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t/t-Diagnosability and t/k-Diagnosability for Augmented Cube Networks

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ABSTRACT Diagnosability is an important parameter for evaluating the reliability of multiprocessor systems. t/t-diagnosability and t/k-diagnosability are both new indexes for measuring the reliability of a system. An n-dimensional augmented cube network (AQ_n) is a variant of an n-dimensional hypercube network. In this paper, we first prove that an n-dimensional augmented cube network is $(4n-8)/(4n-8)$-diagnosable, which implies that the t/t-diagnosability of AQ_n is approximately two times larger than its classical t-diagnosability. Some useful properties of AQ_n not reported by previous studies are proposed. By employing these new properties, we prove that AQ_n is t/k-diagnosable, which implies that the t/k-diagnosability is approximately $(k+1)$ times larger than $2n-1$, i.e., the t-diagnosability of AQ_n, where $t = 2(k+1)n - ((3(k+1)(k+2))/2) + 1$, $k \leqslant (4n/9) - (13/9)$, and $n > 5$.

INDEX TERMS System-level diagnosis, t/t-diagnosability, t/k-diagnosability, PMC model, augmented cube networks.

I. INTRODUCTION

It is an indisputable fact that a multiprocessor system incorporates a very large number of processors (or nodes/units) due to the development of integration technology. With the increase in the number of processors in a multiprocessor system, system designers should consider some novel issues caused by this increase. One of the most important among them is the reliability of the system. Naturally, the faulty nodes in a system are an important reason for the decrease in system reliability. To maintain the reliability of a system, the system should be designed such that it contains automated fault tolerant features. Fault tolerance mainly involves two steps. The first step is called fault identification, in which the faulty processors are diagnosed. The second step is called system configuration [1], in which either faulty processors (previously diagnosed in the first step) are replaced with additional processors or other good processors in the system are redistributed to run those tasks that were running on the faulty processors.

In automated fault diagnosis, there exist two diagnosis models. One of them is called the logic-circuit-level model, in which each unit of a system can be tested solely by utilizing precalculated test data. The shortcoming of this model is that it requires a large amount of data. The other model, called the system-level diagnosis model or the Preparata, Metze, and Chien (PMC) model, was first proposed by [2]. In the PMC model, for a system consisting of n nodes, its diagnostic graph can be represented by a directed graph $G(V, E)$, an edge $(i, j) \in E$ denotes that node u tests node v, and each edge of $G(V, E)$ is endowed with a testing result of either 1 or 0. The collection of all testing results is called a syndrome, represented by σ. $\sigma(i, j)$ denotes the result of node i testing node j. The PMC model assumes that if node i judges node j to be faulty (respectively, fault-free), then $\sigma(i, j) = 1$ (respectively, $\sigma(i, j) = 0$); it also assumes that the result of i testing j is reliable if and only if node i is fault-free. This test invalidation rule is shown in Table 1. Many results of the automated fault diagnosis of multiprocessor systems under the PMC model have been obtained (see [3]–[10]).

TABLE 1. Invalidation rule of the PMC model.

Testing node	Tested node	Test outcome
Fault-free	Fault-free	0
Fault-free	Faulty	1
Faulty	Fault-free	0 or 1
Faulty	Faulty	0 or 1

A complete diagnosis refers to one in which all faulty nodes can be identified. A correct diagnosis refers to one in which no fault-free node can be diagnosed as faulty. In automated fault diagnosis, a complete and correct diagnosis is undeniably an ideal diagnosis for a system. According to the test results obtained by the testing nodes, the diagnosis automatically implemented by a t-diagnosable system is a complete and correct diagnosis. However, for most t-diagnosable systems with N nodes, the value of the diagnosability t, which is the maximal number of faulty nodes that a system can guarantee to diagnose, is much smaller than N. In other words, if the number of faulty nodes exceeds the diagnosability of a system, then the diagnosis implemented by the t-diagnosable system may not be effective. To resolve this issue, many methods have been suggested over the past few decades [25], [32]–[35]. One of them is to increase the diagnosability of a system by allowing some nodes to be incorrectly identified. For example, [25] introduced the t/t-diagnosability; its corresponding diagnosis is called a t/t-diagnosis. A system is said to be t/t-diagnosable if it can locate a t-node set containing all faulty nodes in the system provided that the number of faulty nodes in the system is no more than t [25]. Reference [14] extended the results presented in [25] and proved that at most one fault-free node can be identified as faulty.

It is worth mentioning that [1] introduced another diagnosability, called t/k-diagnosability; its corresponding diagnosis is called a t/k-diagnosis ($1 \leqslant k \leqslant t$). Different from the t/t-diagnosis, the t/k-diagnosis allows at most k nodes to be diagnosed as faulty, which results in the t/k-diagnosability of most regular interconnection networks being larger than the t/t-diagnosability. For example, the t/k-diagnosability of an n-dimensional hypercube network, denoted by Q_n, is $4n-9$, $n \geqslant 4, k = 3$ [1], which is approximately two times larger than $2n-2$, the t/t-diagnosability of Q_n. The latter is approximately two times larger than n, the classical t-diagnosability of Q_n.

Several topologies can be employed to model multiprocessor systems. Among them, the hypercube network is one of the most useful topologies. The reason for this is that the hypercube network has many advantages, such as symmetry, regularity, recursion, having a simple and optimal routing algorithm, and so on. The hypercube network has been used as the topology structure in many parallel processor systems (such as CM-2, ip-SC-806, and nCUBE) and may be the base structure of nanometer computers in the future [36]. However, Q_n is not the best topology. Some variants of Q_n have been introduced, for example, the locally twisted cube [18], the BC graph [26], augmented cubes [29], the exchanged hypercube [37], the crossed cube [17], and so on. As a variant of Q_n, augmented cubes have been widely studied. For example, Hong and Hsieh [38] discussed the strong diagnosability and conditional diagnosability of augmented cubes; the distinguishing number of augmented cubes was studied by Chan [39]; and Hsieh and Shiu [40] presented the cycle embedding of augmented cubes. In addition, results corresponding to the connectivity and panconnectivity of augmented cubes can be found in [41]–[43]. The reason that augmented cubes attract considerable attention may be that they have some properties superior to those of Q_n; for example, the diameter of an augmented cube is approximately half of n, i.e., the diameter of Q_n, and an augmented cube possesses a few embedding properties that are not possessed by Q_n.

In the paper, we will discuss the t/t-diagnosability and t/k-diagnosability of augmented cubes, which have not been studied in previous research. The remainder of this paper is as follows: After introducing some novel properties of the augmented cube network, we propose and prove the t/t-diagnosability of an augmented cube network in section 3. In section 4, we first derive a low bound of the cardinality of the neighbors of the k-node set in augmented cubes and then present the t/k-diagnosability of an augmented cube network. In section 5, we draw our conclusions and present some final remarks.

II. PRELIMINARIES

The fault diagnostic graph of a multiprocessor system is often represented by an undirected graph $G(V, E)$, where the vertices in V and the edges in E correspond to processors and communication links in the network, respectively. For $V_1 \subseteq V$, $G - V_1$ is a subgraph of G induced by all the nodes in V but not in V_1, and $G[V_1]$ is a subgraph of G induced by all the nodes in V_1. $|G|$ or $|V|$ is the number of nodes contained in G. The length of a path $P(u, v)$ in G is the number of edges in the path $P(u, v)$. The neighborhood set of a node v is $N_G(v) = \{u|(u, v) \in E\}$ (in brief, $N(v)$), and the neighborhood set of V_1 is $N_G(V_1) = \{u|(u, v) \in E, v \in V_1\} - V_1$ (in brief, $N(V_1)$). The distance between u and v, denoted by $d(u, v)$, refers to the length of the minimum path between u and v. A node w is called a common neighbor of the nodes $w_1, w_2, \cdots, w_k$ if and only if $d(w, w_i) = 1$ for each $i \in \{1, 2, \cdots, k\}$. Suppose that $S \subset V$, $v \in S$; let $PN_S(v)$ (in brief, PN(v), no confusion) represent the set of the private neighbors of v, $PN_S(v) = \{u \in V - S|(u, v) \in E, \forall w \in S, (u, w) \notin E\}$.

For all terms and notations not defined in this paper, please see [27]. The concept of the n-dimensional augmented cube AQ_n is provided below. Here, we adopt the definition of [29].

Definition 1: The one-dimensional augmented cube AQ_1 is a complete graph with two vertices. For $n \geqslant 2$, AQ_n can be obtained by taking two copies of the augmented cube AQ_{n-1}, denoted by AQ^0_{n-1} and AQ^1_{n-1}, and adding $2 \times 2^{n-1}$ links between them as follows: Let $V(AQ^0_{n-1}) = \{0x_2 \cdots x_{n-1}x_n : x_i \in \{0, 1\}\}$ and $V(AQ^1_{n-1}) = \{1x_2 \cdots x_{n-1}x_n : x_i \in \{0, 1\}\}$. A node $X = 0x_2 \cdots x_{n-1}x_n$ of AQ^0_{n-1} is adjacent to a node $Y = 1y_2 \cdots y_{n-1}y_n$ of AQ^1_{n-1} if and only if either (1) $x_i = y_i$ for $2 \leqslant i \leqslant n$ or (2) $x_i = \overline{y}_i$ for $2 \leqslant i \leqslant n$.

By Definition 1, we have that for each node $v \in V(AQ_n)$, the degree of v, denoted as $deg(v)$, is $2n - 1$. For the sake of convenience, we introduce the following notation $AQ_n = AQ^0_{n-1} \odot AQ^1_{n-1}$, which denotes the recursive construction of n-dimensional augmented cubes AQ_n.

Lemma 1 [29]: AQ_n is $(2n - 1)$-regular, and its connectivity is $\kappa(AQ_n) = 2n - 1, n \geq 1$.

The t/t-diagnosability of the diagnostic capability of a graph is defined as follows.

Definition 2: A graph G is t/t-diagnosable if all the faulty nodes can be isolated to within a set of at most t nodes having at most one fault-free node, provided that the number of faults at any given time is at most t. The t/t-diagnosability of G is the maximum number t such that G is t/t-diagnosable.

Lemma 2 [25]: Let S be a system with n nodes. Then, S is t/t-diagnosable if and only if for each $X \subset V(S)$ with $|X| = 2i$ and $i \in \{1, 2, \cdots, t\}$, $|N(X)| \geqslant t - i + 1$.

III. t/t-DIAGNOSABILITY OF AUGMENTED CUBES

Before proving the t/t-diagnosability of augmented cubes, some results need to be established.

Lemma 3 [28]: Let $G = (V, E)$ be a k-connected graph, $S \subseteq V, S \neq \phi$.

(i) If $|V - S| \leq k - 1$, then $N_G(S) = V - S$.

(ii) If $|V - S| \geq k$, then $|N_G(S)| \geq k$.

Lemma 4 [42]: For two nodes x, y in AQ_n, $|N_{AQ_n}(\{x, y\})| \geqslant 4n - 8$. In particular, if $x = a_1a_2 \cdots a_{n-1}a_n$ and $y = a_1a_2 \cdots \overline{a}_i \cdots \overline{a}_{n-1}a_n (2 \leqslant i \leqslant n - 1)$, then $|N_{AQ_n}(\{x, y\})| = 4n - 8$.

Lemma 5 [30]: Let $V_1 \subset V(AQ_n)$, with $|V_1| = 3$ (respectively, 4); then, $|N(V_1)| \geqslant 6n - 17$ (respectively, $8n - 28$).

Theorem 1: For $n \geqslant 6$, AQ_n is $(4n - 8)/(4n - 8)$-diagnosable.

Proof: By Lemma 2, we need to prove only the following result: for each $i \in \{1, 2, \cdots, 4n - 8\}$ and each $X \subset V(AQ_n)$ with $|X| = 2i$, $|N(X)| \geqslant (4n - 8) - i + 1$. Let $AQ_n = AQ^0_{n-1} \odot AQ^1_{n-1}$, $X_0 = X \cap V(AQ^0_{n-1})$, $X_1 = X \cap V(AQ^1_{n-1})$. Without loss of generality, $|X_1| \leqslant |X_0|$.

Case 1: $i = 1$.

In this case, $|X| = 2$; by Lemma 4, the result is true.

Case 2: $2n - 6 \leqslant i \leqslant 4n - 8$.

Since $|V(AQ_n) - X| = 2^n - 2i \geqslant 2^n - 2(4n - 8) \geqslant 2n-1 (n \geqslant 6)$, by Lemma 1 and Lemma 3, $|N(X)| \geqslant 2n-1 = (4n-8)-(2n-6)+1 \geqslant (4n-8)-i+1$. The result is true.

Case 3: $2 \leqslant i \leqslant 2n - 7$.

We prove that the result is true in this case by induction on n. For $n = 6$, $2 \leqslant i \leqslant 5$, we need to prove that $|N(X)| \geqslant 17 - i$. When $i = 2$, $|X| = 4$, by Lemma 5, we have that $|N(X)| \geqslant 20 \geqslant 17 - i$. When $i \geqslant 3$, consider the following cases:

Case 3.1.1: $X_1 \neq \varnothing$.

Since $|X_1| + |X_0| = 2i$ and $2^5 - |X_r| \geqslant 9$, by Lemma 3, we have that $|N_{AQ^r_5}(X_r)| \geqslant 9$, $r = 0, 1$; hence, $|N_{AQ_6}(X)| \geqslant |N_{AQ^1_5}(X_1)| + |N_{AQ^0_5}(X_0)| \geqslant 18 > 17 - i$.

Case 3.1.2: $X_1 = \varnothing$.

