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On the Estimation and Control of Nonlinear Systems With Parametric Uncertainties and Noisy Outputs

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ABSTRACT In real-time problems, the possibilities of having a precise mathematical model describing the dynamics of the nonlinear system are scarce. Besides, the measurements invariably are tainted with noise which makes the problem of estimating the actual states of the system more difficult. The most common way of solving this issue involves the application of the Kalman Filter (KF) or the Extended Kalman Filter (EKF), for linear and nonlinear systems, respectively; although in both cases, the estimation heavily relies on linear techniques. In a different way, the James-Stein Filter provides a robust approach to estimate linear and nonlinear systems under parametric uncertainties of the mathematical model. In this brief note, a slightly different James-Stein State Estimator (JSSE), named Modified James-Stein State Estimator (JSSE-M), is presented as an alternative to filtering the states of nonlinear systems within a control scheme. The main contribution of this paper is the comparison of performance between KF, EKF, JSSE, and JSSE-M when they are used on a relatively complex nonlinear system which is extremely dependent on its parameters, namely the quadrotor. In this sense, some interesting comparisons focused on both, the effectiveness and processing time are provided.

INDEX TERMS Control systems, nonlinear systems, stochastic systems, state estimation, filtering.

I. INTRODUCTION

Estimation of heart rate [1], of brain activity [2], as well as the behavior of financial markets [3], and many other real-life dynamics are very difficult to achieve due to the lack of accurate mathematical models and because of the presence of noise in the measurements also. When the linear or nonlinear model is available, the problem of estimating the states in the presence of additive Gaussian noise can be solved satisfactorily by the Kalman Filter (KF) or by the Extended Kalman Filter (EKF), respectively [4]. However, the performance, of the KF and EKF algorithms, decreases as the approximation of the corresponding model is reduced, which is generally due to parametric uncertainties. Fortunately, there are works aimed to overcome this problem. Among them, the reader can refer to [5] where linear systems in discrete-time with uncertainties are considered, both in the state matrix and in the output matrix. On this basis, a linear filter is developed to maintain the variance of the estimation error within a certain limit for all admissible uncertainties. In [6], a version of the Kalman Filter is presented to estimate the state of descriptive systems with uncertainties, based on the convergence of robust Riccati equations. In [7], the Extended Kalman Filter (EKF), the Unscented Kalman Filter (UKF), the Gauss-Hermite Quadrature Filter (GHF) and the Quadrature Kalman Filter (QKF) are compared during the estimation of chaotic systems. In [8], a combination of the Kalman filter and Takagi-Sugeno fuzzy modeling is used to obtain an efficient state estimator for nonlinear systems, at least within the fuzzy approximation region. More recently, in [9] an Unscented Kalman Filter is used to protect user privacy in cloud environments.

On the other hand, in [10] there is an alternative to the Kalman filter called James-Stein State Estimator (JSSE). This algorithm allows estimating the states of linear and nonlinear systems, even in the presence of parametric uncertainties in the mathematical model. Unfortunately, the Mean Squared Error (MSE) of the JSSE grows as the standard deviation of the measurement noise increases. To deal with such a disadvantage, in next section a simple modification to the JSSE is proposed in order to provide the algorithm with a more controlled estimation process, while the computational cost remains almost the same. In the following, such a filter will be named Modified James-Stein State Estimator (JSSE-M).

Thus, in present work, the KF, the EKF the JSSE, and the JSSE-M are used to estimate the state of a sufficiently complex nonlinear system with parametric uncertainties and comparisons on effectiveness and processing time are provided.

As in [10], the following notation is considered: $(x)^+ = \max(0, x)$, $N(\mu, \Sigma^2)$ denotes a normal distribution with mean μ, standard deviation Σ and variance Σ^2, $tr(\Omega)$ is the trace of matrix Ω and $\lambda_{max}(\Omega)$ is the greatest eigenvalue of Ω.

The rest of the work is organized as follows. In section II, the estimators considered in this analysis are briefly described; namely, KF, EKF, JSSE, and JSSE-M, where the latter is the one proposed as a new alternative to filter the states of nonlinear systems. In section III, the nonlinear system regarded as the benchmark is presented. The results are given in section IV. Finally, in section V some conclusions are drawn.

