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The Conditional Diagnosability of Exchanged Crossed Cube

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ABSTRACT Conditional diagnosability is widely accepted as an important measure in determining the reliability of an interconnection network. The conditional diagnosability of many well-known interconnection networks has been investigated. Exchanged crossed cube(ECQ(s,t)) is a novel variant of hypercube, which retains the advantages of exchanged hypercube and crossed cube in terms of the smaller diameter, fewer links, and lower cost factor, and indicates more balanced consideration among performance and cost. In this paper, several topological properties of ECQ(s,t) are derived. On this basis, the conditional diagnosability of ECQ(s,t) under the PMC model is shown to be $4(s-1)+1$ for $t \geq s > 2$, which is almost two times larger than its classical diagnosability and also is larger than its conditional diagnosability under the MM model.

INDEX TERMS Exchanged crossed cube, conditionally t-diagnosable, conditional diagnosability.

NOMENCLATURE

ABBREVIATIONS AND ACRONYMS

PMC	Preparata, Metze, Chien
MM	Maeng, Malek
MM*	Maeng, Malek*

NOTATIONS

$G(V, E)$	A multi-processor system
$V(G)$	The vertex set of G
$E(G)$	The edge set of G
$N(u)$	The neighbors of vertex u
$N(X)$	$\bigcup_{v\in X} N(v) - X$
$\deg_G(u)$	The degrees of vertex u in G
$\deg_H(u)$	The degrees of vertex u in H
$P_g(G)$	Every vertex in G has at least g neighbors in G
$k(G)$	The vertex connectivity of G
$k_c(G)$	The conditional vertex connectivity of G
$k_r(G)$	The restricted vertex connectivity of G
$F_1 \triangle F_2$	$(F_1 - F_2) \cup (F_2 - F_1)$
$ECQ(s, t)$	Exchanged crossed cube
$t(G)$	The classical diagnosability of G
$t_c(ECQ(s, t))$	The conditional diagnosability of $ECQ(s, t)$

I. INTRODUCTION

With the continuous development of very large scale integration (VLSI) technology, a multi-processor system has the capacity to contain hundreds, thousands, even tens-of-thousands of processors. However, the high complexity of these systems may threaten their reliability. As a result, the problem of fault location is becoming a matter of major importance in such a system.

System-level diagnosis is a rapid and reliable tool for fault location. It proceeds under a special diagnosis model such as the PMC(Preparata, Metze and Chien) model [1] and the MM(Maeng and Malek) model [2]. The PMC model assumes that each processor is able to test the adjacent processors and determine them to be faulty or fault-free. The test outcomes are considered reliable if the tester is fault-free. In the MM model, a processor, called a comparator, sends the same testing task to each pair of its distinct neighboring processors and then compares their responses. A disagreement over a comparison performed by a fault-free comparator indicates the existence of at least a faulty processor, whereas the test outcome of a comparison performed by a faulty comparator is unreliable.

As we know, the classical connectivity of Menger [3], in which any processor subsets can potentially fail simultaneously, is an important measure of the fault-tolerance ability. However, in classical connectivity, it has generally been assumed that all neighbors of a processor can potentially fail at the same time, which is almost impossible in a real large-scale multi-processor system. To compensate for this shortcoming, Harary [4] introduced the definition of conditional connectivity. Following this trend, restricted connectivity was proposed in [5] and [6].

There are two commonly used fault models: random model and conditional model. The random model is assumed that faults can occur everywhere, while the conditional model is assumed that the distribution of faults must satisfy certain constraints(e.g., any faulty set cannot contain all neighbors of any vertex) [7]. A system is said to be t-diagnosable if all faulty vertices can be identified under the random model, provided the fault bound is t. The classical diagnosability of G, denoted by $t(G)$, is the maximum value of t such that G is t-diagnosable. The classical diagnosability is used to measure the diagnostic ability of multiprocessor systems. The classical diagnosability of many interconnection networks, including hypercube(denoted by Q_n for short), folded hypercube(denoted by FQ_n for short), crossed cube(denoted by CQ_n for short), m?bius cube(denoted by MQ_n for short), twisted cube(denoted by TQ_n for short), exchanged hypercube(denoted by $EH(s, t)$ for short) and locally twisted cube(denoted by LTQ_n for short), based on PMC and MM* model have been proposed [8]–[15]. In applications of diagnosability, it indicates the low possibility that the adjacent processors of any processor are faulty simultaneously. Motivated by this, a novel measure of diagnosability, named conditional diagnosability, was introduced by Lai *et al.* [16], under the conditional fault model. The conditional diagnosability is an important diagnostic strategies that can significantly enhance the system's diagnostic capability and ensure the reliable parallel operation of system. The conditional diagnosability of many novel interconnection networks are studied in the literature [16]–[36].

In order to obtain better topological properties, basing on exchanged hypercube and crossed cube, a new variation of hypercube called exchanged crossed cube(denoted by $ECQ(s, t)$ for short) was proposed by Li *et al.* [37] in 2013. Some basic properties and characteristics of $ECQ(s, t)$ were studied [37]–[42].

In this paper, after exploring various desire properties of exchanged crossed cube $ECQ(s, t)$, we evaluate the conditional diagnosability of exchanged crossed cube $ECQ(s, t)$. We show that the conditional diagnosability of $ECQ(s, t)$ under the PMC model is $4(s-1)+1$ for $t \geq s > 2$. The conditional diagnosability of ECQ(s,t) under the PMC model is almost two times as large as its classical diagnosability [41], and also is larger than its conditional diagnosability under the MM model [42], which indicates stronger diagnostic ability.

The remainder of this paper is divided into four sections. Section 2 introduces some terminologies and preliminaries. In section 3, we propose some important topological properties of $ECQ(s, t)$. Section 4 discusses the conditional diagnosability of $ECQ(s, t)$ under the PMC model. Our conclusions are in section 5.

II. TERMINOLOGIES AND PRELIMINARIES

A multi-processor system can be modeled as a graph $G(V, E)$, where $V(G)$ and $E(G)$ denote vertex and edge sets of G, respectively. $N(u)$ in graph $G(V, E)$ is the set of all neighbors of a vertex u, and $N[u] = N(u) \cup \{u\}$. For an arbitrary vertex set X, $N(X) = \bigcup_{v \in X} N(v) - X$ and $N[X] = N(X) \cup X$. For brevity, $N(\{u, v\}) = N(u, v) = N(u) \cup N(v) - \{u, v\}$ and $N[\{u, v\}] = N[u, v] = N(u, v) \cup \{u, v\}$. $\deg_G(u)$ and $\deg_H(u)$ are the degrees of vertex u in $G(V, E)$ and in subgraph H, respectively. The property $P_g(G)$ holds for $G(V, E)$ if and only if every vertex in G has at least g neighbors in G.

If G is connected and $G - F$ is disconnected, where F is a set of vertices, then we say that F is a vertex cut. The classical vertex connectivity $k(G)$ of a graph G can be denoted by $k(G) = min\{|F| : F \subset V(G)$ and F is a vertex cut$\}$. A conditional vertex cut of a graph G is a vertex cut and $N(u) \not\subset F$ for each vertex $u \in V(G) - F$. The conditional vertex connectivity, $k_c(G)$, of a graph G, is the minimum cardinality of a conditional vertex cut of G, denoted by $k_c(G) = min\{|F| : F \subset V(G)$ and F is a conditional vertex cut $\}$. Following this trend, in [5] and [6], restricted connectivity is introduced by imposing some conditions or restrictions on F. A restricted vertex cut F of a graph G is a vertex cut of G and $N(u) \not\subset F$ for any $u \in V(G)$. The restricted vertex connectivity, $k_r(G)$, of a graph G, is defined as the minimum cardinality of a restricted vertex cut of G, denoted by $k_r(G) = min\{|F| : F \subset V(G)$ and F is a restricted vertex cut $\}$.

TABLE 1. Invalidation rules for the PMC model.

fault-free	fault-free	0
fault-free	faulty	1
faulty	fault-free	0 or 1
faulty	faulty	0 or 1

A. THE PMC MODEL

In the PMC model, a multi-processor system can be modeled as a graph $G(V, E)$ and each vertex is able to test another vertex if there is a link between them. $(u, v) \in E(G)$ means there is a test performed by u on v. Each vertex has two states: fault-free and faulty. The outcome $\sigma(u, v)$, of a test (u, v), equal 0 if u evaluates v as a pass and 1 otherwise. Table 1 summarizes the invalidation rules for the PMC model. The collection of all test outcomes in $G(V, E)$ is called a syndrome, denoted by σ.

B. CLASSICAL DIAGNOSABILITY AND CONDITIONAL DIAGNOSABILITY

In a t-diagnosable system $G(V, E)$, any vertex subset of V can potentially fail simultaneously. As is well known, it is impossible to identify whether a vertex v is fault-free or faulty when $N(v)$ are simultaneously faulty. As a result, the classical diagnosability is no more than its minimum degree. But, in real applications, the probability that all neighbors of a vertex fail at the same time is usually very small. Motivated by this, Lai *et al.* [16] proposed a new measure of diagnosability which is called conditional diagnosability, by claiming the property that each vertex has at least one fault-free neighbor. A conditional fault set F is a fault set and each vertex of the system has at least one neighbor not in F. Lai et al. also introduced an important theorem to identify whether a given system is conditionally t-diagnosable or not as follow.

Theorem 1 [16]: A system $G(V, E)$ is conditionally t-diagnosable if (F_1, F_2) is distinguishable, for each pair of distinct conditional faulty sets $F_1, F_2 \subset V(G)$ with $|F_1| \leq t$ and $|F_2| \leq t$.