Let $X_0 = \{x_j | j = 1, 2, \cdots, 2i\}$. By Lemma 5, we have that $|N_{AQ_6}(X)| \geqslant |N_{AQ^0_5}(X_0)| + |N_{AQ^1_5}(X_0)| \geqslant |N_{AQ^1_5}(\{x_1, x_2, x_3, x_4\})| - (2i - 4) + 2i \geqslant 16 \geqslant 17 - i$.

Suppose that $n - 1 \geqslant 6$; our claim is true for AQ_{n-1}. Next, we will show that the result is true for n. Consider the following cases.

Case 3.2.1: $X_1 = \varnothing$.

Let $X_0 = \{x_j | j = 1, 2, \cdots, 2i\}$. If $2n - 8 = 2(n - 1) - 6 \leqslant i \leqslant 2n - 7 \leqslant 4(n - 1) - 8$, by Case 2, we have that $|N_{AQ^0_{n-1}}(X_0)| \geqslant (4(n - 1)) - 8 - i + 1$. If $2 \leqslant i \leqslant 2n - 9 = 2(n - 1) - 7$, by the induction assumption, we have that $|N_{AQ^0_{n-1}}(X_0)| \geqslant (4(n-1))-8-i+1$. Note that $|N_{AQ^1_{n-1}}(\{x_1, x_2, x_3, x_4\})| \geqslant 4$. Then, $|N_{AQ_n}(X)| = |N_{AQ_n}(X_0)| = |N_{AQ^0_{n-1}}(X_0)| + |N_{AQ^1_{n-1}}(X_0)| \geqslant |N_{AQ^0_{n-1}}(X_0)| + |N_{AQ^1_{n-1}}(\{x_1, x_2, x_3, x_4\})| \geqslant (4(n - 1) - 8 - i + 1) + 4 = (4n - 8) - i + 1$.

Case 3.2.2: $X_1 \neq \varnothing$.

Let $X = \{x_j | j = 1, 2, \cdots, 2i\}$. Consider the following cases:

Case 3.2.2.1: $|X_1| = 1$.

Without loss of generality, let $X_1 = \{x_1\}, X_0 = \{x_j | j = 2, 3, \cdots, 2i\}$; then, $|N_{AQ^1_{n-1}}(X_1)| = 2(n - 1) - 1$. If $i = 2n - 7$, by the induction assumption, we have that $|N_{AQ^0_{n-1}}(\{x_5, x_6, \cdots, x_{2i}\})| \geqslant 4(n - 1) - 8 - (i - 2) + 1 = (4n - 8) - i - 1$, and then $|N_{AQ_n}(X)| \geqslant |N_{AQ^1_{n-1}}(X_1)| + |N_{AQ^0_{n-1}}(X_0)| \geqslant |N_{AQ^1_{n-1}}(X_1)| + |N_{AQ^0_{n-1}}(\{x_5, x_6, \cdots, x_{2i}\})| \geqslant 2(n - 1) - 1 + (4n - 8) - i - 1 - 3 = (4n - 8) - i + 1 + (2n - 8) \geqslant (4n - 8) - i + 1$. If $2 \leqslant i \leqslant 2n - 8$, then $1 \leqslant i - 1 \leqslant 2(n - 1) - 7$; by Case 1 and the induction assumption, we have that $|N_{AQ^0_{n-1}}(\{x_3, x_4, \cdots, x_{2i}\})| \geqslant 4(n-1)-8-(i-1)+1 = (4n-8)-i-2$. Then, we determine that $|N_{AQ_n}(X)| \geqslant |N_{AQ^1_{n-1}}(X_1)| + |N_{AQ^0_{n-1}}(\{x_3, x_4, \cdots, x_{2i}\})| \geqslant 2(n-1)-1+(4n-8)-i-2 = (4n-8)-i+1+(2n-6) \geqslant (4n - 8) - i + 1$.

Case 3.2.2.2: $|X_1| > 1$.

Let $|X_1| = 2j + r_1, |X_1| = 2k + r_2, r_1, r_2 \in \{0, 1\}, 1 \leqslant j \leqslant k < i, 2i = 2(j + k) + r_1 + r_2$. Then, $1 \leqslant j \leqslant k \leqslant 2(n - 1) - 6$. We claim that $|N_{AQ^1_{n-1}}(X_1)| \geqslant 4(n - 1) - 8 - j + 1 - r_1$. To prove that the claim is true, let $X_1 = \{x_1, x_2, \cdots, x_{r_1}, y_1, y_2, \cdots, y_{2j}\}$; then, $|N_{AQ^1_{n-1}}(X_1)| \geqslant |N_{AQ^1_{n-1}}(\{y_1, \cdots, y_{2j}\})| - r_1$. When $j = 1$ (respectively, $j = 2(n - 1) - 6$), by Case 1 (respectively, Case 2), we have that $|N_{AQ^1_{n-1}}(\{y_1, \cdots, y_{2j}\})| \leqslant 4(n-1)-8-j+1$; when $2 \leqslant j \leqslant 2(n - 1) - 7$, according to the induction assumption, we also have that $|N_{AQ^1_{n-1}}(\{y_1, \cdots, y_{2j}\})| \geqslant 4(n-1)-8-j+1$. Hence, our claim is true. Similarly, we have that $|N_{AQ^0_{n-1}}(X_0)| \geqslant 4(n-1)-8-k+1-r_2$. Then, $|N_{AQ_n}(X)| \geqslant |N_{AQ^1_{n-1}}(X_1)| + |N_{AQ^0_{n-1}}(X_0)| \geqslant (4(n-1)-8-j+1-r_1)+(4(n-1)-8-k+1-r_2) = (4n-8)-i+1+(4n-15-\frac{1}{2}(r_1+r_2)) \geqslant (4n-8)-i+1$. ■

Theorem 2: For $n \geqslant 6$, the t/t-diagnosability of AQ_n is $4n - 8$.

Proof: We need only to prove that AQ_n is not $(4n - 7)/(4n - 7)$-diagnosable. Consider a pair complement nodes $x = x_nx_{n-1} \cdots x_1, y = x_n \cdots \bar{x_i} \cdots \bar{x_1} (2 \leqslant i \leqslant n - 1)$; by Lemma 4, we have that $|N_{AQ_n}(\{x, y\})| = 4n - 8 < (4n - 7 - 1 + 1)$. By Lemma 2, it is true that AQ_n is not $(4n - 7)/(4n - 7)$-diagnosable. ■

IV. t/k-DIAGNOSABILITY OF AQ_n

In this section, the following terminologies and notations are used. For AQ_n, suppose that $X, Y, Z \in V(AQ_n)$. Let $X = x_1x_2\cdots x_n$, $Y = y_1y_2\cdots y_n$, $Z = z_1z_2\cdots z_n$, where x_i (y_i, z_i) is 0 or 1 ($i = 1, 2, \cdots, n$). Define an operator $\oplus$ as follows: $Z = X \oplus Y$ if and only if $z_i = x_i \oplus y_i$, where $0 \oplus 0 = 0, 1 \oplus 0 = 1, 0 \oplus 1 = 1, 1 \oplus 1 = 0$. For the sake of convenience, we use 0^i (respectively, 1^i) to denote $\overbrace{0\cdots0}^{i}$ (respectively, $\overbrace{1\cdots1}^{i}$). Let $O_i = 0^{i-1}10^{n-i}$ and $S_i = 0^{i-1}1^{n-i+1}$. Suppose that $A = a_1a_2\cdots a_ia_{i+1}\cdots a_n$; we use A_i (respectively, $\overline{A_i}$) to denote $a_1a_2a_3\cdots a_{i-1}\overline{a_i}a_{i+1}\cdots a_n$ (respectively, $a_1a_2\cdots a_{i-1}\overline{a_i\cdots a_n}$). Obviously, $A_i = A\oplus O_i$, $\overline{A_i} = A \oplus S_i$.

To present the t/k-diagnosability of AQ_n, the properties of AQ_n described below are necessary. For the sake of convenience, for the three nodes $A, B, C \in V(AQ_n)$, let $N_{AB} = N(A) \cap N(B)$, $N_{AC} = N(A) \cap N(C)$, $N_{BC} = N(B) \cap N(C)$, $N_{AB}(C) = (N(A) \cap N(C)) \cup (N(B) \cap N(C)) - N(A) \cap N(B) \cap N(C) - \{A, B\}$.

The following Property 1 follows [31].

Property 1: Suppose that $A, B \in V(AQ_n)$, $d(A, B) = 1$; then,

1) If $B = A \oplus O_i$, then $|N_{AB}| = 2$.

2) If $B = A \oplus S_i$, then if $i \in \{1, n\}$, then $|N_{AB}| = 2$. Otherwise, $|N_{AB}| = 4$.

Since the proof of the following Property 2 is easily obtained from [31], we omit it.

Property 2: Suppose that $A, B \in V(AQ_n)$, $d(A, B) = 2$; then,

1) If $B = A \oplus O_i \oplus O_j$, then if $i + 1 = j (i \leqslant n - 2)$, then $|N_{AB}| = 4$. Otherwise, $|N_{AB}| = 2$.

2) If $B = A \oplus O_i \oplus S_j$, $i + 2 \leqslant j$, then if $i + 2 = j$, then $|N_{AB}| = 4$. Otherwise, $|N_{AB}| = 2$.

3) If $B = A \oplus O_i \oplus S_j (i \geqslant j + 1)$, then if $i - 1 = j$, then $|N_{AB}| = 4$. Otherwise, $|N_{AB}| = 2$.

4) If $B = A \oplus S_i \oplus S_j (|i - j| > 1)$, then if $|i - j| = 2$, then $|N_{AB}| = 4$. Otherwise, $|N_{AB}| = 2$.

Property 3: Let $A, B, C \in V(AQ_n)$; if $d(A, B) = 1$, $d(A, C) = 1$, $d(B, C) = 1$, then $|N_{AB}(C)| \leqslant 2$.

Proof: *Case 1: $B = A \oplus O_i$.*

Case 1.1: $i = 1$. According to Property 1, A, B share 2 common neighbors: $\overline{A}_1, \overline{A}_2$. Then, $C = \overline{A}_1$ or $C = \overline{A}_2$. If $C = \overline{A}_1$, then $N_{AC} = \{\overline{A}_2, B\}$ and $N_{BC} = \{\overline{A}_2, A\}$; the result is true. If $C = \overline{A}_2$, then $N_{AC} = \{\overline{A}_1, B\}$ and $N_{BC} = \{\overline{A}_1, A\}$; the result is also true.

Case 1.2: $2 \leqslant i \leqslant n - 1$. According to Property 1, A, B share 2 common neighbors: $\overline{A}_i, \overline{A}_{i+1}$. If $C = \overline{A}_i$, then $N_{AC} = \{\overline{A}_{i+1}, B, A_{i-1}, \overline{A}_{i-1}\}$ and $N_{BC} = \{\overline{A}_{i+1}, A\}$; the result is true. Similarly, we conclude that when $C = \overline{A}_{i+1}$, the result is true.

Case 1.3: $i = n$.

According to Property 1, A, B share 2 common neighbors: $A_{n-1}, \overline{A}_{n-1}$. If $C = \overline{A}_{n-1}$, then $N_{AC} = \{\overline{A}_{i-2}, B\}$ and $N_{BC} = \{A, A_{i-1}\}$; the result is true. Similarly, we conclude that when $C = A_n$, the result is true.

Case 2: $B = A \oplus S_i$.

Case 2.1: $i = 1$.

According to Property 1, A, B share 2 common neighbors $\overline{A}_2, A_1$. If $C = \overline{A}_2$, then $N_{AC} = \{A_1, \overline{A}_2, \overline{A}_3, B\}$ and $N_{BC} = \{A, A_1\}$, and then $N_{AB}(C) = \{A_2, \overline{A}_3\}$; the result is true. Similarly, we conclude that when $C = A_1$, the result is true.

Case 2.2: $2 \leqslant i \leqslant n - 1$.

According to Property 1, A, B share 4 common neighbors: $A_i, A_{i-1}, \overline{A}_{i-1}, \overline{A}_{i+1}$. If $C = A_i$, then $N_{AC} = \{\overline{A}_{i+1}, B\}$ and $N_{BC} = \{A, \overline{A}_{i+1}\}$; the result is true. Similarly, we conclude that when C is one of $\{A_{i-1}, \overline{A}_{i-1}, \overline{A}_{i+1}\}$, the result is true. ■

Property 4: For any three nodes $A, B, C \in V(AQ_n)$, if $d(A, B) = 1$, $d(A, C) = 2$ and $d(B, C) = 1$, then $|N_{AB}(C)| \leqslant 5$.

Proof: *Case 1: $B = A \oplus O_i$.*

Case 1.1: $C = B \oplus O_j$. Clearly, $N_{BC} = \{\overline{B}_{j+1}, \overline{B}_j\} (j \leqslant n - 1)$ or $N_{BC} = \{\overline{B}_{n-1}, B_{n-1}\} (j = n)$. By $d(A, C) = 2$, we have $i \neq j$. Consider the following cases:

Case 1.1.1: $|j - i| = 1$. If $j = i + 1 \leqslant n - 1$, then $N_{AC} = \{B, \overline{A}_{j+1}, A_j, \overline{A}_i\}$ and $N_{AB}(C) = \{\overline{B}_{j+1}, A_j, \overline{A}_{j+1}\}$. If $j = i + 1 = n$, then $N_{AC} = \{A_{n-2}, \overline{A}_{n-2}, B, \overline{B}_{n-1}\}$ and $N_{AB}(C) = \{A_{n-2}, \overline{A}_{n-2}\}$. Hence, when $j = i + 1$, the result is true.

