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Soft Incomplete Discernibility Matrix for Decision-Making Problems

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ABSTRACT Many decision-making processes are determined in an incomplete data environment. The analysis of decision-making problems on incomplete soft set data is usually performed by first estimating incomplete values. In this paper, we analyze the decision-making problems of incomplete soft sets with incomplete data without turning them into complete data. We define a soft incomplete discernibility matrix and a soft parameter dominant incomplete discernibility matrix to solve the application of incomplete soft sets in decision-making problems. We focus on the classification ability of the corresponding parameter for a given incomplete soft set. The novel method maintains the original data state and successfully constructs a decision-making approach that can be applied to incomplete soft sets. Finally, the weighted soft incomplete discernibility matrix is defined for application in weighted parameter decision-making issues.

INDEX TERMS Incomplete soft sets, decision making, soft incomplete discernibility matrix.

I. INTRODUCTION

Decision making is a common activity in people's daily work and life. The decision-making process can be considered as selecting the (possible) best alternatives or ranking (possible) alternatives from the given data of complete or incomplete information. Furthermore, in the process of collecting and processing data, many real-life problems in economics, medicine, engineering and other fields involve incomplete, unknown, unclear or missing data from the observed information system. Uncertain data frequently appear in observed information systems. To satisfy the acquisition of incomplete knowledge, we discuss the incomplete soft set theory and its decision-making problems in this work.

Uncertainty is widespread because objective data are imprecise and complex, so the collection and modeling of uncertain and complex information are of paramount importance to the acquisition of optimization solutions in decision-making problems. Increasingly many researchers have focused on "uncertainty, imprecision and vagueness" useful information to describe uncertainty, such as the theories of fuzzy sets [1], rough sets [2], probability and other mathematical theories [3], [4]. Meanwhile, Molodtsov [5] sponsored soft set research as a different theorem of mathematics to handle imprecise and uncertain environments. This new theory has established a theoretical framework that contains other theoretical problems of uncertainty. Many scholars believe that soft sets are parametric tools, but we believe that the importance of soft sets is reflected in the parameterization and different functions of the mapping. Different function mapping forms can cause different uncertain problems in research. For example, if the function mapping is an equivalence class relation, the problem is transformed into a rough set [6]. If the function is a membership relation study, the problem is transformed into a fuzzy set of research [7]. The soft set theory is easy to apply in many different areas such as operations research, game theory, and probability theory [5], [8].

With many important application results, one of the most important aspects is the application of (incomplete) soft sets in decision making. Imprecise objects can be approximately described in the (incomplete) soft set theory. There is no restriction on the description of the object. Decision makers can select their own form and parameterized set of values according to their requirements. In fact, the set parameters are not binding, which can greatly simplify the decision-making process, and we can still make a valid decision with the lack of partial information.

Soft set decision-making problems [9] can generally be divided into two classes from the data source viewpoint:

complete data decision issues and incomplete decision-making problems.

Many scholars have studied the soft set theory with complete data and its applications for decision-making issues. Maji *et al.* [10] gave an application to decision-making problems with soft set theory and defined a similar knowledge reduction for soft sets from the point of rough sets. Chen *et al.* [11] improved the parameter reduction and presented an application to the decision-making problem with all possible optimal choices, but they did not consider problems with the suboptimal choice object. Fundamentally, [10] and [11] are from the points of rough sets for the parameter knowledge reduction and decision making. Differently, Kong *et al.* [12] defined the new concept of abnormal parameter reduction, which considered all optimal and suboptimal object decisions. Using the new parameter reduction, all suboptimal levels are preserved. The new parameter reduction approach has shown the significant and different capability to reduce the dimensionality with rough sets. The authors noted that the difference reductions between rough sets and soft sets were essential. Ma *et al.* [13] improved the normal parameter reduction as an oriented parameter sum algorithm. Numerical experiment results showed that the new reduction algorithm had less computational complexity than that in [12]. Han *et al.* [14] examined the normal parameter reduction with a new 0-1 linear programming algorithm, and the numerical experiment proved that the new efficient algorithm had a shorter computational time than that in [13]. Danjuma *et al.* [15] addressed the normal parameter reduction using an alternative method and decision making. Cağman and Enginoğlu [16] presented soft matrices, studied their operations and described the four products of soft matrices and their properties. Then, the algorithm of max-min matrices was applied to handle the decision-making problem. Cağman and Enginoğlu [17] initiated a uni-int operation and defined a uni-int decision function; then, they applied the uni-int operations and decision function to the decision-making problem.

