such that the family of matrices in (28) is Metzler. This condition has been studied in [1]. In this subsection, we recall some important results.

The matrix inequality in (28), in general, can be seen as a (infinite) set of LMI's in the variables $(L,\ N)$, since the solutions depend on each value of the Jacobian matrix $D_\sigma \xi(\sigma + z;\ t,\ y,\ u)$. Defining a set, that contains all values of $D_\sigma \xi(\sigma + z;\ t,\ y,\ u)$, as

$$\mathcal{J} = \left\{ \Gamma \in \mathbb{R}^{m\times r} \middle| \Gamma = D_\sigma \xi\left(\sigma;\ t,\ y,\ u\right), \forall \sigma,\ t,\ y,\ u \right\}, \tag{36}$$

it is possible to rewrite the cooperativity condition as

$$A_L + GD_\sigma \xi\left(\sigma + z;\ t,\ y,\ u\right) H_N \succeq 0, \quad \forall D_\sigma \xi(\sigma + z) \in \mathcal{J} \tag{37}$$

Now, the work is focused on reducing the infinite number of LMI's. It is assumed that the set $\mathcal{J}$ is bounded and the nonlinearity $\psi\left(\sigma,\ z;\ t,\ y,\ u\right)$ in (22) belongs to the sector $[-K\ ,\ K]$ with the value $K = \gamma I$ (see page 52804), that is the inequality

$$[\psi(\sigma,\ z;\ t,\ y,\ u) - (-K)\, z]^T\, [Kz - \psi(\sigma,\ z;\ t,\ y,\ u)] \geq 0 \tag{38}$$

is satisfied with the constant matrix $K \in \mathbb{R}^{m\times r}$. From the Mean Value Theorem, it is easy to prove that $\mathcal{J}$ is contained in the convex set Υ, given by

$$\Upsilon \triangleq \left\{ \Gamma \in \mathbb{R}^{m\times r} \middle| [\Gamma - K]^T\, [K + \Gamma] \leq 0 \right\}. \tag{39}$$

Since inequality (37) is convex, it suffices to solve it on the boundary of the set $\mathcal{J}$, which reduces the set of LMI's.

Proposition 23: We consider the multivariable nonlinear function $\xi(z;\ t,\ y,\ u)\ :\ \mathbb{R}^r \rightarrow \mathbb{R}^m$. Assume that the nonlinearity $\psi\left(\sigma,\ z;\ t,\ y,\ u\right) = \xi\left(\sigma;\ t,\ y,\ u\right) - \xi\left(z + \sigma;\ t,\ y,\ u\right)$ belongs to the sector $[-K,\ K]$ with the value $K = \gamma I$. If the set of matrices

$$A_L + G\Delta H_N \overset{M}{\succeq} 0 \tag{40}$$

is Metzler $\forall \Delta \in \Upsilon_{\mathrm{F}}$, where

$$\Upsilon_{\mathrm{F}} = \left\{ \Gamma \in \mathbb{R}^{m\times r} \middle|\ (\Gamma - K)^T\, (K + \Gamma) = 0 \right\}, \tag{41}$$

then the cooperativity requirement, given in (28) is fulfilled $\forall\, z \in \mathbb{R}^r$, and for all $t,\ y,\ u$.

Remark 24: According to the proposition 23 the number of LMI's to check the cooperativity condition is reduced to the set of boundary values Υ_{F}. However, in general Υ_{F} is still an infinite set. In particular, for a scalar nonlinearity $\sigma \rightarrow \psi(\sigma;\ t,\ y,\ u) : \mathbb{R} \rightarrow \mathbb{R}$, the set of matrices (41) is only defined by two points of $\mathcal{J} \subset \Upsilon_{\mathrm{F}}$, which correspond to the sector extremes $\Upsilon_{\mathrm{F}} = \{-K,\ K\}$ with the value $K = \gamma I$. In the diagonal case $\sigma \rightarrow \psi(\sigma;\ t,\ y,\ u) : \mathbb{R}^m \rightarrow \mathbb{R}^m$, with $\sigma_i \rightarrow \psi_i(\sigma_i;\ t,\ y,\ u) : \mathbb{R} \rightarrow \mathbb{R}$ for $i = 1, \ldots, m$, the set Υ_{F} is given by a finite number of points.

