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Fast Approach for Analysis Windows Computation of Multiwindow Discrete Gabor Transform

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ABSTRACT By using the biorthogonal analysis approach, an effective algorithm based on factorization for solving the analysis/dual windows in multiwindow discrete Gabor transform (M-DGT) is presented for arbitrary given synthesis windows. The constraint condition matrix of the M-DGT between analysis/dual windows and synthesis windows is proved to be equivalent to a fixed number of independent orthogonal relationship matrixes of discrete Gabor transform (DGT), which can be quickly and efficiently solved by using sub-equation sets. The analysis and comparison of computational complexity of related algorithms have indicated that the proposed algorithm provided a faster computational approach for computing analysis/dual windows as compared with that of the existing algorithms, which can save amount of computation and memory.

INDEX TERMS Bi-orthogonal analysis approach, multiwindow discrete Gabor transform (M-DGT), analysis windows, discrete Gabor transform (DGT).

I. INTRODUCTION

The M-DGT [1]–[8], which extended from DGT [9]–[15], has become a valuable time-frequency analysis technology for diverse areas, such as signal processing [16]–[19], evolutionary spectral analysis [20], image watermarking [21], ultrasonic echo signal [22], [23], image processing [24], [25], and so forth. Due to the uncertainty principle [26], the traditional DGT, a type of canonical linear transform with single window, suffers a limit of time-frequency resolution [27] in time-frequency plane, which could not adaptively process the non-stationary signals with different frequency components. To overcome this gab, the M-DGT can be able to utilize multiple analysis/dual windows of different shape and bandwidth to extract or/and detect the local time-frequency information of the analyzed signal in an efficient way, in which narrower time window is in charge of detecting and localizing the fast-changing/transient features of the analyzed signal and wider time window is invented to process the slow changes of a signal.

The analysis windows of M-DGT, called the dual windows in Gabor frame theory [14], [15], can be obtained directly by applying a linear operator [11], which involved the bi-orthogonal relationship between analysis/dual windows and synthesis windows [13] or frame matrix operator [28], [29]. However, with more windows used, the computation of analysis/dual windows increases rapidly as the number of windows applying to application, which requires a great deal of computational time and memory space and would be possible to result in computational instability. To solve this problem, a faster method, based on factorization, for calculating analysis/dual windows of M-DGT, is presented in this paper. Firstly, the orthogonal relationship of M-DGT can be proved to equal to a certain number of orthogonal relationship of DGT, which solving the analysis windows problem of M-DGT can convert into obtaining the analysis windows of DGT. Secondly, the orthogonal relationship matrix of DGT, generated by the orthogonal relationship of M-DGT, can be simplified and separated it into a set of unrelated sub-orthogonal relationship matrix between analysis window and synthesis window. Finally, the analysis windows of M-DGT can be fast and separably processed by sub-equation sets. The analysis of computational complexity has been given to demonstrate the efficiency and the merits of decreasing the computation of proposed approach in comparison to existing algorithms.

The rest of the paper is organized as follows. The background of traditional M-DGT will be reviewed in section 2. In section 3, we develop an efficient approach to calculate the analysis windows of M-DGT. In section 4, the computation complexity analysis and comparison between the proposed algorithm and existing algorithms are discussed and some experiments are given to validate the proposed approach. Finally, the paper is concluded in Section 5.

II. REVIEW OF M-DGT

Let $f(k)$ represents a discrete-time signal of periodic length L, the multiple window version of discrete Gabor expansion (M-DGE) [7] is provided by the follows

$$f(k) = \sum_{r=0}^{R-1}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} c^{(r)}(m,n)h^{(r)}(k-ma)\exp\left(\frac{j2\pi nk}{N}\right), \tag{1}$$

and defining $c^{(r)}(m,n)$ as the multiwindow Gabor coefficients, they can be calculated according to

$$c^{(r)}(m,n) = \sum_{k=0}^{L-1} f(k)g^{(r)}(k-ma)\exp\left(-\frac{j2\pi nk}{N}\right), \tag{2}$$

where $j = \sqrt{-1}$ is the imaginary unit. The M-DGT can be obtained by (2) corresponding to its expansion (or inverse transform) in (1). Let $0 \le r < R$ denotes an index of windows that use in M-DGT. In expansion and transform, $L = Ma = N\omega$, let the positive integers M, N are the sampling points of time and frequency, and let ω, a denote the modulation/shift step in frequency domain and shift/translation interval in time domain. For a numerically stable reconstruction, the constrained condition by $L \le MN$ (or $L \ge \omega a$) has to be satisfied. By selecting proper parameters M and N, the Gabor oversampling ratio $\beta = \frac{MN}{L}$ is a positive integer. Since (1) and (2) constrained by completeness condition, the bi-orthogonal relationship of $h^{(r)}$ and $g^{(r)}$ should be satisfied as follows:

