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Ensemble Interval-Valued Fuzzy Cognitive Maps

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ABSTRACT As a novel generalization of fuzzy cognitive map (FCM), interval-valued FCM (IVFCM) can provide more flexibility in modeling that increasingly complex system with uncertainty. However, the problem of aggregating IVFCMs has not been considered to this day. Concerning this key point, we propose an ensemble IVFCMs via evidential reasoning (ER) approach. First, we give a detailed analysis of IVIFS in terms of evidence theory and introduce the concept of augmented connection matrix within the framework of IVFCMs. Second, we present a theory of ensemble IVFCMs using the former work and ER approach, particular emphases are put on assessing the weights of different IVFCMs and aggregating them. Both theoretical analysis and practical examples show that the ensemble IVFCMs not only reflects the importance levels of different maps but also can achieve the goal of merging of information from different maps in system modeling.

INDEX TERMS System modeling, ensemble interval-valued fuzzy cognitive maps, multi-objective optimization, spectrum sensing.

I. INTRODUCTION

In 1948, Tolman [1] firstly introduced the concept of cognitive map (CM). A CM can be regarded as a type of mental expression of physical locations by processing relevant information from daily life or some spatial environment. The term was later generalized by some scholars, as a kind of semantic network representing some knowledge systems [2]. In the past decades, CMs have been studied and utilized in lots of aspects, such as political selection, planning, urban planning, management and history [2]. However, CMs have a limitation that the uncertainty in the inference process can not be described accurately. To overcome the limitation of CM, Kosko [3] introduced fuzzy cognitive map (FCM) as a generalization of CM. FCM is a signed fuzzy digraph within which the relations between the concepts of mental landscape can be utilized to determine the strength of these concepts. The same as CMs, FCMs have gained considerable research interest due to their ability in representing structured knowledge and system modeling in a number of fields including business [4], ecological engineering [5], management [6], system modeling [7], risk assessment [8], machine learning [9], etc. More information on FCMs, please refer to [10]–[14]. This growing interest leads to the demand for establishing more effective models which can better describe the complex real situations.

Regarding FCM and its generalizations, the existing studies mainly include three aspects [10], i.e., generating cognitive models from input historical data, quantifying the state of concepts and the connections between concepts, and combining multiple maps. For the first research point, a number of learning algorithms have been successfully utilized for constructing maps including Hebbian learning, genetic algorithms, memetic algorithms, imperialist competitive algorithms, evolutionary algorithms [10], [16], etc. To satisfy the demand of system modeling regarding complex situations, various extensions of FCMs have been presented one after another from different perspectives including extended FCMs [15], dynamic CMs [17], fuzzy grey cognitive maps (FGCMs) [18], evidential cognitive maps (ECMs) [19], intuitionistic fuzzy cognitive maps (IFCMs) [20], granular cognitive maps (GCMs) [21], interval-valued fuzzy cognitive maps (IVFCMs) [22], extended evidential cognitive maps [24], etc. As pointed out by Pedrycz [23], most generalizations which apply information granules to depict the state of concepts and the connection matrix can be regarded as some special cases of GCMs. As system modeling goes, combining knowledge plays an important role for whether FCMs or high-order FCMs. As described in [3], any number of FCMs can be naturally aggregated into a single FCM through additively combining augmented connection matrices. In contrast with the theory of FCMs, the problem of combining knowledge has not been fully considered in a number of GCMs. Among numerous GCMs, the theory of IVFCM was introduced in 2016 and offers much more flexibility for representing the uncertainty from the state of concepts and the connection matrices. For the actuality of IVFCMs, the relevant study is still in the early stage, especially for establishing ensemble IVFCMs. As proved in [18], [19], and [24]–[26], the ensemble map not only plays a key role to better model complex system or to reflect the inference reasoning from multiple experts [19], [24]–[26], but also can effectively avoid unreasonable fuzzy reasoning results which may derived from individual cognitive model. By comparison with the comprehensive theory of aggregating FCMs, there are some key problems of establishing ensemble model to be solved including constructing augmented connection matrices, assessing the weights and determining the ensemble model with respect to lots of high-order cognitive maps.

In order to develop and perfect the theory of IVFCMs, we propose ensemble IVFCMs to satisfy the demand of modeling complex systems. Simply stated, the main merits of ensemble IVFCMs can be represented as below.

- We give a detailed analysis of interval-valued fuzzy sets in terms of evidence theory.
  As discussed in [27], an interval-valued fuzzy number (IVFN) corresponds to an intuitionistic fuzzy number (IFN). We should also note that an IFN is a probability distribution as proved in [28]. In essence, a probability distribution is a piece of evidence [29]. Thus, the relationship between IVFN and evidence theory can be established via intuitionistic fuzzy sets (IFSs).
- We propose a method to define the augmented connection matrices.
  Quantifying the augmented connection matrices is the prerequisite to establish ensemble IVFCMs. From the perspective of IVFSs, we propose a method to augment the connection matrices, particular emphasis is put on quantifying the default connections.
- We propose an approach based on ER approach to construct ensemble IVFCMs.
  ER approach [30] belongs to a universal evidence-based multiple attribute decision analysis algorithm for solving those problems with both quantitative and qualitative attributes under an uncertain and random environment. To this day, ER approach has been widely and successfully utilized in dealing with information fusion problems with uncertainty [31]–[33]. Considering its predominance in aggregating uncertain information, we propose an approach using ER approach to aggregate all the augmented connection matrices into a matrix which denotes the ensemble IVFCMs. Meanwhile, we present an optimization model to assess the importance levels of different maps with constraint conditions.

It is clear that ensemble IVFCMs not only enables to model the complex systems with uncertainty but also represents the weights of different maps.

The rest of this paper includes four sections. Some relevant concepts will be presented in Section II. In Section III, we propose ensemble IVFCMs using ER approach. Next, we employ three examples to validate the performance of this theory in Section IV. Finally, Section V summarizes the whole paper.

II. PRELIMINARY

Both IVFSs [34] and intuitionistic fuzzy sets (IFSs) [35] constitute extensions of conventional fuzzy sets [34]. An IVFS is equivalent to an IFS in the sense of *Lattice* [27]. In this section, we recall two concepts and some operations on them.

A. IVFSS AND IFSS

Definition 1 [34]: An IVFS A in a universe Ω is a mapping

$$A : \Omega \to Int([0, 1]) : x \mapsto [\underline{A}(x), \overline{A}(x)] \subseteq [0, 1], \quad (1)$$

for any element $x \in \Omega$, where $Int([0, 1])$ denotes the set including all closed subintervals of $[0, 1]$.

Definition 2 [35]: An IFS A on an universe Ω is with the form $A = \{(x, \mu_A(x), \nu_A(x), \pi_A(x))|x \in \Omega\}$, where $\mu_A(x) + \nu_A(x) + \pi_A(x) = 1$ and $0 \leq \mu_A(x), \nu_A(x), \pi_A(x) \leq 1$ hold.

To simplify thing, we utilize $[\underline{\alpha}, \overline{\alpha}]$ and (a, b) (or (a, b, c)) to respectively denote an IVFN and an IFN, where $0 \leq \underline{\alpha} \leq \overline{\alpha} \leq 1$, $0 \leq a, b, c \leq 1$, $\underline{\alpha} + \overline{\alpha} = 1$ and $a + b + c = 1$.