On the other hand, if $j = i - 1 \leqslant n - 2$, then $N_{AC} = \{B, A_{i-1}, \overline{A}_{i-1}, \overline{A}_{i+1}\}$ and $N_{AB}(C) = \{\overline{B}_j, A_{i-1}, \overline{A}_{i-1}\}$. If $j = i - 1 = n - 1$, then $N_{AC} = \{A_{n-2}, \overline{A}_{n-2}, B, \overline{B}_{n-1}\}$ and $N_{AB}(C) = \{A_{n-2}, \overline{A}_{n-2}, B, B_{n-1}\}$. Hence, when $j = i - 1$, the result is true.

Case 1.1.2: $|j - i| \neq 1$. If $j = i - 2 = n - 2$, then $N_{AC} = \{B, A_{n-2}, \overline{A}_{n-2}, A_{n-1}\}$ and $N_{AB}(C) = \{\overline{B}_{n-2}, \overline{A}_{n-2}, A_{n-2}\}$. If $i = j - 2 = n - 2$, then $N_{AC} = \{B, A_n, \overline{A}_{n-2}, A_{n-1}\}$ and $N_{AB}(C) = \{B_{n-1}, A_n, A_{n-1}\}$. For other situations, $N_{AC} = \{A_i, A_j\}$, and then $|N_{AB}(C)| \leqslant 4$. Hence, the result is true.

Case 1.2: $C = B \oplus S_j$. When $j = n$, the situation is the same as that discussed in Case 1.1. We need to consider only the situation where $j \leqslant n - 1$. By Property 2, we have $N_{BC} = \{B_j, B_{j-1}, \overline{B}_{j+1}, \overline{B}_{j-1}\} (2 \leqslant j \leqslant n - 1)$ or $N_{BC} = \{\overline{B}_2, B_1\}$ $(j = 1)$. By $d(A, C) = 2$, we have that $j \neq i$ and $j \neq i + 1$. Now, consider the following cases:

Case 1.2.1: $j = 1$. Then, $N_{AC} = \{B, \overline{A}_1\}$ and $N_{AB}(C) = \{\overline{A}_1, \overline{B}_2, B_1\}$; the result is true.

Case 1.2.2: $2 \leqslant j \leqslant n - 1$.

Case 1.2.2.1: If $j = i - 1 \leqslant n - 2$, then $N_{AC} = \{B, \overline{A}_{i-1}, A_{i-1}, \overline{A}_{i+1}\}$ and $N_{AB}(C) = \{B_j, B_{j-1}, A_{i-1}, \overline{A}_{i-1}, \overline{B}_{j-1}\}$; the result is true.

Case 1.2.2.2: If $j = i - 1 = n - 1$, then $C = A \oplus O_{n-1}$. This is a contradiction to the assumption that $d(A, C) = 2$; hence, this situation is impossible.

Case 1.2.2.3: If $j - i = 2$, then $N_{AC} = \{B, A_{i+1}, \overline{A}_i, \overline{A}_{i+2}\}$ and $N_{AB}(C) = \{B_j, B_{j-1}, A_{i+1}, \overline{A}_{i+2}, \overline{B}_{j+1}\}$; the result is true.

Case 1.2.2.4: If $j - i > 2$ or $i - j > 2$, then $N_{AC} = \{B, \overline{A}_j\}$, and then $|N_{AB}(C)| \leqslant |N_{AC} \cup N_{BC} - \{B\}| = 6 - 1 = 5$; the result is true.

Case 1.2.3: $j = n$. This case is identical to the case in the proof of Property 3.

Case 2: $B = A \oplus S_i$ $(i \neq n)$.

Case 2.1: $C = B \oplus O_j$. Clearly, $N_{BC} = \{B_1, \overline{B}_2\}$ $(j = 1)$ or $N_{BC} = \{\overline{B}_{n-1}, B_{n-1}\}$ $(2 \leqslant j \leqslant n-1)$. By $d(A, C) = 2$, we have that $i \neq j$ and $j \neq i-1$. Consider the following cases:

Case 2.1.1: $j \leqslant n-1$. If $j = i-2$, then $N_{AC} = \{B, \overline{A}_{i-2}, A_{i-1}, A_{i-2}\}$ and $N_{AB}(C) = \{\overline{A}_{i-2}, A_{i-2}, B_j\}$. If $j \leqslant i-3$, then $N_{AC} = \{B, A_j\}$ and $N_{AB}(C) = \{\overline{B}_{j+1}, \overline{B}_j, A_j\}$. If $j = i+1$, then $N_{AC} = \{B, A_{i+1}, \overline{A}_{i+2}, A_i\}$ and $N_{AB}(C) = \{\overline{B}_{j+1}, \overline{A}_{i+2}\}$. If $j \geqslant i+2$, then $N_{AC} = \{B, A_j\}$, and then $|N_{AB}(C)| \leqslant |N_{AC} \cup N_{BC} - \{A, B\}| \leqslant 4 - 1 = 3$. Hence, the result is true.

Case 2.1.2: $j = n$. By $d(A, C) = 2$, we have that $i \neq n-1$ and $i \neq n$. Now, we need to consider only the situation where $i \leqslant n-2$. If $i = n-2$, then $N_{AC} = \{B, A_n, A_{n-1}, A_{n-2}\}$ and $N_{AB}(C) = \{A_n, A_{n-1}, B_{n-1}\}$. If $i \leqslant n-3$, then $N_{AC} = \{B, A_n\}$ and $N_{AB}(C) = \{\overline{B}_{n-1}, B_{n-1}, A_n\}$. Hence, when $j = n$, the result is true.

Case 2.2: $C = B \oplus S_j (j \neq n)$. By $d(A, C) = 2$, we have that $|i-j| > 1$. Since $B \in N_{AC}$, by Property 1 and Property 2, we need only the following situation: $|i-j| = 2$ and $2 \leqslant j \leqslant n-1$. By Property 1 and the condition $2 \leqslant j \leqslant n-1$, we have that $N_{BC} = \{\overline{B}_{j-1}, B_{j-1}, B_j, \overline{B}_{j+1}\}$. If $j - i = 2$, then $N_{AC} = \{A_j, A_{j+1}, \overline{A}_j, \overline{A}_i\}$ and $N_{AB}(C) = \{\overline{B}_{j-1}, B_{j-1}, B_j, A_j, \overline{A}_j\}$. If $i - j = 2$, then $N_{AC} = \{A_i, A_{i+1}, \overline{A}_i, \overline{A}_j\}$ and $N_{AB}(C) = \{\overline{B}_{j+1}, B_{j-1}, B_j, A_{i+1}, \overline{A}_j\}$. Hence, the result is true. ■

Property 5: Let $A, B, C \in V(AQ_n)$. Suppose that $d(A, B) = 1, d(A, C) = 2$, and $d(B, C) = 2$. Then, $|N_{AB}(C)| \leqslant 5$.

Proof: *Case 1:* $B = A \oplus O_i$.

Case 1.1: $i = 1$.

Case 1.1.1: $C = B \oplus O_j \oplus O_k$. Without loss of generality, assume that $j < k$. Then, $C = A \oplus O_i \oplus O_j \oplus O_k$. By $d(A, C) = 2$, we have that any two of i, j, k are different. When $j = 2, k = 3$, we have that $N_{AC} = \{\overline{A}_1, \overline{A}_4\}$, $N_{BC} = \{B_2, B_3, \overline{B}_2, \overline{B}_4\}$ and $N_{AB}(C) = \{B_2, B_3, \overline{A}_4, \overline{B}_4\}$. Hence, the result is true. When $j \neq 2$ or $k \neq 3$, $N_{AC} = \phi$, and when B and C share 2 or 4 common neighbors by Property 2, then $|N_{AB}(C)| \leqslant 4$.

Case 1.1.2: $C = B \oplus O_j \oplus S_k$. Clearly, $j \neq k, k-1$ (otherwise, $d(B, C) = 1$). $d(A, C) = 2$ implies that $j \neq 1$.

Case 1.1.2.1: $j = 2$. $d(A, C) = 2$ implies that $k = 4$; then, $N_{AC} = \{\overline{A}_1, A_3\}$, $N_{BC} = \{B_2, B_3, \overline{B}_4, \overline{B}_2\}$ and $N_{AB}(C) = \{B_2, B_3, \overline{B}_4, A_3\}$. Hence, the result is true.

Case 1.1.2.2: $j \neq 2$. We claim that $k \leqslant 2$. To the contrary, if $k > 2$, then $N_{AC} = \phi$; this is a contradiction to $d(A, C) = 2$. Now, we need to consider only the situations of $k = 1$ and $k = 2$.

Case 1.1.2.2.1: $k = 1$. $d(A, C) = 2$ implies that $j \neq 1, 2$. If $j = 3$, then $N_{AC} = \{A_2, \overline{A}_2, A_3, \overline{A}_4\}$, $N_{BC} = \{\overline{B}_1, B_3\}$ and $N_{AB}(C) = \{A_2, B_3, A_3, \overline{A}_4\}$. Hence, the result is true. If $j \geqslant 4$, then $N_{AC} = \{\overline{A}_2, A_j\}$, $N_{BC} = \{\overline{B}_1, B_j\}$ and $N_{AB}(C) = \{B_j, A_j\}$. Hence, the result is true.

Case 1.1.2.2.2: $k = 2$. We have that $N_{AC} = \{A_j, \overline{A}_1\}$. If $j = 3$, then $N_{BC} = \{\overline{B}_2, B_3, B_2, \overline{B}_4\}$ and $N_{AB}(C) = \{B_3, B_2, \overline{B}_4, A_3\}$. If $j > 3$, then $N_{BC} = \{\overline{B}_2, B_j\}$ and $N_{AB}(C) = \{B_j, A_j\}$.

Case 1.1.3: $C = B \oplus S_j \oplus S_k$. $d(B, C) = 2$ implies that |j-k| $\neq 0, 1$. When $|k-j| = 2$, the case is the same as that discussed in Case 1.1.1. Without loss of generality, assume that $j < k$. We claim that $j \leqslant 2$. To the contrary, suppose that $j \geqslant 3$; then, $N_{AC} = \phi$, a contradiction to $d(A, C) = 2$. Now, we need to consider only the following situation: $j \leqslant 3, k - j \geqslant 3$.

Case 1.1.3.1: $j = 1$. If $k = 4$, then $N_{AC} = \{\overline{A}_2, \overline{A}_4, A_2, A_3\}$, $N_{BC} = \{\overline{B}_1, \overline{B}_4\}$ and $N_{AB}(C) = \{\overline{A}_4, A_2, A_3, \overline{B}_4\}$. If $k \geqslant 5$, then $N_{AC} = \{\overline{A}_2, \overline{A}_k\}, N_{BC} = \{\overline{B}_1, \overline{B}_k\}$ and $N_{AB}(C) = \{\overline{A}_k, \overline{B}_k\}$. Hence, the result is true.

Case 1.1.3.2: $j = 2$. We have that $N_{AC} = \{\overline{A}_1, \overline{A}_k\}, N_{BC} = \{\overline{B}_2, \overline{B}_k\}$ and $N_{AB}(C) = \{\overline{A}_k, \overline{B}_k\}$. Hence, the result is true.

Case 1.2: $i \in \{2, 3, 4, \cdots, n-3\}$.

Case 1.2.1: $C = B \oplus O_j \oplus O_k$. Without loss of generality, suppose that $j < k$. Similar to Case 1.1.1, we have that any two of i, j, k are different.

Case 1.2.1.1: $i = n-3, j = n-2, k = n$ or $j = n-1, k = n$. If $i = n-3, j = n-2, k = n$, we have that $N_{AC} = \{\overline{A}_{n-3}, A_{n-1}\}$, $N_{BC} = \{B_{n-1}, \overline{B}_{n-2}, B_{n-2}, B_n\}$, and $N_{AB}(C) = \{B_{n-1}, A_{n-1}, B_{n-2}, B_n\}$. If $i = n-3$, $j = n-1, k = n$, we have that $N_{AC} = \{B, \overline{A}_{n-3}, A_{n-2}, \overline{A}_{n-1}\}$, $N_{BC} = \{B_{n-1}, \overline{B}_{n-2}, B_{n-2}, B_n\}$, and $N_{AB}(C) = \{B_{n-1}, A_{n-2}, \overline{A}_{n-1}, B_{n-2}, B_n\}$. If $i \neq n-3, j = n-1, k = n$, we have that $N_{AC} = \{B, \overline{A}_{n-1}\}$, $N_{BC} = \{B_{n-1}, \overline{B}_{n-2}, B_{n-2}, B_n\}$, and $N_{AB}(C) = \{B_{n-1}, \overline{B}_{n-2}, B_{n-2}, \overline{A}_{n-1}\}$. Hence, the result is true.

For other situations, not including Case 1.2.1.1, $d(A, C) = 2$ implies that i, j, k are three consecutive integers. Now, we need to discuss only the following situations in which i, j, k are three consecutive integers.

Case 1.2.1.2: $i = j-1$. Since $N_{AC} = \{\overline{A}_{i+3}, \overline{A}_i\}$ and $N_{BC} = \{B_{i+1}, B_{i+2}, \overline{A}_i, \overline{B}_{i+3}\}$, $N_{AB}(C) = \{B_{i+1}, B_{i+2}, \overline{A}_{i+3}, \overline{B}_{i+3}\}$. Hence, the result is true.

Case 1.2.1.3: $j = i-1 = k-2$. Since $N_{AC} = \{\overline{A}_{i+2}, \overline{A}_{i-1}\}$ and $N_{BC} = \{B_{i-1}, B_{i+1}\}$, $N_{AB}(C) = \{\overline{A}_{i+2}, \overline{A}_{i-1}, B_{i-1}, B_{i+1}\}$. Hence, the result is true.

Case 1.2.1.4: $i = k+1$. Since $N_{AC} = \{\overline{A}_{i+1}, \overline{A}_{i-2}\}$ and $N_{BC} = \{B_{i-1}, B_{i-2}, \overline{B}_{i-2}, \overline{B}_i\}$, $N_{AB}(C) = \{B_{i-1}, B_{i-2}, \overline{B}_{i-2}, \overline{A}_{i-2}\}$. Hence, the result is true.