$$\frac{L}{MN}\delta(\bar{m})\delta(\bar{n}) = \sum_{r=0}^{R-1}\sum_{\bar{k}=0}^{L-1}\left(h^{(r)}(\bar{k}+\bar{m}N)g^{(r)}(\bar{k})\right) \cdot \exp\left(-\frac{j2\pi\bar{n}\bar{k}}{a}\right), \tag{3}$$

where $\delta(k)$ is the discrete version of delta function, $0 \le \bar{m} < \omega$, and $0 \le \bar{n} < a$. The time-frequency representation (TFR) of M-DGT is defined as

$$\bar{c}^{(r)}(m,n) = \left|c^{(r)}(m,n)\right|^2, \tag{4}$$

and the combined time-frequency spectrum of M-DGT can be defined as the arithmetic average of $\bar{c}^{(r)}(m,n)$

$$S(m,n) = \frac{1}{R}\sum_{r=0}^{R-1}\bar{c}^{(r)}(m,n), \tag{5}$$

or the geometric average of $\bar{c}^{(r)}(m,n)$

$$S(m,n) = \left(\prod_{r=0}^{R-1}\bar{c}^{(r)}(m,n)\right)^{\frac{1}{R}}, \tag{6}$$

where $0 \le m \le M-1$ and $0 \le n \le N-1$.

III. FAST ALGORITHM FOR CALCULATING ANALYSIS WINDOWS OF M-DGT

The bi-orthogonal condition in (3) can be converted into matrix form

$$\boldsymbol{Hg} = \boldsymbol{v}, \tag{7}$$

where

$$\boldsymbol{H} = \left[\boldsymbol{H}^{(0)}, \boldsymbol{H}^{(1)}, \cdots, \boldsymbol{H}^{(R-1)}\right], \tag{8}$$

$$\boldsymbol{g} = \left[\boldsymbol{g}^{(0)T}, \boldsymbol{g}^{(1)T}, \cdots, \boldsymbol{g}^{(R-1)T}\right]^T, \tag{9}$$

$\boldsymbol{g}^{(r)}$ is a real vector with length $L-1$

$$\boldsymbol{g}^{(r)} = \left[g^{(r)}(0), g^{(r)}(1), \cdots, g^{(r)}(L-1)\right]^T, \tag{10}$$

$\boldsymbol{v}$ is a wa-long real vector with the first element being L/MN and others being zeros

$$\boldsymbol{v} = \left[\frac{L}{MN}, 0, \cdots, 0\right]^T, \tag{11}$$

and defining $\boldsymbol{H}^{(r)}$ in (8) as the $(\omega a)\times L$ matrix, it is composed by

$$\boldsymbol{H}^{(r)} = \begin{bmatrix} h^0_{r,0,0} & h^1_{r,0,0} & \cdots & h^{L-1}_{r,0,0} \\ h^0_{r,1,0} & h^1_{r,1,0} & \cdots & h^{L-1}_{r,1,0} \\ \vdots & \vdots & \ddots & \vdots \\ h^0_{r,\omega-1,a-1} & h^1_{r,\omega-1,a-1} & \cdots & h^{L-1}_{r,\omega-1,a-1} \end{bmatrix}, \tag{12}$$

where $h^k_{r,m,n} = h^{(r)}(mN+k)\exp\left(-\frac{j2\pi nk}{a}\right), 0 \le k \le L-1$, $0 \le m \le \omega-1$, and $0 \le n \le a-1$.

Theorem 1: Let $\boldsymbol{h}^{(r)}$ and $\bar{\boldsymbol{g}}^{(r)}$, which represent the synthesis and analysis/dual window of DGT with single window, satisfy following bi-orthogonal relationship

$$\boldsymbol{H}^{(r)}\bar{\boldsymbol{g}}^{(r)} = \boldsymbol{v} \quad 0 \le r \le R-1. \tag{13}$$