In real-life application, we sometimes require incomplete data information to make decisions. Many papers studied incomplete soft sets to solve decision-making problems. Zou and Xiao [18] initiated data analysis under incomplete information and defined incomplete soft sets for decision-making problems. The incomplete information decision values were computed from the average value of all possible choice values. Kong *et al.* [19] showed efficient incomplete data filling following the simplified probability approach for incomplete soft sets, and the numerical results showed that the computations were less than those in [18]. Muhammad *et al.* [20] presented an alternative data-filling method based on the reliability of association parameters in incomplete soft sets. Alcantud and Santos-Garcia [21] proposed two modified algorithms to analyze incomplete soft sets and compared them with earlier solutions. Han *et al.* [22] developed and compared several elicitation criteria in analyzing incomplete soft-set-based decision making.

As far as we know, the main ideas of all papers to solve incomplete soft set decision-making problems are data filling the incomplete information from some reasonable aspects and subsequent decision making. The incomplete soft set data filling fills the unknown parameter mapping values, which have no or low effect on decision-making problems.

In this paper, we do not directly operate on and analyze the incomplete soft set data. By establishing an incomplete discernibility matrix and a soft parameter dominant incomplete discernibility matrix, we find the cardinality of the soft parameter dominant incomplete discernibility matrix and the order decision-making relation to find the optimal objects.

The remainder of the paper is organized as follows. Section 2 reviews the concepts of incomplete information systems and incomplete soft sets. Section 3 introduces the soft incomplete discernibility matrix and soft parameter dominant incomplete discernibility matrix for incomplete soft sets and discusses some properties of the soft incomplete discernibility matrix and soft parameter dominant incomplete discernibility matrix. Finally, the cardinality of the soft parameter dominant incomplete discernibility matrix is presented to find the optimal objects. In section 4, the weighted incomplete soft set is proposed and applied to decision-making problems. In the final section, the conclusions and future works are presented.

II. PRELIMINARIES

A. INFORMATION SYSTEM AND INCOMPLETE INFORMATION SYSTEM

The definitions of information systems and incomplete information systems are reviewed as follows.

Definition 1 [23]: A complete information system or a complete information table system is defined as a 4-tuple S=(U,A,V,f), where $U = \{x_i | x_i \in U\}$ *denotes a non-empty finite set of objects, every* $x_i \in U$ $(i \leq n)$ *denotes one object in the object universe,* $A = \left\{a_j | a_j \in A\right\}$ *is the attributes in a universe set, each* $a_j \in A\,(j \leq m)$ *is an attribute, and* $V = \cup_{a \in A} V_a$*, where* V_a *is the value set of attribute a and* $f : U \times A \to V$ *is a complete information function such that* $f(x, a) \in V_a$ *for every* $x \in U, a \in A$.

Definition 2 [24]: An incomplete information system indicates that the attribute values of interest V_a *for some objects are unknown, where* $U = \{x_i | x_i \in U\}$ *is a non-empty finite set of objects,* $A = \left\{a_j | a_j \in A\right\}$ *is the attributes in a universe set, and the special symbol "∗" is the unknown value. For example, if* $f(a, x) = *$*, the value of attribute a and object x is unknown.*

Definition 3 [24]: A similar equivalence relation is defined as $SIM(A)$ *over U; the classification of U, which is determined by a similar indiscernibility relation, is denoted by* $SIM(A) = \{(x, y) \in U \times U | \forall a \in A, f(a, x) = f(a, y) \text{ or } f(a, x) = * \text{ or } f(a, y) = *\}$*, where the similar indiscernibility relation is called the tolerance relation in rough sets. The unknown value is equivalent to any domain value with the corresponding object and attribute.*

The incomplete rough set is a special mathematical set to address the incomplete information system.There are interesting connections between the incomplete rough set and the incomplete information system. We also note that the incomplete information system and incomplete soft set are closely related.

B. SOFT SETS AND INCOMPLETE SOFT SETS

We adopt the general statement of soft sets and incomplete soft sets:

U is a nonempty initial universal set of objects; E is a set of parameters related to the objects in U. As is well known, soft sets can be represented in tabular form. The rows denote the objects in U, and the columns denote the parameters in E.