II. STATE ESTIMATORS

A. KALMAN FILTER (KF)

The Kalman filter consists of a set of recursive equations allowing to compute an optimal estimator of states of a linear system on the basis the least squares method [4]. Consider the system

$$x_{k+1} = A_k x_k + B_k u_k + \Gamma_k \xi_k, \tag{1}$$
$$y_k = C_k x_k + \Pi_k \eta_k, \tag{2}$$

where k is the discrete-time, $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the input vector, $y_k \in \mathbb{R}^p$ is the output vector, $\xi_k \in \mathbb{R}^q$ is the dynamic noise with zero mean and variance $Q_k \in \mathbb{R}^{q\times q}$, and $\eta_k \in \mathbb{R}^\ell$ is the measurement noise with zero mean and variance $R_k \in \mathbb{R}^{\ell\times\ell}$. Besides, $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times m}$, $C_k \in \mathbb{R}^{p\times n}$, $\Gamma_k \in \mathbb{R}^{n\times q}$, and $\Pi_k \in \mathbb{R}^{p\times\ell}$ are known matrices. Thus, the Kalman filter (KF) is described by the following equations:

$$\hat{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1|k-1} + B_{k-1}u_{k-1}, \tag{3}$$
$$P_{k|k-1} = A_{k-1}P_{k-1|k-1}A_{k-1}^T + \Gamma_{k-1}Q_{k-1}\Gamma_{k-1}^T, \tag{4}$$
$$G_k = P_{k|k-1}C_k^T\left(C_k P_{k|k-1}C_k^T + R_k\right)^{-1}, \tag{5}$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + G_k(y_k - C_k\hat{x}_{k|k-1}), \tag{6}$$
$$P_{k|k} = (I_{n\times n} - G_k C_k)\, P_{k|k-1}, \tag{7}$$

where $\hat{x}_{k-1|k-1}$ is the estimation for the state x_{k-1} at iteration $k-1$, $\hat{x}_{k|k-1}$ is the prediction for state x_k at iteration k and $\hat{x}_{k|k}$ is the corrected estimation for state x_k at iteration k. In the same way, $P_{k-1|k-1}$ is the estimation for the error variance at iteration $k-1$, $P_{k|k-1}$ is the prediction for error variance at iteration k and $P_{k|k}$ is the corrected estimation for the error variance at iteration k. Both estimations, $\hat{x}_{k|k}$ and $P_{k|k}$ are corrected through G_k, which is known as the Kalman gain.

B. EXTENDED KALMAN FILTER (EKF)

The natural extension of the Kalman filter for nonlinear systems consists of linearizing the nonlinear system at each iteration and applying equations (3)-(7) on such a linearization. To illustrate the above mentioned, consider the nonlinear system

$$x_{k+1} = f(x_k, u_k) + M(x_k)\xi_k, \tag{8}$$
$$y_k = h(x_k) + N(x_k)\eta_k, \tag{9}$$

with x_k, y_k, ξ_k, and η_k defined as before. Thus, the Extended Kalman filter (EKF) is defined by the following equations:

$$\hat{x}_{k|k-1} = f(\hat{x}_{k-1|k-1}, u_{k-1}), \tag{10}$$
$$P_{k|k-1} = Jf_k P_{k-1|k-1} Jf_k^T + M(\hat{x}_{k-1|k-1})Q_{k-1}M(\hat{x}_{k-1|k-1})^T, \tag{11}$$
$$G_k = P_{k|k-1}Jh_k^T\left(Jh_k P_{k|k-1}Jh_k^T + R_k\right)^{-1}, \tag{12}$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + G_k(y_k - Jh_k\hat{x}_{k|k-1}), \tag{13}$$
$$P_{k|k} = (I_{n\times n} - G_k Jh_k)\, P_{k|k-1}, \tag{14}$$

with

$$Jf_k = \left(\frac{\partial f}{\partial \hat{x}_{k-1|k-1}}(\hat{x}_{k-1|k-1})\right), \tag{15}$$

and

$$Jh_k = \left(\frac{\partial h}{\partial \hat{x}_{k-1|k-1}}(\hat{x}_{k-1|k-1})\right). \tag{16}$$

A thorough derivation of both, the KF and the EKF can be found in [4].