For any choice of $\alpha_r \in (0,1)$

$$\sum_{r=0}^{R-1}\alpha_r = 1, \tag{14}$$

and

$$\boldsymbol{g}^{(r)} = \alpha_r\bar{\boldsymbol{g}}^{(r)}, \tag{15}$$

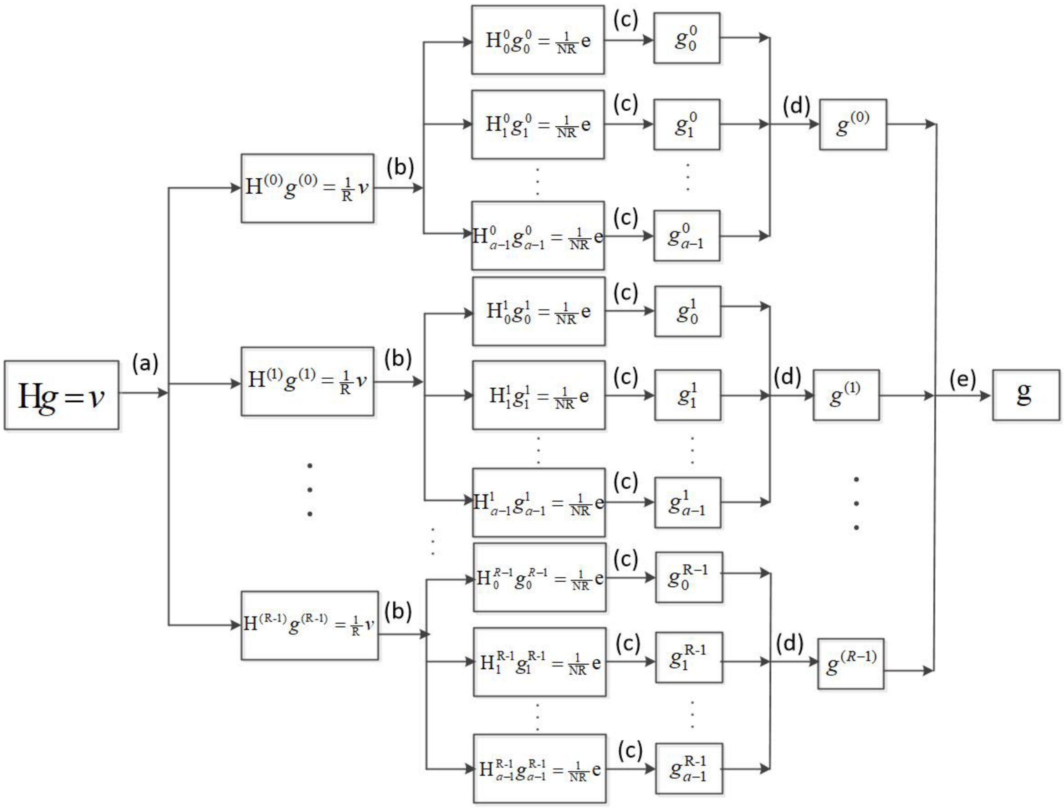


FIGURE 1. Flow processing of proposed approach.

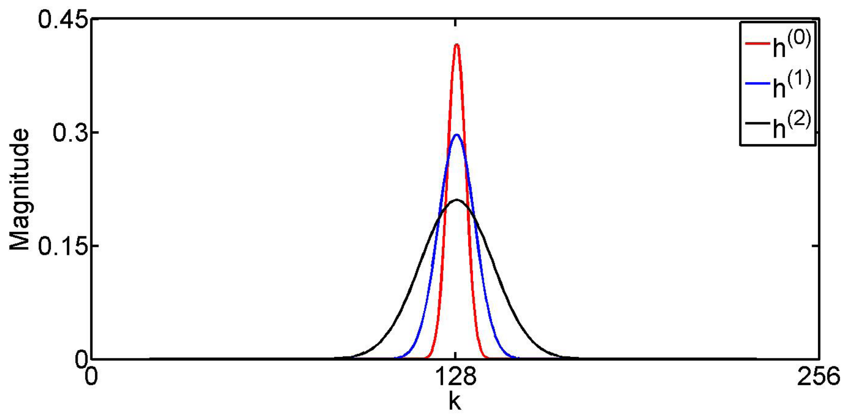


FIGURE 2. Three Gaussian synthesis windows ($R = 3$, $L = 256$).

then $\alpha_r \bar{\boldsymbol{g}}^{(r)}$ also satisfies the bi-orthogonality condition of (7) in M-DGT.

Proof: Substituting (15) into (7) leads to

$$\begin{aligned}
\boldsymbol{v} &= \sum_{r=0}^{R-1} \alpha_r \boldsymbol{H}^{(r)} \bar{\boldsymbol{g}}^{(r)} \\
&= \alpha_0 \boldsymbol{H}^{(0)} \bar{\boldsymbol{g}}^{(0)} + \alpha_1 \boldsymbol{H}^{(1)} \bar{\boldsymbol{g}}^{(1)} + \cdots + \alpha_{R-1} \boldsymbol{H}^{(r)} \bar{\boldsymbol{g}}^{(R-1)} \\
&= \alpha_0 \boldsymbol{v} + \alpha_1 \boldsymbol{v} +, \cdots, +\alpha_{R-1} \boldsymbol{v} \\
&= (\alpha_0 + \alpha_1 +, \cdots, +\alpha_{R-1})\, \boldsymbol{v} \\
&= \boldsymbol{v}. \qquad (16)
\end{aligned}$$

Remark 1: In theorem 1, without loss of generality, let $\alpha_0 = \alpha_1 = \cdots = \alpha_{R-1} = 1/R$, one can simplify the original biorthogonal relationship of M-DGT in (7) and separate it into R unrelated sub-biorthogonal relationship of DGT.