The approach is geometrically presented in the vectorial case $\Gamma \in \mathbb{R}^m$, which can be easily extended to matrix case with $\Gamma \in \mathbb{R}^{m\times m}$. It is then possible to write the set Υ in (39) as

$$\Upsilon = \{\Gamma \in \mathbb{R}^m |\ \Gamma^T Q \Gamma - 2\Gamma^T S + R \geq 0\} \tag{42}$$

which stands for an ellipsoid in $\mathbb{R}^m$ [1], [44] with $Q < 0$. It is now possible to find a finite set of LMI's to check (37) constructing polytopes [1], [45]. The cooperativity inequality in (37) is satisfied iff (37) is examined in a set of points, which become the vertices of two polytopes, described by

- A closed convex polytope $P_I \subset \mathbb{R}^m$ as

$$P_I = \left\{ \sum_{i=1}^{k} \alpha_i \Delta_i \in \mathbb{R}^m \middle| \alpha_i \geq 0, \sum_{i=1}^{k} \alpha_i = 1,\ \ i = 1, \ldots, k \right\}. \tag{43}$$

defined as the convex hull of $k > 1$ points $\Delta_i \in \Upsilon_{\mathrm{F}} \subset \mathbb{R}^m$.

- A polytope circumscribing to Υ_{F}, defined by

$$P_C = \left\{ \sum_{i=1}^{\kappa} \alpha_i \Omega_i \in \mathbb{R}^m \middle| \alpha_i \geq 0, \sum_{i=1}^{\kappa} \alpha_i = 1,\ \ i = 1, \ldots, \kappa \right\}. \tag{44}$$

$\Omega_i, i = 1, \ldots, \kappa$ are the vertices of P_C.

P_I is a polytope inscribed in Υ, whose vertices are given by the points $\Delta_i \in \Upsilon_F$. Increasing the number of points, we can obtain an appropriate refinement of P_I. The polytope P_C is defined as the intersection of the (closed half-spaces) *supporting hyperplanes* H_i, which are tangent hyperplanes to Υ_{F}, and depend on the vertices Δ_i.

Remark 25: The polytopes P_I and P_C represent the necessary and sufficient conditions, respectively, to examine (37). Since $P_I \subset \Upsilon$ and $\Upsilon \subset P_C$, the cooperativity inequality (27) is fulfilled if the following expressions are satisfied.

(i) If $A_L + GD_\sigma \xi\left(\sigma + z\right) H_N \overset{M}{\succeq} 0$ is satisfied for every $D_\sigma \xi\left(\sigma + z\right) \in \mathcal{J}$, then $A_L + G\Delta_i H_N \overset{M}{\succeq} 0$ is satisfied for every vertex $\{\Delta_1, .., \Delta_k\}$ of P_I.

(ii) If $A_L + G\Omega_i H_N \succeq 0$ is satisfied at each vertex $\{\Omega_1, \ldots, \Omega_k\}$ of P_C, then $A_L + GD_\sigma \xi\left(\sigma + z\right) H_N \overset{M}{\succeq} 0$ is satisfied for every $D_\sigma \xi\left(\sigma + z\right) \in \mathcal{J}$.

Remark 26: In order to find a solution to the design of the preserving order and interval observers, we add the previous cooperativity conditions in the requirements defined in the Step 1 of the foregoing subsection.

VI. ILLUSTRATIVE EXAMPLES

The effectiveness of design method of the interval and preserving order observers has been tested in some numerical simulations.