Let (F_1, F_2) be a pair of distinct faulty sets. (F_1, F_2) is indistinguishable if and only if there exists no edge between $V(G) - (F_1 \cup F_2)$ and $F_1 \triangle F_2$ ($F_1 \triangle F_2 = (F_1 - F_2) \cup (F_2 - F_1)$) [16]. The conditional diagnosability of a graph $G(V, E)$, denoted by $t_c(G)$, is the maximum value of t such that G is conditionally t-diagnosable. According to the definition of conditional diagnosability, a useful lemma about conditional faulty sets is described as follows.

Lemma 1 [16]: Let $G(V, E)$ be the graph representation of a system G. If (F_1, F_2) is a pair of distinct indistinguishable conditional faulty sets, the following two conditions hold:

(1) $|N(u) \cap (V - (F_1 \cup F_2))| \geq 1$ for $u \in (V - (F_1 \cup F_2))$ and

(2) $|N(v) \cap (F_1 - F_2)| \geq 1$ and $|N(v) \cap (F_2 - F_1)| \geq 1$ for $v \in F_1 \Delta F_2$

C. EXCHANGED CROSSED CUBE

Exchanged crossed cube is a new variant of hypercube, which retains many advantages of exchanged hypercube and crossed cube such as recursive structure, high partitionability and strong connectivity.

Let $T = \{(00, 00), (10, 10), (01, 11), (11, 01)\}$. Two binary strings $X = x_1x_0$ and $Y = y_1y_0$ are pair related iff $(X, Y) \in T$, denoted by $X \sim Y$.

An exchanged crossed cube $ECQ(s, t)$ with $s \geq 1$ and $t \geq 1$ can be modeled as an undirected graph $G(V, E)$, where $V = \{a_{s-1}a_{s-2} \dots a_0b_{t-1}b_{t-2} \dots b_0c | a_i, b_j, c \in \{0, 1\}, i \in [o, s), j \in [o, t)\}$, $E = \{(u, v)|(u, v) \in V \times V\}$. There are three types of edges, i.e., E_1, E_2 and E_3, as described below:

E_1: $u[0] \neq v[0]$, $u \oplus v = 1$, where $u[i]$ denotes the *ith* bit of vertex u and $\oplus$ is the exclusive-OR operator.

E_2: $u[0] = v[0] = 0$, $u[1 : t] = v[1 : t]$, where $u[x : y]$ denotes the bit pattern of u between dimensions x and y, inclusive. For all $s \geq 1$, if and only if there exists a positive integer l, $s + t \geq l > t$, such that $u[l : s + t] = v[l : s + t]$, $u[l - 1] \neq v[l - 1]$, $u[l - 2] = v[l - 2]$ if $l - t$ is even, and $u[t + 2i + 2 : t + 2i + 1] \sim v[t + 2i + 2 : t + 2i + 1]$ for $(l - t - 1)/2 > i \geq 0$.

E_3: $u[0] = v[0] = 1$, $u[t + 1 : s + t] = v[t + 1 : s + t]$. For all $t \geq 1$, if and only if there exists a positive integer l, $t \geq l \geq 1$, such that $u[l : t] = v[l : t]$, $u[l - 1] \neq v[l - 1]$, $u[l - 2] = v[l - 2]$ if l is even, and $u[2i + 2 : 2i + 1] \sim v[2i + 2 : 2i + 1]$ for $\lfloor (l - 1)/2 \rfloor > i \geq 0$.

By the definition of $ECQ(s, t)$, the total number of vertices in $ECQ(s, t)$ is 2^{s+t+1}, the number of edges in $ECQ(s, t)$ is $(s + t + 2)2^{s+t-1}$. The definition of $ECQ(s, t)$ also reveals that the number of edges in E_1 is 2^{s+t}, the number of edges in E_2 is $t \times 2^{s+t-1}$, the number of edges in E_3 is $s \times 2^{s+t-1}$ [37].

Figure 1 shows an illustration of $ECQ(s, t)$ with $s = 2$ and $t = 2$, where the dashed links, solid heavy links and solid thin links correspond to E_1, E_2 and E_3, respectively [37].

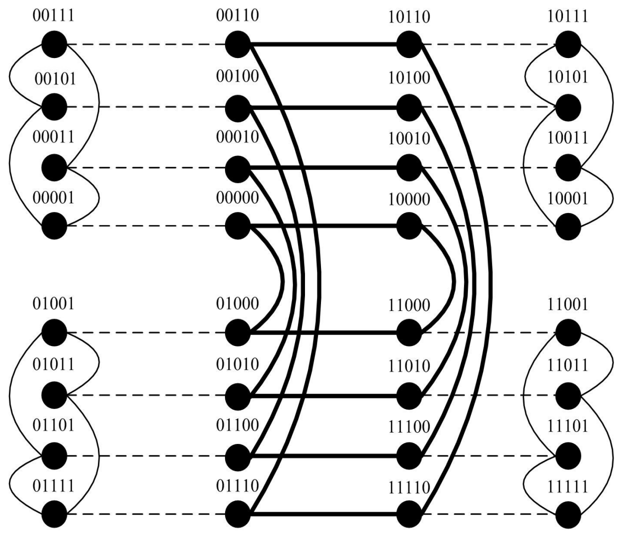

FIGURE 1. **An exchanged crossed cube *ECQ*(2, 2).**

There are some important topological properties of exchanged crossed cube $ECQ(s, t)$ as follows.

Lemma 2 [37]: The degree of vertices in $V(ECQ(s, t))$ whose bit addresses end in 0 is $s + 1$, while the degree of vertices in $V(ECQ(s, t))$ whose bit addresses end in 1 is $t + 1$.

By Lemma 2, we can show the minimum degree of $ECQ(s, t)$, denoted by $\delta(ECQ(s, t))$, is $s+1$, where $t \geq s \geq 1$.

Lemma 3 [37]: An exchanged crossed cube $ECQ(s, t)$ can be decomposed into two copies of $ECQ(s - 1, t)$ or $ECQ(s, t - 1)$.

By Lemma 3, an $ECQ(s, t)$ can be partitioned into two subgraphs L and R, where $V(L) = \{0a_{s-2} \dots a_0b_{t-1} \dots b_0c | a_i, b_j, c \in \{0, 1\}, i \in [0, s - 2], j \in [0, t - 1]\}$, $V(R) = \{1a_{s-2} \dots a_0b_{t-1} \dots b_0c | a_i, b_j, c \in \{0, 1\}, i \in [0, s - 2], j \in [0, t - 1]\}$, $L \cong ECQ(s - 1, t)$ and $R \cong ECQ(s - 1, t)$. Moreover, $V(L)$ can be subdivided into two vertex sets A and B, and $V(R)$ can be subdivided into two vertex sets C and D, where $A = \{0a_{s-2} \dots a_0b_{t-1} \dots b_00 | a_i, b_j \in \{0, 1\}, i \in [0, s - 2], j \in [0, t - 1]\}$, $B = \{0a_{s-2} \dots a_0b_{t-1} \dots b_01 | a_i, b_j \in \{0, 1\}, i \in [0, s - 2], j \in [0, t - 1]\}$, $C = \{1a_{s-2} \dots a_0b_{t-1} \dots b_00 | a_i, b_j \in \{0, 1\}, i \in [0, s - 2], j \in [0, t - 1]\}$, $D = \{1a_{s-2} \dots a_0b_{t-1} \dots b_01 | a_i, b_j \in \{0, 1\}, i \in [0, s - 2], j \in [0, t - 1]\}$ [38].

A shown in Figure 2, the edges between A and B and the edges between C and D belong to E_1. The edges between A and C belong to E_2. By the definition of A, B, C and D, there are three perfect matchings of subgraphs induced by $A \cup B$, $A \cup C$ and $C \cup D$ [38]. The edges between two distinct vertices in B (or D) belong to E_3. Similarly, the edge between two distinct vertices in A(or C) belong to E_2. As shown in Figure 2, a path $u_1 - u_2 - u_3 - u_4$ of length 3 with $u_1 \in B$, $u_2 \in A$, $u_3 \in C$ and $u_4 \in D$, such that $(u_1, u_2), (u_2, u_3), (u_3, u_4) \in E(ECQ(s, t))$, is a horizontal straight line. There are 2^{s+t-1} horizontal straight lines in $ECQ(s, t)$.

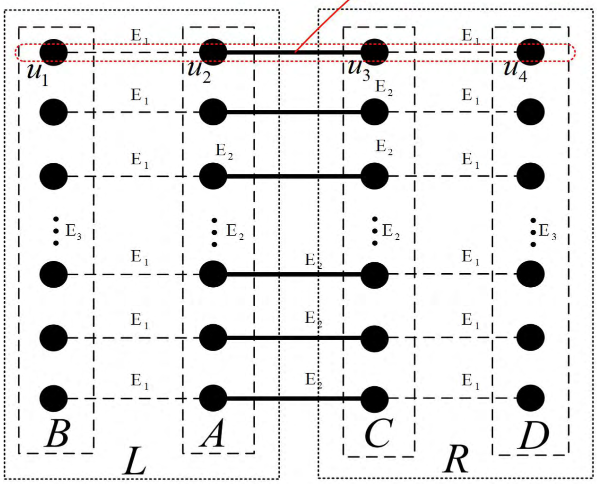

FIGURE 2. The vertex sets of A, B, C and D.

Lemma 4 [37]: $ECQ(s,t)$ and $ECQ(t,s)$ are isomorphic, denoted by $ECQ(s,t) \cong ECQ(t,s)$.

By Lemma 4, without loss of generality, we can assume $t \geq s > 0$ in the following discussion.

Lemma 5 [38]: $k(ECQ(s,t)) = s+1$, where $t \geq s$.

Lemma 6 [39]: $k_r(ECQ(s,t)) = 2s$, where $t \geq s > 0$.

Lemma 7 [38]: There is no triangle in exchanged crossed cube $ECQ(s,t)$.

Lemma 8 [37]: An exchanged crossed cube $ECQ(s,t)$ can be decomposed into 2^s topological networks of CQ_t and 2^t topological networks of CQ_s.