Remark 1: As can be observed from *Definition* 1 and *Definition* 2, both IVFS and IFS are with more flexibility and freedom to quantify those incomplete and uncertain information.

B. OPERATIONS ON IVFSS

Definition 3 [36]: For two IVFNs $a = [\underline{a}, \overline{a}]$ and $b = [\underline{b}, \overline{b}]$, the addition, the multiplication and the subtraction between a and b are with the following forms,

$$a \oplus b = \left[\min(\underline{a} + \overline{b}, \overline{a} + \underline{b}), \overline{a} + \overline{b}\right], \quad (2)$$

$$a \otimes b = \left[\underline{ab}, \max(\underline{a}\overline{b}, \overline{a}\underline{b})\right], \quad (3)$$

$$a \ominus b = \left[\underline{a} - \overline{b}, \max(\underline{a} - \underline{b}, \overline{a} - \overline{b})\right]. \quad (4)$$

Clearly, the following proposition holds.

Proposition 1: For two IVFNs a and b, we have

$$a \oplus b = a, \quad (5)$$

$$a \ominus b = a, \quad (6)$$

$$a \otimes b = b, \quad (7)$$

when $b = [0, 0]$.

Definition 4 [36]: Let $\Upsilon(X)$ be all the IVFSs on $X = \{x_1, x_2, \ldots, x_n\}$. For two IVFSs A and A' $(A, A' \in \Upsilon(X))$, then the distance between A and A' is with the following form.

$$\vartheta(A, A') = \sqrt{\frac{1}{2n} \sum_{i=1}^{n} \left((\underline{A}(x_i) - \underline{A'}(x_i))^2 + (\overline{A}(x_i) - \overline{A'}(x_i))^2\right)}. \quad (8)$$

Remark 2: As defined in *Definition* 4, the proposed distance measure provides an efficient way to represent the divergence degree between two IVFNs.

III. ENSEMBLE IVFCMS

The objective of this section is to construct ensemble IVFCMs. Firstly, we recall the concept of IVFCM. Secondly, we present a detailed analysis of IVFS w.r.t evidence theory. Finally, we focus on establishing ensemble IVFCMs, which is the core of this section.

A. IVFCM

Similarly to FCM [3], an IVFCM is a graph-oriented map representing causal connections between a number of concepts N ($N = \{N_1, N_2, \ldots, N_n\}$) [22]. Different from conventional CM and FCM, the causal weights and the state of concepts are expressed with IVFNs. For above n concepts involved, the dynamics of the map is denoted by

$$\theta_i(t+1) = f\left(\theta_i(t) \oplus (\oplus_{j=1, j\neq i}^{n}(\theta_j(t) \otimes w_{ji}))\right), \tag{9}$$

which includes recurring connection on $t \geq 0$ between $\theta(t+1)$ and $\theta(t)$ ($i \in \{1, 2, \ldots, n\}$), where $\theta_i(t)$ is the state value of the concept N_i at t and f is a nonlinear threshold function.

Let $\theta_i(t) = [\underline{\theta}_i(t), \overline{\theta}_i(t)]$. Assume that $\theta_i(t) \oplus (\oplus_{j=1, j\neq i}^{n}(\theta_j(t) \otimes w_{ji})) = [\underline{\gamma}, \overline{\gamma}]$. Then equation (9) can be expressed by

$$\begin{aligned}\theta_i(t+1) &= [\underline{\theta}_i(t+1), \overline{\theta}_i(t+1)]\\ &= f\left(\theta_i(t) \oplus (\oplus_{j=1, j\neq i}^{n}(\theta_j(t) \otimes w_{ji}))\right)\\ &= [\min\{f(\underline{\gamma}), f(\overline{\gamma})\}, \max\{f(\underline{\gamma}), f(\overline{\gamma})\}]. \end{aligned}\tag{10}$$

To get a better understanding of IVFCM, we present an example as below.

Example 1: Figure 1 illustrates an example of IVFCM [22]. The IVFCM's connection matrix is equation (11), and its threshold function is $\frac{1}{1+\exp(-x)}$.

$$\Gamma = \begin{pmatrix} 0 & [0.4, 0.5] & [0.1, 0.3] \\ 0 & 0 & [0.7, 0.9] \\ 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

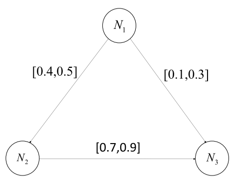

FIGURE 1. A three-node IVFCM.

Let $\theta_1(0) = [0.5, 0.6]$, $\theta_2(0) = [0.2, 0.4]$ and $\theta_3(0) = [0.3, 0.4]$ be the initial state values of three concepts. When $t = 0$, the concept values $\theta_1(1)$, $\theta_2(1)$ and $\theta_3(1)$ are calculated as equations (12)-(14); (14) is shown at the bottom of this page.

$$\begin{aligned}\theta_1(1) &= f(\theta_1(0))\\ &= f([0.5, 0.6])\\ &= [0.6225, 0.6457], \end{aligned}\tag{12}$$

$$\begin{aligned}\theta_2(1) &= f\left(\theta_2(0) \oplus (\theta_1(0) \otimes w_{12})\right)\\ &= f\left([0.2, 0.4] \oplus ([0.5, 0.6] \otimes [0.4, 0.5]\right)\\ &= f([0.45, 0.65])\\ &= [0.6106, 0.6570], \end{aligned}\tag{13}$$

Remark 3: As can be seen from Example 1, the default connection is denoted by 0 which is not convenient for augmenting the connection matrices to construct ensemble IVFCMs. Proposition 1 implies that $[0, 0]$ can replace 0 to quantify the default connections. In other words, Proposition 1 provides a tool to define the augmented connection matrices for the establishment of ensemble IVFCMs.

B. IVFS IN TERMS OF EVIDENCE THEORY

As proved in [27], an IVFN $[\underline{a}, \overline{a}]$ is equivalent to an IFN $(\underline{a}, 1-\overline{a}, \overline{a}-\underline{a})$, where $0 \leq \underline{a} \leq \overline{a} \leq 1$. Obviously, $(\underline{a}, 1-\overline{a}, \overline{a}-\underline{a})$ is a probability distribution. Let $\Theta = \{H_1, H_0, \{H_1, H_0\}\}$ be a framework of discernment. Then we define

$$\eta(H_1) = \underline{a}, \tag{15}$$

$$\eta(H_0) = 1 - \overline{a}, \tag{16}$$

$$\eta(\{H_1, H_0\}) = \overline{a} - \underline{a}, \tag{17}$$

where $\eta(H_1)$, $\eta(H_0)$ and $\eta(\{H_1, H_0\})$ constitute a piece of evidence from the perspective of ER approach, and respectively represent the positive degree, the negative degree and the neutral degree.

Remark 4: As stated above, we have presented a detailed analysis of IVFS in terms of evidence theory. Thus, we can utilize the relevant approaches of evidence theory to extend some theories on IVFSs. What's more, these theories can be employed to construct ensemble IVFCMs.