Case 1.2.2: $C = B \oplus O_j \oplus S_k (k \neq n)$. $d(B, C) = 2$ implies that $j \neq k, k-1$. At the same time, by $d(B, C) = 2$, we conclude that $k = n-1$ implies $j \neq n$. $d(A, C) = 2$ implies that $j \neq i$. We claim that $k \leqslant i+3$. In contrast, if $k > i+3$, then $N_{AC} = \phi$; this is a contradiction to $d(A, C) = 2$. Now, we need to consider only the situations in which $k \leqslant i+3$.

Case 1.2.2.1: $k = i+1$. Since $d(A, C) = 2$, we have that $j \leqslant i-2$ or $j \geqslant i+2$. If $j = i-2$, we have that $N_{AC} = \{A_{i-2}, \overline{A}_i, A_{i-1}, \overline{A}_{i-2}\}$, $N_{BC} = \{B_{i-2}, \overline{B}_{i+1}\}$ and $N_{AB}(C) = \{A_{i-2}, A_{i-1}, \overline{A}_{i-2}, B_{i-2}\}$. If $j < i-2$, we have that $N_{AC} = \{A_j, \overline{A}_i\}$, $N_{BC} = \{B_j, \overline{B}_{i+1}\}$ and $N_{AB}(C) = \{A_j, B_j\}$.

Similarly, if $j \geqslant i+2$, we have that $N_{AC} = \{A_j, \overline{A}_i\}$, $N_{BC} = \{B_j, \overline{B}_{i+1}\}(j > i+2)$ or $N_{BC} = \{B_{i+1}, \overline{B}_{i+1}, B_{i+2}, \overline{B}_{i+3}\}(j = i+2)$. Then, $N_{AB}(C) = \{A_j, B_j\}(j > i+2)$ or $N_{AB}(C) = \{B_{i+1}, A_{i+2}, B_{i+2}, \overline{B}_{i+3}\}(j = i+2)$. Hence, the result is true.

Case 1.2.2.2 : $k = i+2$. $d(A, C) = 2$ implies that $j = i-1$. Then, $N_{AC} = \{\overline{A}_{i-1}, A_{i+1}\}$ and $N_{BC} = \{B_{i-1}, \overline{B}_{i+2}\}$. Subsequently, $|N_{AB}(C)| \leqslant 4$. Hence, the result is true.

Case 1.2.2.3: $k = i+3$. $d(A, C) = 2$ implies that $j = i+1$. Then, $N_{AC} = \{\overline{A}_i, A_{i+2}\}$ and $N_{BC} = \{B_{i+1}, \overline{B}_{i+3}, \overline{B}_{i+1}, B_{i+2}\}$. Subsequently, $N_{AB}(C) = \{B_{i+1}, \overline{B}_{i+3}, A_{i+2}, B_{i+2}\}$. Hence, the result is true.

Case 1.2.2.4: $k = i$. $d(A, C) = 2$ implies that $j \neq i+1$. If $j \leqslant i-2$, then $N_{AC} = \{A_j, \overline{A}_{i+1}\}$ and $N_{BC} = \{B_{i-2}, \overline{B}_i, B_{i-1}, \overline{B}_{i-2}\}(j = i-2)$ or $N_{BC} = \{B_j, \overline{B}_i\}(j < n-2)$. Subsequently, $N_{AB}(C) = \{B_{i-2}, A_{i-2}, B_{i-1}, \overline{B}_{i-2}\}(j = i-2)$ or $N_{AB}(C) = \{A_{i-2}, A_{i-1}, \overline{A}_{i-2}, B_{i-2}\}(j < n-2)$. If $j = i+2$, then $N_{AC} = \{\overline{A}_{i+3}, A_{i+1}, A_{i+2}, \overline{A}_{i+1}\}$, $N_{BC} = \{B_{i+2}, \overline{B}_i\}$ and $N_{AB}(C) = \{\overline{A}_{i+3}, A_{i+1}, A_{i+2}, B_{i+2}\}$. If $j > i+2$, then $N_{AC} = \{\overline{A}_{i+1}, A_j\}$, $N_{BC} = \{B_j, \overline{B}_i\}$ and $N_{AB}(C) = \{A_j, B_j\}$. Hence, the result is true.

Case 1.2.2.5: $k < i$. $d(A, C) = 2$ implies that i, j, k are three consecutive integers. When $k = i-1 = j-2$, we have that $N_{AC} = \{A_{i-1}, \overline{A}_{i+2}\}$, $N_{BC} = \{B_{i+1}, \overline{B}_{i-1}\}$ and $N_{AB}(C) = \{A_{i-1}, \overline{A}_{i+2}, B_{i+1}, \overline{B}_{i-1}\}$. When $k = j-1 = i-2$, we have that $N_{AC} = \{A_{i-2}, \overline{A}_{i+1}\}$, $N_{BC} = \{B_{i-2}, \overline{B}_i, \overline{B}_{i-2}, B_{i-1}\}$ and $N_{AB}(C) = \{B_{i-2}, A_{i-2}, \overline{B}_{i-2}, B_{i-1}\}$. Hence, the result is true.

Case 1.2.3: $C = B \oplus S_j \oplus S_k$. $d(B, C) = 2$ implies that |j-k| $\neq 0, 1$. When $|k-j| = 2$, the case is the same as that discussed in Case 1.2.1. Now, we need to consider only the following situation: $|j-k| > 2$. Without loss of generality, assume that $j < k-2$. We claim that $j \leqslant i+1$. In contrast, suppose that $j \geqslant i+2$; then, $N_{AC} = \phi$, which contradicts $d(A, C) = 2$. Next, we need to discuss only the following situations.

Case 1.2.3.1: $j = i+1$. Then, $N_{AC} = \{\overline{A}_i, \overline{A}_k\}$, $N_{BC} = \{\overline{B}_{i+1}, \overline{B}_k\}$ and $N_{AB}(C) = \{\overline{A}_k, \overline{B}_k\}$. Hence, the result is true.

Case 1.2.3.2: $j = i$. If $k = j+3$, then $N_{AC} = \{A_{i+1}, A_{i+2}, \overline{A}_{i+3}, \overline{A}_{i+1}\}$, $N_{BC} = \{\overline{B}_i, \overline{B}_{i+3}\}$ and $N_{AB}(C) = \{A_{i+1}, A_{i+2}, \overline{A}_{i+3}, \overline{B}_{i+3}\}$. If $k > j+3$, then $N_{AC} = \{\overline{A}_{i+1}, \overline{A}_k\}$, $N_{BC} = \{\overline{B}_i, \overline{B}_k\}$ and $N_{AB}(C) = \{\overline{A}_k, \overline{B}_k\}$. Hence, the result is true.

Case 1.2.3.3: $j = i-1$. $d(A, B) = 2$ implies that $k = i+2$. Then, $N_{AC} = \{A_{i-1}, A_{i+1}\}$, $N_{BC} = \{\overline{B}_{i-1}, \overline{B}_{i+2}\}$ and $N_{AB}(C) = \{A_{i-1}, A_{i+1}, \overline{B}_{i-1}, \overline{B}_{i+2}\}$. Hence, the result is true.

Case 1.2.3.4: $j < i-1$. $d(A, C) = 2$ implies that $k = i$ or $k = i+1$. If $k = i$, then $N_{AC} = \{\overline{A}_j, \overline{A}_{i+1}\}$, $N_{BC} = \{\overline{B}_j, \overline{B}_i\}$ and $N_{AB}(C) = \{\overline{A}_j, \overline{B}_j\}$. If $k = i+1$ and $j = i-2$, then $N_{AC} = \{\overline{A}_{i-2}, \overline{A}_i, A_{i-2}, A_{i-1}\}$, $N_{BC} = \{\overline{B}_{i-2}, \overline{B}_{i+1}\}$ and $N_{AB}(C) = \{\overline{A}_{i-2}, \overline{B}_{i-2}, A_{i-2}, A_{i-1}\}$. If $k = i+1$ and $j < i-2$, then $N_{AC} = \{\overline{A}_j, \overline{A}_i\}$, $N_{BC} = \{\overline{B}_j, \overline{B}_{i+1}\}$ and $N_{AB}(C) = \{\overline{A}_j, \overline{B}_j\}$. Hence, the result is true.

Case 1.3: $i = n-2$.

Case 1.3.1: $C = B \oplus O_j \oplus O_k$. Without loss of generality, suppose that $j < k$. Similar to Case 1.1.1, we have that any two of i, j, k are different.

Case 1.3.1.1: $j = n-3, k = n$. We have that $N_{AC} = \{\overline{A}_{n-3}, A_{n-1}\}$, $N_{BC} = \{B_{n-3}, B_n\}$, and $N_{AB}(C) = \{\overline{A}_{n-3}, A_{n-1}, B_{n-3}, B_n\}$.

For other situations, not including Case 1.3.1.1, $d(A, C) = 2$ implies that i, j, k are three consecutive integers. Now, we need to discuss only the situation in which i, j, k are three consecutive integers.

Case 1.3.1.2: $j = k-1 = n-1$. Then, $N_{AC} = \{B, \overline{A}_{n-1}, A_{n-3}, \overline{A}_{n-3}\}$ and $N_{BC} = \{B_{n-1}, B_n, B_{n-2}, \overline{B}_{n-2}\}$. Subsequently, $N_{AB}(C) = \{B_{n-1}, B_n, B_{n-2}, A_{n-3}, \overline{A}_{n-3}\}$. Hence, the result is true.

Case 1.3.1.3: $j = n-3, k = n-1$. Since $N_{AC} = \{\overline{A}_{n-3}, A_n\}$ and $N_{BC} = \{B_{n-3}, B_{n-1}\}$, $N_{AB}(C) = \{\overline{A}_{n-3}, A_n, B_{n-3}, B_{n-1}\}$. Hence, the result is true.

Case 1.3.1.4: $j = n-4, k = n-3$. Then, $N_{AC} = \{\overline{A}_{n-4}, \overline{A}_{n-1}\}$ and $N_{BC} = \{B_{n-4}, B_{n-3}, \overline{B}_{n-4}, \overline{B}_{n-2}\}$. Subsequently, $N_{AB}(C) = \{B_{n-4}, B_{n-3}, \overline{B}_{n-4}, \overline{A}_{n-4}\}$. Hence, the result is true.

Case 1.3.2: $C = B \oplus O_j \oplus S_k (k \neq n)$. $d(B, C) = 2$ implies that $j \neq k, k-1$. $d(A, C) = 2$ implies that $j \neq i$. We claim that $k \geqslant n-4$. In contrast, suppose that $k < n-4$; then, $N_{AC} = \phi$, which contradicts $d(A, C) = 2$. Next, we need to consider only the following situations.

Case 1.3.2.1: $k = n-1$. Since $d(A, C) = 2$ and $d(B, C) = 2$, we have that $j \leqslant n-4$. Then, $N_{AC} = \{A_{n-4}, \overline{A}_{n-2}, A_{n-3}, \overline{A}_{n-4}\}(j = n-4)$ or $N_{AC} = \{A_j, \overline{A}_{n-2}\}(j < n-4)$, while $N_{BC} = \{B_j, \overline{B}_{n-1}\}$. Subsequently, $N_{AB}(C) = \{A_{n-4}, B_{n-4}, A_{n-3}, \overline{A}_{n-4}\}$ or $N_{AB}(C) = \{A_j, B_j\}$. Hence, the result is true.

Case 1.3.2.2: $k = i = n-2$. $d(A, C) = 2$ and $d(B, C) = 2$ imply that $j \leqslant n-4$. Then, $N_{AC} = \{A_{n-4}, \overline{A}_{n-2}, A_{n-3}, \overline{A}_{n-4}\}(j = n-4)$ or $N_{AC} = \{A_j, \overline{A}_{n-2}\}(j < n-4)$, while $N_{BC} = \{B_j, \overline{B}_{n-1}\}$. Subsequently, $N_{AB}(C) = \{A_{n-4}, B_{n-4}, A_{n-3}, \overline{A}_{n-4}\}$ or $N_{AB}(C) = \{A_j, B_j\}$. Hence, the result is true.

Case 1.3.2.3 $k = n-3$. By $d(A, C) = d(B, C) = 2$, we have that $j = n$ or $j = n-1$. If $j = n$, then $N_{AC} = \{A_{n-3}, A_{n-1}\}$, $N_{BC} = \{B_n, \overline{B}_{n-3}\}$, and thus $N_{AB}(C) = \{A_{n-1}, A_{n-3}, B_n, \overline{B}_{n-3}\}$. If $j = n-1$, then $N_{AC} = \{A_{n-3}, A_n\}$, $N_{BC} = \{B_{n-1}, \overline{B}_{n-3}\}$, and thus $N_{AB}(C) = \{A_n, A_{n-3}, B_{n-1}, \overline{B}_{n-3}\}$. Hence, the result is true.

Case 1.3.2.4: $k = n-4$. By $d(A, C) = d(B, C) = 2$, we have that $j = n-3$. Then, $N_{AC} = \{A_{n-4}, \overline{A}_{n-1}\}$, $N_{BC} = \{B_{n-4}, \overline{B}_{n-2}, B_{n-3}, \overline{B}_{n-4}\}$, and thus $N_{AB}(C) = \{B_{n-4}, A_{n-4}, B_{n-3}, \overline{B}_{n-4}\}$.

Case 1.3.3: $C = B \oplus S_j \oplus S_k$. $d(B, C) = 2$ implies that |j-k| $\neq 0, 1$. When $|k-j| = 2$, the case is the same as that discussed in Case 1.3.1. Without loss of generality, assume that $j < k$. When $k = n$, the proof of the case can be obtained by the proof of Case 1.3.2. Now, we need to consider only the following situation: $j \leqslant k-3$ and $k \neq n$.