C. JAMES-STEIN STATE ESTIMATOR (JSSE)

The James-Stein state estimator (JSSE) was proposed in [10] and it is the result of iteratively applying the James-Stein estimator given in [11]. By considering a random vector X of dimension s, which has a multivariate normal distribution with mean $\mu \in \mathbb{R}^s$ and covariance equal to the identity of dimension $s\times s$, i.e., $X \sim N(\mu, I_{s\times s})$, in [11], James and Stein showed that if the dimension s of X is greater than 2, then the estimator for μ with the Least Mean Squared Error (MSE) is:

$$\mu^{JS} = \left(1 - \frac{s-2}{\|X\|^2}\right)X. \tag{17}$$

This estimator is known as the James-Stein filter (JS) and in [10] it was taken into account as the foundation to produce an algorithm that allows estimating the state vector of linear and nonlinear systems, even in the presence of parametric uncertainties in the models.

To do this, the authors in [10] considered the following system

$$x_{k+1} = A_k x_k + B_k u_k + \Gamma_k \xi_k, \tag{18}$$
$$y_k = C_k x_k + \Pi_k \eta_k, \tag{19}$$

where k is the discrete-time, $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the input vector, $y_k \in \mathbb{R}^p$ is the output vector. With $\xi_k \in \mathbb{R}^q$ and $\eta_k \in \mathbb{R}^p$ as random vectors satisfying: $\xi_k \sim N(0, Q_k)$ and $\eta_k \sim N(0, r^2 I_{p\times p})$; and with $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times m}$, $\Gamma_k \in \mathbb{R}^{n\times q}$, $C_k \in \mathbb{R}^{p\times n}$ and $\Pi_k \in \mathbb{R}^{p\times p}$ as known matrices. Under these assumptions, the JSSE was defined as follows:

$$\hat{x}^{JS}_{k|k-1} = \hat{x}^{JS}_{k-1|k-1} + (1 - r^2 \cdot \alpha_k)^+ \times (\hat{x}^{ML}_k - \hat{x}^{JS}_{k-1|k-1}), \tag{20}$$
$$\hat{x}^{JS}_{k|k} = A_k \hat{x}^{JS}_{k|k-1} + B_k u_{k-1|k-1}. \tag{21}$$

With

$$\alpha_k = \frac{(\min\{(n-2), 2(n^*-2)\})^+}{\left\| \Pi_k^{-1} C_k \left(\hat{x}^{ML}_k - \hat{x}^{JS}_{k-1|k-1} \right) \right\|^2}, \tag{22}$$
$$n^* = \frac{tr\left\{ \left(C_k^T \left(\Pi_k \Pi_k^T \right)^{-1} C_k \right)^{-1} \right\}}{\lambda_{max}\left\{ \left(C_k^T \left(\Pi_k \Pi_k^T \right)^{-1} C_k \right)^{-1} \right\}}, \tag{23}$$
$$\hat{x}^{ML}_k = \left(C_k^T \left(\Pi_k \Pi_k^T \right)^{-1} C_k \right)^{-1} C_k^T \left(\Pi_k \Pi_k^T \right)^{-1} y_k. \tag{24}$$

D. MODIFIED JAMES-STEIN STATE ESTIMATOR (JSSE-M)

From a number of applications of the JSSE, it has been observed that the mean of the estimations provided by the JSSE is very close to the actual value. However, the MSE increases as the standard deviation of the measurement noise grows. For that reason, a slightly different algorithm of the JSSE with a more controlled correction of its estimation is proposed.