C. METHODOLOGY OF ENSEMBLE IVFCMS

As pointed in [19] and [24]–[26], every expert may provide his or her own cognitive model. These individual models could then be aggregated into a representative one. In general, larger expert sample sizes can present much more reasonable and efficient cognitive model. However, it is necessary to solve the individual cognitive models with different size and importance levels. In other words, defining augmented connection matrices, assessing the weights of different models and aggregating them are three key challenges in the establishment of ensemble map. Up to now, how to establish ensemble IVFCMs is still an open problem. To satisfy the demand in the field of system modeling, we propose a methodology of ensemble IVFCMs. Firstly, augment all the

$$\begin{aligned}\theta_3(1) &= f\left(\theta_3(0) \oplus ((\theta_1(0) \otimes w_{13}) \oplus (\theta_2(0) \otimes w_{23}))\right)\\ &= f\left([0.3, 0.4] \oplus (([0.5, 0.6] \otimes [0.1, 0.3]) \oplus ([0.2, 0.4] \otimes [0.7, 0.9]))\right)\\ &= f([0.69, 0.83])\\ &= [0.6660, 0.6964]. \end{aligned}\tag{14}$$

IVFCMs into same-sized maps. Secondly, assess the weights of different augmented maps. Finally, establish ensemble IVFCMs based on the known weights and all the augmented maps.

There are K ($K \geq 2$) IVFCMs to be utilized to construct ensemble maps. Let ξ_k ($k = 1, 2, \ldots, K$) be the weights of all K maps satisfying $\sum_{k=1}^{K} \xi_k = 1$ and $\xi_k \in [0, 1]$. The connection matrices of all K maps are shown as

$$\Gamma_k = \begin{pmatrix} \alpha_{11}^{(k)} & \alpha_{12}^{(k)} & \cdots & \alpha_{1\mathbb{N}_k}^{(k)} \\ \alpha_{21}^{(k)} & \alpha_{22}^{(k)} & \cdots & \alpha_{2\mathbb{N}_k}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{\mathbb{N}_k 1}^{(k)} & \alpha_{\mathbb{N}_k 2}^{(k)} & \cdots & \alpha_{\mathbb{N}_k \mathbb{N}_k}^{(k)} \end{pmatrix}, \tag{18}$$

where $\alpha_{ij}^{(k)}$ ($i, j = 1, 2, \ldots, \mathbb{N}_k$) is an IVFN.

Step 1: Augment connection matrices of all K IVFCMs into same-size matrices.

Due to the limited knowledge, these connection matrices Γ_k ($k = 1, 2, \ldots, K$) are unlikely to be aggregated directly. It is clear that ensemble IVFCMs' priority is rightly to make all the augmented matrices with same size. Suppose that one map includes a concept N_1 which is not employed in another one. It means that there are not connections between N_1 and every concept in another map. Clearly, N_1 can be regarded as a new concept of the second map. Next, the causal weight between N_1 and other concepts is defined by $[0, 0]$. Assume that all K maps include $\mathbb{N}$ different concepts. Thus, Γ_k can be augmented to a $\mathbb{N} \times \mathbb{N}$ matrix.

$$\bar{\Gamma}_k = \begin{pmatrix} \dot{\alpha}_{11}^{(k)} & \dot{\alpha}_{12}^{(k)} & \cdots & \dot{\alpha}_{1\mathbb{N}}^{(k)} \\ \dot{\alpha}_{21}^{(k)} & \dot{\alpha}_{22}^{(k)} & \cdots & \dot{\alpha}_{2\mathbb{N}}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \dot{\alpha}_{\mathbb{N}1}^{(k)} & \dot{\alpha}_{\mathbb{N}2}^{(k)} & \cdots & \dot{\alpha}_{\mathbb{N}\mathbb{N}}^{(k)} \end{pmatrix}, \tag{19}$$

where $\dot{\alpha}_{ij}^{(k)}$ is an IVFN.

Step 2: Assess the weights of different maps.

As pointed in [37], how to assess the importance levels of different maps is the key point to construct ensemble IVFCMs. Next, we present an optimization model to quantify the weights ξ_k ($k = 1, 2, \ldots, K$) of different maps under constraint conditions Δ. Before representing the model, we firstly introduce the distance between two matrices in which all the elements are expressed with IVFNs on the basis of Definition 4.

Definition 5: For two $n \times q$ interval-valued fuzzy matrices A and B,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1q} \\ a_{21} & a_{22} & \cdots & a_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nq} \end{pmatrix}, \tag{20}$$

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nq} \end{pmatrix}, \tag{21}$$

$a_{ij} = [\underline{a}_{ij}, \overline{a}_{ij}]$ and $b_{ij} = [\underline{b}_{ij}, \overline{b}_{ij}]$ are IVFNs for all $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, q\}$. Then the distance measure between two matrices $\vartheta(A, B)$ is quantified as

$$\vartheta(A, B) = \sqrt{\frac{1}{2nq} \sum_{i=1}^{n} \sum_{j=1}^{q} \left((\underline{a}_{ij} - \underline{b}_{ij})^2 + (\overline{a}_{ij} - \overline{b}_{ij})^2\right)}. \tag{22}$$

Theorem 1: For two interval-valued fuzzy matrices A and B, $\vartheta(A, B)$ satisfies the following three properties: $0 \leq \vartheta(A, B) \leq 1$; $\vartheta(A, B) = 1$ if and only if $A = B$; $\vartheta(A, B) = \vartheta(B, A)$.

Proof: Definition 4 implies that this theorem holds. ∎

Following Definition 5, a model is proposed to assess the weights of different maps when establishing ensemble IVFCMs. Let $\dot{\alpha}_{ij}^{(k)} = [\underline{c}_{ij}^{(k)}, \overline{c}_{ij}^{(k)}]$, where $0 \leq \underline{c}_{ij}^{(k)} \leq \overline{c}_{ij}^{(k)} \leq 1$. Based on $\bar{\Gamma}_k$ and ξ_k, we get the following matrix:

$$E_k = \begin{pmatrix} e_{11}^{(k)} & e_{12}^{(k)} & \cdots & e_{1\mathbb{N}}^{(k)} \\ e_{21}^{(k)} & e_{22}^{(k)} & \cdots & e_{2\mathbb{N}}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ e_{\mathbb{N}1}^{(k)} & e_{\mathbb{N}2}^{(k)} & \cdots & e_{\mathbb{N}\mathbb{N}}^{(k)} \end{pmatrix}, \tag{23}$$

where $e_{ij}^{(k)} = [\xi_k \underline{c}_{ij}^{(k)}, 1 - \xi_k + \xi_k \overline{c}_{ij}^{(k)}]$. Depending on E_k ($k = 1, 2, \ldots, K$), we introduce a function as

$$f_k(\xi) = \sum_{k'=1, k' \neq k}^{K} \vartheta(E_k, E_{k'})^2. \tag{24}$$

It is clear that the smaller the value $f_k(\tau)$, the greater consensus between the kth IVFCM with others. In other words, the kth one occupies a more important proportion of ensemble IVFCMs. To maximize the effect of the kth one in ensemble IVFCMs, we can construct the following optimal model:

$$\begin{aligned} &\min f_k(\xi) \\ &s.t.\ \xi \in \Delta \\ &\qquad \sum_{k=1}^{K} \xi_k = 1 \\ &\qquad \xi_k \geq 0 \quad k = 1, 2, \ldots, K. \end{aligned} \tag{25}$$