Definition 4 [5]: Let U be the objects in a nonempty universe and E be the parameters in a universal set. When $A \subseteq E$, a pair (F, A) is a soft set over U, and F is a mapping given by $F : A \to P(U)$, where $P(U)$ is the power sets of U approximated by parameter e.

From definition 4, in the corresponding universe U, the pair (F, E) can be considered a parameterized family of subsets in U. In the classical soft set theory, parameters that structure into a particular object can separate objects into two classes represented by 0 or 1. For each parameter $e \in A$, subset $F(e) \subseteq U$ may be considered the set of e or e-approximate elements in the soft set.

Proposition 1 [13]: A pair (F, E) is a soft set; (F, E) can be represented as a Boolean information system.

Let (F, E) be a soft set on universe U; we define mapping $F = \{f_1, f_2, \ldots f_n\}$,where $f_i : U \to V_i$ and $f_i(x) = \begin{cases} 1, x \in F(a_i) \\ 0, x \notin F(a_i) \end{cases}$. If the parameter set E may be considered the attribute set $A = E$, the universe in (F, E) is considered the same universe in information system $V = U_{e_i \in A}$, where $V_{e_i} = \{0, 1\}$. Hence, the pair (F, E) may be represented as a Boolean information system $S = (U, A, V, f)$.

There are interesting relationships between soft sets and Boolean information systems. We note that soft sets and Boolean information systems are closely related. Then, can we find the relation between incomplete soft sets and incomplete information systems? In the following part, we will answer this question.

Definition 5 [22]: A pair (F, E) is defined as an incomplete soft set over U. In this case, A is a subset of E and $F : A \to \{0, 1, *\}^U$, where $\{0, 1, *\}^U$ is the mapping of parameters from U to $\{0, 1, *\}$. In this paper, the "$*$" symbol denotes 0 or 1 in the mapping (class) value of the incomplete soft sets.

Obviously, soft sets can be considered a special case of incomplete soft sets. To some extent, every soft set can be considered incomplete. In definition 5, the "$*$" symbol captures uncertain information. The parameters e belong or do not belong to one special object and are unknown.

Proposition 2: A pair (F, E) is an incomplete soft set; (F, E) can be represented as an incomplete Boolean information system.

Proof: The pair (F, E) is a soft set in the universe U. We define the mapping $F = \{f_1, f_2, \ldots f_n\}$, where

$$f_1 : U \to V_1 \text{ and } f_1(x) = \begin{cases} 1 \; or \; *, & x \in F(e_1) \\ 0 \; or \; *, & x \notin F(e_1) \end{cases}$$
$$f_2 : U \to V_2 \text{ and } f_2(x) = \begin{cases} 1 \; or \; *, & x \in F(e_2) \\ 0 \; or \; *, & x \notin F(e_2) \end{cases}$$
$$\ldots$$
$$f_n : U \to V_n \text{ and } f_n(x) = \begin{cases} 1 \; or \; *, & x \in F(e_n) \\ 0 \; or \; *, & x \notin F(e_n) \end{cases}$$

If the parameter set E may be considered the attribute set $A = E$, the universe U in an incomplete soft set is considered the same universe in incomplete information system $V = U_{e_i \in A}$, where $V_{e_i} = \{0, 1, *\}$. We can conclude that an incomplete soft set may be represented as an incomplete Boolean information system $S = (U, A, V, f)$.

Ali [25], [26] constructed a soft binary relation for complete soft sets. We expand this definition to incomplete soft sets.

Definition 6: If $F : A \to \{0, 1, *\}^{U \times U}$ is a mapping from a subset of parameters with $A \subseteq E$ for all subsets of $U \times U$, then the incomplete soft set (F, E) over $U \times U$ is defined as an incomplete soft binary relation over U.

Definition 7: An incomplete soft binary relation over (F, E) is defined as a similar equivalence relation over U if $A \subseteq E$, $F(\alpha) \neq \emptyset$ is a similar equivalence relation over U for all $\alpha \in A$.

Each similar equivalence relation on the incomplete soft sets partitions the sets into different classes. Then, each different class of the partitions constructs a similar equivalence relation for incomplete soft sets. Therefore, a similar equivalence relation provides us a parametrized collection of partitions in incomplete soft sets.