The main idea behind such a correction is to produce estimations nearer to the mean because it has been proven in [11] that such a mean is very close to the actual value. So, in this paper, an exponential correction is included in (20) while the rest of the algorithm remains intact, i.e., expression (20) is substituted by:

$$\hat{x}^{JS}_{k|k-1} = \hat{x}^{JS}_{k-1|k-1} + \frac{(1 - r^2 \cdot \alpha_k)^+ \cdot (\hat{x}^{ML}_k - \hat{x}^{JS}_{k-1|k-1})}{\exp\left(\frac{\gamma}{r^2}\right)}, \tag{25}$$

where r^2 is the variance of the measurement noise, and γ is a tuning parameter, which affects the performance of the JSSE-M. In this work, it is suggested to choose γ such that the exponential term takes large values for "small" values of r and that the exponential term takes values close to one when the measurement noise is "large."

Considering that the system will be stabilized by a controller, then the rationale behind this proposal is that, when the measurement noise is "small" the stability provided by the controller helps to tolerate big changes in the estimation of the JSSE-M. On the other hand, when the measurement noise is "large," the exponential term must be close to one in order to avoid more disturbances within the closed-loop system. In the former, the performance of the JSSE-M is very different from the performance of the JSSE, while in the latter, the performance of the JSSE-M tends to the performance of the JSSE.

III. THE BENCHMARK MODEL

In order to test the efficacy and efficiency of the afore mentioned state estimators, consider the problem of stabilizing the quadrotor depicted in Fig. 1, whose dynamics are approximated by the following equations [12]:

$$\dot{x}(t) = f(x(t), u(t)), \tag{26}$$
$$y(t) = h(x(t)), \tag{27}$$

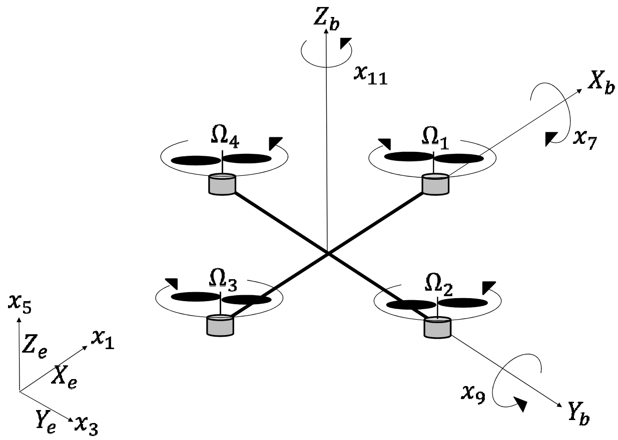

FIGURE 1. Schematics of the quadrotor.

where t is the continuous time,

$$x(t) = [x_1(t) \ldots x_{12}(t)]^T, \quad u(t) = [u_1(t) \ \ldots \ u_4(t)]^T,$$

$$f(x,u) = \begin{bmatrix} x_2, \\ (\sin(x_{11})\sin(x_7) + \cos(x_{11})\sin(x_9) \\ \cos(x_7))\frac{\beta_1}{m}, x_4, \\ (-\cos(x_{11})\sin(x_7) + \sin(x_{11})\sin(x_9) \\ \cos(x_7))\frac{\beta_1}{m}, x_6, \\ -g + (\cos(x_9)\cos(x_7)\frac{\beta_1}{m}, \\ x_8, \\ x_{10}x_{12}\frac{I_{yy} - I_{zz}}{I_{xx}} - \frac{J_{tp}}{I_{xx}}x_{10}\Omega + \frac{l\beta_2}{I_{xx}}, \\ x_{10}, \\ x_8 x_{12}\frac{I_{zz} - I_{xx}}{I_{yy}} + \frac{J_{tp}}{I_{yy}}x_8\Omega + \frac{l\beta_3}{I_{yy}}, \\ x_{12}, \\ x_8 x_{10}\frac{I_{xx} - I_{yy}}{I_{zz}} + \frac{\beta_4}{I_{zz}} \end{bmatrix}, \tag{28}$$

with

$$\begin{aligned}\beta_1 &= b(u_1^2+u_2^2+u_3^2+u_4^2),\\ \beta_2 &= b(u_4^2+u_3^2-u_1^2-u_2^2),\\ \beta_3 &= b(u_2^2+u_3^2-u_1^2-u_4^2),\\ \beta_4 &= d(u_1^2+u_3^2-u_2^2-u_4^2),\\ \Omega &= u_1-u_2+u_3-u_4,\end{aligned} \tag{29}$$