Clearly, it is necessary to solve the following model to assess the weights of all maps.

$$\begin{aligned} &\min f_1(\xi) \\ &\min f_2(\xi) \\ &\vdots \\ &\min f_K(\xi) \\ &s.t.\ \xi \in \Delta \\ &\qquad \sum_{k=1}^{K} \xi_k = 1 \\ &\qquad \xi_k \geq 0 \quad k = 1, 2, \ldots, K. \end{aligned} \tag{26}$$

Here we consider all the objective functions having equal importance levels. Then equation (26) is equivalent to the following equation:

$$\begin{aligned} &\min f(\xi) \\ &s.t.\ \xi \in \Delta \\ &\quad \sum_{k=1}^{K} \xi_k = 1 \\ &\quad \xi_k \geq 0 \quad k = 1, 2, \ldots, K. \end{aligned} \tag{27}$$

where $f(\xi) = \sum_{k=1}^{K} f_k(\xi)$.

Solving the above optimal model, we can get the weights ξ.

Step 3: Establish ensemble IVFCMs.

Based on $\bar{\Gamma}_k$ and ξ_k $(k = 1, 2, \ldots, K)$, we determine the ensemble IVFCMs. Let Γ be the aggregated connection matrix as below:

$$\Gamma = \begin{pmatrix} \delta_{11} & \delta_{12} & \cdots & \delta_{1\mathbb{N}} \\ \delta_{21} & \delta_{22} & \cdots & \delta_{2\mathbb{N}} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{\mathbb{N}1} & \delta_{\mathbb{N}2} & \cdots & \delta_{\mathbb{N}\mathbb{N}} \end{pmatrix}, \tag{28}$$

where $\delta_{ij} = [\underline{g}_{ij}, \overline{g}_i j]$ is an IVFN for $i, j \in \{1, 2, \ldots, \mathbb{N}\}$.

Take δ_{11} for instance. Let $\eta_{1,k} = \xi_k \underline{c}_{11}^{(k)}$, $\eta_{2,k} = \xi_k(1-\overline{c}_{11}^{(k)})$, $\bar{\eta}_{H,k} = 1 - \xi_k$, $\tilde{\eta}_{H,k} = \xi_k(\overline{c}_{11}^{(k)} - \underline{c}_{11}^{(k)})$, $\eta_{H,k} = \bar{m}_{H,k} + \tilde{\eta}_{H,k}$ $(k = 1, 2, \ldots, K)$, $\eta_{1,I(1)} = \eta_{1,1}$, $\eta_{2,I(1)} = \eta_{2,1}$, $\eta_{H,I(1)} = \eta_{H,1}$, $\bar{\eta}_{H,I(1)} = \bar{m}_{H,1}$ and $\tilde{\eta}_{H,I(1)} = \tilde{\eta}_{H,1}$. Do the following mathematical operations as equations (29)-(34),

$$\begin{aligned} \eta_{1,I(k+1)} &= R_{I(k+1)}[\eta_{1,I(k)}\eta_{1,k+1} \\ &\quad + \eta_{H,I(k)\eta_{1,k+1}} + \eta_{1,I(k)}\eta_{H,k+1}], \end{aligned} \tag{29}$$

$$\begin{aligned} \eta_{2,I(k+1)} &= R_{I(k+1)}[\eta_{2,I(k)}\eta_{2,k+1} \\ &\quad + \eta_{H,I(k)\eta_{2,k+1}} + \eta_{2,I(k)}\eta_{H,k+1}], \end{aligned} \tag{30}$$

$$\eta_{H,I(k)} = \bar{\eta}_{H,I(k)} + \tilde{\eta}_{H,I(k)}, \tag{31}$$

$$\begin{aligned} \tilde{\eta}_{H,I(k+1)} &= R_{I(k+1)}[\tilde{\eta}_{H,I(k)}\tilde{\eta}_{H,k+1} \\ &\quad + \bar{\eta}_{H,I(k)}\tilde{\eta}_{H,k+1} + \tilde{\eta}_{H,I(k)}\bar{\eta}_{H,k+1}], \end{aligned} \tag{32}$$

$$\bar{\eta}_{H,I(k+1)} = R_{I(k+1)}[\bar{\eta}_{H,I(k)}\bar{\eta}_{H,k+1}], \tag{33}$$

$$R_{I(k+1)} = \frac{1}{1 - \eta_{1,I(k)}\eta_{2,k+1} - \eta_{2,I(k)}\eta_{1,k+1}}, \tag{34}$$

where $\eta_{1,I(k+1)} + \eta_{2,I(k+1)}) + \eta_{H,I(k+1)} = 1$. Perform the recurrent procedures until $k = K - 1$. Finally, δ_{11} is defined as follows:

$$\delta_{11} = [\frac{\eta_{1,I(K)}}{1 - \bar{\eta}_{H,I(K)}}, 1 - \frac{\eta_{2,I(K)}}{1 - \bar{\eta}_{H,I(K)}}]. \tag{35}$$

Same as δ_{11}, we get Γ.

Remark 5: As mentioned above, the theory of ensemble IVFCMs has been completely established via the ER theory within the framework of IVFSs. The theory not only provides a method to aggregate different maps but also shows their importance levels. What's more, how to assess the weights of different maps have been solved by a multi-objective optimization model.

D. EVALUATION OF ENSEMBLE IVFCMS

Concerning the topic of establishing ensemble cognitive maps or high-order maps, there are a number of aggregation methods are proposed in succession from different perspectives. Roughly speaking, existing methods mainly cover two classes. The first class just considers non-default connections when quantifying the aggregated connections, while the second class fully takes into account the influences from all maps. As discussed in [19] and [24]–[26], above two classes are chosen and utilized in view of the actual demands, that is to say, there are not complete and quantitative evaluation methods in establishing ensemble maps or high-order maps.

As described in this Section, the proposed method in this paper belongs to the second class, which is different from ECM and FGCM [18], [19].

IV. EXPERIMENTS AND ANALYSIS

In this section, we utilize three examples to validate the performance of the proposed ensemble IVFCMs and have a detailed analysis.

A. SOCIO-ECONOMIC PROBLEM MODELING

Here we consider a socio-economic inference model as discussed in [19]. This model constitutes Population (N_1), Crime (N_2), Economic condition (N_3), Poverty (N_4), and Unemployment (N_5) as partial nodes of three IVFCMs shown as Figures 2(a)–2(c) and their connection matrices are defined as equations (36)–(38).