According to the definition of exchanged crossed cube $ECQ(s,t)$, when $c = 0$ and have the same value in bit addresses $[1 : t]$, there are 2^s vertices which can compose a crossed cube CQ_s. Similarly, when $c = 1$ and have the same value in bit addresses $[1+t : s+t]$, there are 2^t vertices which can compose a CQ_t. Thus an $ECQ(s,t)$ can be decomposed into 2^s topological networks of CQ_t and 2^t topological networks of CQ_s [39]. Each pair of distinct $CQ_s s$ (or $CQ_t s$) is not connected directly [39]. For CQ_n, two extra but important properties are presented below.

Lemma 9 [20]: For any two distinct vertices u and v of CQ_n, they share at most 2 common neighbors, denoted by $|N(u) \cap N(v)| \leq 2$.

Lemma 10: Let (u,v) be an arbitrary edge of crossed cube CQ_n and w be another vertex of CQ_n with $n \geq 2$. Then $|N(u,v) \cap N(w)| \leq 3$.

Proof: This is clearly true for the case $n = 2$ and $n = 3$. Assume this to be true for CQ_{n-1}. We show that this is true for CQ_n. For $n \geq 2$, By the definition of CQ_n, it can be partitioned into two copies of CQ_{n-1}, denoted by CQ^0_{n-1} and CQ^1_{n-1}, such that CQ^0_{n-1} and CQ^1_{n-1} are connected by a perfect matching, i.e., every vertex of CQ^0_{n-1}(or CQ^1_{n-1}) is adjacent to exactly one vertex of CQ^1_{n-1}(or CQ^0_{n-1}) [43]. Without the loss of generality, three possibilities need to be investigate.

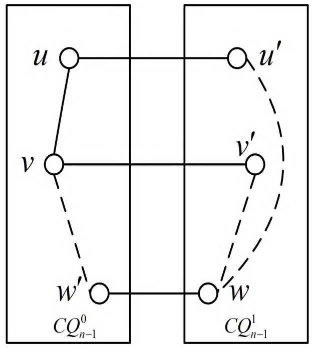

FIGURE 3. Illustration for Case 2.

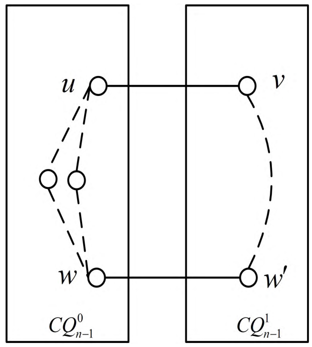

FIGURE 4. Illustration for Case 3.

Case 1 ($(u,v) \in E(CQ^0_{n-1})$ and $w \in V(CQ^0_{n-1})$): By the induction assumption, we have $|N(u,v) \cap N(w)| \leq 3$.

Case 2 ($(u,v) \in E(CQ^0_{n-1})$ and $w \in V(CQ^1_{n-1})$): Let $u' = N(u) \cap V(CQ^1_{n-1})$, $v' = N(v) \cap V(CQ^1_{n-1})$ and $w' = N(w) \cap V(CQ^0_{n-1})$, as shown in Figure 3. Since $CQ^0_{n-1} \cup CQ^1_{n-1}$ is a perfect matching, $N(u,v) \cap N(w) \subseteq \{u', v', w'\}$. Therefore, $|N(u,v) \cap N(w)| \leq 3$.

Case 3 ($u \in V(CQ^0_{n-1})$, $v \in V(CQ^1_{n-1})$ and $w \in V(CQ^0_{n-1})$): Let $w' = N(w) \cap V(CQ^1_{n-1})$, $N(w) \cap N(v) \subseteq \{u, w'\}$. If $(u,w) \in E(CQ^0_{n-1})$, we have $N(u) \cap N(w) = \varnothing$ because there is no triangle in CQ_n. Then $N(u,v) \cap N(w) \subseteq \{w'\}$. Otherwise, $(u,w) \notin E(CQ^0_{n-1})$. By Lemma 9, we have $|N(u) \cap N(w)| \leq 2$, as shown in Figure 4. Therefore, $|N(u,v) \cap N(w)| \leq 3$. □

III. THE TOPOLOGICAL PROPERTIES OF EXCHANGED CROSSED CUBE

This section presents some useful topological properties of exchanged crossed cube $ECQ(s,t)$.

Theorem 2: Let a, b, c and d be four arbitrary vertices of $ECQ(s,t)$ where $a \in A$, $b \in B$, $c \in B$ and $d \in A$. Then, a-b-c-d-a is not a cycle of length four in $ECQ(s,t)$.

Proof: As shown in Figure 5, we assume a-b-c-d-a is a cycle of length four in $ECQ(s,t)$ where $a \in A$, $b \in B$, $c \in B$ and $d \in A$. Then, we have $(a,b) \in E_1$, $(b,c) \in E_3$, $(c,d) \in E_1$, and $(d,a) \in E_3$. Let $a = \{0a_{s-2} \ldots a_0 b_{t-1} \ldots b_0 0\}$. By the definition of E_1, we have $b = \{0a_{s-2} \ldots a_0 b_{t-1} \ldots b_0 1\}$. By the definition of E_3, we

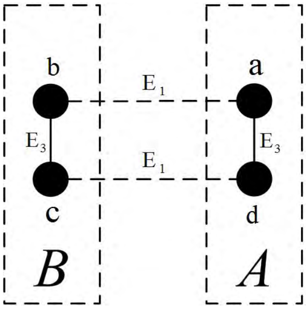

FIGURE 5. A cycle of length four in $ECQ(s,t)$ with $a \in A$, $b \in B$, $c \in B$ and $d \in A$.

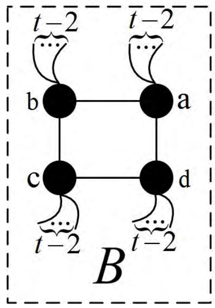

FIGURE 6. A cycle C_H of length four in H with $V(H) \subset B$.

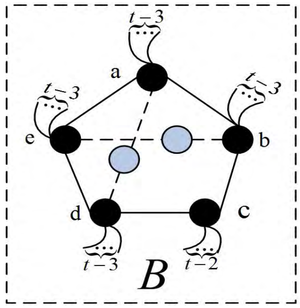

FIGURE 7. A cycle C_H of length five in H with $V(H) \subset B$.

have $c = \{0a_{s-2}\dots a_0 d_{t-1}\dots d_0 1\}$ and $d_{t-1}\dots d_0$ exists at least one bit different from $b_{t-1}\dots b_0$. By the definition of E_1, we have $d = \{0a_{s-2}\dots a_0 d_{t-1}\dots d_0 0\}$. Thus, a and d cannot be connected by a edge because $d_{t-1}\dots d_0$ exists at least one bit different from $b_{t-1}\dots b_0$, which contradicts the assumption. □

Theorem 3: For any two distinct vertices u and v of $ECQ(s,t)$, they share at most 2 common neighbors, denoted by $|N(u)\cap N(v)| \le 2$.

Proof: By induction. Clearly, the theorem holds for $ECQ(1,1)$. Assume true for $ECQ(s-1,t)$ (or $ECQ(s,t-1)$). According to Lemma 3, we decompose $ECQ(s,t)$ into L and R, L and R are both isomorphic to $ECQ(s-1,t)$ (or $ECQ(s,t-1)$). Without loss of generality, we assume $L \cong ECQ(s-1,t)$ and $R \cong ECQ(s-1,t)$. When $u, v \in V(L)$ (or $u, v \in V(R)$) we have $N(u)\cap N(v) \subset V(L)$ (or $N(u)\cap N(v) \subset V(R)$). By the induction hypothesis, we have $|N(u)\cap N(v)| \le 2$. When $u \in L$ and $v \in R$ (or $u \in R$ and $v \in L$), by the fact that $A\cup C$ contains a perfect matching, we have $|N(u)\cap N(v)| \le 2$.

Hence, the theorem holds. □

Theorem 4: Partition an $ECQ(s,t)$ into two subgraphs L and R, $L \cong ECQ(s-1,t)$ and $R \cong ECQ(s-1,t)$. Let F be a set of vertices, $F \subset V(ECQ(s,t))$. We set $F_0 = F\cap L$ and $F_1 = F\cap R$. Suppose that $ECQ(s,t)-F$ is disconnected and there exists a component H of $ECQ(s,t)-F$, such that $V(H)\cap(V(R)-F_1) = \varnothing$ and $\deg_H(v) \ge 2$ for any vertex v in H, denoted by $P_2(H)$. Then, $|F| \ge 4s-4$ for $t \ge s \ge 2$.

Proof: Since $\deg_H(v) \ge 2$ for any vertex v in H, there exists a cycle in H [19]. Since $ECQ(s,t)$ is triangle-free, there exists a cycle in H with minimum length 4. Let C_H be a cycle in H with minimum length, then we have $|V(C_H)| \ge 4$. Without loss of generality, there are 3 cases to be considered.

Case 1 ($V(H) \in B$): $V(C_H) \in B$ because $V(H) \in B$. Let $N_B(C_H) = N(C_H)\cap B$, we have $N_B(C_H) \subseteq (F\cap B)\cup(V(H)-V(C_H))$. Since $V(H) \in B$, each edge in H lies in E_3. According to the definition of E_3 in $ECQ(s,t)$, all vertices in H have $c = 1$ and have the same value in bit addresses $[1+t : s+t]$. Similarly, all vertices in $N_B(C_H)$ have $c = 1$ and have the same value in bit addresses $[1+t : s+t]$. By Lemma 8, all vertices in H and $N_B(C_H)$ are in the same CQ_t, denoted by Y. Because $N_A(H) = N(H)\cap A \subseteq (F\cap A)$, we have $|F\cap A| \ge |N_A(H)|$. By the fact that $A\cup B$ contains a perfect matching and $V(H) \in B$, we have $|N_A(H)| = |V(H)|$. Then, $|F| \ge |F\cap A| + |F\cap B| \ge |N_A(H)| + |N_B(C_H)| - |V(H)-V(C_H)| = |V(H)| + |N_B(C_H)| - |V(H)| + |V(C_H)| = |N_B(C_H)| + |V(C_H)|$.