Definition 8: Suppose that an incomplete soft set (F, E) is defined as a similar soft equivalence relation over U with $A \subseteq E$. For each equivalence relation $F(e), e \in A$, the notion of a similar equivalence relation can be represented by $SIM\ F(e_i)|_{[x]} = \{y : (x, y) \in F(e_i) \; or \; (x, *) \in F(e_i) \; or \; (*, y) \in F(e_i), y \in U\}$.

Definition 9: Two incomplete soft set parameters $\alpha \neq \beta$ are called an equivalent classification if $SIM\ F(\alpha) = SIM\ F(\beta)$ for all $x \in U$.

From the above definition, we can find an indiscernibility relation derived from the similar soft equivalence relation of incomplete soft sets. The incomplete indiscernibility relation (IIR) can be defined as the intersection of all similar equivalence relations, which is denoted by

$$IIR(F, E) = \cap_{e_i \in A} SIM\ F(e_i)$$

Suppose that (F, E) is an incomplete soft set on U and $h_i \in U (i = 1, 2, \ldots m)$. From the similar indiscernibility relation $SIM(F, E)$, the classification of U can be represented by $U\,|IIR(F,E)$, and

$$U\,|IIR(F,E)=\{C_1,C_2,C_3,\ldots C_i\}\quad (i\le m)$$

where $C_i=\left\{\left[h_j\right]_{IIR(F,E)}:h_j\in U\right\}$.

For decision-making problems in complete soft sets, the decision value is $d_i=\sum_j h_{ij}$, h_{ij} is the value for the ith object corresponding to the jth parameter e_j, and the optimal choice is $\max(d_i)$. There is a significant relationship between condition parameters and decision values in the classic soft sets. The optimal objective is to select the object with the maximum choice value in the complete soft set. However, for decision making with incomplete soft sets,we cannot directly make a decision with the optimal object by following the max choice value approach.

III. SOFT INCOMPLETE DISCERNIBILITY MATRIX AND DECISION MAKING

The discernibility matrix is applied for the attribute reduction and decision analysis based on rough sets [27]. The advantages of the discernibility matrix are conciseness, ease and convenience of understanding. The theory has also been extended to attribute reduction and decision analysis in incomplete rough sets [28], [29]. Meanwhile, each incomplete soft set can be represented as an incomplete Boolean information system. Therefore, we want to define the soft incomplete discernibility matrix to solve decision-making problems.

A. THE INCOMPLETE DISCERNIBILITY RELATION

In this section, to solve decision-making problems of incomplete soft sets, we present the following definitions.

Definition 10: A pair (F,E) is an incomplete soft set over U. U is a universe of objects, and E is a universal set of parameters. Then, when $A\subseteq E$, F is a function from $U\times A$ to V; we construct $F : U\times A\rightarrow V$ such that function $F(h_i,e_l)\in V$, $V=\{0,1,*\}$, where $U=\{h_1,h_2,\ldots,h_{|U|}\}$ and $A=\{e_1,e_2,\ldots,e_{|A|}\}$. The "$*$" symbol denotes 0 or 1.

Definition 11: The pair (F,E) is an incomplete soft set over U. $R(h_i,h_j)$ is defined as a covering discernibility relation to (F,E), where

$$R\left(h_i,h_j\right)=\big\{e_l\in A:F(h_i,e_l)=(1,0,*)\neq F(h_j,e_l),\\ =(1,0,*)h_i,h_j\in U\big\}$$

denotes the relation set of object parameters between h_i and h_j. $F(h_i,e_l)$ represents the value of object h_i corresponding to parameter e_l. $F(h_j,e_l)$ represents the value of object h_j corresponding to parameter e_l.

From definition 11, there are several potential unequal relations between h_i and h_j. The asterisk $*$ corresponding parameters determined by F may be either 0 or 1. We can obtain an identical decision classification that maps the objects to the same covering discernibility relation by F. In other words, they may have identical objects. Thus, for incomplete (F,E), the discernibility relation based on covering classification can reduce the dimension of the discernibility relation if we can provide the concept of the classification discernibility relation on the incomplete soft set (F,E).