By using theorem 1, the property of $\delta(k)$, and the discrete poisson-sum theory, (3) can be rewritten as

$$\begin{aligned}
\frac{a}{RN}\delta(\bar{m}) &= \frac{L}{RMN} \sum_{\bar{n}=0}^{a-1} \delta(\bar{m})\delta(\bar{n}) \exp\left(\frac{j2\pi \bar{n} k'}{a}\right) \\
&= \sum_{\bar{n}=0}^{a-1} \left(\sum_{\bar{k}=0}^{L-1} h^{(r)}\left(\bar{k} + \bar{m}N\right) g^{(r)}\left(\bar{k}\right) \right)
\end{aligned}$$

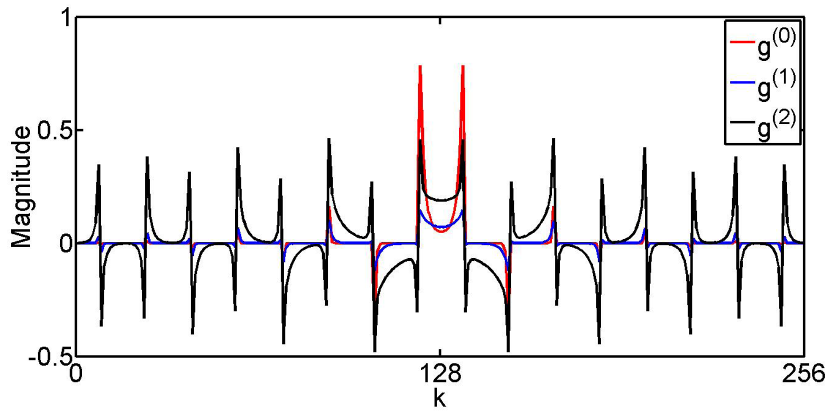


FIGURE 3. Analysis windows corresponding to Gaussian synthesis windows under critical sampling ($M = 16$, $N = 16$).

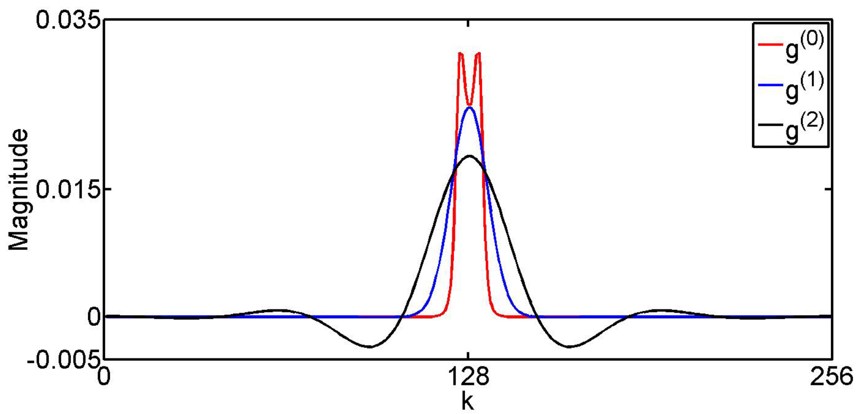


FIGURE 4. Analysis windows corresponding to Gaussian synthesis windows under oversampling ($M = 32$, $N = 32$).

$$\cdot \exp\left(-\frac{j2\pi\bar{n}\bar{k}}{a}\right)\exp\left(\frac{j2\pi\bar{n}k'}{a}\right)$$
$$= \sum_{\bar{k}=0}^{L-1} h^{(r)}\left(\bar{k}+\bar{m}N\right)g^{(r)}\left(\bar{k}\right) \cdot a\sum_{q=0}^{M-1}\delta\left(\bar{k}-k'-qa\right), \tag{17}$$

where $0 \leq r < R$, $0 \leq \bar{m} < \omega$, and $0 \leq k' < L$. For the sake of simplicity, (17) can be explicitly expressed as

$$\frac{\delta(\bar{m})}{RN} = \sum_{q=0}^{M-1} g^{(r)}(k'+qa)h^{(r)}(k'+qa+\bar{m}N)$$
$$= \sum_{q=0}^{M-1} g^{(r)}(s+ta+qa)h^{(r)}(s+ta+qa+\bar{m}N)$$
$$= \sum_{\bar{q}=0}^{M-1} g^{(r)}(s+\bar{q}a)h^{(r)}(s+\bar{q}a+\bar{m}N), \tag{18}$$

where $k' = s + ta$, $s = 0, 1, \cdots, a-1$, and $t = 0, 1, \cdots, M-1$. Equation (18) can be rewritten and expressed in matrix form, which is obviously constructed by Ra independent linear equation sets