and

$$h(x) = Cx, \text{ with } C = I_{12\times 12}, \tag{30}$$

where for the sake of space, $x_* \equiv x_*(t)$ and $u_* \equiv u_*(t)$ for adequate values of "$*$," and where the effective control inputs u_1, u_2, u_3, and u_4 are the frequency of rotors 1, 2, 3, and 4 respectively (given in radians per second). The state variables, x_1, x_3, and x_5 are in meters and they represent the linear displacements along the earth fixed axes X_e, Y_e, and Z_e, respectively. While x_7, x_9, and x_{11} are in radians and they describe the angular displacements around the body axes X_b, Y_b, and Z_b, respectively. The rest of the state variables are the corresponding velocities, which can be easily inferred from (26)-(30).

The parameters considered are: $b = 54.2\times 10^{-6} N\cdot s^2$ as the thrust factor, $d = 1.1\times 10^{-6} N\cdot m\cdot s^2$ as the drag factor, $l = 0.24m$ as the distance from the center of the quadrotor to the center of the rotors, $m = 1kg$ as the mass of the quadrotor, $g = 9.81\frac{m}{s^2}$ as the acceleration of gravity, $J_{tp} = 104\times 10^{-6} N\cdot m\cdot s^2$ as the total moment of inertia for the rotors, and $Ixx = 8.1\times 10^{-3} N\cdot m\cdot s^2$, $Iyy = 8.1\times 10^{-3} N\cdot m\cdot s^2$, and $Izz = 14.2\times 10^{-3} N\cdot m\cdot s^2$, as the moments of inertia respect to axes x, y and z, respectively. With this parameters, the rotors' frequency needed to maintain the quadrotor in hover position is $u_{o1} = u_{o2} = u_{o3} = u_{o4} = 212.718305490559\frac{rad}{s}$.

At this point, a discrete-time approximation for system (26)-(27) can be obtained by means of the Euler discretization method [4]. Considering the expression for the first-order derivative as:

$$\dot{x}(t) = \lim_{\Delta t\to 0}\frac{x(t+\Delta t)-x(t)}{\Delta t}, \tag{31}$$

which can be approximated by

$$\dot{x}(t) \approx \frac{x(t+T)-x(t)}{T}, \tag{32}$$

for sufficiently small T, with T as the sampling time, results:

$$x(t+T) \approx x(t) + T\dot{x}(t). \tag{33}$$

Thus, according to the Euler discretization method, the discrete-time approximation for (26)-(27) is

$$x_{k+1} = x_k + Tf(x_k, u_k), \tag{34}$$
$$y_k = h(x_k), \tag{35}$$

with $f(\cdot,\cdot)$ as in (28). Therefore, the discrete-time model given by equations (34)-(35) will be used to benchmark the aforementioned estimation approaches with $T = \frac{1}{40}s$.

In order to simulate the dynamic uncertainties and measurement noise, the following model is considered:

$$x_{k+1} = f_d(x_k, u_k) + q\Gamma_k\xi_k, \tag{36}$$
$$y_k = h_d(x_k) + \Pi_k\eta_k, \tag{37}$$

where $f_d(x_k,u_k) = x_k + Tf(x_k,u_k)$, $\Gamma_k = I_{12\times 12}$, $h_d(x_k) = x_k$, $\Pi_k = I_{12\times 12}$, $\xi_k \sim N(0, Q_k)$ and $\eta_k \sim N(0, r^2 I_{p\times p})$, with q as the standard deviation of the dynamic noise and r as the standard deviation of the measurement noise, i.e., $Q_k = q^2\Gamma_k$ and $R_k = r^2\Pi_k$.