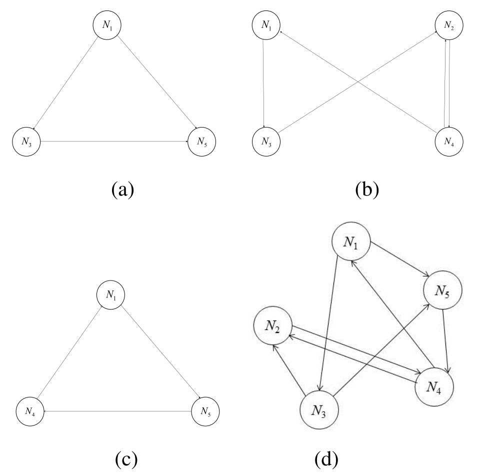

FIGURE 2. The respective IVFCMs and ensemble IVFCMs. (a) IVFCM 1. (b) IVFCM 2. (c) IVFCM 3. (d) Ensemble IVFCMs.

$$\Gamma_1 = \begin{pmatrix} & N_1 & N_3 & N_5 \\ N_1 & 0 & [0.2, 0.4] & [0.5, 0.8] \\ N_3 & 0 & 0 & [0.2, 0.5] \\ N_5 & 0 & 0 & 0 \end{pmatrix}, \tag{36}$$

$$\Gamma_2 = \begin{pmatrix} & N_1 & N_2 & N_3 & N_4 \\ N_1 & 0 & 0 & [0.3, 0.5] & 0 \\ N_2 & 0 & 0 & 0 & [0.1, 0.6] \\ N_3 & 0 & [0.1, 0.3] & 0 & 0 \\ N_4 & [0.3, 0.5] & [0.6, 0.9] & 0 & 0 \end{pmatrix}, \tag{37}$$

$$\Gamma_3 = \begin{pmatrix} & N_1 & N_4 & N_5 \\ N_1 & 0 & 0 & [0.6, 0.8] \\ N_4 & [0.1, 0.4] & 0 & 0 \\ N_5 & 0 & [0.6, 0.7] & 0 \end{pmatrix}. \quad (38)$$

Assume that the constraint conditions of three maps are denoted by $\Delta = \{\xi_1 - \xi_2 > 0.1,\ \xi_3 \geq 0.3,\ \xi_2 + \xi_3^2 \leq 0.4\}$. Next, we present the ensemble IVFCMs via the three maps.

Step 1: On the basis of Γ_1, Γ_2 and Γ_3, we get the augmented connection matrices as equations (39)-(41).

$$\bar{\Gamma}_1 = \begin{pmatrix} [0,0] & [0,0] & [0.2,0.4] & [0,0] & [0.5,0.8] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0.2,0.5] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix}, \quad (39)$$

$$\bar{\Gamma}_2 = \begin{pmatrix} [0,0] & [0,0] & [0.3,0.5] & [0,0] & [x0,0] \\ [0,0] & [0,0] & [0,0] & [0.1,0.6] & [0,0] \\ [0,0] & [0.1,0.3] & [0,0] & [0,0] & [0,0] \\ [0.3,0.5] & [0.6,0.9] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix}, \quad (40)$$

$$\bar{\Gamma}_3 = \begin{pmatrix} [0,0] & [0,0] & [0,0] & [0,0] & [0.6,0.8] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0.1,0.4] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0.6,0.7] & [0,0] \end{pmatrix}. \quad (41)$$

Step 2: Depending on $\bar{\Gamma}_1$, $\bar{\Gamma}_2$ and $\bar{\Gamma}_3$ and the constraint conditions Δ, a optimization model is established as equation (42),

$$\begin{aligned} & \min f(\xi) \\ & s.t.\ \xi_1 - \xi_2 \geq 0.1, \\ & \quad \xi_3 \geq 0.3, \\ & \quad \xi_2 + \xi_3^2 \leq 0.4, \\ & \quad \xi_1 + \xi_2 + \xi_3 = 1, \\ & \quad \xi_1 \geq 0, \\ & \quad \xi_2 \geq 0, \\ & \quad \xi_3 \geq 0, \end{aligned} \quad (42)$$

where $f(\xi) = 1.8384\xi_1^2 + 1.7376\xi_2^2 + 1.8576\xi_3^2 - 1.6608\xi_1\xi_2 - 1.7872\xi_1\xi_3 - 1.6424\xi_2\xi_3$. Solving this equation,

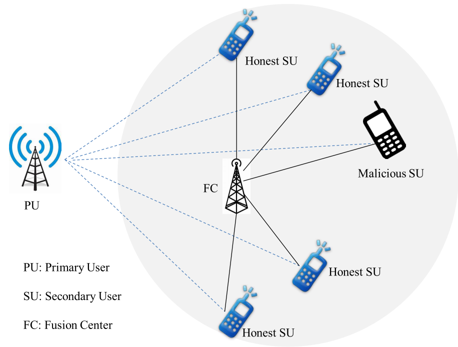

FIGURE 3. System model of cooperative sensing.

we get

$$\xi = [0.3843 \quad 0.2843 \quad 0.3313]. \quad (43)$$

Step 3: Based on the obtained $\bar{\Gamma}_1$, $\bar{\Gamma}_2$, $\bar{\Gamma}_3$ and ξ, we get the aggregated connection matrix as equation (44), as shown at the top of the next page.

Step 4: Let $\theta(0) = [[0.9, 1]\ [0, 0]\ [0, 0]\ [0, 0]\ [0, 0]]$ be the initial concept values of the ensemble IVFCMs, and $\frac{1}{1+e^{-x}}$ be the threshold function. If $|\overline{\theta}_i(t+1) - \overline{\theta}_i(t)| < 10^{-4}$ and $|\underline{\theta}_i(t+1) - \underline{\theta}_i(t)| < 10^{-4}$ $(i = 1, 2, 3, 4, 5)$ $(\theta_i(t) = [\underline{\theta}_i(t), \overline{\theta}_i(t)]$ and $\theta_i(t+1) = [\underline{\theta}_i(t+1), \overline{\theta}_i(t+1)])$, we think the ensemble map reaches the steady states. The results of the reasoning process are shown as Table 1 and the steady state is equation (45), as shown at the top of the next page.

Remark 6: As discussed in this example, the IVFN $[0, 0]$ has been employed to define the default connections and the augmented connection matrices. What's more, the multi-objective optimization model has opened a new theoretical analysis perspective for assessing the weights of different maps, which is the basis of establishing ensemble IVFCMs.

B. MODELING OF COOPERATIVE SENSING IN COGNITIVE RADIO

Cognitive radio [38] has been an effective tool to strengthen spectrum applications. Cooperative sensing plays an important role in cognitive radio which is shown as Figure 3. Different SUs have mutual relations in cognitive radio network. Meanwhile, it should be noted that there exist both honest and malicious SUs in the whole sensing system. Some SUs of a local sensing area forms a cluster in which the variation

TABLE 1. Reasoning using the ensemble IVFCMs.