Definition 12: A pair (F,E) is an incomplete soft set over U, and $A\subseteq E$. Classification $U|IIR(F,E)=\{C_i : i\le|U|\}$. $R=R\left(C_i,C_j\right)$ is defined as the classification discernibility relation on incomplete (F,E), where

$$R\left(C_i,C_j\right)=\{e_l\in A:F\left(h_i,e_l\right)=(1,0,*)\neq F\left(h_j,e_l\right)\\ =(1,0,*),\forall h_i\in C_i,\forall h_j\in C_j\}$$

denotes the relation set of the classification discernibility parameter between C_i and C_j, $F(h_i,e_l)$ denotes the value of objects h_i corresponding to parameter e_l, and $F(h_j,e_l)$ denotes the value of objects h_j corresponding to parameter e_l.

To elaborate this concept, we will give an example of incomplete soft sets.

Example 1: A company recruits new employees, and 6 people are applying for a job. Assume an incomplete soft set (F,E) with the tabular representation shown in Table 1, the universe of applications is $U=\{p_1,p_2,\ldots,p_6\}$, and $E=\{e_1,e_2,\ldots,e_7\}$ are the candidate condition parameters "management experience", "good communication skills", "good language skills", "get married", "salary requirements", "corporate culture" and "team spirit" shown in Table 1. According to the collected data, we construct the covering discernibility relation associated with definition 11.

TABLE 1. Incomplete soft set in Example 1.

U	e_1	e_2	e_3	e_4	e_5	e_6	e_7
p_1	1	0	1	1	$*$	1	0
p_2	1	1	$*$	1	1	0	0
p_3	0	$*$	1	1	$*$	0	0
p_4	0	1	1	1	0	1	0
p_5	0	1	0	0	0	1	1
p_6	1	1	$*$	1	1	0	0

For $F(e_1)$, the covering equivalence classification is $\{p_1,p_2,p_6\},\{p_3,p_4,p_5\}$.

For $F(e_2)$, the covering equivalence classification is $\{p_1,p_3\},\{p_2,p_3,p_4,p_5,p_6\}$.

For $F(e_3)$, the covering equivalence classification is $\{p_1,p_2,p_3,p_4,p_6\},\{p_2,p_5,p_6\}$.

For $F(e_4)$, the covering equivalence classification is $\{p_1,p_2,p_3,p_4,p_6\},\{p_5\}$.

For $F(e_5)$, the covering equivalence classification is $\{p_1,p_2,p_3,p_6\},\{p_1,p_3,p_4,p_5\}$.

For $F(e_6)$, the covering equivalence classification is $\{p_1,p_4,p_5\},\{p_2,p_3,p_6\}$.

For $F(e_7)$, the covering equivalence classification is $\{p_1,p_2,p_3,p_4,p_6\},\{p_5\}$.

We can obtain an indiscernibility relation by intersecting all covering equivalence relations with parameters, i.e., $IIR(F,E)=\cap_{e_i\in E}SIMF(e_i)$, so we can obtain $IIR(F,E)=\{(p_1,p_1),(p_2,p_2),(p_3,p_3),(p_4,p_4),(p_5,p_5),(p_6,p_6),(p_2,p_6),(p_6,p_2)\}$. Thus, the classification determined by the indiscernibility relation is $C_1=\{p_1\}$; $C_2=\{p_2,p_6\}$; $C_3=\{p_3\}$; $C_4=\{p_4\}$; $C_5=\{p_5\}$.

TABLE 2. Incomplete classification-based discernibility matrix of Table 1.

C_i	C_1	C_2	C_3	C_4	C_5
C_1	$\emptyset$				
C_2	$\{e_2, e_3, e_5, e_6\}$	$\emptyset$			
C_3	$\{e_1, e_2, e_5, e_6\}$	$\{e_1, e_2, e_3, e_5\}$	$\emptyset$		
C_4	$\{e_1, e_2, e_5\}$	$\{e_1, e_3, e_5, e_6\}$	$\{e_2, e_5, e_6\}$	$\emptyset$	
C_5	$\{e_1, e_2, e_3, e_4, e_5, e_7\}$	$\{e_1, e_3, e_4, e_5, e_6, e_7\}$	$\{e_2, e_3, e_4, e_5, e_6, e_7\}$	$\{e_3, e_4, e_7\}$	$\emptyset$

The classification of U indiscernibility relation $IIR(F, E)$ over incomplete (F, E) divides U into five classes: $C_1 = \{p_1\}$;
$C_2 = \{p_2, p_6\}$; $C_3 = \{p_3\}$; $C_4 = \{p_4\}$; $C_5 = \{p_5\}$. Thus, the incomplete discernibility matrix based on definition 12 is shown in tabular form in Table 2.