Since $d(A, C) = 2$, we have that $k = i = n-2$ or $k = i+1 = n-1$. If $k = n-2$, then $N_{AC} = \{\overline{A}_j, \overline{A}_{i+1}\}$, $N_{BC} = \{\overline{B}_j, \overline{B}_i\}$ and $N_{AB}(C) = \{\overline{A}_j, \overline{B}_j\}$. If $k = i+1$ and $j = k-3$, then $N_{AC} = \{\overline{A}_{n-4}, \overline{A}_{n-2}, A_{n-4}, A_{n-3}\}$, $N_{BC} = \{\overline{B}_{n-4}, \overline{B}_{n-1}\}$ and $N_{AB}(C) = \{\overline{A}_{n-4}, \overline{B}_{n-4}, A_{n-4}, A_{n-3}\}$. If $k = i+1$ and

$j < k - 3$, then $N_{AC} = \{\overline{A}_j, \overline{A}_i\}$, $N_{BC} = \{\overline{B}_j, \overline{B}_k\}$ and $N_{AB}(C) = \{\overline{A}_j, \overline{B}_j\}$. Hence, the result is true.

Case 1.4: $i = n - 1$.

Case 1.4.1: $C = B \oplus O_j \oplus O_k$. Without loss of generality, suppose that $j < k$. Similar to Case 1.1.1, we have that any two of i, j, k are different. Since $d(A, C) = 2$, the case of $j < n - 3$ and $k = n - 2$ is impossible, and the case of $k < n - 2$ is also impossible. Thus, we need to consider only the following subcases.

Case 1.4.1.1: $k = n$. If $j = n - 2$, then $N_{AC} = \{A_{n-3}, \overline{A}_{n-3}, \overline{A}_{n-1}, A_{n-2}\}$ and $N_{BC} = \{B_{n-2}, B_n, \overline{B}_{n-2}, A\}$; subsequently, $N_{AB}(C) = \{A_{n-3}, \overline{A}_{n-3}, \overline{A}_{n-1}, B_{n-2}, \overline{B}_{n-2}\}$. If $j = n - 3$, then $N_{AC} = \{A_{n-3}, \overline{A}_{n-3}, \overline{A}_{n-1}, A_{n-2}\}$ and $N_{BC} = \{B_{n-3}, B_n\}$; subsequently, $N_{AB}(C) = \{A_{n-3}, \overline{A}_{n-3}, B_{n-3}, A_{n-2}\}$. If $j < n - 3$, then $N_{AC} = \{A_j, \overline{A}_{n-1}\}$ and $N_{BC} = \{B_j, B_n\}$; subsequently, $N_{AB}(C) = \{A_j, B_j\}$. Hence, the result is true.

Case 1.4.1.2: $j = n - 3, k = n - 2$. We have that $N_{AC} = \{\overline{A}_{n-3}, A_n\}$, $N_{BC} = \{B_{n-3}, B_{n-2}, \overline{B}_{n-3}, \overline{B}_{n-1}\}$, and $N_{AB}(C) = \{B_{n-3}, B_{n-2}, \overline{B}_{n-3}, \overline{A}_{n-3}\}$. Hence, the result is true.

Case 1.4.2: $C = B \oplus O_j \oplus S_k (k \neq n)$. $d(B, C) = 2$ implies that $j \neq k, k - 1$. $d(A, C) = 2$ implies that $j \neq i$. Consider the following cases:

Case 1.4.2.1: $k < j$. Since $d(A, C) = 2$, the case of $k < n - 3$ and $j = n - 2$ is impossible; the case of $j < n - 2$ is also impossible. Now, we consider the following cases:

Case 1.4.2.1.1: $j = n$. By $d(A, C) = 2$, we have that $k \neq n - 2$. If $k = n - 3$, then $N_{AC} = \{A_{n-3}, \overline{A}_{n-1}, A_{n-2}, \overline{A}_{n-3}\}$, $N_{BC} = \{\overline{B}_{n-3}, B_n\}$ and $N_{AB}(C) = \{A_{n-3}, \overline{B}_{n-3}, A_{n-2}, \overline{A}_{n-3}\}$. If $k < n - 3$, then $N_{AC} = \{\overline{A}_{n-1}, \overline{A}_k\}$, $N_{BC} = \{\overline{B}_k, B_n\}$ and $N_{AB}(C) = \{\overline{A}_k, \overline{B}_k\}$. Hence, the result is true.

Case 1.4.2.1.2: $k = n - 3, j = n - 2$. Then, $N_{AC} = \{A_{n-3}, A_n\}$, $N_{BC} = \{B_{n-2}, \overline{B}_{n-3}, B_{n-3}, \overline{B}_{n-1}\}$ and $N_{AB}(C) = \{A_{n-3}, B_{n-2}, B_{n-3}, \overline{B}_{n-3}\}$. Hence, the result is true.

Case 1.4.3: $C = B \oplus S_j \oplus S_k$. $d(B, C) = 2$ implies that |j-k| $\neq 0, 1$. When $|k - j| = 2$, the case is the same as that discussed in Case 1.4.1. Without loss of generality, assume that $j < k$. When $k = n$, the proof of the case can be obtained by the proof of Case 1.4.2. Now, we need to consider only the following situation: $j \leqslant k - 3$ and $k \neq n$. Since $d(A, C) = 2$, we have that $k = i = n - 1$. Then, $N_{AC} = \{\overline{A}_j, \overline{A}_n\}$, $N_{BC} = \{\overline{B}_j, \overline{B}_{n-1}\}$ and $N_{AB}(C) = \{\overline{A}_j, \overline{B}_j\}$.

Case 1.5: $i = n$.

Case 1.5.1: $C = B \oplus O_j \oplus O_k$. Without loss of generality, suppose that $j < k$. Similar to Case 1.1.1, we have that any two of i, j, k are different. Since $d(A, C) = 2$, the following cases are impossible: (1) $j = n - 2, k = n - 1$, (2) $k = n - 2, j = n - 4$, (3) $k \leqslant n - 3$, and (4) $k - j \geqslant 3$. Thus, we need to consider only the following subcases.

Case 1.5.1.1: $j = n - 3, k = n - 1$. We have that $N_{AC} = \{\overline{A}_{n-3}, A_{n-3}, A_{n-2}, \overline{A}_{n-1}\}$ and $N_{BC} = \{B_{n-3}, B_{n-1}\}$. Subsequently, $N_{AB}(C) = \{\overline{A}_{n-3}, A_{n-3}, A_{n-2}, B_{n-3}\}$. Hence, the result is true.

Case 1.5.1.2: $j = n - 3, k = n - 2$. We have that $N_{AC} = \{\overline{A}_{n-3}, A_{n-1}\}$, $N_{BC} = \{\overline{B}_{n-3}, B_{n-2}\}$, and $N_{AB}(C) = \{\overline{A}_{n-3}, A_{n-1}, \overline{B}_{n-3}, B_{n-2}\}$. Hence, the result is true.

Case 1.5.2: $C = B \oplus O_j \oplus S_k (k \neq n)$. $d(B, C) = 2$ implies that $j \neq k, k - 1$. $d(A, C) = 2$ implies that $j \neq i$. We need to consider only the following cases.

Case 1.5.2.1: $k < j$. Since $d(A, C) = 2$, the following cases are impossible: (1) $j = n - 1, k = n - 2$, (2) $j = n - 2, k < n - 3$ and (3) $j \leqslant n - 3$. Thus, we need to consider only the following subcases.

Case 1.5.2.1.1: $k = n - 3, j = n - 1$. Then, $N_{AC} = \{A_{n-3}, \overline{A}_{n-1}, A_{n-2}, \overline{A}_{n-3}\}$, $N_{BC} = \{\overline{B}_{n-3}, B_{n-1}\}$ and $N_{AB}(C) = \{A_{n-3}, \overline{B}_{n-3}, A_{n-2}, \overline{A}_{n-3}\}$. Hence, the result is true.

Case 1.5.2.1.2: $k < n - 3, j = n - 1$. Then, $N_{AC} = \{\overline{A}_{n-1}, \overline{A}_k\}$, $N_{BC} = \{\overline{B}_k, B_{n-1}\}$ and $N_{AB}(C) = \{\overline{A}_k, \overline{B}_k\}$. Hence, the result is true.

Case 1.5.2.1.3: $k = n - 3, j = n - 2$. Then, $N_{AC} = \{A_{n-3}, A_{n-1}\}$, $N_{BC} = \{B_{n-2}, \overline{B}_{n-3}, B_{n-3}, \overline{B}_{n-1}\}$ and $N_{AB}(C) = \{B_{n-2}, \overline{B}_{n-3}, B_{n-3}, A_{n-3}\}$. Hence, the result is true.

Case 1.5.2.1.4: $j < k$. By $j \neq k - 1$ and $d(A, C) = 2$, we conclude that this case is impossible.

Case 1.5.3: $C = B \oplus S_j \oplus S_k$. $d(B, C) = 2$ implies that |j-k| $\neq 0, 1$. When $|k - j| = 2$, the case is the same as that discussed in Case 1.5.1. Without loss of generality, assume that $j < k$. $d(A, C) = 2$ means that $k \neq n$. Now, we need to consider only the following situation: $j \leqslant k - 3$ and $k \neq n$. Since $d(A, C) = 2$, we have that $k = n - 1$. Then, $N_{AC} = \{\overline{A}_j, A_{n-1}\}$, $N_{BC} = \{\overline{B}_j, \overline{B}_{n-1}\}$, and thus $N_{AB}(C) = \{\overline{A}_j, \overline{B}_j\}$. Hence, the result is true.

Case 2: $B = A \oplus S_i$.

Case 2.1: $i = 1$.

Case 2.1.1: $C = B \oplus O_j \oplus O_k$. Without loss of generality, suppose that $j < k$. By $d(A, C) = 2$ and $d(B, C) = 2$, we have that $j = 1$ and $k > 2$. If $k = 3$, then $N_{AC} = \{A_2, \overline{A}_2, A_3, \overline{A}_4\}$, $N_{BC} = \{B_1, B_3\}$ and $N_{AB}(C) = \{A_2, B_3, A_3, \overline{A}_4\}$. If $k > 3$, then $N_{AC} = \{\overline{A}_2, A_k\}$, $N_{BC} = \{B_1, B_k\}$ and $N_{AB}(C) = \{A_k, B_k\}$. Hence, the result is true.

Case 2.1.2: $C = B \oplus O_j \oplus S_k (k \neq n)$. $d(B, C) = 2$ implies that $j \neq k, k - 1$. Consider the following cases:

Case 2.1.2.1: $k \leqslant j - 1$. By $d(A, C) = 2$ and $d(B, C) = 2$, we have that $k = 2$. Then, $N_{AC} = \{A_1, A_j\}$, $N_{BC} = \{B_j, \overline{B}_2\}$ and $N_{AB}(C) = \{A_j, B_j\}$. Hence, the result is true.

Case 2.1.2.2: $k > j + 1$. $d(A, C) = 2$ implies that $j = 1$ and $k \geqslant 4$. If $k = 4$, then $N_{AC} = \{A_2, A_3, \overline{A}_2, \overline{A}_4\}$, $N_{BC} = \{B_1, \overline{B}_4\}$, and thus $N_{AB}(C) = \{A_2, A_3, \overline{B}_4, \overline{A}_4\}$. If $k > 4$, then $N_{AC} = \{\overline{A}_2, \overline{A}_k\}$, $N_{BC} = \{B_1, \overline{B}_k\}$, and thus $N_{AB}(C) = \{\overline{A}_k, \overline{B}_k\}$. Hence, the result is true.

Case 2.1.3: $C = B \oplus S_j \oplus S_k$. $d(B, C) = 2$ implies that |j-k| $\neq 0, 1$. When $|k - j| = 2$, the case is the same as that discussed in Case 2.1.1. Without loss of generality, assume that $j < k$. When $k = n$, the case is the same as that discussed in Case 2.1.1. Now, we need to consider only the following situation in which $j \leqslant k - 3$ and $k \neq n$. By $d(A, C) = 2$ and

$j \leqslant k-3$, we conclude that $j = 2$. Then, $N_{AC} = \{A_1, A_k\}$, $N_{BC} = \{\overline{B}_2, \overline{B}_k\}$ and $N_{AB}(C) = \{A_k, \overline{B}_k\}$.

Case 2.2: $i \in \{2, 3, \cdots, n-1\}$.

Case 2.2.1: $C = B \oplus O_j \oplus O_k$. Without loss of generality, suppose that $j < k$. By $d(A, C) = 2$ and $d(B, C) = 2$, we need to consider only the following cases:

Case 2.2.1.1: $j = i$ and $k > i+1$. If $k = i+2$, then $N_{AC} = \{\overline{A}_{i+1}, A_{i+1}, A_{i+2}, \overline{A}_{i+3}\}$, $N_{BC} = \{B_i, B_{i+2}\}$, and thus $N_{AB}(C) = \{B_{i+2}, A_{i+1}, A_{i+2}, \overline{A}_{i+3}\}$. If $k > i+2$, then $N_{AC} = \{\overline{A}_{i+1}, A_k\}$, $N_{BC} = \{B_i, B_k\}$, and thus $N_{AB}(C) = \{B_k, A_k\}$. Hence, the result is true.

Case 2.2.1.2: $j = i-1$. If $k = i$, then $N_{AC} = \{A_{i_1}, \overline{A}_{i-1}, A_i, \overline{A}_{i+1}\}$, $N_{BC} = \{B_{i-1}, B_i\}$, and $N_{AB}(C) = \{A_{i_1}, B_i, A_i, \overline{A}_{i+1}\}$. If $k > i$, then $N_{AC} = \{\overline{A}_{i-1}, A_k\}$, $N_{BC} = \{B_{i-1}, B_k\}$, and $N_{AB}(C) = \{A_k, B_k\}$. Hence, the result is true.