IV. STATE ESTIMATION

Because the KF is a linear estimator, a linear approximation of the nonlinear system is needed. From equations (34)-(35), the linear approximation for the dynamics of the quadrotor at rest, in horizontal position and with a constant altitude $\hat{d}$, i.e., at operation point $x_o = [0,0,0,0,\hat{d},0,0,0,0,0,0,0]^T$ is given by equations

$$x_{k+1} = Ax_k + Bu_k, \tag{38}$$
$$y_k = Cx_k, \tag{39}$$

with A and B in (40) and (41), respectively, as shown at the bottom of the next page, and

$$C = I_{12\times 12}. \tag{42}$$

EKF, JSSE, and JSSE-M do not require special preprocessing, because they can be applied on the nonlinear system (34)-(35) in a straightforward way. In the following, the desired altitude of the quadrotor is 30 meters, i.e., $\hat{d} = 30m$. The stability control for the nonlinear systems (36)-(37) is designed on the basis of the linear model (38)-(39) as a feedback controller of the form:

$$u_k = -F(x_k - x_o) + u_o, \tag{43}$$

which is obtained by means of the Robust Pole Placement approach given in [13], and where $u_o = [u_{o1}\ u_{o2}\ u_{o3}\ u_{o4}]^T$, with u_{o1}, u_{o2}, u_{o3}, and u_{o4} as above. Because (38)-(39) is a discrete-time linear system, the set of desired eigenvalues considered during the pole placement method is:

$$\lambda = [0.84\ 0.85\ 0.86\ 0.87\ 0.88\ 0.89\ 0.9\ 0.91\ 0.92\ 0.93\ 0.94\ 0.95],$$

and the corresponding control gain F turns out to be:

$$F = \begin{pmatrix} -25.0 & 411.0 & -8.9 & 399.0\\ -4.3 & 333.0 & 13.0 & 311.0\\ -311.0 & -20.0 & -322.0 & -32.0\\ -277.0 & -22.0 & -299.0 & -37.0\\ 199.0 & 200.0 & 199.0 & 200.0\\ 93.0 & 95.0 & 93.0 & 95.0\\ 711.0 & 59.0 & 766.0 & 122.0\\ 56.0 & 2.9 & 67.0 & 14.0\\ 28.0 & 844.0 & 97.0 & 766.0\\ 3.4 & 71.0 & 15.0 & 58.0\\ 177.0 & -188.0 & 177.0 & -188.0\\ 74.0 & -76.0 & 74.0 & -76.0 \end{pmatrix} \tag{44}$$

Thus, the numerical simulations are obtained using equations (36)-(37), with $r = 0.1$, $q = 0.001$ and $\gamma = 0.15$ where the initial conditions are taken at random within a neighborhood of x_o. In those simulations, the estimated states provided by KF, EKF, JSSE, and JSSE-M are used in (43) correspondingly. The results are given below.

However, from Figs. 2 and 3, it is not easy to determine which estimator provides a better performance. For that reason, the Mean Squared Error (MSE) is considered:

$$MSE = \frac{1}{N}\Sigma_{i=1}^{N}\left(x_o - \hat{x}_i\right)^2, \tag{45}$$

where x_o is the operation point defined above, $\hat{x}_i$ is the state estimation at instant i and N is the total number of samples. In this case, $N = 2000$ because $T = \frac{1}{40}s$ and the simulation time is $t = 50s$. As before, the initial conditions are taken at random within a neighborhood of x_o.

It is important mentioning that the results given in Fig. 4 and Table 1 were obtained as the average after running 100 simulations for each estimator on an Intel Core i7-4720HQ 2.6GHz CPU with 8Gb of RAM running Matlab© r2016b.

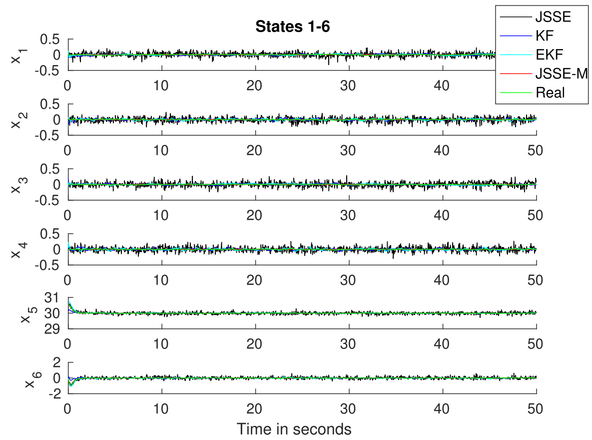

FIGURE 2. First six states of the quadrotor.