For p_2, p_6, we find that the corresponding parameters have identical covering classifications. Thus, we only need to select one object for comparison with other classes.

As observed in the above definition, the incomplete covering and classification discernibility relation is similar to the basic relation set of the discernibility matrix in information systems. Then, can we use the incomplete covering and classification discernibility relation to construct the incomplete discernibility matrix to solve the decision-making problem under an incomplete soft set? We attempt to answer this question in the remainder of this article.

In a complete soft set for decision making, the decision value is determined by $d_i = \sum_j h_{ij}$, where h_{ij} is the value of corresponding parameters with values of 1, and the optimal choice is determined by the object with the maximum choice value $\max(d_i)$. In other words, the best object(s) is(are) selected by the object(s) with the maximal sum value of corresponding parameters with values of 1. However, Table 2 shows that it is impossible to solve decision-making problems using an incomplete soft set. We find that the value of the corresponding parameter of incomplete soft sets is decided not just by the corresponding parameters with value "1, 0" but also by the potential corresponding parameters with unknown asterisk value "$1, *$", "$*, 0$" and "$*, *$". How do we select the best object(s) with the corresponding parameter values of "1, 0" or the potential corresponding parameter values of "$1, *$", "$*, 0$" and "$*, *$" in an incomplete soft set? The values of known and unknown parameters should be compared between all corresponding parameters of objects in the discernibility relation of the incomplete soft set.

To fully use the discernibility relation of incomplete sets to solve the problem of decision making, using the results of pairwise comparisons, we establish a soft incomplete discernibility matrix by matching the comparison method as follows.

B. INCOMPLETE DISCERNIBILITY MATRIX

Definition 13: A pair (F, E) is an incomplete soft set over U when $A \subseteq E$. Classification $U|IIR(F, A) = \{C_i : i \leq |U|\}$. We define the soft incomplete discernibility matrix $D = \left(D\left(C_i, C_j\right)\right)_{i,j \leq |U|}$, *where*

$$\begin{aligned} &D\left(C_i, C_j\right) \\ &\quad = \{E^i \cup E^j : i, j \leq |U|.\} \\ &\quad = \{E^{i(1)} \cup E^{i(*)} \cup E^{j(1)} \cup E^{j(*)} \cup E^{**} : i, j \leq |U|.\} \end{aligned}$$

is called the set of soft incomplete discernibility parameters between C_i and C_j. For $E^{i(1)} = \{e_l^{i(1)} : F(h_i, e_l) = 1$ and $F(h_j, e_l) = 0, \forall h_i \in C_i, \forall h_j \in C_j, e_l \in A\}$, $E^{i(*)} = \{e_l^{i(*)} : F(h_i, e_l) = 1$ and $F(h_j, e_l) = *$, or $F(h_i, e_l) = *$ and $F(h_j, e_l) = 0, \forall h_i \in C_i, \forall h_j \in C_j, e_l \in A\}$, $E^{j(1)} = \{e_l^{j(1)} : F(h_j, e_l) = 1$ and $F(h_i, e_l) = 0$, $\forall h_i \in C_i, \forall h_j \in C_j, e_l \in A\}$, $E^{j(*)} = \{e_l^j : F(h_j, e_l) = 1$ and $F(h_i, e_l) = *$ or $F(h_j, e_l) = *$ and $F(h_i, e_l) = 0, \forall h_i \in C_i, \forall h_j \in C_j, e_l \in A\}$ and $E^{**} = \{e_l^{**} : F(h_i, e_l) = F(h_j, e_l) = *, \forall h_i \in C_i, \forall h_j \in C_j, e_l \in A\}$.

The symbol $E^{i(1)}(or E^{j(1)})$ indicates the objects in $C_i(or\ C_j)$ for parameter e_l with value 1, which make $F(h_i, e_l) > F(h_j, e_l)$, and the symbol $E^{i(*)}(or E^{j(*)})$ indicates the objects in $C_i(or\ C_j)$ with the potential relation that $F(h_i, e_l) > F(h_j, e_l)$ for parameter e_l. The symbol E^{**} indicates that the objects in $C_i(or\ C_j)$ have the uncertain value of 0 or 1 for parameter e_l.