t	$\theta_1(t)$	$\theta_2(t)$	$\theta_3(t)$	$\theta_4(t)$	$\theta_5(t)$
0	[0.9,1]	[0,0]	[0,0]	[0,0]	[0,0]
1	[0.7109,0.7311]	[0.5000,0.5000]	[0.5297,0.5519]	[0.5000,0.5000]	[0.5885,0.6215]
2	[0.6848,0.6964]	[0.6422,0.6489]	[0.6561,0.6719]	[0.6479,0.6575]	[0.7265,0.7477]
3	[0.6801,0.6953]	[0.6808,0.6889]	[0.6818,0.6964]	[0.6886,0.7001]	[0.7522,0.7713]
4	[0.6807,0.6968]	[0.6908,0.6992]	[0.6870,0.7014]	[0.6987,0.7107]	[0.7569,0.7756]
5	[0.6812,0.6975]	[0.6933,0.7018]	[0.6881,0.7025]	[0.7011,0.7133]	[0.7579,0.7765]
6	[0.6814,0.6978]	[0.6939,0.7024]	[0.6883,0.7027]	[0.7017,0.7139]	[0.7581,0.7767]
7	[0.6814,0.6978]	[0.6940,0.7026]	[0.6884,0.7028]	[0.7018,0.7140]	[0.7582,0.7768]
8	[0.6815,0.6979]	[0.6941,0.7026]	[0.6884,0.7028]	[0.7019,0.7141]	[0.7582,0.7768]
9	[0.6815,0.6979]	[0.6941,0.7026]	[0.6884,0.7028]	[0.7019,0.7141]	[0.7582,0.7768]

$$\Gamma = \begin{pmatrix} [0,0] & [0,0] & [0.1320, 0.2316] & [0,0] & [0.3973, 0.5508] \\ [0,0] & [0,0] & [0,0] & [0.0170, 0.1020] & [0,0] \\ [0,0] & [0.0170, 0.0510] & [0,0] & [0,0] & [0.0553, 0.1382] \\ [0.0893, 0.1985] & [0.1161, 0.1741] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0.1474, 0.1720] & [0,0] \end{pmatrix}. \quad (44)$$

$$\theta(8) = [[0.6815, 0.6979]\ [0.6941, 0.7026]\ [0.6884, 0.7028]\ [0.7019, 0.7141]\ [0.7582, 0.7768]]. \quad (45)$$

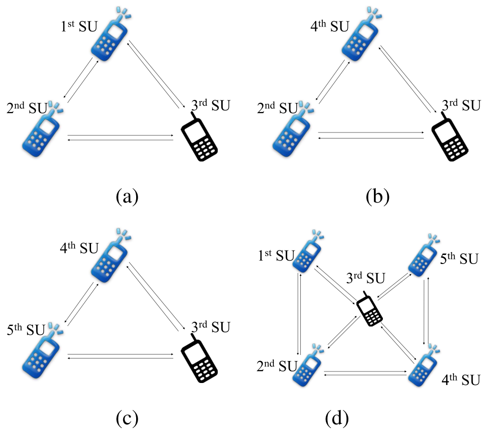

FIGURE 4. **The respective IVFCMs and ensemble IVFCMs. (a) IVFCM 1. (b) IVFCM 2. (c) IVFCM 3. (d) Ensemble IVFCMs.**

tendency of each SU and the connections among them can be understood as an IVFCM. Here we consider a cognitive radio network including five SUs (SU_i, $i = 1, 2, 3, 4, 5$) belonging to three local sensing areas. The three areas respectively constitute three clusters C_1 (SU_1, SU_2, SU_3), C_2 (SU_2, SU_3, SU_4) and C_3 (SU_3, SU_4, SU_5). Based on the three clusters, we construct the following three IVFCMs as Figure 4 and equations (46)-(48).

$$\Gamma_1 = \begin{pmatrix} & N_1 & N_2 & N_3 \\ N_1 & 0 & [0.8, 0.9] & [0.1, 0.2] \\ N_2 & [0.8, 1.0] & 0 & [0.2, 0.3] \\ N_3 & [0.7, 0.8] & [0.8, 0.9] & 0 \end{pmatrix}, \quad (46)$$

$$\Gamma_2 = \begin{pmatrix} & N_2 & N_3 & N_4 \\ N_2 & 0 & [0, 0.1] & [0.6, 0.9] \\ N_3 & [0.7, 0.8] & 0 & [0.6, 0.7] \\ N_4 & [0.8, 0.9] & [0.1, 0.2] & 0 \end{pmatrix}, \quad (47)$$

$$\Gamma_3 = \begin{pmatrix} & N_3 & N_4 & N_5 \\ N_3 & 0 & [0.6, 0.7] & [0.9, 1.0] \\ N_4 & [0.1, 0.3] & 0 & [0.8, 0.9] \\ N_5 & [0.1, 0.2] & [0.7, 0.7] & 0 \end{pmatrix}. \quad (48)$$

There are five SUs including one malicious one (the 3rd SU) and other four honest ones. Here three IVFCMs are with identical weights, i.e., $\xi = [\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}]$. Let $\frac{1}{1+\exp(-x)}$ be the threshold function. From the PU, we get the initial state values of five SUs are as below:

$$\theta(0) = \big[[0.8, 0.9]\ [0.7, 0.8]\ [0, 0.1]\ [0.6, 0.8]\ [0.8, 0.9]\big]. \quad (49)$$

The inference process of ensemble IVFCMs is as follows:

Step 1: From Γ_1, Γ_2 and Γ_3, we get the following augmented connection matrices:

$$\bar{\Gamma}_1 = \begin{pmatrix} [0,0] & [0.8, 0.9] & [0.1, 0.2] & [0,0] & [0,0] \\ [0.8, 1.0] & [0,0] & [0.2, 0.3] & [0,0] & [0,0] \\ [0.7, 0.8] & [0.8, 0.9] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix}, \quad (50)$$

$$\bar{\Gamma}_2 = \begin{pmatrix} [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0, 0.1] & [0.6, 0.9] & [0,0] \\ [0,0] & [0.7, 0.8] & [0,0] & [0.6, 0.7] & [0,0] \\ [0,0] & [0.8, 0.9] & [0.1, 0.2] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \end{pmatrix}, \quad (51)$$

$$\bar{\Gamma}_3 = \begin{pmatrix} [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0,0] & [0,0] \\ [0,0] & [0,0] & [0,0] & [0.6, 0.7] & [0.9, 1.0] \\ [0,0] & [0,0] & [0.1, 0.3] & [0,0] & [0.8, 0.9] \\ [0,0] & [0,0] & [0.1, 0.2] & [0.7, 0.7] & [0,0] \end{pmatrix}. \quad (52)$$

Step 2: Based on ξ, $\bar{\Gamma}_1$, $\bar{\Gamma}_2$ and $\bar{\Gamma}_3$, we get the aggregated connection matrix as equation (53), as shown at the bottom of the next page.

Step 3: Determine the steady state of the ensemble IVFCMs. If $|\overline{\theta}_i(t+1) - \overline{\theta}_i(t)| < 10^{-4}$ and $|\underline{\theta}_i(t+1) - \underline{\theta}_i(t)| < 10^{-4}$ $(i = 1, 2, 3, 4, 5)$ $(\theta_i(t) = [\underline{\theta}_i(t), \overline{\theta}_i(t)]$ and $\theta_i(t+1) = [\underline{\theta}_i(t+1), \overline{\theta}_i(t+1)])$, we think the ensemble map reaches the steady states. The results of the reasoning process are shown as Table 2 and the steady state is equation (54), as shown at the bottom of the next page.