Case 2.2.1.3: $j \leqslant i-2$ and $k = i$. Then, $N_{AC} = \{\overline{A}_{i+1}, A_j\}$, $N_{BC} = \{B_j, B_k\}$ and $N_{AB}(C) = \{A_j, B_j\}$ (if $i = 2$, the case does not exist and thus is not considered here). Hence, the result is true.

Case 2.2.1.4: $j = i-3$ and $k = i-1$. Then, $N_{AC} = \{\overline{A}_{i-3}, A_{i-2}, \overline{A}_{i-1}, A_{i-3}\}$, $N_{BC} = \{B_{i-3}, B_{i-1}\}$ and $N_{AB}(C) = \{\overline{A}_{i-3}, B_{i-3}, A_{i-2}, A_{i-3}\}$ (if $i < 3$, the case does not exist and thus is not considered here). Hence, the result is true.

Case 2.2.1.5: $j < i-3$ and $k = i-1$. Then, $N_{AC} = \{A_j, \overline{A}_{i-1}\}$, $N_{BC} = \{B_j, B_{i-1}\}$ and $N_{AB}(C) = \{\overline{A}_j, B_j\}$ (if $i < 4$, the case does not exist and thus is not considered here). Hence, the result is true.

Case 2.2.1.6 $j = i-3$ and $k = i-2$. Then, $N_{AC} = \{\overline{A}_{i-3}, A_{i-1}\}$, $N_{BC} = \{\overline{B}_{i-3}, \overline{B}_{i-1}, B_{i-3}, B_{i-2}\}$ and $N_{AB}(C) = \{\overline{A}_{i-3}, B_{i-3}, B_{i-3}, B_{i-2}\}$ (if $i < 4$ or $i = n-1$, the case does not exist and thus is not considered here).

Case 2.2.2: $C = B \oplus O_j \oplus S_k (k \neq n)$. $d(A, C) = 2$ implies that $k \neq i$. $d(B, C) = 2$ implies that $j \neq k, k-1$. By $d(A, C) = 2$ and $d(B, C) = 2$, we need to consider only the following cases:

Case 2.2.2.1: $i < k \leqslant j-1$. By $d(A, C) = 2$ and $d(B, C) = 2$, we have that $k = i+1$. $d(B, C) = 2$ implies that the situation in which $k = j-1 = n-1$ is impossible. If $k = j-1 < n-1$, then $N_{AC} = \{A_i, A_{i+2}\}$, $N_{BC} = \{\overline{B}_{i+1}, B_{i+2}, B_{i+1}, \overline{B}_{i+3}\}$, and $N_{AB}(C) = \{B_{i+2}, B_{i+1}, \overline{B}_{i+3}, A_{i+2}\}$. If $k < j-1$, then $N_{AC} = \{A_i, A_j\}$, $N_{BC} = \{\overline{B}_{i+1}, B_j\}$ and $N_{AB}(C) = \{A_j, B_j\}$. Hence, the result is true.

Case 2.2.2.2: $k < i \leqslant j-1$. By $d(A, C) = 2$ and $d(B, C) = 2$, we have that $k = i-1$.

If $i = j-1 = n-1$, then $N_{AC} = \{A_{n-2}, \overline{A}_{n-2}, A_{n-1}, A_n\}$, $N_{BC} = \{\overline{B}_{n-2}, B_{n-1}, B_n, B_{n-2}\}$, and $N_{AB}(C) = \phi$. If $i = j-1 < n-1$, then $N_{AC} = \{A_{i-1}, A_j\}$, $N_{BC} = \{\overline{B}_{i-1}, B_j\}$ and $N_{AB}(C) = \{A_j, B_j\}$. Hence, the result is true.

Case 2.2.2.3: $k \leqslant j-1 < i$. By $d(A, C) = 2$ and $d(B, C) = 2$, we conclude that one of the following three conditions holds: 1) $i = j$; 2) $j = i-1$ and $k \leqslant j-2$; or 3) $j = i-2$ and $k = j-1$.

1) If $i = j$ and $k = j-1$, then $N_{AC} = \{A_{i-1}, A_i, \overline{A}_{i-1}, \overline{A}_{i+1}\}$, $N_{BC} = \{B_i, \overline{B}_{i-1}, B_{i-1}, \overline{B}_{i+1}\}$, and $N_{AB}(C) = \phi$. Hence, the result is true.

If $i = j$ and $k < j-1$, then $N_{AC} = \{\overline{A}_k, \overline{A}_{i+1}\}$, $N_{BC} = \{B_i, \overline{B}_k\}$ and $N_{AB}(C) = \{\overline{A}_k, \overline{B}_k\}$. Hence, the result is true.

2) $j = i-1$ and $k \leqslant j-2$. If $k = j-2$, then $N_{AC} = \{A_{i-3}, A_{i-2}, \overline{A}_{i-3}, \overline{A}_{i-1}\}$, $N_{BC} = \{B_{i-1}, \overline{B}_{i-3}\}$ and $N_{AB}(C) = \{A_{i-3}, A_{i-2}, \overline{A}_{i-3}, \overline{B}_{i-3}\}$. If $k < j-2$, then $N_{AC} = \{\overline{A}_k, \overline{A}_{i-1}\}$, $N_{BC} = \{B_{i-1}, \overline{B}_k\}$ and $N_{AB}(C) = \{A_k, \overline{B}_k\}$. Hence, the result is true.

3) If $j = i-2$ and $k = j-1$, then $N_{AC} = \{A_{i-3}, A_{i-1}\}$, $N_{BC} = \{B_{i-3}, \overline{B}_{i-3}, B_{i-2}, \overline{B}_{i-1}\}$, and $N_{AB}(C) = \{A_{i-3}, B_{i-3}, \overline{B}_{i-3}, B_{i-2}, \}$. Hence, the result is true.

Case 2.2.2.4: $i \leqslant j+1 < k$. By $d(A, C) = 2$ and $d(B, C) = 2$, we conclude that either $i = j$ and $k \geqslant j+3$ or $j = i+1$ and $k = j+2$. We discuss both situations in the following.

1) $i = j, k \geqslant j+3$. If $k = j+3$, then $N_{AC} = \{A_{i+1}, A_{i+2}, \overline{A}_{i+1}, \overline{A}_{i+3}\}$, $N_{BC} = \{B_i, \overline{B}_{i+3}\}$, and $N_{AB}(C) = \{A_{i+1}, A_{i+2}, \overline{B}_{i+3}, \overline{A}_{i+3}\}$. Hence, the result is true.

If $k > j+3$, then $N_{AC} = \{\overline{A}_{i+1}, \overline{A}_k\}$, $N_{BC} = \{B_i, \overline{B}_k\}$, and $N_{AB}(C) = \{\overline{B}_k, \overline{A}_k\}$. Hence, the result is true.

2) If $j = i+1$ and $k = j+2$, then $N_{AC} = \{A_i, A_{i+2}\}$, $N_{BC} = \{B_{i+1}, B_{i+2}, \overline{B}_{i+1}, \overline{B}_{i+3}\}$, and $N_{AB}(C) = \{B_{i+1}, B_{i+2}, A_{i+2}, \overline{B}_{i+3}\}$. Hence, the result is true.

Case 2.2.2.5: $j+1 < i < k$. By $d(A, C) = 2$ and $d(B, C) = 2$, we conclude that $k = i+1$. Then, $N_{AC} = \{A_i, A_j\}$, $N_{BC} = \{B_j, \overline{B}_{i+1}\}$, and $N_{AB}(C) = \{A_j, B_j\}$. Hence, the result is true.

Case 2.2.2.6: $j+1 < k < i$. By $d(A, C) = 2$ and $d(B, C) = 2$, we conclude that $k = i-1$. Then, $N_{AC} = \{A_{i-1}, A_j\}$. If $k = j+2$, then $N_{BC} = \{B_j, \overline{B}_{i-1}, \overline{B}_j, B_{i-2}\}$ and $N_{AB}(C) = \{B_j, A_j, \overline{B}_j, B_{i-2}\}$. If $k > j+2$, then $N_{BC} = \{B_j, \overline{B}_{i-1}\}$ and $N_{AB}(C) = \{B_j, A_j\}$. Hence, the result is true.

Case 2.2.3: $C = B \oplus S_j \oplus S_k$. $d(B, C) = 2$ implies that |j-k| $\neq 0, 1$. By $d(A, C) = 2$, we conclude that any two of i, j, k are different. When $|k-j| = 2$, the case is the same as that discussed in Case 2.1.1. Without loss of generality, assume that $j < k$. When $k = n$, the case is the same as that discussed in Case 2.2.2. Now, we need to consider only the following subcases in which $j \leqslant k-3$ and $k \neq n$.

Case 2.2.3.1: $i < j \leqslant k-3$. By $d(A, C) = 2$, we have that $i = j-1$. Then, $N_{AC} = \{A_i, \overline{A}_k\}$, $N_{BC} = \{\overline{B}_j, \overline{B}_k\}$, and $N_{AB}(C) = \{\overline{A}_k, \overline{B}_k\}$. Hence, the result is true.

Case 2.2.3.2: $j < i$ and $j \leqslant k-3$. By $d(A, C) = 2$, we have that either $j = i-1$ or $|i-k| = 1$. If $j = i-1$, then $N_{AC} = \{A_j, \overline{A}_k\}$, $N_{BC} = \{B_j, \overline{B}_k\}$ and $N_{AB}(C) = \{A_j, B_j, \overline{A}_k, \overline{B}_k\}$. If $i = k-1$, then $N_{AC} = \{A_i, \overline{A}_j\}$, $N_{BC} = \{\overline{B}_j, \overline{B}_k\}$ and $N_{AB}(C) = \{\overline{B}_j, \overline{A}_j\}$. If $i = k+1$, then $N_{AC} = \{A_k, \overline{A}_j\}$, $N_{BC} = \{\overline{B}_j, \overline{B}_k\}$ and $N_{AB}(C) = \{\overline{B}_j, \overline{A}_j\}$. Hence, the result is true.

Case 2.3: $i = n$. This case is included in Case 1. ■

Property 6: Let $A, B, C \subseteq V(AQ_n)$. Suppose that $d(A, B) = 2, d(A, C) = 2, d(B, C) = 2$. Then, $|N_{AB}(C)| \leqslant 6$.

Proof:

Case 1: $C = A \oplus O_i \oplus O_j (i < j)$.

Case 1.1: $B = C \oplus O_k \oplus O_l (k < l)$. By Property 2 and the symmetry of A, B, C, we need to consider only the following three cases: 1) $j = i+1 (i \leqslant n-2)$ and $l = k+1 (k \leqslant n-2)$; 2) $j = i+1 (i \leqslant n-2)$ and $l = k+2 = n$; and 3) $j = i+2 = n$ and $l = k+2 = n$.

Case 1.1.1: $j = i + 1(i \leqslant n - 2)$ and $l = k + 1(k \leqslant n - 2)$. Then, $N_{AC} = \{A_i, A_{i+1}, \overline{A}_{i+2}, \overline{A}_i\}$ and $N_{BC} = \{C_k, C_{k+1}, \overline{C}_{k+2}, \overline{C}_k\}$. Without loss of generality, suppose that $j \leqslant k$. By $d(A, B) = 2$, we conclude that either $j = k$ or $j = k - 1$. Then, either $N_{AB}(C) = \{A_{i+1}, \overline{A}_{i+2}, \overline{A}_i, C_{k+1}, \overline{C}_{k+2}, \overline{C}_k\}(j = k)$ or $N_{AB}(C) = \{A_i, A_{i+1}, \overline{A}_{i+2}, C_k, C_{k+1}, \overline{C}_{k+2}\}(j = k - 1)$. Hence, the result is true.

Case 1.1.2: $j = i + 1(i \leqslant n - 2)$ and $l = k + 2 = n$. Then, $N_{AC} = \{A_i, A_{i+1}, \overline{A}_{i+2}, \overline{A}_i\}$ and $N_{BC} = \{C_n, C_{n-2}, \overline{C}_{n-2}, \overline{C}_{n-1}\}$. By $d(A, B) = 2$, we conclude that $n - 3 \leqslant j \leqslant n - 1$. Then, either $N_{AB}(C) = \{A_{n-4}, A_{n-3}, \overline{A}_{n-2}, C_n, C_{n-2}, \overline{C}_{n-1}\}(j = n - 3)$, $N_{AB}(C) = \{A_{n-2}, \overline{A}_{n-1}, C_n, \overline{C}_{n-2}\}(j = n - 2)$ or $N_{AB}(C) = \{A_{n-2}, \overline{C}_{n-1}\}(j = n - 1)$. Hence, the result is true.

Case 1.1.3: $j = i + 2 = n$ and $l = k + 2 = n$. By $d(A, B) = 2$, we conclude that this case is impossible.

Case 1.2: $B = C \oplus O_k \oplus S_l(l \neq n)$. By Property 2, we need to consider only the following four cases: 1) $j = i + 1(i \leqslant n - 2)$ and $l = k + 2$; 2) $j = i + 1(i \leqslant n - 2)$ and $l = k - 1$; 3) $j = i + 2 = n$ and $l = k + 2$; and 4) $j = i + 2 = n$ and $l = k - 1$.