From definition 13, we find that the union set E^i is determined by objects with (potential) value 1 in C_i and the corresponding objects with (potential) value 0 in C_j. Similarly, the union set E^j is determined by objects with (potential) value 1 in C_j and the corresponding objects with (potential) value 0 in C_i. If we can determine the cardinality of $\left|E^i\right|$ and $\left|E^j\right|$ in $D(C_i, C_j)$, we can easily calculate the order relation between C_i and C_j of incomplete soft sets and decision making.

Next, we will continue analyzing Example 1. From definition 13, the soft incomplete discernibility matrix can be expressed in the tabular form shown in Table 3.

Here, we only analyze $D(C_1, C_2)$, and other soft incomplete discernibility parameter sets can be obtained using the same method. In $D(C_1, C_2)$, there is only one object $p_1 \in C_1$ and two objectives $p_2, p_6 \in C_2$. Then, from Table 2, we have $D(C_1, C_2) = \{e_2, e_3, e_5, e_6\}$, and from Table 1, we have $E^{i(1)} = \{F(p_1, e_6) = 1$ *and* $F(p_2, e_6) = 0\}$ and $E^{i(*)} = \{F(p_1, e_3) = 1$ *and* $F(p_2, e_3) = *\}$. Therefore, $E^1 = \{e_3^{i(*)}, e_6^{i(1)}\}$. Moreover, $E^{j(1)} = \{F(p_2, e_2) = 1$ *and* $F(p_1, e_2) = 0\}$ and $E^{j(*)} = \{F(p_2, e_5) = 1$ *and* $F(p_1, e_5) = *\}$. Therefore, $E^2 = \{e_2^{j(1)}, e_5^{j(*)}\}$. Based on definition 13, we can obtain $D(C_1, C_2) = E^1 \cup E^2 = \left\{e_2^{j(1)}, e_3^{i(*)}, e_5^{j(*)}, e_6^{i(1)}\right\}$.

TABLE 3. Soft incomplete discernibility matrix of Table 1.

C_i	C_1	C_2	C_3	C_4	C_5
C_1	$\emptyset$				
C_2	$\left\{e_2^{j(1)}, e_3^{i(*)}, e_5^{j(*)}, e_6^{i(1)}\right\}$	$\emptyset$			
C_3	$\left\{e_1^{i(1)}, e_2^{j(*)}, e_5^{**}, e_6^{i(1)}\right\}$	$\left\{e_1^{i(1)}, e_2^{i(*)}, e_3^{j(*)}, e_5^{i(*)}\right\}$	$\emptyset$		
C_4	$\left\{e_1^{i(1)}, e_2^{j(1)}, e_5^{i(*)}\right\}$	$\left\{e_1^{i(1)}, e_3^{j(*)}, e_5^{i(1)}, e_6^{j(1)}\right\}$	$\left\{e_2^{j(*)}, e_5^{i(*)}, e_6^{j(1)}\right\}$	$\emptyset$	
C_5	$\left\{e_1^{i(1)}, e_2^{j(1)}, e_3^{i(1)}, e_4^{i(1)}, e_5^{i(*)}, e_7^{j(1)}\right\}$	$\left\{e_1^{i(1)}, e_3^{i(*)}, e_4^{i(1)}, e_5^{i(1)}, e_6^{j(1)}, e_7^{j(1)}\right\}$	$\left\{e_2^{j(*)}, e_3^{i(1)}, e_4^{i(1)}, e_5^{i(*)}, e_6^{j(1)}, e_7^{j(1)}\right\}$	$\left\{e_3^{i(1)}, e_4^{i(1)}, e_7^{j(1)}\right\}$	$\emptyset$

TABLE 4. Soft parameter dominant incomplete discernibility matrix of Table 1.