Remark 7: In order to improve spectrum utilization, it is of great importance to consider the connections among different SUs and to detect malicious SUs in cognitive radio network. The ensemble map can more accurately describe the connections among different SUs, and the the variation tendency of concept values is helpful to detect malicious SUs.

C. MONITORING SYSTEM OF AIR QUALITY

Here we consider a problem of monitoring system of air quality in *Gaoming District, Foshan, Guangdong Province, China.* In a $10km \times 10km$ area, there have been three stations as Figure 5. This station is with four sensors that can be configured to measure: *Carbon monoxide* (N_1), *Nitrogen dioxide* (N_2), *Sulfur dioxide* (N_3) and *Ozone* (N_4). From the historical data, we get three IVFCMs of three stations and

TABLE 2. Reasoning using the ensemble IVFCMs.

t	$\theta_1(t)$	$\theta_2(t)$	$\theta_3(t)$	$\theta_4(t)$	$\theta_5(t)$
0	[0.8,0.9]	[0.7,0.8]	[0,0.1]	[0.6,0.8]	[0.8,0.9]
1	[0.7320,0.7511]	[0.7528,0.7709]	[0.5508,0.5754]	[0.7228,0.7610]	[0.7302,0.7494]
2	[0.7360,0.7438]	[0.8022,0.8070]	[0.6776,0.6884]	[0.7773,0.7870]	[0.7418,0.7455]
3	[0.7417,0.7498]	[0.8197,0.8253]	[0.7034,0.7144]	[0.7922,0.8020]	[0.7499,0.7538]
4	[0.7446,0.7528]	[0.8250,0.8307]	[0.7095,0.7205]	[0.7972,0.8069]	[0.7534,0.7573]
5	[0.7456,0.7539]	[0.8265,0.8322]	[0.7109,0.7220]	[0.7986,0.8084]	[0.7546,0.7585]
6	[0.7459,0.7542]	[0.8269,0.8326]	[0.7113,0.7224]	[0.7991,0.8088]	[0.7549,0.7588]
7	[0.7460,0.7543]	[0.8270,0.8327]	[0.7114,0.7225]	[0.7992,0.8089]	[0.7550,0.7589]
8	[0.7460,0.7543]	[0.8271,0.8327]	[0.7114,0.7225]	[0.7992,0.8090]	[0.7550,0.7590]
9	[0.7460,0.7543]	[0.8271,0.8327]	[0.7114,0.7225]	[0.7992,0.8090]	[0.7550,0.7590]

FIGURE 5. A multi-parameter air quality station.

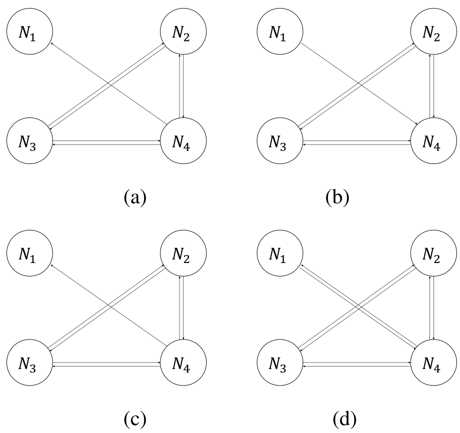

FIGURE 6. The respective IVFCMs and ensemble IVFCMs. (a) IVFCM 1. (b) IVFCM 2. (c) IVFCM 3. (d) Ensemble IVFCMs.

their respective connection matrices are shown as equations (55)-(57) and Figure 6.

$$\Gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & [0.4, 0.5] & [0.3, 0.4] \\ 0 & [0.3, 0.5] & 0 & [0.2, 0.3] \\ [0.2, 0.3] & [0.1, 0.2] & [0.3, 0.4] & 0 \end{pmatrix}, \tag{55}$$

$$\Gamma_2 = \begin{pmatrix} 0 & 0 & 0 & [0, 0.1] \\ 0 & 0 & [0.5, 0.6] & [0.4, 0.4] \\ 0 & [0.3, 0.5] & 0 & [0.2, 0.3] \\ 0 & [0.2, 0.3] & [0.2, 0.3] & 0 \end{pmatrix}, \tag{56}$$

$$\Gamma_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & [0.4, 0.5] & [0.2, 0.3] \\ 0 & [0.5, 0.6] & 0 & [0.3, 0.4] \\ [0, 0.1] & [0.1, 0.3] & [0.2, 0.4] & 0 \end{pmatrix}. \tag{57}$$

The inference process of ensemble IVFCMs is as follows:

Step 1: From Γ_1, Γ_2 and Γ_3, we get the following augmented connection matrices:

$$\bar{\Gamma}_1 = \begin{pmatrix} [0, 0] & [0, 0] & [0, 0] & [0, 0] \\ [0, 0] & [0, 0] & [0.4, 0.5] & [0.3, 0.4] \\ [0, 0] & [0.3, 0.5] & [0, 0] & [0.2, 0.3] \\ [0.2, 0.3] & [0.1, 0.2] & [0.3, 0.4] & [0, 0] \end{pmatrix}, \tag{58}$$

$$\bar{\Gamma}_2 = \begin{pmatrix} [0, 0] & [0, 0] & [0, 0] & [0, 0.1] \\ [0, 0] & [0, 0] & [0.5, 0.6] & [0.4, 0.4] \\ [0, 0] & [0.3, 0.5] & [0, 0] & [0.2, 0.3] \\ [0, 0] & [0.2, 0.3] & [0.2, 0.3] & [0, 0] \end{pmatrix}, \tag{59}$$

$$\bar{\Gamma}_3 = \begin{pmatrix} [0, 0] & [0, 0] & [0, 0] & [0, 0] \\ [0, 0] & [0, 0] & [0.4, 0.5] & [0.2, 0.3] \\ [0, 0] & [0.5, 0.6] & [0, 0] & [0.3, 0.4] \\ [0, 0.1] & [0.1, 0.3] & [0.2, 0.4] & [0, 0] \end{pmatrix}. \tag{60}$$

Step 2: In the monitoring process, the three stations are with equal weights, i.e., $\xi = [\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}]$. Based on ξ, $\bar{\Gamma}_1$, $\bar{\Gamma}_2$ and $\bar{\Gamma}_3$, we get the aggregated connection matrix as equation (61), as shown at the top of the next page.

$$\Gamma = \begin{pmatrix} [0, 0] & [0.2133, 0.2400] & [0.0216, 0.0432] & [0, 0] & [0, 0] \\ [0.2133, 0.2667] & [0, 0] & [0.0466, 0.0920] & [0.1500, 0.2250] & [0, 0] \\ [0.1806, 0.2065] & [0.5082, 0.5644] & [0, 0] & [0.3830, 0.4375] & [0.2483, 0.2759] \\ [0, 0] & [0.2133, 0.2400] & [0.0486, 0.1171 & [0, 0] & [0.2133, 0.2400] \\ [0, 0] & [0, 0] & [0.0216, 0.0432] & [0.1806, 0.1806] & [0, 0] \end{pmatrix}. \tag{53}$$

$$\theta(8) = [[0.7460, 0.7543]\ [0.8271, 0.8327]\ [0.7114, 0.7225]\ [0.7992, 0.8090]\ [0.7550, 0.7590]]. \tag{54}$$

$$\Gamma = \begin{pmatrix} [0, 0] & [0, 0] & [0, 0] & [0, 0.0211] \\ [0, 0] & [0, 0] & [0.4399, 0.5192] & [0.2753, 0.3263] \\ [0, 0] & [0.3731, 0.5058] & [0, 0] & [0.2069, 0.2828] \\ [0.0466, 0.0920] & [0.1107, 0.2081] & [0.2122, 0.3140] & [0, 0] \end{pmatrix}. \quad (61)$$