Case 1.2.1: $j = i + 1(i \leqslant n - 2)$ and $l = k + 2$. Then, $N_{AC} = \{A_i, A_{i+1}, \overline{A}_{i+2}, \overline{A}_i\}$ and $N_{BC} = \{C_k, C_{k+1}, \overline{C}_{k+2}, \overline{C}_k\}$. By $d(A, B) = 2$, we conclude that either $j = k - 1$, $j = k$, $j = k + 2$ or $j = k + 3$, and then either $N_{AB}(C) = \{A_i, A_{i+1}, \overline{A}_{i+2}, C_k, C_{k+1}, \overline{C}_{k+2}\}(j = k - 1)$, $N_{AB}(C) = \{A_{i+1}, \overline{A}_{i+2}, \overline{A}_i, C_{k+1}, \overline{C}_{k+2}, \overline{C}_k\}(k = j)$, $N_{AB}(C) = \{A_i, \overline{A}_i, \overline{A}_{i+2}, C_k, \overline{C}_k, \overline{C}_{k+2}\}(j = k + 2)$ or $N_{AB}(C) = \{A_i, A_{i+1}, \overline{A}_i, C_k, C_{k+1}, \overline{C}_k\}(j = k + 3)$. Hence, the result is true.

Case 1.2.2: $j = i + 1(i \leqslant n - 2)$ and $l = k - 1$. A similar argument to that made in Case 1.2.1 can be used here.

Case 1.2.3: $j = i + 2 = n$ and $l = k + 2$. Then, $N_{AC} = \{A_{n-2}, A_{n-1}, \overline{A}_{n-2}, A_n\}$ and $N_{BC} = \{C_k, C_{k+1}, \overline{C}_{k+2}, \overline{C}_k\}$. By $d(A, B) = 2$, we conclude that either $k = n - 4$ or $k = n - 3$, and then either $N_{AB}(C) = \{A_{n-2}, \overline{A}_{n-2}, A_n, C_k, C_{k+1}, \overline{C}_k\}(k = n - 4)$ or $N_{AB}(C) = \{A_{n-2}, \overline{A}_{n-2}, A_{n-1}, C_k, \overline{C}_{k+2}, \overline{C}_k\}(k = n - 3)$. Hence, the result is true.

Case 1.2.4: $j = i + 2 = n$ and $l = k - 1$. Then, $N_{AC} = \{A_{n-2}, A_{n-1}, \overline{A}_{n-2}, A_n\}$ and $N_{BC} = \{C_k, C_{k-1}, \overline{C}_{k-1}, \overline{C}_{k+1}\}$. By $d(A, B) = 2$, we conclude that either $k = n - 3$ or $k = n - 2$, and then either $N_{AB}(C) = \{A_{n-2}, \overline{A}_{n-2}, A_n, C_k, C_{k-1}, \overline{C}_{k-1}\}(k = n - 3)$ or $N_{AB}(C) = \{A_{n-2}, \overline{A}_{n-2}, A_{n-1}, C_{k-1}, \overline{C}_{k-1}, \overline{C}_{k+1}\}(k = n - 2)$. Hence, the result is true.

Case 1.3: $B = C \oplus S_k \oplus S_l(l < k)$. By Property 2, we need to consider only the following two cases: 1) $j = i + 1(i \leqslant n - 2)$ and $l = k + 2$; and 2) $j = i + 2 = n$ and $l = k + 2$.

Case 1.3.1: $j = i + 1(i \leqslant n - 2)$ and $l = k + 2$. Note that when $l = k + 2$, we have that $B = C \oplus S_k \oplus S_l = C \oplus O_k \oplus O_{k+1}$, which implies that an argument similar to that made in Case 1.1.1 can be used in this case.

Case 1.3.2: $j = i+2 = n$ and $l = k+2$. A similar argument to that made in Case 1.1.2 can be used to prove that the result is true in this case.

Case 2: $C = A \oplus O_i \oplus S_j(j \neq n)$.

Case 2.1: $B = C \oplus O_k \oplus O_l(k < l)$. Noting that $A = C \oplus O_i \oplus S_j$ and $C = B \oplus O_k \oplus O_l(k < l)$, by Case 1.2, we conclude that the result is true.

Case 2.2: $B = C \oplus O_k \oplus S_l(l \neq n)$. By Property 2, we need to consider only the following three cases: 1)$j = i + 2$ and $l = k + 2$; 2) $j = i - 1$ and $l = k + 2$; and 3) $j = i - 1$ and $l = k - 1$.

Case 2.2.1: $j = i + 2$ and $l = k + 2$. Then, $N_{AC} = \{A_i, A_{i+1}, \overline{A}_{i+2}, \overline{A}_i\}$ and $N_{BC} = \{C_k, C_{k+1}, \overline{C}_{k+2}, \overline{C}_k\}$. By $d(A, B) = 2$, we conclude that either $k = i - 1$, $k = i - 2$, $k = i + 1$ or $k = i + 2$. Then, either $N_{AB}(C) = \{A_i, A_{i+1}, \overline{A}_i, C_k, \overline{C}_{k+2}, \overline{C}_k\}(k = i - 1)$, $N_{AB}(C) = \{A_i, \overline{A}_{i+2}, \overline{A}_i, C_k, C_{k+1}, \overline{C}_k\}(k = i - 2)$, $N_{AB}(C) = \{A_i, A_{i+1}, \overline{A}_{i+2}, C_{k+1}, \overline{C}_{k+2}, \overline{C}_k\}(k = i - 1)$ or $N_{AB}(C) = \{A_{i+1}, \overline{A}_{i+2}, \overline{A}_i, C_k, C_{k+1}, \overline{C}_{k+2}\}(k = i + 2)$.

Case 2.2.2: $j = i - 1$ and $l = k + 2$. Note that if $j = i - 1$, then $C = A \oplus O_j \oplus S_{j+2}$. Hence, we can use an argument similar to that made in Case 2.2.1 (respectively, Case 1.2.3) to prove the case where $j + 2 \neq n$ (respectively, $j + 2 = n$).

Case 2.2.3: $j = i - 1$ and $l = k - 1$. By the assumption, we have that $C = A \oplus O_j \oplus S_{j+2}$ and $B = C \oplus O_l \oplus S_{l+2}$. We can use a similar argument to that made in Case 2.2.1 to prove this case.

Case 2.3: $B = C \oplus S_k \oplus S_l(k < l, l \neq n)$. By Property 2, we need to consider only the situation where $l = k+1$. On the other hand, $l = k + 1$ implies that $B = C \oplus O_k \oplus O_{k+1}$. Furthermore, by Case 2.1, we conclude that the result is true.

Case 3: $C = A \oplus S_i \oplus S_j(i < j)$. By Property 2, we need to consider only the situation where $j = i + 1$. On the other hand, $j = i+1$ implies that $C = A \oplus O_i \oplus O_{i+1}$. Furthermore, by Case 2, we conclude that the result is true in this case. ■

Theorem 3: For any subset $S \subset V(AQ_n)$ with $|S| = \alpha$, where $1 \leqslant \alpha \leqslant 2n - 1$, the following conditions hold.

i) If there exist two nodes $x, y \in S$ such that $d(x, y) = 1$, then there exists a node $v \in \{x, y\}$ such that $|PN(v)| \geqslant 2n - 2.5\alpha - 1$. Otherwise,

ii) There exists a node $v \in S$ such that $|PN(v)| \geqslant 2n - 3\alpha + 1$.

Proof: For the sake of convenience, we introduce the following notations: suppose that $u, v, w \in V(AQ_n)$; let $N_{uv}(w) = (N(u) \cap N(w)) \cup (N(v) \cap N(w)) - N(u) \cap N(v) \cap N(w) - \{u, v\}$.

i) There exist $x, y \in S$ such that $d(x, y) = 1$. By Property 1, we have that x and y share two or four common neighbors. Hence, they have $2(2n-1-1-2)$ or $2(2n-1-1-4)$ neighbors that are not shared by them. For each node $z \in S - \{x, y\}$, consider the following cases:

(1) $d(x, z) = 1$ and $d(y, z) = 1$. In this case, by Property 3, we have that $|N_{xy}(z)| \leqslant 2$.

(2) $d(x, z) = 1$ and $d(y, z) = 2$. In this case, by Property 4, we have that $|N_{xy}(z)| \leqslant 5$.

(3) $d(x, z) = 2$ and $d(y, z) = 2$. In this case, by Property 5, we have that $|N_{xy}(z)| \leqslant 5$.

(4) $d(x, z) = 2$ and $d(y, z) > 2$. In this case, by Property 2, we have that $N(x) \cap N(z) \leqslant 4$. On the other hand, $d(y, z) > 2$ implies that $N(y) \cap N(z) = \phi$. Hence, we have that $|N_{xy}(z)| \leqslant 4$.

In summary, for every $z \in S - \{x, y\}$, we have that $|N_{xy}(z)| \leqslant 5$. Furthermore, $|PN(x)| + |PN(y)| \geqslant 2(2n - 1 - 1 - 4) - 5(\alpha - 2) = 4n - 5\alpha - 2$. Hence, either $|PN(x)| \geqslant 2n - 2.5\alpha - 1$ or $|PN(y)| \geqslant 2n - 2.5\alpha - 1$ (or both of them) holds.

ii) We distinguish between two subcases:

Case 1: There exist two nodes $x, y \in S$ such that $d(x, y) = 2$. By Property 2, we have that x and y share two or four common neighbors. Hence, they have $2(2n-1-2)$ or $2(2n - 1 - 4)$ neighbors that are not shared by them. For each node $z \in S - \{x, y\}$, consider the following cases:

(1) $d(x, z) = d(y, z) = 2$. In this case, according to Property 6, we have that $|N_{xy}(z)| \leqslant 6$.

(2) $d(x, z) = 2$ and $d(y, z) > 2$. In this case, by Property 2, we have that $N(x) \cap N(z) \leqslant 4$. On the other hand, $d(y, z) > 2$ implies that $N(y) \cap N(z) = \phi$. Hence, we have that $|N_{xy}(z)| \leqslant 4$.

(3) $d(x, z) > 2$ and $d(y, z) > 2$. In this case, z has no common neighbors to share with either x or y, which implies that $|N_{xy}(z)| = 0$.

In summary, for every $z \in S - \{x, y\}$, we have that $|N_{xy}(z)| \leqslant 6$. Furthermore, $|PN(x)| + |PN(y)| \geqslant 2(2n - 1 - 4) - 6(\alpha - 2) = 4n - 6\alpha + 2$. Hence, either $|PN(x)| \geqslant 2n - 3\alpha + 1$ or $|PN(y)| \geqslant 2n - 3\alpha + 1$ (or both of them) holds.

Case 2: For each pair $u, v \in S$, $d(u, v) > 2$. Choose a pair of nodes $x, y \in X$; since they have no common neighbor, $|PN(x)| = |PN(y)| = 2n - 1 > 2n - 3\alpha + 1$. ∎

The following Lemma 6 follows [30].

Lemma 6: Let $AQ_n = AQ^0_{n-1} \odot AQ^1_{n-1}$ and $x, y \in V(AQ^0_{n-1})$ (respectively, $x, y \in V(AQ^1_{n-1})$). Then, $x = \bar{y}_2$ if and only if they have exactly two common neighbors that belong to $V(AQ^1_{n-1})$ (respectively, $V(AQ^0_{n-1})$). Moreover, if $x \neq \bar{y}_2$, then they have no common neighbors in $V(AQ^1_{n-1})$ (respectively, $V(AQ^0_{n-1})$).

The following Lemma 7 follows [1].

Lemma 7: Suppose that $G(X, E)$ is the diagnostic graph of a system H with N nodes. Then, H is t/k-diagnosable if the following two conditions hold:

(1) For each $X' \subset X$ with $|X'| = k + 1$, $|N_G(X')| > t - 1$.

(2) For each $X' \subset X$ with $|X'| = 2(k + q)$, $|N_G(X')| > t - (k + q)$, where q is an integer, with $1 \leqslant q \leqslant t - k$.

The following Lemma 8 follows [30].

Lemma 8: Let $G(X, E)$ be the diagnostic graph of an n-dimensional augmented cube AQ_n and $X' \subset X$, with $|X'| = k$. Then:

(1) If $k = 1$, then $|N_G(X')| = 2n - 1$.

(2) If $k = 2$, then $|N_G(X')| \geqslant 4n - 8$.

(3) If $k = 3$, then $|N_G(X')| \geqslant 6n - 17$.

(4) If $k = 4$, then $|N_G(X')| \geqslant 8n - 28$.

Lemma 9: Let $G(X, E)$ be the diagnostic graph of an n-dimensional augmented cube AQ_n and $X' \subset X$, with $|X'| = k, 0 < k \leqslant 2n - 1$. Then, $|N_G(X')| \geqslant 2kn - \frac{3k(k+1)}{2} + 1$.

Proof: We prove that the result is true by induction on k. According to Lemma 8, it is easily seen that the result is true when $1 \leqslant k \leqslant 4$. Next, suppose that the result is true for $4 \leqslant k < 2n - 1$. We will show that it is also true for $k + 1$. By contradiction, suppose that there exists $X' \subset X$, with $|X'| = k + 1$, such that $|N_G(X')| < 2(k+1)n - \frac{3(k+1)(k+2)}{2} + 1$. Consider the following cases:

Case 1: There exists two nodes $x, y \in X'$ such that $(x, y) \in E$. By Theorem 3, one of x and y, say x, satisfies the following: $|PN(x)| \geqslant 2n - 2.5\alpha - 1$. Let $X'' = X' - \{x\}$. Then, $N_G(X'') = (N_G(X') - PN(x)) \cup \{x\}$; subsequently, $|N_G(X'')| = |N_G(X')| - |PN(x)| + 1 < 2(k + 1)n - \frac{3(k+1)(k+2)}{2} + 1 - (2n - 2.5(k + 1) - 1) + 1 < 2kn - \frac{3k(k+1)}{2} + 1$, which is a contradiction.