We illustrate our theoretical results in the following academic system

$$\Upsilon_S : \begin{cases} \dot{x}_1 = -4x_1 + 3x_2 + \xi(x_2), \\ \dot{x}_2 = 5x_1 - 6x_2, \\ y = x_1, \end{cases} \quad (45)$$

where $x = [x_1, x_2]^T$ is the state vector and x_1 is available for the on-line measurement y. Moreover, $\xi(x_2) = \frac{1}{1+x_2^2}$ is a nonlinear function depending on the unmeasured state variable x_2. It is worth to show that the system Υ_S can be expressed in the form Ξ_S in (19), using the following matrices

$$A = \begin{bmatrix} -4 & 3 \\ 5 & -6 \end{bmatrix}, \quad G = C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \; H = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and the nonlinear function $\xi(\sigma) = \dfrac{1}{1+\sigma^2}$.

The objective of the observation problem is then to estimate on-line the variable x_2 from the variable x_1 using the proposed methodology for designing the preserving and interval observers.

A. SCENARIO 1: OBSERVERS DESIGN FOR NOMINAL SYSTEM

For the system Υ_S, we get the preserving order observer

$$\Upsilon_O : \begin{cases} \dot{\widehat{x}}_1 = (l_1 - 4)\widehat{x}_1 - l_1 y + 3\widehat{x}_2 \\ \qquad + \xi(\widehat{\sigma} + N(\widehat{x}_1 - y)), \\ \dot{\widehat{x}}_2 = (l_2 + 5)\widehat{x}_1 - 6\widehat{x}_2 - l_2 y, \\ \widehat{\sigma} = \widehat{x}_2, \\ \widehat{y} = \widehat{x}_1, \end{cases} \quad (46)$$

where the gains l_1 and l_2 constitute the design matrix $L = [l_1, l_2]^T$ in Ξ_O in (20). Hence, the incremental nonlinearity is given by,

$$\psi(\sigma, z) = \frac{1}{1+\sigma^2} - \frac{1}{1+(\sigma+z)^2}$$

and belongs to the sector $[-0.6495, 0.6495]$. Moreover, $\psi(\sigma, z)$ satisfies the Lipschitz condition in (11) with $\gamma = 0.6495$. The proposed design matrices, for the preserving order observer,

$$L = \begin{bmatrix} -18.6846 \\ -4.9540 \end{bmatrix} \quad \text{and} \quad N = 0.000465,$$

were found solving the conditions of Theorem (13). It is important to mention that the stability radius using the previous matrices, is significantly incremented with respect to the matrices for an open-loop (ol) observer ($L = 0$, $N = 0$), that is, we have

$$\left\| H_N (A_L)^{-1} G \right\|^{-1} = 2.7868 \times 10^3 > \left\| H A^{-1} G \right\|^{-1} = 1.8 > \gamma = 0.6495.$$

This comparative allows to ensure, in terms of stability radii, that the design of the output injection observer is more robust than in the open-loop observer. Additionally, the eigenvalues associated with the linear part of the observers lie in left regions of the complex plane,

$$\lambda(A_L) = [-5.9917329, \; -22.692867],$$
$$\lambda(A) = [-1, \; -9],$$

which guarantees that the convergence of the output injection observer is faster than the open-loop observer.

Using the design of the preserving order observer, we construct an output injection interval observer (Γ_O^-, Γ_O^+), selecting the design matrices $L = L^+ = L^-$ and $N = N^+ = N^-$. A similar situation has also been considered for a open-loop (ol) interval observer with matrices $L = 0$ and $N = 0$.

The simulation results are presented in Figure 2. From this figure, it is shown the evolution of the trajectories of the system real state as well as of the upper and lower estimations of the interval observers. It is clear that output injection interval observer converges faster than the open-loop interval observer. This is of course expected from the previous paragraphs. It is easy to see the typical behavior of the interval observers for systems in absence of unknown signals: (i) the partial ordering is guaranteed between the trajectories of the state and the estimations from the partial