$$\begin{bmatrix} \boldsymbol{H}_0^r & 0 & \cdots & 0 \\ 0 & \boldsymbol{H}_1^r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{H}_{a-1}^r \end{bmatrix} \begin{bmatrix} \boldsymbol{g}_0^r \\ \boldsymbol{g}_1^r \\ \vdots \\ \boldsymbol{g}_{a-1}^r \end{bmatrix} = \frac{1}{RN}\begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{e} \\ \vdots \\ \boldsymbol{e} \end{bmatrix}, \tag{19}$$

where $\boldsymbol{g}_s^r$ is a M-long vector, $\boldsymbol{H}_s^r$ is a real matrix of size $\omega \times M$, $s = 0, 1, \cdots, a-1$, and $\boldsymbol{e}$ is a ω-long unit vector

$$\boldsymbol{e} = [1, 0, \cdots, 0]^T, \tag{20}$$
$$\boldsymbol{g}_s^r = \left[g^{(r)}(s), g^{(r)}(s+a), \cdots, g^{(r)}(s+(M-1)a)\right]^T, \tag{21}$$
$$\boldsymbol{H}_s^r = \begin{bmatrix} h_{0,0}^{r,s} & h_{0,1}^{r,s} & \cdots & h_{0,M-1}^{r,s} \\ h_{1,0}^{r,s} & h_{1,1}^{r,s} & \cdots & h_{1,M-1}^{r,s} \\ \vdots & \vdots & \ddots & \vdots \\ h_{w-1,0}^{r,s} & h_{w-1,1}^{r,s} & \cdots & h_{w-1,M-1}^{r,s} \end{bmatrix}, \tag{22}$$

where $h_{u,v}^{r,s} = h^{(r)}(s + uN + va)$, $0 \leq u \leq \omega - 1$, and $0 \leq v \leq M - 1$. For $r = 0, 1, \cdots, R-1$, (19) can be split into a

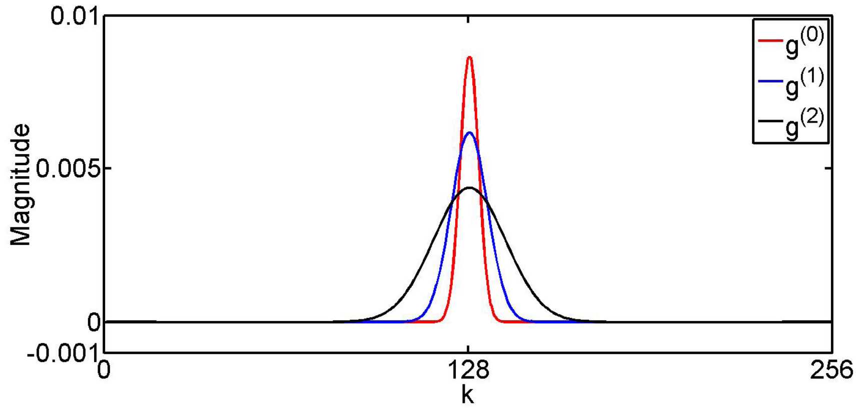


FIGURE 5. Analysis windows corresponding to Gaussian synthesis windows under oversampling ($M = 64, N = 64$).

Algorithm 1 Factorization for Calculating Analysis Windows of M-DGT

1: **Input:** $R, L, N, a, \boldsymbol{h}^{(0)}, \boldsymbol{h}^{(1)}, \cdots, \boldsymbol{h}^{(R-1)}$.
2: **Output:** $\boldsymbol{g}^{(0)}, \boldsymbol{g}^{(1)}, \cdots, \boldsymbol{g}^{(R-1)}$.
3: **for** $r = 0 : R-1$ **do**
4: **for** $s = 0 : a-1$ **do**
5: (1) Compute inverse matrix $\boldsymbol{I}_s^r = \left(\boldsymbol{H}_s^r \boldsymbol{H}_s^{rT}\right)^{-1}$.
6: (2) Calculate the matrix $\boldsymbol{R}_s^r = \boldsymbol{H}_s^{rT} \cdot \boldsymbol{I}_s^r$.
7: (3) Solve the vector $\boldsymbol{g}_s^r = \frac{1}{RN} \cdot \boldsymbol{R}_s^r \cdot \boldsymbol{e}$.
8: **end for**
9: **end for**

independent sub-matrix sets

$$\boldsymbol{H}_s^r \boldsymbol{g}_s^r = \frac{1}{RN}\boldsymbol{e}. \tag{23}$$

The least ℓ_2 norm solution of (23) can be obtained by [30] in following

$$\boldsymbol{g}_s^r = \frac{1}{RN}\boldsymbol{H}_s^{rT}\left(\boldsymbol{H}_s^r \boldsymbol{H}_s^{rT}\right)^{-1}\boldsymbol{e}, \tag{24}$$

where $0 \le 0 \le R-1$ and $0 \le s \le a-1$. In summarize, the following procedure in Alg. 1 can be concluded from the basic idea of the above description, which the flow processing shown in Fig. 1.