Step 3: Determine the steady state of the ensemble IVFCMs. If $|\overline{\theta}_i(t+1) - \overline{\theta}_i(t)| < 10^{-4}$ and $|\underline{\theta}_i(t+1) - \underline{\theta}_i(t)| < 10^{-4}$ $(i = 1, 2, 3, 4, 5)$ $(\theta_i(t) = [\underline{\theta}_i(t), \overline{\theta}_i(t)]$ and $\theta_i(t+1) = [\underline{\theta}_i(t+1), \overline{\theta}_i(t+1)])$, we think the ensemble map reaches the steady states. Let tanh be the threshold function. Here $\theta(0) = \big[[0.6, 0.7]\ [0.8, 0.9]\ [0.5, 0.6]\ [0.7, 0.8]\big]$. The results of the reasoning process derived from three respective IVFCMs and ensemble IVFCMs are shown as Table 3 and Table 4.

Remark 8: Up to now, air quality has been a serious issue for everyone, especially in some developing countries. However, some sensors or certain station may lose efficacy or generate abnormal data. To avoid the negative influence from those invalid sensors or stations, it is necessary to synthetically consider all the stations in a local area. As illustrated Table 4, the ensemble IVFCMs is helpful to detect abnormal sensors or station by comparing the respective ones and the ensemble one.

D. ANALYSIS

From the above three examples, we can get the following interesting results:

- As indicated in the above three examples, the problems of quantifying both default connections and augmented connection matrices have been successfully solved. The proposed method has delivered a new vision to assess the importance levels of different maps when establishing ensemble IVFCMs.

TABLE 3. Reasoning using the ensemble IVFCMs.

t	$\theta_1(t)$	$\theta_2(t)$	$\theta_3(t)$	$\theta_4(t)$
0	[0.6000,0.7000]	[0.8000,0.9000]	[0.5000,0.6000]	[0.7000,0.8000]
1	[0.5813,0.6437]	[0.8332,0.8614]	[0.8128,0.8441]	[0.8059,0.8382]
2	[0.5753,0.6155]	[0.8698,0.8937]	[0.8952,0.9104]	[0.8593,0.8746]
3	[0.5746,0.6009]	[0.8868,0.9096]	[0.9128,0.9263]	[0.8754,0.8904]
4	[0.5661,0.5925]	[0.8922,0.9144]	[0.9174,0.9304]	[0.8811,0.8957]
5	[0.5605,0.5873]	[0.8937,0.9157]	[0.9187,0.9316]	[0.8829,0.8973]
6	[0.5570,0.5840]	[0.8941,0.9161]	[0.9191,0.9319]	[0.8834,0.8978]
7	[0.5547,0.5819]	[0.8943,0.9162]	[0.9192,0.9320]	[0.8835,0.8979]
8	[0.5532,0.5805]	[0.8943,0.9162]	[0.9192,0.9320]	[0.8835,0.8979]
9	[0.5523,0.5795]	[0.8943,0.9162]	[0.9192,0.9321]	[0.8835,0.8979]
10	[0.5516,0.5789]	[0.8943,0.9162]	[0.9192,0.9321]	[0.8835,0.8979]
11	[0.5512,0.5785]	[0.8943,0.9162]	[0.9192,0.9321]	[0.8835,0.8979]
12	[0.5509,0.5782]	[0.8943,0.9162]	[0.9192,0.9321]	[0.8835,0.8979]
13	[0.5507,0.5780]	[0.8943,0.9162]	[0.9192,0.9321]	[0.8835,0.8979]
14	[0.5506,0.5779]	[0.8943,0.9162]	[0.9192,0.9321]	[0.8835,0.8979]
15	[0.5505,0.5778]	[0.8943,0.9162]	[0.9192,0.9321]	[0.8835,0.8979]
16	[0.5504,0.5778]	[0.8943,0.9162]	[0.9192,0.9321]	[0.8835,0.8979]
17	[0.5504,0.5777]	[0.8943,0.9162]	[0.9192,0.9321]	[0.8835,0.8979]
18	[0.5504,0.5777]	[0.8943,0.9162]	[0.9192,0.9321]	[0.8835,0.8979]

TABLE 4. Monitoring system of air quality: Summary of steady state results and iterative times using the ensemble IVFCMs.

Models	Concepts	Initial State	Steady state	Iterative times
IVFCM 1	N_1	$[0.6, 0.7]$	[0.74480.7821]	9
	N_2	$[0.8, 0.9]$	[0.87940.9152]	
	N_3	$[0.5, 0.6]$	[0.92860.9400]	
	N_4	$[0.7, 0.8]$	[0.89570.9126]	
IVFCM 2	N_1	$[0.6, 0.7]$	[0.04810.0530]	496
	N_2	$[0.8, 0.9]$	[0.90160.9311]	
	N_3	$[0.5, 0.6]$	[0.93060.9417]	
	N_4	$[0.7, 0.8]$	[0.90170.9177]	
IVFCM 3	N_1	$[0.6, 0.7]$	[0.53290.5942]	15
	N_2	$[0.8, 0.9]$	[0.91930.9429]	
	N_3	$[0.5, 0.6]$	[0.91890.9427]	
	N_4	$[0.7, 0.8]$	[0.89890.9152]	
Ensemble IVFCMs	N_1	$[0.6, 0.7]$	$[0.5504, 0.5777]$	17
	N_2	$[0.8, 0.9]$	$[0.8943, 0.9162]$	
	N_3	$[0.5, 0.6]$	$[0.9192, 0.9321]$	
	N_4	$[0.7, 0.8]$	$[0.8835, 0.8979]$	

- As shown in equation (44), equation (53), equation (61), Table 1, Table 2 and Table 3, both the connection matrix and the state values of the ensemble IVFCMs reflect the causal relationship among concepts and their variation tendency.
- As can be observed from Table 4, the ensemble IVFCMs can more accurately reflect the connections between concepts than the respective maps. From the variation tendency of the concepts' state, the abnormal concept or stations can be detected by comparing the respective IVFCMs and the ensemble one.

V. CONCLUSION

We have studied the challenging problem of establishing ensemble IVFCMs in this study, which includes three key points, i.e., quantifying the default connections and the augmented connection matrices, assessing the weights of different maps, and aggregating all the maps. Firstly, we have redefined the default connections and presented the augmented connection matrices as the basis of establishing ensemble IVFCMs. Then, a multi-objective model has been presented for assessing the importance levels of different maps. Finally, we have proposed a scheme based on ER theory to aggregate a number of maps with different weights. The results of three examples indicated that the ensemble IVFCMs provides an effective way to model complex systems with uncertainty from numerous fields, such as social systems, medical decision making and supplier selection.

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