When $|V(C_H)| = 4$, let $C_H = a-b-c-d-a$, as shown in Figure 6, $|N_B(C_H)| = |N_Y(C_H)| = |N_Y(a,b,c,d)| = 4t-8$. So $|F| \ge |N_B(C_H)| + |V(C_H)| \ge 4t-8+4 = 4t-4 \ge 4s-4$.

When $|V(C_H)| = 5$, let $C_H = a-b-c-d-e-a$, we have $t \ge 3$ because $V(CQ_t) = 2^t \ge 5$. Because there is no triangle in CQ_n, we have $N_Y(a)\cap N_Y(b) = \varnothing$, $N_Y(b)\cap N_Y(c) = \varnothing$, $N_Y(c)\cap N_Y(d) = \varnothing$, $N_Y(d)\cap N_Y(e) = \varnothing$, and $N_Y(e)\cap N_Y(q) = \varnothing$, as shown in Figure 7. By lemma 10, we have $|N_Y(a)\cap N_Y(c,d) - \{e,b\}| \le 1$ and $|N_Y(e)\cap N_Y(b,c) - \{a,d\}| \le 1$. If $|N_Y(a)\cap N_Y(d) - \{e\}| = 1$, then $N_Y(a)\cap N_Y(c) - \{d\} = \varnothing$ and $N_Y(d)\cap N_Y(b) - \{c\} = \varnothing$. Similarly, if $|N_Y(b)\cap N_Y(e) - \{a\}| = 1$, then $N_Y(c)\cap N_Y(e) - \{d\} = \varnothing$. Hence, As shown in Figure 7, $|N_B(C_H)| = |N_Y(C_H)| = |N_Y(a,b,c,d,e)| \ge 5t-12$. So $|F| \ge |N_B(C_H)| + |V(C_H)| \ge 5t-12+5 = 4t-4+(t-3) \ge 4t-4 \ge 4s-4$.

When $|V(C_H)| = 6$, let $C_H = a-b-c-d-e-f-a$, we also have $t \ge 3$ by $V(CQ_t) = 2^t \ge 6$. If $t = 3$, then each vertex in C_H has 3 neighbors in B. Therefore, as shown in Figure 8, each vertex in C_H has at most one common neighbor with other nonadjacent vertices in C_H. Hence, $|N_B(C_H)| = |N_Y(C_H)| = |N_Y(a,b,c,d,e,f)| \ge 6t-15$. So $|F| \ge |N_B(C_H)| + |V(C_H)| \ge 6t-15+6 = 4t-4+(2t-5) \ge 4t-4 \ge 4s-4$. If $t = 4$, then each vertex in C_H has 4 neighbors in B. Therefore, as shown in Figure 9, each vertex in C_H has at most 2 common neighbor

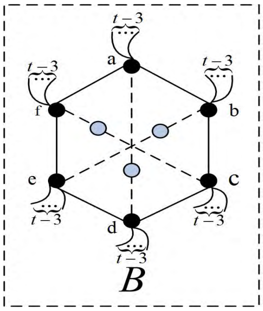

FIGURE 8. A cycle C_H of length six in H with $V(H) \subset B$ and $t = 3$.

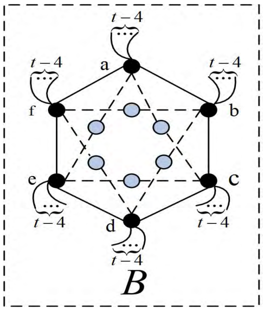

FIGURE 9. A cycle C_H of length six in H with $V(H) \subset B$ and $t = 4$.

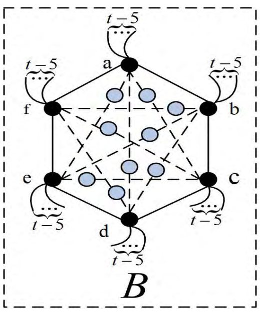

FIGURE 10. A cycle C_H of length six in H with $V(H) \subset B$ and $t \geq 5$.

with other nonadjacent vertices in C_H. Hence, $|N_B(C_H)| = |N_Y(C_H)| = |N_Y(a, b, c, d, e, f)| \geq 6t - 18$. So $|F| \geq |N_B(C_H)| + |V(C_H)| \geq 6t - 18 + 6 = 4t - 4 + (2t - 8) \geq 4t - 4 \geq 4s - 4$. If $t = 5$, then each vertex in C_H has at most 5 neighbors in B. In an extreme case, as shown in Figure 10, each vertex in C_H has at most 3 common neighbor with other nonadjacent vertices in C_H. But as we know, CQ_5 does not contain a subgraph isomorphic to Figure 10. Hence, when $t = 5$ $|N_B(C_H)| = |N_Y(C_H)| = |N_Y(a, b, c, d, e, f)| > 6t - 21$. Then, we have $|N_B(C_H)| \geq 6t - 20$. So $|F| \geq |N_B(C_H)| + |V(C_H)| \geq 6t - 20 + 6 = 4t - 4 + (2t - 10) \geq 4t - 4 \geq 4s - 4$ for $t = 5$. If $t \geq 6$, we have $|N_B(C_H)| = |N_Y(C_H)| = |N_Y(a, b, c, d, e, f)| \geq 6t - 21$ as shown in Figure 10. So $|F| \geq |N_B(C_H)| + |V(C_H)| \geq 6t - 21 + 6 = 4t - 4 + (2t - 11) \geq 4t - 4 \geq 4s - 4$ for $t \geq 6$.

When $|V(C_H)| \geq 7$, we also have $t \geq 3$ by $V(CQ_t) = 2^t \geq 7$. There exists a path $\ldots - a - b - c - d - \ldots$ in C_H, where a, b, c, d are 4 vertices in C_H and $(a, b), (b, c), (c, d) \in$

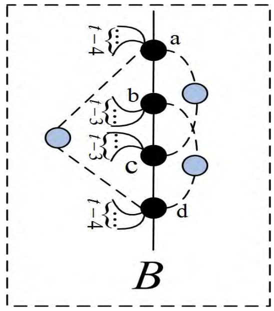

FIGURE 11. A cycle C_H in H with $|V(C_H)| \geq 7$ and $V(H) \subset B$.

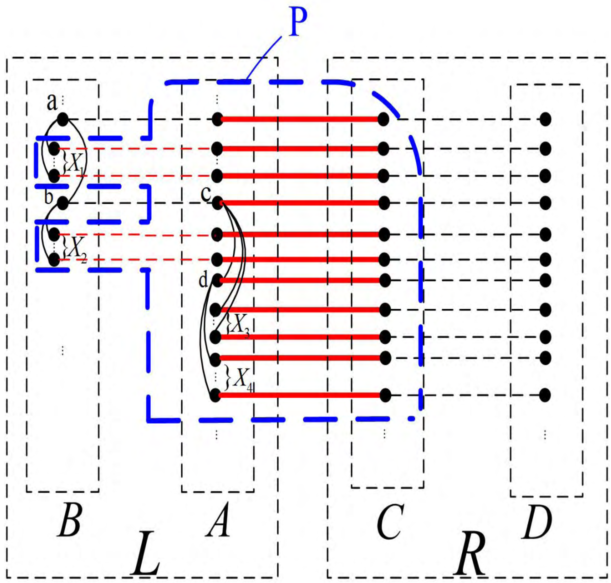

FIGURE 12. Illustration for Case 2.

$E(C_H)$. By lemma 10, we have $|N_Y(a) \cap N_Y(c, d) - \{b\}| \leq 2$ and $|N_Y(d) \cap N_Y(a, b) - \{c\}| \leq 2$. There is an extreme case as shown in Figure 11. Hence, $|N_B(C_H)| = |N_Y(C_H)| \geq |N_Y(a, b, c, d) - V(C_H)| \geq 4t - 11$. So $|F| \geq |N_B(C_H)| + |V(C_H)| \geq 4t - 11 + 7 = 4t - 4 \geq 4s - 4$.

Case 2 ($V(H) \cap A \neq \emptyset$ and $V(H) \cap B \neq \emptyset$): By $P_2(H)$ and $A \cup B$ contains a perfect matching, there exists a path $a - b - c - d$ in H, where $a, b \in B$ and $c, d \in A$. According the definition of $ECQ(s, t)$, we have $a[t+1 : s+t] = b[t+1 : s+t]$, $b[1 : s+t] = c[1 : s+t]$ and $c[1 : t] = d[1 : t]$. As shown in Figure 12, let $N(a) \cap B - \{b\} = X_1$, $N(b) \cap B - \{a\} = X_2$, $N(c) \cap A - \{d\} = X_3$, $N(d) \cap A - \{c\} = X_4$. Note that $|X_1| = t - 1$, $|X_2| = t - 1$, $|X_3| = s - 2$, and $|X_4| = s - 2$. Because there is no triangle in $ECQ(s, t)$, we have $X_1 \cap X_2 = \emptyset$, $X_2 \cap X_3 = \emptyset$, and $X_3 \cap X_4 = \emptyset$. By Theorem 2, all the vertices in $N[a, b, c, d]$ will at least appear in $t - 1 + t - 1 + s - 2 + s - 2 + 3 \geq 4s - 3$ horizontal straight lines as shown in Figure 12. Each horizontal straight line in subgraph P has at least one vertex in F because $V(H) \cap (V(R) - F_1) = \emptyset$. Therefore, $|F| \geq 4s - 3$.