Note that: (1) the symbol (a) in Fig. 1 represents the original orthogonal relationships of M-DGT with the number of R analysis windows can prove to be equivalent to R independent orthogonal relationship of DGT; (2) the symbol (b) in Fig. 1 denotes the each independent orthogonal relationship of DGT can be decomposed into a certain number of unrelated sub-orthogonal relationships; (3) the symbol (c) in Fig. 1 indicates the s-th sub-sequences of r-th sub-analysis window can be computed by unrelated sub-orthogonal relationships; (4) the symbol (d) in Fig. 1 shows the r-th analysis window can be solved by a sub-analysis window sequences; (5) the symbol (e) in Fig. 1 represents the analysis windows of M-DGT can be obtained by R independent analysis window of DGT.

TABLE 1. A detailed comparison of computational complexity between proposed approach and others.

References	Computational complexity
[4]	$R \times \left(MN + 2L^2 + L^3\right)$
[7]	$R\left(\frac{L}{N}\frac{L}{M}\right) + 2RL\left(\frac{L}{M}\frac{L}{N}\right)^2 + \left(\frac{L}{M}\frac{L}{N}\right)^3$
Proposed approach	$R \times \frac{L}{M} \times \left(2M\left(\frac{L}{N}\right)^2 + \left(\frac{L}{N}\right)^3 + M\frac{L}{N}\right)$

IV. COMPLEXITY COMPARISON AND EXPERIMENTS

A. COMPUTATIONAL COMPLEXITY ANALYSIS

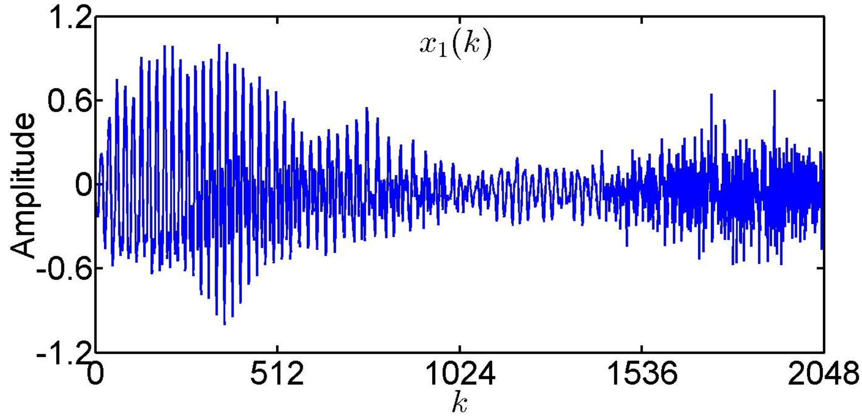
Because the original orthogonal relationship of M-DGT can be decomposed into R independent orthogonal relationship of DGT, the computational complexity of proposed approach for solving analysis windows in M-DGT is equivalent to the complexity of the total R unrelated orthogonal relationship equation sets of DGT for computing analysis windows. A detailed comparison of the proposed algorithm and the existed canonical algorithms [4], [7] has been given in Table 1, which obviously shows that the computational complexity of the proposed approach is lower than that of the others under the critical sampling case and the oversampling case. By using the formula in Table 1. the total number of multiplication times between proposed algorithm and existing canonical algorithms are compared in Table 2.

B. NUMERICAL EXAMPLES

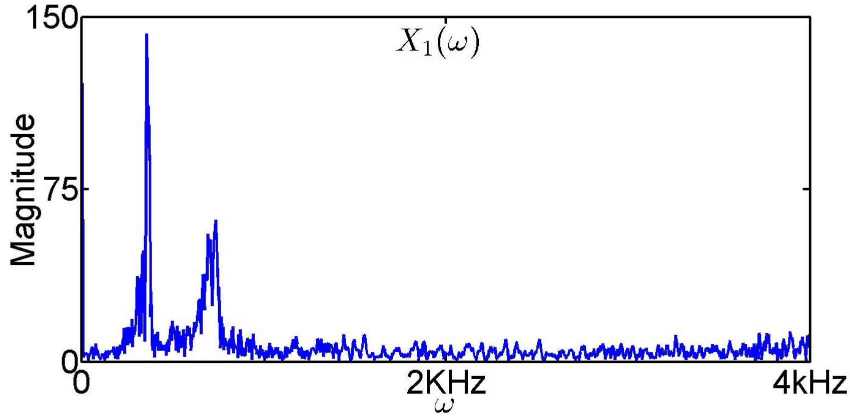
Example 1: To verify the effectiveness of the proposed algorithm for computing analysis windows of M-DGT, three Gaussian synthesis windows $h^{(r)}(k)$ is given in (25) and shown in Fig. 2

$$h^{(r)}(k) = \frac{1}{{\sigma_r}^{0.25}} \exp\left(-\frac{[k-0.5(L-1)]^2}{2\sigma_r}\right), \tag{25}$$