From Fig. 4 and Table 1, it can be readily observed that the JSSE-M is a good choice to estimate the states of the

$$A = \begin{bmatrix} 1 & \frac{1}{40} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{981}{4000} & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{40} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{981}{4000} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{40} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{40} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{40} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{40} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{40}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 5.76 \cdot 10^{-4} & 5.76 \cdot 10^{-4} & 5.76 \cdot 10^{-4} & 5.76 \cdot 10^{-4} \\ 0 & 0 & 0 & 0 \\ -0.0171 & -0.0171 & 0.0171 & 0.0171 \\ 0 & 0 & 0 & 0 \\ -0.0171 & 0.0171 & 0.0171 & -0.0171 \\ 0 & 0 & 0 & 0 \\ 8.24 \cdot 10^{-4} & -8.24 \cdot 10^{-4} & 8.24 \cdot 10^{-4} & -8.24 \cdot 10^{-4} \end{bmatrix}, \tag{41}$$

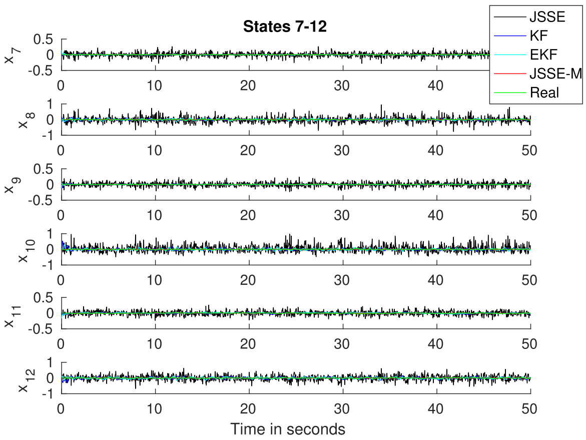

FIGURE 3. Last six states of the quadrotor.

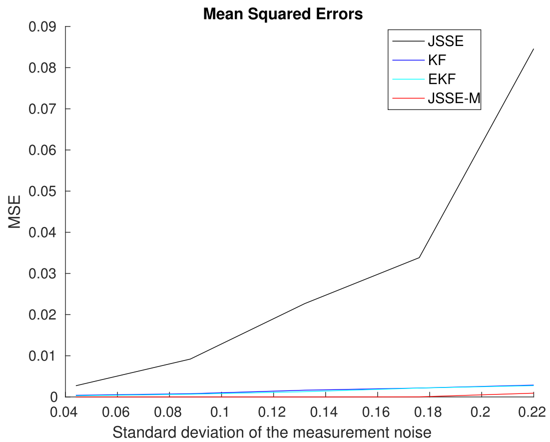

FIGURE 4. MSE for different values of r.

TABLE 1. Processing time for KF, EKF, JSSE, and JSSE-M in seconds.

KF	EKF	JSSE	JSSE-M
0.15871633	0.86501887	0.26077425	0.2632239

quadrotor for purposes of controlling such a system. Notice that the measurement noise affects the behavior of the system because the state estimators require of the measurements to estimate the "actual" states and such an estimation is included in the feedback controller. Therefore, poor estimations may cause instabilities in the closed-loop system.

V. CONCLUSION

In this brief note, an alternative to estimate the states of a dynamical nonlinear system has been presented. It can be seen, that at least for the system considered, the JSSE-M can be used as a good choice to filter the states during a control scheme. It can also be concluded that the KF and the EKF present a very good performance when the nonlinear system has a linear or quasi-linear behavior, but their results degrade conforms the measurement noise provokes more complex behavior. Besides, notice that the JSSE is only suggested to be used when the measurement noise is "small." It is important mentioning that any other type of controller can be used; however, this work was intended to evaluate the estimators instead of the controllers.

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