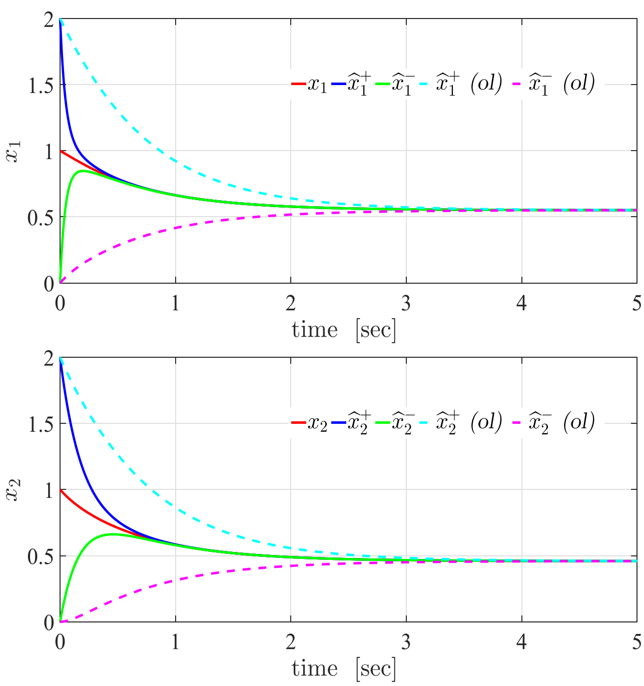

FIGURE 2. Evolution of the estimations of the output injection interval observer and the open-loop (*ol*) interval observer.

ordering of the initial conditions, and (ii) the upper and lower estimations (exponentially) converge to their true values.

B. SCENARIO 2:INTERVAL OBSERVER DESIGN WITH NONNEGATIVE GAINS

In this case, we include the non-negativity constraint in the design matrices. Taking into account the similar characteristics of the previous design, we now propose the following nonnegative matrices for the preserving order observer

$$L = \begin{bmatrix} 0 \\ 1.4564 \end{bmatrix} \quad \text{and } N = 0.00365.$$

Note that the stability radius, using the nonnegative matrices, is decremented with respect to the matrices for an open-loop (ol) observer ($L = 0,\ N = 0$)

$$\left\| HA^{-1}G \right\|^{-1} = 1.8 > \left\| H_N (A_L)^{-1} G \right\|^{-1} = 0.714817 > \gamma = 0.6495,$$

indicating that the design of the output injection interval observer achieves the exponential convergence. Moreover, the eigenvalues associated with the linear part of the observers are

$$\lambda (A) = \begin{bmatrix} -1, & -9 \end{bmatrix},$$
$$\lambda (A_L) = \begin{bmatrix} -0.486774, & -9.513225 \end{bmatrix},$$

which guarantees that the convergence rate of the output injection interval observer is slower than the open-loop observer, due to that $\mu(A) \leq \mu(A_L)$, see Remark 15.

Figure 3 shows the evolution of the trajectories of the real state as well as of the upper and lower estimations of the interval observers. It is easy to see the convergence of the output injection interval observer is slower than that of the open-loop interval observer due to the non-negativity in the design matrices.

C. SCENARIO 3: INTERVAL OBSERVER IN PRESENCE OF DISTURBANCES

In this scenario, we consider that the system Υ_S is affected by the presence of the disturbance $w_2(t)$. The system can be then rewritten to the form Ψ_S with the disturbance vector $w(t) = [0,\ w_2(t)]^T$. It is assumed that the disturbance is upper and lower bounded by $w_2^{\pm} = 10 \exp(-t) (\sin(10t) \pm 0.61)$, which fulfill the constraint in (24).

In this design, we have considered the above matrices A, G, C, and the nonlinearities $\xi(\sigma)$ for the systems Ψ_S, Ψ_{O^-}, Ψ_{O^+}, Ψ_{E^-}, and Ψ_{E^+}. The design matrices, for the interval observer, are proposed as

$$L^+ = \begin{bmatrix} -12.7649 \\ -5 \end{bmatrix} \quad \text{and } N^+ = 0.002786,$$
$$L^- = \begin{bmatrix} -10.8497 \\ -4.9892 \end{bmatrix} \quad \text{and } N^- = 0.006597,$$

for satisfying the conditions of Theorem 21.