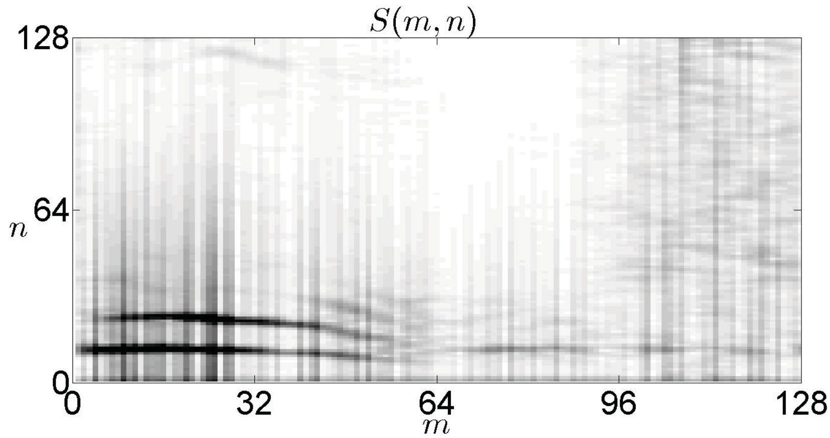
where $0 \le r \le R-1$, $0 \le k \le L-1$, $L = 256$, and $\boldsymbol{\sigma} = [\sigma_0, \sigma_1, \sigma_2] = [16, 64, 512]$. The analysis windows in Fig. 3-5, corresponding to the synthesis windows in Fig. 2,



(a)



(b)



(c)

FIGURE 6. (a) A speech signal $x_1(k)$. (b) The Fourier spectrum $X_1(\omega)$ of $x_1(k)$. (c) The combined Gabor time-frequency spectrum of $x_1(k)$ by using a narrower analysis window and a wider analysis window.

were computed under the critical sampling case and the oversampling case. Fig. 3 shows the analysis windows under the critical sampling case with $M = 16$ and $N = 16$. Fig. 4-5 show the analysis windows under the oversampling case with the oversampling rate $\beta = 4$ ($M = 32$, $N = 32$) and $\beta = 16$ ($M = 64$, $N = 64$).

Example 2: Fig. 6(a) and Fig. 6(b) show a speech signal $x_1(k)$ with $L = 2048$ samples, 256ms of the word ''yes'' spoken by a man, and its Fourier spectrum $X_1(\omega)$. Because the speech signal $x_1(k)$ contains multiple time-varying frequency including transient and tone components, the traditional Fourier transform can not process it in an