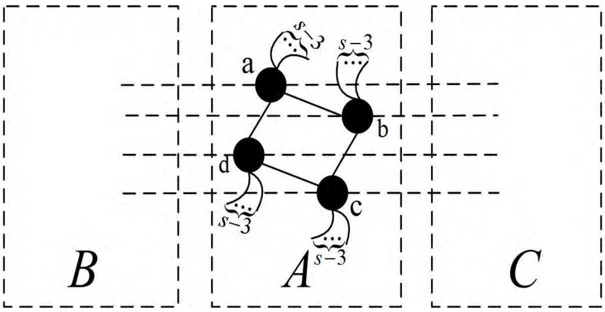

FIGURE 13. A cycle C_H in H with $V(H) \subset A$ and $|V(C_H)| = 4$.

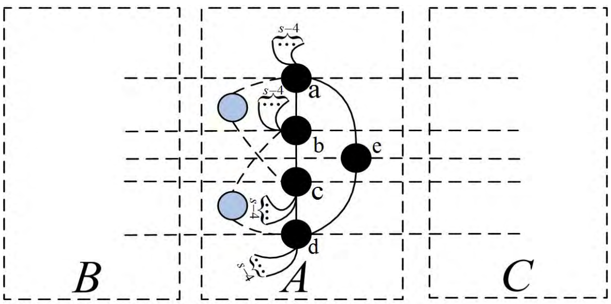

FIGURE 14. A cycle C_H in H with $V(H) \subset A$ and $|V(C_H)| = 5$.

Case 3 ($V(H) \in A$): By $V(H) \in A$, we have $V(C_H) \in A$. By the fact that there are two perfect matchings of subgraphs induced by $A \cup B$, $A \cup C$, each vertex in H has exactly one neighbor in B, one neighbor in C, and $s - 1$ neighbors in A, then $|F| \geq |N(H)| \geq |N_A(C_H) - (V(H) - V(C_H))| + 2|V(H)| \geq |N_A(C_H)| + |V(C_H)| + |V(H)|$.

When $|V(C_H)| = 4$, as shown in Figure 13, we have $|N_A(C_H)| = |N_A(a, b, c, d)| = 4s - 12$. Hence, $|F| \geq |N_A(C_H)| + |V(C_H)| + |V(H)| \geq 4t - 12 + 4 + 4 \geq 4s - 4$.

When $|V(C_H)| = 5$, let $C_H = a - b - c - d - e - a$. In an extreme case, as shown in Figure14, by lemma 9, lemma 10 and there is no triangle in $ECQ(s, t)$, we have $|N_A(C_H)| \geq |N_A(a, b, c, d) - \{e\}| \geq 4s - 14$. Hence, $|F| \geq |N_A(C_H)| + |V(C_H)| + |V(H)| \geq 4s - 14 + 5 + 5 = 4s - 4$.

When $|V(C_H)| \geq 6$, as shown in Figure 15, there exists a path $\ldots - a - b - c - d - \ldots$ in C_H, where a, b, c, d are 4 vertices in C_H and $(a, b), (b, c), (c, d) \in E(C_H)$. In an extreme case, as shown in Figure15, by lemma 9, lemma 10 and there is no triangle in $ECQ(s, t)$, we have $|N_A(C_H)| \geq |N_A(a, b, c, d) - V(C_H)| \geq 4s - 15$. Hence, $|F| \geq |N_A(C_H)| + |V(C_H)| + |V(H)| \geq 4t - 15 + 6 + 6 \geq 4s - 3$.

The proof is complete. □

Theorem 5: For any edge (u, v) of $ECQ(s, t)$, where $u \in A$ and $v \in C$, $|N(w) \cap N(u, v)| \leq 3$ for any vertex w of $ECQ(s, t)$.

Proof: There are 2 cases to be considered.

Case 1 ($w \in A$ (Similarly, $w \in C$)): We have $N(w) \cap N(u, v) \subset A \cup C$. By theorem 3, we have $|N(w) \cap N(u)| \leq 2$ and $|N(w) \cap N(v)| \leq 2$. When $|N(w) \cap N(v)| = 2$, we have $u \in N(w) \cap N(v)$. Then we also have $|N(w) \cap N(v)| - |N(w) \cap \{u, v\}| \leq 1$. Hence, $|N(w) \cap N(u, v)| = |N(w) \cap N(u)| + |N(w) \cap N(v)| - |N(w) \cap \{u, v\}| \leq 2 + 2 - 1 = 3$.

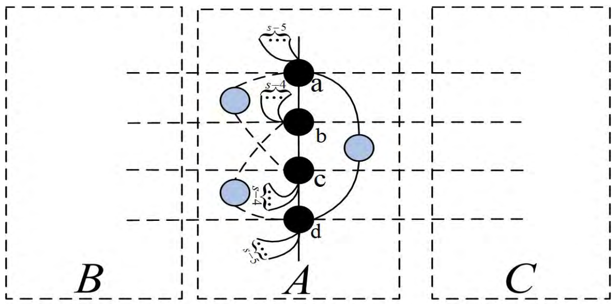

FIGURE 15. A cycle C_H in H with $V(H) \subset A$ and $|V(C_H)| \geq 6$.

Case 2 ($w \in B$ (Similarly, $w \in D$)): w and v have exactly one common neighbor if and only if u, v and w are in a horizontal straight line. Then we have $|N(w) \cap N(u)| = 0$ because $ECQ(s, t)$ is triangle-free. Therefore, $|N(w) \cap N(u, v)| \leq 2$.

When u, v and w are not in a horizontal straight line, we have $|N(w) \cap N(v)| = 0$. By theorem 3, we have $|N(w) \cap N(u)| \leq 2$. Thus, $|N(w) \cap N(u, v)| \leq 2$.

The proof is complete. □

IV. THE CONDITIONAL DIAGNOSABILITY OF EXCHANGED CROSSED CUBE UNDER THE PMC MODEL

In this section, we will give a general method to investigate the conditional diagnosability of $ECQ(s, t)$ under the PMC model. Before discussing this, we introduce some useful theorems as follows.

Theorem 6: Let F be a vertex set of exchanged crossed cube $ECQ(s, t)$ with $|F| \leq 2s - 1$. Then, one of the following two conditions hold:

(1) $ECQ(s, t) - F$ is connected or

(2) $ECQ(s, t) - F$ has exactly two components, one is trivial and the other is nontrivial.

Proof: By Lemma 3, we partition an $ECQ(s, t)$ into two $ECQ(s - 1, t)$ subgraphs, denoted by L and R, where $V(L) = \{0a_{s-2} \ldots a_0 b_{t-1} \ldots b_0 c\}$, $V(R) = \{1a_{s-2} \ldots a_0 b_{t-1} \ldots b_0 c\}$, $L \cong ECQ(s - 1, t)$ and $R \cong ECQ(s - 1, t)$. Let $F_0 = F \cap L$ and $F_1 = F \cap R$, we have $F_0 \cap F_1 = \emptyset$. Because $F_0 \cap F_1 = \emptyset$ and $|F| \leq 2s - 1$, either $|F_0| < s$ or $|F_1| < s$. Without loss of generality, we may assume that $|F_1| < s$. Since $R \cong ECQ(s - 1, t)$, we have $k(R) = s$ by Lemma 5. Then, by $k(R) = s$ and $|F_1| < s$, we know $R - F_1$ is connected. In the following proof, we investigate two cases.

Case 1: There exists a vertex $u \in V(L) - F_0$ such that $N(u) \subset F$.

Let v be an arbitrary vertex of $V(L) - F_0 - \{u\}$, denoted by $v \in V(L) - F_0 - \{u\}$. We consider the following two subcases: $v \in A$ and $v \in B$.

Subcase 1.1 ($v \in A$): Let $v' = N(v) \cap C$.

Subcase 1.1.1 ($v' \notin F$): Since $v' \notin F$, v can be connected to $R - F_1$ by edge (v, v').

Subcase 1.1.2 ($v' \in F$): If $N(v) \subset F$, then $|F| \geq |N(u) \cup N(v)| = |N(u)| + |N(v)| - |N(u) \cap N(v)| \geq s + 1 + s + 1 - 2 = 2s$, which contradicts the condition of $|F| \leq 2s - 1$. Therefore, $N(v) \not\subset F$ which implies $N(v) \cap (A \cup B) \neq \emptyset$.

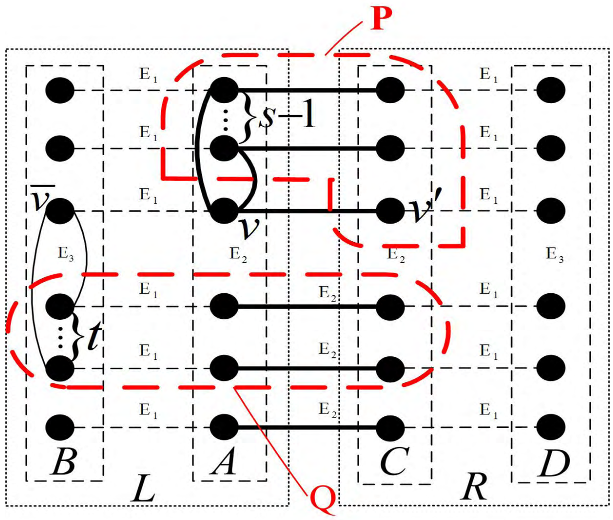

FIGURE 16. Illustration for Subcase 1.1.2.1 in theorem 6.

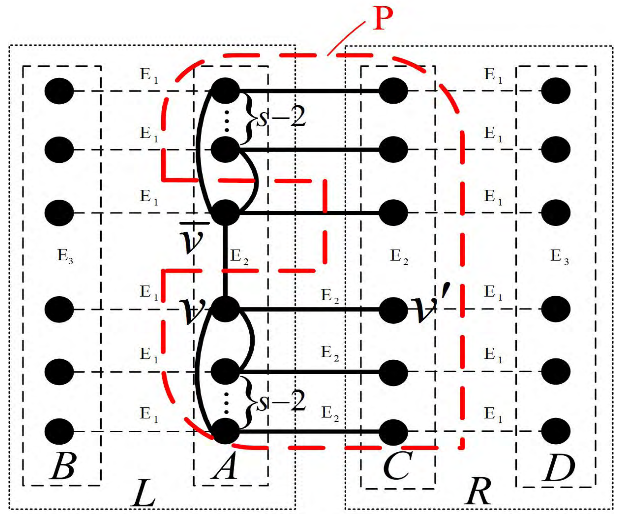

FIGURE 17. Illustration for Subcase 1.1.2.2 in theorem 6.