Figure 4 presents the evolution of the trajectories of the real state of the system Υ_S affected by the presence of a time-varying disturbance. Moreover, it is easy to see that the upper and lower estimations of the interval observer preserve the partial ordering even if the perturbations are present in the system Υ_S. In addition, these estimations converge (exponentially) to their real values as the disturbance vanishes.

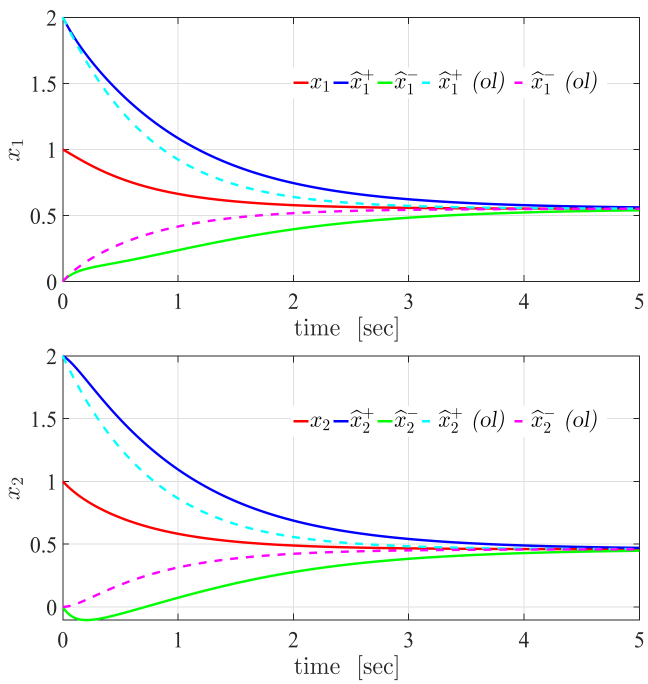

FIGURE 3. Evolution of the estimations of the open-loop (*ol*) interval observer and the output injection interval observer with nonnegative gains.

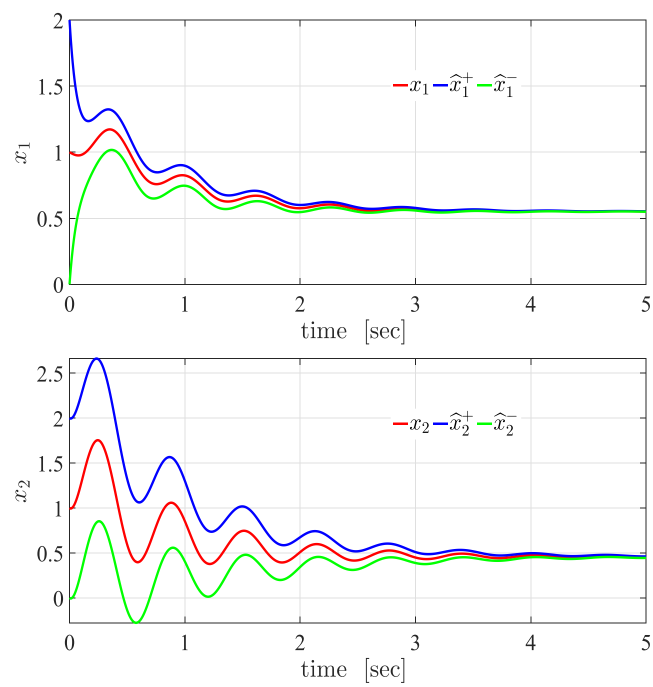

FIGURE 4. Evolution of the interval observer for Υ_S affected by the presence of a time-varying disturbance.

VII. CONCLUSIONS

The paper is devoted to the design of preserving order and interval observers for a family of nonlinear systems. The method is based on combining the cooperativity property and the stability radii, which are applied to the estimation error systems. The proposed approach is valid for Lipschitz nonlinearities. The design conditions, depending on the matrix gains of the observers, can be converted under some conditions to Linear Matrix Inequalities (LMI's). The relaxation of the stability radius constraints are the future issues of research.

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