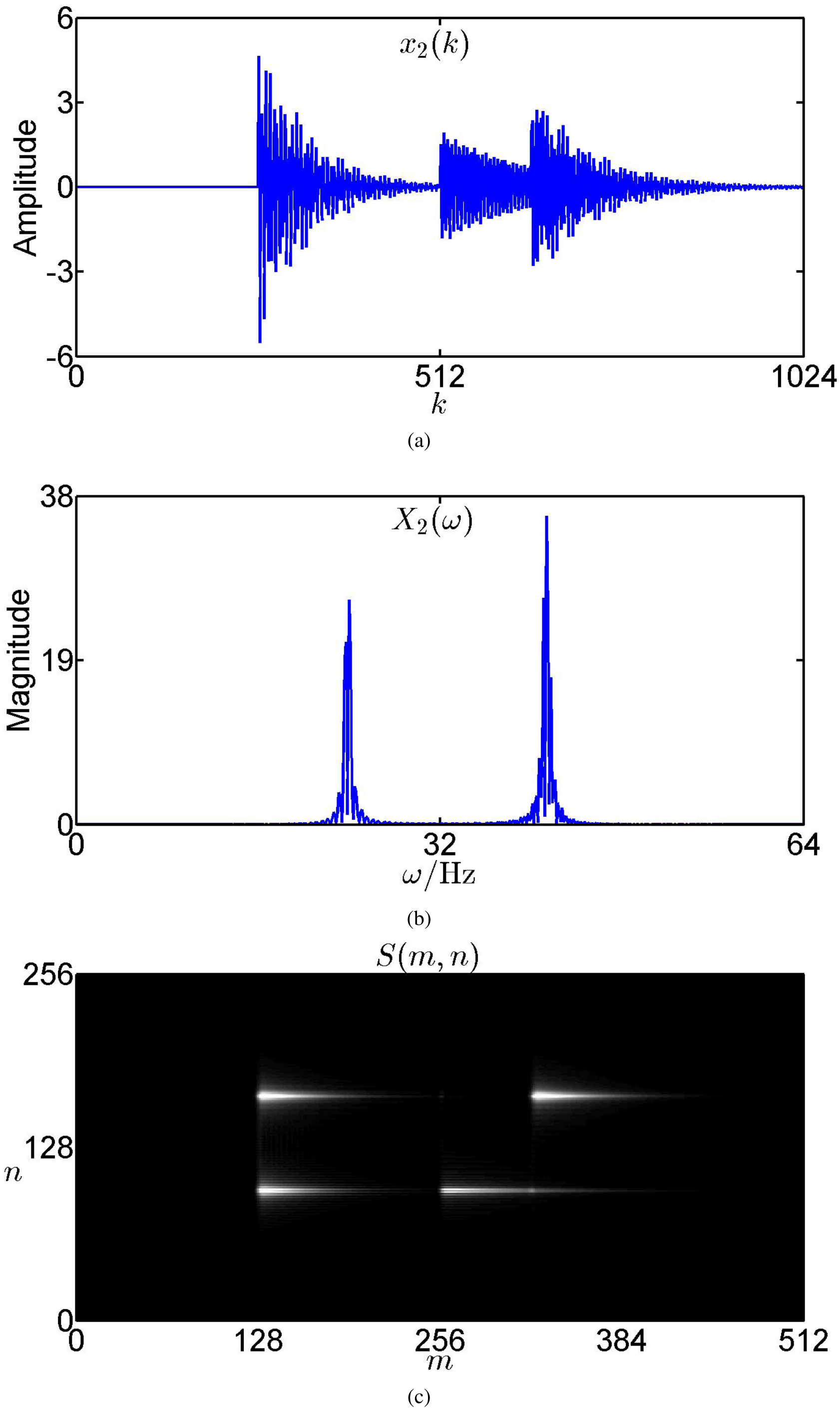


FIGURE 7. (a) A transient signal $x_2(k)$ composed of four transients. (b) The Fourier spectrum $X_2(\omega)$ of $x_2(k)$. (c) The combined time-frequency spectrum of $x_2(k)$ by using a narrower analysis window and a wider analysis window.

effective manner as displayed in Fig. 6(b). The combined time-frequency spectrum of M-DGT of $x_1(k)$ shown in Fig. 6(c), computed by M-DGT with the narrower analysis window and the wider analysis window, permits good time resolution and high frequency resolution, where the parameters are set as $R = 2$, $M = 128$, and $N = 256$.

Example 3: Fig. 7(a) shows a transient signal $x_2(k)$ consisting of four transients with length $L = 1024$ points which is sampled at 128Hz and its Fourier spectrum $X_2(\omega)$ given in Fig. 7(b). Because the transient signal $x_2(k)$ contains four transients and each of them has a single frequency and an arrival time, the traditional Fourier transform insuf-

TABLE 2. A detailed numeric comparison of multiplication times between proposed approach and others.

R	L	M	N	β	Total number of multiplications		
					[4]	[7]	Proposed approach
2	128	16	8	1	4260096	10518528	200704
		32	16	4	4260864	565248	38912
		32	64	16	4263936	35328	2624
4	256	16	16	1	67634176	151257088	802816
		32	32	4	67637248	8716288	155648
		64	64	16	67649536	544768	37888
8	512	16	32	1	107794022	2283798528	3211264
		32	64	4	1077952512	136839168	622592
		64	128	16	1078001664	8552448	151552

ficiently analyzes it as shown in Fig. 7(b), which only detects two frequencies information but fails to extract local time-varying frequencies. By using a narrower analysis window and a wider analysis window in M-DGT, the combined time-frequency spectrum of M-DGT of $x_2(k)$, shown in Fig. 7(c), is displayed with high time and frequency resolutions, where the parameters are set as $R = 2$, $M = 512$, and $N = 512$.

From the above examples, we can reach the conclusion that the analysis windows in M-DGT play an important role in time-frequency analysis, which is essential to study an efficient method for computing the analysis windows of M-DGT.

V. CONCLUSION

By converting the original bi-orthogonal relationship matrix of M-DGT into several independent orthogonal relationship matrix of DGT, an effective algorithm for computing analysis/dual windows of M-DGT was presented. The orthogonal relationship matrix of DGT, derived from bi-orthogonal relationship of M-DGT, can be simplified and expressed as equivalent form of a group of linear equations, which can be fast solved by sub-equation sets. The computational complexity analysis and comparison indicate that the proposed algorithm for computing analysis/dual windows of M-DGT is more competitive against others. Numerical results show that the proposed algorithm can provide an efficient and fast method for calculating analysis/dual windows of M-DGT, which can save a large amount of computational time and memory.

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