Subcase 1.1.2.1 ($N(v) \cap B \notin F$): Let $\overline{v} = N(v) \cap B$. As shown in Figure 16, v has $s-1$ neighbors in A and $\overline{v}$ has t neighbors in B by the definition of $ECQ(s,t)$. By theorem 2, each vertex in $N(v) \cap A$ and each vertex in $N(\overline{v}) \cap B$ are not in a horizontal straight line. If v cannot connect to $R - F_1$, then each horizontal straight line of subgraph P and Q has at least one vertex in F. There are $s-1+1+t$ horizontal straight lines in subgraph P and Q. Therefore, $|F| \geq s-1+1+t \geq 2s$, which contradicts the condition of $|F| \leq 2s-1$.

Subcase 1.1.2.2 ($N(v) \cap B \in F$ and $N(v) \cap A \not\subset F$): Let $\overline{v} \in N(v) \cap A - F$. As shown in Figure 17, $\overline{v}$ has $s-2$ neighbors in A besides v, and v has $s-2$ neighbors in A besides $\overline{v}$. We have $|N(v) \cap N(\overline{v})| = 0$ because $ECQ(s,t)$ is triangle-free. If v cannot connect to $R - F_1$, then each horizontal straight line in subgraph P has at least one vertex in F (see Figure 17). If $N(\overline{v}) \cap B \in F$, we have $|F| \geq s-2+1+s-2+1+|N(v) \cap B|+|N(\overline{v}) \cap B| = 2s$, which contradicts the condition of $|F| \leq 2s-1$. If $N(\overline{v}) \cap B \notin F$, with the proof of Subcase 1.1.2.1, we have $|F| \geq 2s$, which also contradicts the condition of $|F| \leq 2s-1$.

Subcase 1.2 ($v \in B$): Let $v' = N(v) \cap A$.

Subcase 1.2.1 ($v' \notin F$): With the proof of Subcase 1.1.2.1, we have $|F| \geq 2s$, which also contradicts the condition of $|F| \leq 2s-1$.

Subcase 1.2.2 ($v' \in F$): If $N(v) \subset F$, we have $|F| \geq |N(u) \cup N(v)| = |N(u)|+|N(v)|-|N(u) \cap N(v)| \geq s+1+s+1-2 = 2s$, which contradicts the condition of $|F| \leq 2s-1$. Therefore, $N(v) \not\subset F$ which implies $N(v) \cap (A \cup B) \not\subset F$.

Since $v' \in F$, we have $N(v) \cap B \not\subset F$. Let $\overline{v}$ be an arbitrary vertex of $N(v) \cap B - F$, denoted by $\overline{v} \in N(v) \cap B - F$. As shown in Figure 18, v has $t-1$ neighbors in B besides $\overline{v}$ and $\overline{v}$ has $t-1$ neighbors in B besides v. Because $ECQ(s,t)$ is triangle-free, we have $|N(v) \cap N(\overline{v})| = 0$. If v cannot connect to $R - F_1$, then each horizontal straight line in subgraph P has at least one vertex in F. We have $|F| \geq t-1+1+t-1+1 = 2t \geq 2s$, which contradicts the condition that $|F| \leq 2s-1$.

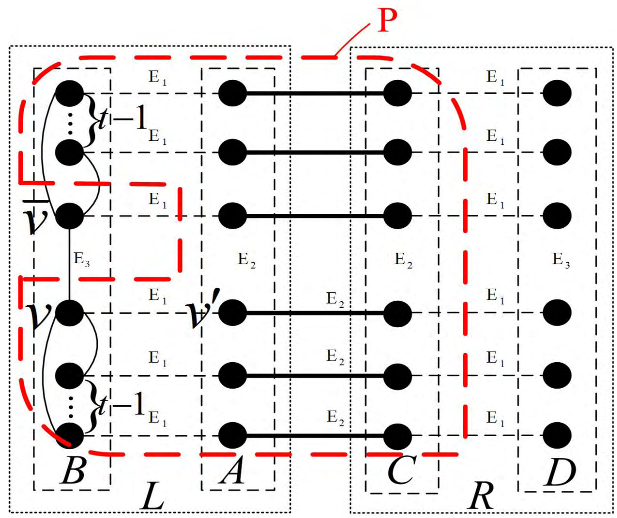

FIGURE 18. Illustration for Subcase 1.2.2 in theorem 6.

Hence, any vertex in $V(L) - F_0 - \{u\}$ is connected to $R - F_1$ when there exists a vertex $u \in V(L) - F_0$ such that $N(u) \subset F$. Since $R - F_1$ is connected, then condition (2) holds.

Case 2 ($N(u) \not\subset F$ for any vertex $u \in V(L) - F_0$): We have the following two subcases.

Subcase 2.1 ($u \in A$): Let $\bar{u} = N(u) \cap C$.

Subcase 2.1.1 ($\bar{u} \notin F$): u can be connected to $R - F_1$ by edge $(u, \bar{u})$.

Subcase 2.1.2 ($\bar{u} \in F$ and $N(u) \cap B \notin F$): Let $u' = N(u) \cap B$. If u cannot connect to $R - F_1$, with the proof of Subcase 1.1.2.1, we have $|F| \geq 2s$, which contradicts the condition of $|F| \leq 2s-1$.

Subcase 2.1.3 ($\bar{u} \in F$ and $N(u) \cap B \in F$): By $N(u) \not\subset F$, we have $N(u) \cap A \not\subset F$. Let $u' = N(u) \cap A - F$. If u cannot connect to $R - F_1$, with the proof of Subcase 1.1.2.2, we have $|F| \geq 2s$, which contradicts the condition of $|F| \leq 2s-1$.

Subcase 2.2 ($u \in B$):

Subcase 2.2.1 ($N(u) \cap A \notin F$): If u cannot connect to $R - F_1$, with the proof of Subcase 1.1.2.1, we have $|F| \geq 2s$, which contradicts the condition of $|F| \leq 2s - 1$.

Subcase 2.2.2 ($N(u) \cap A \in F$): Since $N(u) \not\subset F$, we have $N(u) \cap B \not\subset F$. Let $u' \in N(u) \cap B - F$. If u cannot connect to $R - F_1$, with the proof of Subcase 1.2.2, we can deduce $|F| \geq 2s$, which contradicts the condition of $|F| \leq 2s - 1$.

Therefore, for any vertex u in $L - F_0$, u is connected to $R - F_1$ when $N(u) \not\subset F$ for any vertex $u \in V(L) - F_0$. Since $R - F_1$ is connected, $ECQ(s, t) - F$ is connected. Condition (1) holds. □

Theorem 7: Let F be a vertex set of exchanged crossed cube $ECQ(s, t)$ with $s \geq 1$ and $t \geq 1$. Suppose that $ECQ(s, t) - F$ is disconnected and every component of $ECQ(s, t) - F$ is nontrivial and suppose that there exists one component H of $ECQ(s, t) - F$ such that $\deg_H(v) \geq 2$ for every vertex v in H, denoted by $P_2(H)$. Then, one of the following two conditions holds:

(1)$|F| \geq 4(s - 1)$ or

(2)$|H| \geq 4(s - 1) - 1$.

Proof: By Lemma 3, we partition an $ECQ(s, t)$ into two $ECQ(s - 1, t)$ subgraphs, denoted by L and R, where $V(L) = \{0a_{s-2} \dots a_0 b_{t-1} \dots b_0 c\}$, $V(R) = \{1a_{s-2} \dots a_0 b_{t-1} \dots b_0 c\}$, $L \cong ECQ(s - 1, t)$ and $R \cong ECQ(s - 1, t)$. Let $F_0 = F \cap L$ and $F_1 = F \cap R$. Two possibilities need to investigated.

Case 1 ($|F_0| \geq 2(s - 1)$ and $|F_1| \geq 2(s - 1)$): We have $|F| = |F_0| + |F_1| \geq 4(s - 1)$, then condition (1) holds.

Case 2 ($|F_0| \leq 2(s - 1) - 1$ or $|F_1| \leq 2(s - 1) - 1$): Without loss of generality, we assume $|F_1| \leq 2(s - 1) - 1$. By Theorem 6, $R - F_1$ is connected or $R - F_1$ is disconnected and has exactly two components, one is trivial and the other is nontrivial.

Subcase 2.1: $R - F_1$ is connected.

Subcase 2.1.1 ($V(H) \cap (V(R) - F_1) \neq \varnothing$): Because $V(H) \cap (V(R) - F_1) \neq \varnothing$ and $R - F_1$ is connected, we have $V(R) - F_1 \subset V(H)$. Therefore, $|V(H)| \geq |V(R)| - |F_1| \geq 2^{s+t} - [2(s - 1) - 1] \geq 4(s - 1) - 1$ when $s \geq 1$ and $t \geq 1$. Then, condition (2) holds.

Subcase 2.1.2 ($V(H) \cap (V(R) - F_1) = \varnothing$): Since $V(H) \cap (V(R) - F_1) = \varnothing$, we have $V(H) \subset V(L) - F_0$. Then, we consider the following two subcases.

Subcase 2.1.2.1 ($L - F_0$ Is Connected): Since $ECQ(s, t) - F$ is disconnected, each edge between A and C has at least one adjacent vertex in F because $L - F_0$ and $R - F_1$ are connected. There are 2^{s+t-1} edges between A and C. Hence, $|F| \geq 2^{s+t-1} \geq 4(s - 1)$ for $s \geq 1$ and $t \geq 1$. Then condition (1) holds.