Case 2: For any two nodes $x, y \in X'$, $(x, y) \notin E$. By Theorem 3, there exists a node $v \in S$ such that $|PN(v)| \geqslant 2n - 3(k + 1)$. Let $X'' = X' - \{v\}$. Then, $N_G(X'') = (N_G(X') - PN(v))$; subsequently, $|N_G(X'')| = |N_G(X')| - |PN(v)| < 2(k + 1)n - \frac{3(k+1)(k+2)}{2} + 1 - (2n - 3(k + 1) + 1) < 2kn - \frac{3k(k+1)}{2} + 1$, which is a contradiction. ∎

Lemma 10: Let $G(X, E)$ be the diagnostic graph of an n-dimensional augmented cube AQ_n and $P' \subset X$, with $|P'| = k+r+1, t = 2(k+1)n - \frac{3(k+1)(k+2)}{2} + 1$, and $F = |N_G(P')| - t$, where $0 \leqslant k \leqslant \frac{4n}{9} - \frac{13}{9}, n > 5, 0 \leqslant r \leqslant k + 1$. Then, the following conditions are true:

(*) If $r = 1$, then $F \geqslant 2$.

(**) If $2 \leqslant r \leqslant k + 1$, then $F \geqslant 1$.

Proof: The assumption that $k \leqslant \frac{4n}{9} - \frac{13}{9}$ and $n > 5$ implies that $k \leqslant \frac{4n}{9} - \frac{13}{9} \leqslant \frac{2n}{3} - \frac{8}{3}$. Moreover, $k + r \leqslant 2k + 1 \leqslant 2n - 1$. By Lemma 9, we have that the following inequality holds: $F = |N_G(P')| - t \geqslant [2(k + 1 + r)n - \frac{3(k+r+1)(k+r+2)}{2} + 1] - [2(k+1)n - \frac{3(k+1)(k+2)}{2} + 1] = \frac{r}{2}(4n - 9 - 3(r + 2k))$. Note that $k \leqslant \frac{2n}{3} - \frac{8}{3}$; when $r = 1$, $F \geqslant \frac{r}{2}(4n - 9 - 3(r + 2k)) = 2n - 3k - 6 \geqslant 2$. Similarly, when $r = 2$, $F \geqslant \frac{r}{2}(4n - 9 - 3(r + 2k)) = 4n - 6k - 15 \geqslant 1$. If $3 \leqslant r \leqslant k + 1$, then $4n - 9 - 3(r + 2k) \geqslant 9(\frac{4n}{9} - \frac{12}{9} - k)$. Then, by the assumption that $k \leqslant \frac{4n}{9} - \frac{13}{9}$, we have that $F \geqslant 1$. ∎

Theorem 4: An n-dimensional augmented cube network AQ_n is t/k-diagnosable for $t = 2(k + 1)n - \frac{3(k+1)(k+2)}{2} + 1$, where $k \leqslant \frac{4n}{9} - \frac{13}{9}$ and $n > 5$.

Proof: Let $G(X, E)$ be the diagnostic graph of AQ_n. (1) For each $X' \subset X$, with $|X'| = k+1$, by Lemma 9, we have that $|N_G(X')| = 2(k + 1)n - \frac{3(k+1)(k+2)}{2} + 1 > t - 1$. Hence, condition (1) in Lemma 7 is satisfied.

(2) Now, for each integer q, with $1 \leqslant q \leqslant t - k$, let $X' \subset X$ be a subset of X, with $|X'| = 2(k + q)$. We will show that $|N_G(X')| > t - (k + q)$. Let $X' = \{x^{(1)}, \cdots, x^{2(k+q)}\}$ denote the subset mentioned above and $x^{(i)}_1 \cdots x^{(i)}_n$ denote the addresses of node $x^{(i)}, i = 1, 2, \cdots, 2(k + q)$. Consider the following cases:

Case 1: There exists at least one bit position, say the ith position, such that the total number of '0's is different from the total number of '1's for the ith position of these $2(k+q)$ node addresses. Without loss of generality, assume that the total number of '0's is different from the total number of '1's for the ith position of these $2(k+q)$ node addresses and that $i=1$; this is always possible to achieve (by renumbering the nodes). Moreover, without loss of generality, assume that the first position of each node of $\{x^{(1)}, \cdots, x^{(k+q+1)}\}$ is '0'.

Let $P=\{x^{(1)}, \cdots, x^{(k+1)}\}$, $Q=\{x^{(k+2)}, \cdots, x^{(k+q+1)}\}$, $R=\{x^{(k+q+2)}, \cdots, x^{2(k+q)}\}$, $L_{PQ}=\{v\in Q|\overline{v}_2\in P\}$, and $r=|L_{PQ}|$. It is obvious that $r\leqslant k+1$. Now, consider the following subcases:

Case 1.1: $r=0$. Let $Q''=\{v=1w_2\cdots w_n|v_1=0w_2\cdots w_n\in Q\}$; then, $|Q''|=q$. By Lemma 6, we have that $N_G(P)\cap Q''=\phi$. Then, $|N_G(X')|\geqslant |N_G(P)\cup Q''|-|Q\cup R|>t+q-1-(k+2q-1)=t-(k+q)$.

Case 1.2: $1\leqslant r\leqslant max\{q,k+1\}$. Let $P'=P\cup L_{PQ}$, $Q'=Q-L_{PQ}$, and $Q''=\{v=1w_2\cdots w_n|v_1=0w_2\cdots w_n\in Q'\}$; then, $X'=P'\cup Q'\cup R$, $Q''\subset N_G(Q')$, and $|Q''|=q-r$. By Lemma 6, we have that $N_G(P')\cap Q''=\phi$. Hence, we have the following inequality: $|N_G(X')|\geqslant |N_G(P')\cup Q''|-|Q'\cup R|=|N_G(P')|+(q-r)-(k+2q-r-1)=|N_G(P')|+1-(k+q)$. By Lemma 10, we have that $|N_G(X')|\geqslant |N_G(P')|+1-(k+q)\geqslant t+1-(k+q)>t-(k+q)$.

Case 2: The total number of '0's at each bit position is exactly the same as the total number of '1's at the same position for each bit position of these $2(k+q)$ node addresses. Consider the following cases:

Case 2.1: $q=1$. By the assumption $k\leqslant \frac{4n}{9}-\frac{13}{9}$, we have that $0<2(k+1)<2n-1$. Then, by Lemma 9, $N_G(X')\geqslant 4(k+1)n-3(k+1)(2(k+1)+1)+1$. Consider a function of the variable k: $f(k)=4(k+1)n-3(k+1)(2(k+1)+1)+1-t+(k+1)$. Next, we need to prove only that $f(k)>0$. In fact, $f(k)=-\frac{9}{2}k^2+(2n-\frac{19}{2})k+(2n-5)>0$ if and only if $g(k)=9k^2-(4n-19)k-(4n-10)<0$. $g(k)$ is a quadratic function. After a simple process of computing, we can determine that the two roots of $g(k)=0$ are as follows: $k_1=\frac{1}{18}[(4n-19)-\sqrt{(4n-19)^2+36(4n-10)}]$ and $k_2=\frac{1}{18}[(4n-19)+\sqrt{(4n-19)^2+36(4n-10)}]$. By the assumption that $k\leqslant \frac{4n}{9}-\frac{13}{9}$ and $n>5$, we can determine that $k_1<k<k_2$, which implies that $g(k)<0$.

Case 2.2: $1<q\leqslant t-k$. Let $P=\{x^{(1)}, \cdots, x^{(k+1)}\}$, $Q=\{x^{(k+2)}, \cdots, x^{(k+q)}\}$, and $R=\{x^{(k+q+1)}, \cdots, x^{2(k+q)}\}$. Consider the following cases:

Case 2.2.1: There do not exist two nodes $x\in P, y\in Q$ such that $x=\overline{y}_2$.

Case 2.2.1.1: There do not exist two nodes $u,v\in Q$ such that $u=\overline{v}_2$. Let $P'=\{x^{(1)}, \cdots, x^{(k+1)}\}\cup\{x^{(k+2)}\}$ and $Q'=Q-\{x^{(k+2)}\}$. Then, $X'=P'\cup Q'\cup R$. Let $Q''=\{v=1w_2\cdots w_n|v_1=0w_2\cdots w_n\in Q'\}$; then, $Q''\subset N_G(Q')$ and $|Q''|=q-2$. By Lemma 6, we have that $N_G(P')\cap Q''=\phi$. Hence, we have that the following inequality holds: $|N_G(X')|\geqslant |N_G(P')\cup Q''|-|Q'\cup R|=|N_G(P')|+(q-2)-(k+2q-2)=|N_G(P')|-(k+q)$. By Lemma 10, we have that $F=|N_G(P')|-t\geqslant 2$, which implies that $|N_G(X')|>t-(k+q)$.

Case 2.2.1.2: There exist two nodes $u,v\in Q$ such that $u=\overline{v}_2$. Let $P'=\{x^{(1)}, \cdots, x^{(k+1)}\}\cup\{u,v\}$ and $Q'=Q-\{u,v\}$. Then, $X'=P'\cup Q'\cup R$. Let $Q''=\{v=1w_2\cdots w_n|v_1=0w_2\cdots w_n\in Q'\}$; then, $Q''\subset N_G(Q')$ and $|Q''|=q-3$. By Lemma 6, we have that $N_G(P')\cap Q''=\phi$. Hence, we have that the following inequality holds: $|N_G(X')|\geqslant |N_G(P')\cup Q''|-|Q'\cup R|=|N_G(P')|+(q-3)-(k+2q-3)=|N_G(P')|-(k+q)$. By Lemma 10, we have that $F=|N_G(P')|-t\geqslant 1$, which implies that $|N_G(X')|>t-(k+q)$.

Case 2.2.2: There exists exactly l pairs of nodes $x\in P$, $y\in Q$ such that $x=\overline{y}_2$, where $1\leqslant l\leqslant max\{q-1,k+1\}$. Without loss of generality, suppose that $x^{(j)}\in Q(k+2\leqslant j\leqslant k+l+1)$ satisfies $(x^{(j)})_2\in P$. Let $Q_0=\{x^{(k+2)}, \cdots, x^{(k+l+1)}\}$, $Q_1=Q-Q_0$.

Case 2.2.2.1: There do not exist two nodes $u,v\in Q_1$ such that $u=\overline{v}_2$. Let $P'=\{x^{(1)}, \cdots, x^{(k+1)}\}\cup Q_0\cup\{x^{(k+l+2)}\}$ and $Q'=Q_1-\{x^{(k+l+2)}\}$. Then, $X'=P'\cup Q'\cup R$. Let $Q''=\{v=1w_2\cdots w_n|v_1=0w_2\cdots w_n\in Q'\}$; then, $Q''\subset N_G(Q')$ and $|Q''|=q-l-2$. By Lemma 6, we have that $N_G(P')\cap Q''=\phi$. Hence, we have that the following inequality holds: $|N_G(X')|\geqslant |N_G(P')\cup Q''|-|Q'\cup R|=|N_G(P')|+(q-l-2)-(k+2q-l-2)=|N_G(P')|-(k+q)$. By Lemma 10, we have that $F=|N_G(P')|-t\geqslant 1$, which implies that $|N_G(X')|>t-(k+q)$.

Case 2.2.2.2: There exist two nodes $u,v\in Q_1$ such that $u=\overline{v}_2$. Let $P'=\{x^{(1)}, \cdots, x^{(k+1)}\}\cup Q_0\cup\{u,v\}$ and $Q'=Q_1-\{u,v\}$. Then, $X'=P'\cup Q'\cup R$. Let $Q''=\{v=1w_2\cdots w_n|v_1=0w_2\cdots w_n\in Q'\}$; then, $Q''\subset N_G(Q')$ and $|Q''|=q-l-3$. By Lemma 6, we have that $N_G(P')\cap Q''=\phi$. Hence, we have that the following inequality holds: $|N_G(X')|\geqslant |N_G(P')\cup Q''|-|Q'\cup R|=|N_G(P')|+(q-l-3)-(k+2q-l-3)=|N_G(P')|-(k+q)$. By Lemma 10, we have that $F=|N_G(P')|-t\geqslant 1$, which implies that $|N_G(X')|>t-(k+q)$. ■

V. CONCLUSIONS

The diagnosability of an interconnection network system based on some diagnosis strategy refers to the maximum number of faulty nodes identified correctly by the system via the diagnosis strategy. The t-diagnosability of AQ_n has been proven to be $2n-1$ in previous studies. In other words, if the number of faulty nodes in AQ_n is more than $2n-1$, then the tools of t-diagnosability do not work.

In our paper, we introduce two new diagnosis strategies to increase the diagnosability of AQ_n. One of them is called the t/t-diagnosis strategy. Under the t/t-diagnosis strategy and the condition that the system has at most t faulty nodes, the system can guarantee the isolation of all faulty nodes to within a set S, with $|S|\leqslant t$. We present and prove the result that under the t/t-diagnosis strategy, the diagnosability of AQ_n is $4n-8$, which is almost 4 times as large as $2n-1$, the classical diagnosability of AQ_n. The other strategy is called the t/k-diagnosis strategy. Under the

t/k-diagnosis strategy and the condition that the system has at most t faulty nodes, the system can guarantee the location of a set S, which contains all faulty nodes in the system and has at most k fault-free nodes. We present and prove the following result: For two integers $n > 5$ and $k(0 \leqslant k \leqslant \frac{4n}{9} - \frac{13}{9})$, the t/k-diagnosability of AQ_n is $2(k+1)n - \frac{3(k+1)(k+2)}{2} + 1$, which is $2kn - \frac{3(k+1)(k+2)}{2} + 2$ times larger than $2n - 1$, the classical diagnosability of AQ_n.

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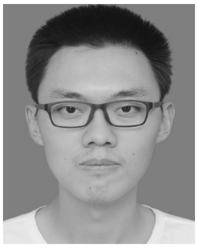

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