Subcase 2.1.2.2 ($L - F_0$ Is Disconnected): Because H is a component of $ECQ(s, t) - F$, such that $\deg_H(v) \geq 2$ for every vertex in H, by Theorem 4, we have $|F| \geq 4s - 4$. Then, condition (1) holds.

Subcase 2.2 ($R - F_1$ Is Disconnected): Let the trivial component of $R - F_1$ be vertex u, the nontrivial component of $R - F_1$ be $R - F_1 - \{u\}$.

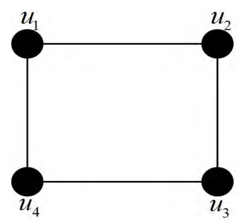

FIGURE 19. **A cycle of length four in $ECQ(s, t)$.**

Subcase 2.2.1: $V(H) \cap (V(R) - F_1) \neq \varnothing$.

Subcase 2.2.1.1 ($V(H) \cap \{u\} \neq \varnothing$): Since $V(H) \cap \{u\} \neq \varnothing$, we have $u \in H$. Because u is a trivial component of $R - F_1$ and every component of $ECQ(s, t) - F$ is nontrivial, we have $u \in C$. By the fact that $A \cup C$ contains a perfect matching, u has no more than one neighbor in H, which contradicts the condition of $P_2(H)$. Hence, $V(H) \cap \{u\} = \varnothing$.

Subcase 2.2.1.2 ($V(H) \cap \{u\} = \varnothing$): Since $V(H) \cap \{u\} = \varnothing$, we have $V(H) \cap (V(R) - F_1 - \{u\}) \neq \varnothing$. Hence, H is connected to $(R - F_1 - \{u\})$ which implies $(R - F_1 - \{u\}) \subseteq H$. Because $|F_1| \leq 2(s - 1) - 1$, we have $|V(H)| \geq |V(R) - F_1 - \{u\}| = |V(R)| - |F_1| - |\{u\}| = 2^{s+t} - |F_1| - 1 \geq 2^{s+t} - (2(s - 1) - 1) - 1 > 4(s - 1)$ for $s \geq 1$ and $t \geq 1$. Then, condition (2) holds.

Subcase 2.2.2 ($V(H) \cap (V(R) - F_1) = \varnothing$): Since H is a component of $ECQ(s, t) - F$ and $\deg_H(v) \geq 2$ for every vertex v in H. By Theorem 4, we have $|F| \geq 4(s - 1)$. Then, condition (1) holds. □

Theorem 8: $t_c(ECQ(s, t)) \leq 4(s - 1) + 1, t \geq s > 2$.

Proof: In order to derive the upper bound of $t_c(ECQ(s, t))$, we may assume that there exists two distinct conditional faulty sets F_1 and F_2, such that $|F_1| = 4(s - 1) + 2$ and $|F_2| = 4(s - 1) + 2$. If (F_1, F_2) is a pair of indistinguishable conditional faulty sets, then $ECQ(s, t)$ is not conditional $4(s - 1) + 2$-diagnosable under the PMC model. Therefore, $t_c(ECQ(s, t)) \leq 4(s - 1) + 1$.

Suppose $u_1 = \{a_{s-1} \dots a_2 00 b_{t-1} \dots b_0 0\}$, $u_2 = \{a_{s-1} \dots a_2 01 b_{t-1} \dots b_0 0\}$, $u_3 = \{a_{s-1} \dots a_2 11 b_{t-1} \dots b_0 0\}$, $u_4 = \{a_{s-1} \dots a_2 10 b_{t-1} \dots b_0 0\}$. As shown in Figure 19, $u_1 - u_2 - u_3 - u_4 - u_1$ is a cycle of length four in $ECQ(s, t)$.

We set $F_1 = N(u_1, u_2, u_3, u_4) \cup \{u_1, u_4\}$ and $F_2 = N(u_1, u_2, u_3, u_4) \cup \{u_2, u_3\}$. It is easy to check that F_1 and F_2 are two distinct conditional faulty sets of $ECQ(s, t)$, such that $|F_1| = |F_2| = 4(s - 1) + 2$, $F_1 \Delta F_2 = \{u_1, u_2, u_3, u_4\}$ and $V(ECQ(s, t)) - (F_1 \cup F_2) = V(ECQ(s, t)) - N(u_1, u_2, u_3, u_4) - \{u_1, u_2, u_3, u_4\}$.

Because $N(F_1 \Delta F_2) = N(u_1, u_2, u_3, u_4)$, any vertex in $F_1 \Delta F_2$ is disconnected to $ECQ(s, t) - (F_1 \cup F_2)$. Hence, (F_0, F_1) is a indistinguishable pair of conditional faulty sets. Then, $ECQ(s, t)$ is not conditional $4(s - 1) + 2$-diagnosable under the PMC model by Theorem 1. Therefore, $t_c(ECQ(s, t)) < 4(s - 1) + 2$, which can deduce $t_c(ECQ(s, t)) \leq 4(s - 1) + 1$ for $t \geq s > 2$. □

Theorem 9: $t_c(ECQ(s, t)) \geq 4(s - 1) + 1, t \geq s > 2$.

Proof: Let F_1 and F_2 be any two distinct conditional faulty sets of $ECQ(s, t)$ with $|F_1| \leq 4(s - 1)$ and

$|F_2| \leq 4(s-1)$. Then, $F_1 \cap F_2$ is also a conditional faulty set of $ECQ(s,t)$ and $|F_1 - F_2| \geq 1$ or $|F_2 - F_1| \geq 1$.

Suppose (F_1, F_2) is an indistinguishable pair. Hence, there is no edge between $F_1 \triangle F_2$ and $V(ECQ(s,t)) - (F_1 \cup F_2)$, which implies that $F_1 \cap F_2$ is a vertex cut of $ECQ(s,t)$. By Lemma 1, any vertex of $F_1 \Delta F_2$ has at least two neighbors in $F_1 \Delta F_2$ and any vertex in $V(ECQ(s,t)) - (F_1 \cup F_2)$ has at least one neighbor in $V(ECQ(s,t)) - (F_1 \cup F_2)$. Because F_1 and F_2 are two distinct conditional faulty sets, any vertex of $F_1 \cap F_2$ has at least one neighbor in $V(ECQ(s,t)) - (F_1 \cap F_2)$. So $F_1 \cap F_2$ is a restricted vertex cut of $ECQ(s,t)$. By Lemma 6, $k_r(ECQ(s,t)) = 2s$ for $t \geq s > 2$. Therefore, $|F_1 \cap F_2| \geq 2s$. Two possibilities need to be investigated.

Case 1 ($V(ECQ(s,t)) = F_1 \cup F_2$): Since $V(ECQ(s,t)) = F_1 \cup F_2$, we have $|V(ECQ(s,t))| = 2^{s+t+1} = |F_1| + |F_2| - |F_1 \cap F_2| \leq |F_1| + |F_2| \leq 8(s-1)$, which contradicts the fact that $8(s-1) < 2^{s+t+1}$ for $t \geq s > 2$. Hence, $V(ECQ(s,t)) \neq F_1 \cup F_2$.

Case 2 ($V(ECQ(s,t)) \neq F_1 \cup F_2$): By Lemma 1, there exists a component H of $ECQ(s,t) - (F_1 \cap F_2)$, such that $V(H) \subset F_1 \Delta F_2$ and $\deg_H(v) \geq 2$ for $v \in V(H)$. Because $V(H) \subset F_1 \Delta F_2$, we have $|H| \leq |F_1 \Delta F_2|$. Since $F_1 \cap F_2$ is a restricted vertex cut of $ECQ(s,t)$, each component of $ECQ(s,t) - (F_1 \cap F_2)$ is nontrivial. By Theorem 7, We have the following two subcases.

Subcase 2.1 ($|F_1 \cap F_2| \geq 4(s-1)$): Since $|F_1 - F_2| \geq 1$ or $|F_2 - F_1| \geq 1$, we have $|F_1| = |F_1 - F_2| + |F_1 \cap F_2| \geq 1 + 4(s-1)$ or $|F_2| = |F_2 - F_1| + |F_1 \cap F_2| \geq 1 + 4(s-1)$, which contradicts the conditions of $|F_1| \leq 4(s-1)$ and $|F_2| \leq 4(s-1)$.

Subcase 2.2 ($|H| \geq 4(s-1)-1$): Since $|F_1 \cap F_2| \geq 2s$ and $|H| \leq |F_1 \Delta F_2|$, we have $|F_1| \geq |F_1 \Delta F_2|/2 + |F_1 \cap F_2| \geq 2(s-1) + 2s = 4s - 2$ or $|F_2| \geq |F_1 \Delta F_2|/2 + |F_1 \cap F_2| \geq 2(s-1) + 2s = 4s - 2$, either of which contradict the conditions of $|F_1| \leq 4(s-1)$ and $|F_2| \leq 4(s-1)$.

Therefore, (F_1, F_2) is an distinguishable pair, then $t_c(ECQ(s,t)) \geq 4(s-1) + 1$ for $t \geq s > 2$. □

Theorem 10: $t_c(ECQ(s,t)) = 4(s-1) + 1$, $t \geq s > 2$.

Proof: By Theorem 8 and Theorem 9, we have $t_c(ECQ(s,t)) = 4(s-1) + 1$ for $t \geq s > 2$. □

V. CONCLUSIONS

The conditional diagnosability of exchanged crossed cube $ECQ(s,t)$ is studied in this paper. By exploring the topological properties of $ECQ(s,t)$, we have successfully demonstrated that the conditional diagnosability of $ECQ(s,t)$ under the PMC model is $4(s-1) + 1$ for $t \geq s > 2$. For further discussion, It is an attractive work to expose the g-good-neighbor conditional diagnosability [44] of $ECQ(s,t)$ under the PMC and MM* model.

REFERENCES

[1] F. P. Preparata, G. Metze, and R. T. Chien, "On the connection assignment problem of diagnosable systems," *IEEE Trans. Electron. Comput.*, vol. EC-16, no. 6, pp. 448–454, Dec. 1967.

[2] J. Maeng and M. Malek, "A comparison connection assignment for self-diagnosis of multiprocessors systems," in *Proc. 11th Int. Symp. Fault-Tolerant Comput.*, pp. 173–175, 1981.

[3] H. Whitney, "Congruent graphs and the connectivity of graphs," in *Hassler Whitney Collected Papers*. Boston, MA, USA: Birkhäuser, 1992, pp. 150–168.

[4] F. Harary, "Conditional connectivity," *Networks*, vol. 13, no. 3, pp. 347–357, 1983.

[5] A.-H. Esfahanian and S. L. Hakimi, "On computing a conditional edge-connectivity of a graph," *Inf. Process. Lett.*, vol. 27, no. 4, pp. 195–199, Apr. 1988.

[6] A.-H. Esfahanian, "Generalized measures of fault tolerance with application to N-cube networks," *IEEE Trans. Comput.*, vol. 38, no. 11, pp. 1586–1591, Nov. 1989.

[7] G.-Y. Chang, "Conditional (t, k)-diagnosis under the PMC model," *IEEE Trans. Parallel Distrib. Syst.*, vol. 22, no. 11, pp. 1797–1803, Nov. 2011.

[8] J. R. Armstrong and F. G. Gray, "Fault diagnosis in a Boolean n cube array of microprocessors," *IEEE Trans. Comput.*, vol. COM-30, no. 8, pp. 587–590, Aug. 1981.

[9] J. Fan, "Diagnosability of the Möbius cubes," *IEEE Trans. Parallel Distrib. Syst.*, vol. 9, no. 9, pp. 923–927, Sep. 1998.

[10] D. Wang, "Diagnosability of hypercubes and enhanced hypercubes under the comparison diagnosis model," *IEEE Trans. Comput.*, vol. 48, no. 12, pp. 1369–1374, Dec. 1999.

[11] J. Fan, "Diagnosability of crossed cubes under the comparison diagnosis model," *IEEE Trans. Parallel Distrib. Syst.*, vol. 13, no. 10, pp. 1099–1104, Oct. 2002.

[12] P.-L. Lai, J. J. M. Tan, C.-H. Tsai, and L.-H. Hsu, "The diagnosability of the matching composition network under the comparison diagnosis model," *IEEE Trans. Comput.*, vol. 53, no. 8, pp. 1064–1069, Aug. 2004.

[13] C.-W. Lee and S.-Y. Hsieh, "Diagnosability of two-matching composition networks under the MM* model," *IEEE Trans. Dependable Secure Comput.*, vol. 8, no. 2, pp. 246–255, Mar. 2009.

[14] C.-W. Lee and S.-Y. Hsieh, "Determining the diagnosability of (1,2)-matching composition networks and its applications," *IEEE Trans. Depend. Sec. Comput.*, vol. 8, no. 3, pp. 353–362, May 2011.

[15] G.-Y. Chang, G. J. Chang, and G.-H. Chen, "Diagnosabilities of regular networks," *IEEE Trans. Parallel Distrib. Syst.*, vol. 16, no. 4, pp. 314–323, Apr. 2005.

[16] P.-L. Lai, J. J. M. Tan, C.-P. Chang, and L.-H. Hsu, "Conditional diagnosability measures for large multiprocessor systems," *IEEE Trans. Comput.*, vol. 54, no. 2, pp. 165–175, Feb. 2005.

[17] G.-H. Hsu, C.-F. Chiang, L.-M. Shih, L.-H. Hsu, and J. J. M. Tan, "Conditional diagnosability of hypercubes under the comparison diagnosis model," *J. Syst. Archit.*, vol. 55, no. 2, pp. 140–146, Feb. 2009.

[18] G.-H. Hsu and J. J. M. Tan, "Conditional diagnosability of the BC networks under the comparison diagnosis model," *Int. Comput. Symp.*, vol. 1, pp. 269–274, Nov. 2008.

[19] Q. Zhu, S.-Y. Liu, and M. Xu, "On conditional diagnosability of the folded hypercubes," *Inf. Sci.*, vol. 178, no. 4, pp. 1069–1077, Feb. 2008.

[20] Q. Zhu, "On conditional diagnosability and reliability of the BC networks," *J. Supercomput.*, vol. 45, no. 2, pp. 173–184, Aug. 2008.

[21] M. Xu, K. Thulasiraman, and X. D. Hu, "Conditional diagnosability of matching composition networks under the PMC model," *IEEE Trans. Circuits Syst., II, Exp. Briefs*, vol. 56, no. 11, pp. 875–879, Nov. 2009.

[22] M.-C. Yang, "Conditional diagnosability of matching composition networks under the MM* model," *Inf. Sci.*, vol. 233, no. 1, pp. 230–243, Jun. 2013.

[23] M.-C. Yang, "Conditional diagnosability of balanced hypercubes under the PMC model," *Inf. Sci.*, vol. 222, pp. 754–760, Sep. 2013.

[24] M.-C. Yang, "Conditional diagnosability of balanced hypercubes under the MM* model," *J. Supercomput.*, vol. 65, pp. 1264–1278, Jan. 2013.

[25] S. Zhou, "The conditional diagnosability of crossed cubes under the comparison model," *Int. J. Comput. Math.*, vol. 87, no. 15, pp. 3387–3396, Nov. 2010.

[26] S. Zhou, "The conditional diagnosability of locally twisted cubes," in *Proc. 4th Int. Conf. Comput. Sci. Educ.*, Jul. 2009, pp. 221–226.

[27] S. Zhou, "The conditional diagnosability of twisted cubes under the comparison model," in *Proc. IEEE Int. Symp. Parallel Distrib. Process. Appl.*, Aug. 2009, pp. 696–701.

[28] S. Zhou, "The conditional diagnosability of Möbius cubes under the comparison model," in *Proc. IEEE Int. Conf. Inf. Automat.*, 2009, pp. 96–100.

[29] S. Y. Hsieh, C. Y. Tsai, and C. A. Chen, "Strong diagnosability and conditional diagnosability of multiprocessor systems and folded hypercubes," *IEEE Trans. Comput.*, vol. 62, no. 7, pp. 1472–1477, Jul. 2013.

[30] N. W. Chang, W. H. Deng, and S.-Y. Hsieh, "Conditional diagnosability of (n,k)-star networks under the comparison diagnosis model," *IEEE Trans. Rel.*, vol. 64, no. 1, pp. 132–143, Mar. 2015.

[31] L. Lin, L. Xu, S. Zhou, and S.-Y. Hsieh, "The extra, restricted connectivity and conditional diagnosability of split-star networks," *IEEE Trans. Parallel Distrib. Syst.*, vol. 27, no. 2, pp. 533–545, Feb. 2016.

[32] M. Xu, K. Thulasiraman, and Q. Zhu, "Conditional diagnosability of a class of matching composition networks under the comparison model," *Theor. Comput. Sci.*, vol. 674, pp. 43–52, Apr. 2017.

[33] L. Lin, L. Xu, R. Chen, S.-Y. Hsieh, and D. Wang, "Relating extra connectivity and extra conditional diagnosability in regular networks," *IEEE Trans. Dependable Secure Comput.*, to be published.

[34] N. Chang, E. Cheng, and S.-Y. Hsieh, "Conditional diagnosability of cayley graphs generated by transposition trees under the PMC model," *ACM Trans. Des. Automat. Electron. Syst.*, vol. 20, no. 2, Feb. 2015, Art. no. 20.

[35] E. Cheng, K. Qiu, and Z. Shen, "A unified approach to the conditional diagnosability of interconnection networks," *J. Interconnection Netw.*, vol. 13, nos. 3–4, p. 1250007, Apr. 2013.

[36] E. Cheng, K. Qiu, and Z. Shen, "On the conditional diagnosability of matching composition networks," *Theor. Comput. Sci.*, vol. 557, pp. 101–114, Nov. 2014.

[37] K. Li, Y. Mu, K. Li, and G. Min, "Exchanged crossed cube: A novel interconnection network for parallel computation," *IEEE Trans. Parallel Distrib. Syst.*, vol. 24, no. 11, pp. 2211–2219, Nov. 2013.

[38] W. Ning, X. Feng, and L. Wang, "The connectivity of exchanged crossed cube," *Inf. Process. Lett.*, vol. 115, no. 2, pp. 394–396, Feb. 2015.

[39] W. Ning, "The h-connectivity of exchanged crossed cube," *Theor. Comput. Sci.*, vol. 696, pp. 65–68, Feb. 2017.

[40] W. Ning, "The super connectivity of exchanged crossed cube," *Inf. Process. Lett.*, vol. 116, no. 2, pp. 80–84, Feb. 2016.

[41] D. Zhou, J. Fan, J. Zhou, Y. Wang, and B. Cheng, "Diagnosability of the exchanged crossed cube," *Int. J. Comp. Math., Comp. Syst. Theory*, vol. 3, no. 2, pp. 1–12, Mar. 2018.

[42] C. Guo, M. Lemg, S. Peng, and B. Wang, "Conditional diagnosability of exchanged crossed cube under the MM model," *J. Commun.*, vol. 38, no. 9, pp. 106–124, Sep. 2017.

[43] R.-W. Hung, "The property of edge-disjoint Hamiltonian cycles in transposition networks and hypercube-like networks," *Discrete Appl. Math.*, vol. 181, pp. 109–122, Jan. 2015.

[44] S.-L. Peng, C.-K. Lin, J. J. M. Tan, and L.-H. Hsu, "The g-good-neighbor conditional diagnosability of hypercube under PMC model," *Appl. Math. Comput.*, vol. 218, no. 21, pp. 10406–10412, 2012.

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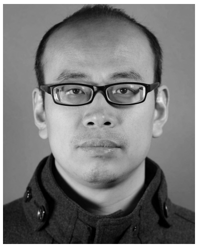

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