C_i	C_1	C_2	C_3	C_4	C_5
C_1	$\emptyset$	$\left\{e_2^{j(1)}, e_5^{j(*)}\right\}$	$\left\{e_2^{j(*)}, e_5^{**}\right\}$	$\left\{e_2^{j(1)}\right\}$	$\left\{e_2^{j(1)}, e_7^{j(1)}\right\}$
C_2	$\left\{e_3^{i(*)}, e_6^{i(1)}\right\}$	$\emptyset$	$\left\{e_3^{j(*)}\right\}$	$\left\{e_3^{j(*)}, e_6^{j(1)}\right\}$	$\left\{e_6^{j(1)}, e_7^{j(1)}\right\}$
C_3	$\left\{e_1^{i(1)}, e_5^{**}, e_6^{i(1)}\right\}$	$\left\{e_1^{i(1)}, e_2^{i(*)}, e_5^{i(*)}\right\}$	$\emptyset$	$\left\{e_2^{j(*)}, e_6^{j(1)}\right\}$	$\left\{e_2^{j(*)}, e_6^{j(1)}, e_7^{j(1)}\right\}$
C_4	$\left\{e_1^{i(1)}, e_5^{i(*)}\right\}$	$\left\{e_1^{i(1)}, e_5^{i(1)}\right\}$	$\left\{e_5^{i(*)}\right\}$	$\emptyset$	$\left\{e_7^{j(1)}\right\}$
C_5	$\left\{e_1^{i(1)}, e_3^{i(1)}, e_4^{i(1)}, e_5^{i(*)}\right\}$	$\left\{e_1^{i(1)}, e_3^{i(*)}, e_4^{i(1)}, e_5^{i(1)}\right\}$	$\left\{e_3^{i(1)}, e_4^{i(1)}, e_5^{i(*)}\right\}$	$\left\{e_3^{i(1)}, e_4^{i(1)}\right\}$	$\emptyset$

To conveniently calculate the cardinality of the soft incomplete discernibility matrix to solve the decision-making issues and to distinguish the element of the soft incomplete discernibility matrix, we establish a soft parameter dominant incomplete discernibility matrix as follows.

Definition 14 (Soft Parameter Dominant Incomplete Discernibility Matrix): The pair (F, E) is a soft set over U when $A \subseteq E$. Partition $U|IIR\,(F, A) = \{C_i : i \leq |U|\}$. Then, we can redefine the soft incomplete discernibility matrix as $D = \left(D\left(C_i, C_j\right)\right)_{i,j \leq |U|}$, where

$$D\left(C_i, C_j\right) = \begin{cases} D(C_i \rightarrow C_j) = \left\{E^{i(1)} \cup E^{i*} \cup E^{**}\right\}, & i > j \\ D(C_j \rightarrow C_i) = \left\{E^{j(1)} \cup E^{j*} \cup E^{**}\right\}, & i < j \\ D\left(C_i, C_j\right) = \emptyset, & i = j \end{cases}$$

denotes the soft parameter dominant matrix between C_i and C_j in an incomplete discernibility matrix.

Next, we continue to analyze Example 1. The soft incomplete discernibility matrix based on definition 14 can be expressed in the tabular form shown in Table 4.

Property 1: The pair (F, E) is an incomplete soft set on U, where $h_i \in U\,(i = 1, 2, \ldots, m)$. Some properties are provided below for the soft incomplete discernibility matrix over incomplete soft set (F, E):

(i) All main diagonal elements in the incomplete discernibility matrix are empty sets, which can be expressed as $D\,(C_i, C_i) = \emptyset$, where $i = 1, 2, \ldots, m$.

However, the following three properties are invalid:

(ii) The matrix elements do not satisfy the commutative law $D(C_i, C_j) = D(C_j, C_i)$, where $i, j = 1, 2, \ldots, m$.

(iii) The matrix elements do not satisfy the inclusion relation. $D(C_i, C_j) \subseteq D(C_i, C_k) \cup D(C_k, C_j)$, where $i, j, k = 1, 2, \ldots, m$.

Definition 15 (Cardinality of the Soft Incomplete Discernibility Matrix): The pair (F, E) is an incomplete soft set on U, where $h_i \in U(i = 1, 2, \ldots m)$. We define $card(C_i \rightarrow C_j) = |D(C_i \rightarrow C_j)| = |E^i|$ and $card(C_j \rightarrow C_i) = |D(C_j \rightarrow C_i)| = |E^j|$, where $card(C_i \rightarrow C_j)$ and $card(C_j \rightarrow C_i)$ denote the cardinalities of $D(C_i, C_j)$ and $D(C_j, C_i)$, and

$$\left|E^i\right| = |E^{i(1)} \cup E^{i(*)} \cup E^{**}| = |E^{i(1)}| + |E^{i(*)}| + |E^{**}|,$$
$$\left|E^j\right| = |E^{j(1)} \cup E^{j(*)} \cup E^{**}| = |E^{j(1)}| + |E^{j(*)}| + |E^{**}|.$$

The cardinality $card(C_i, C_j)$ has the following properties:

(i) The cardinality of one to oneself is 0, which can be expressed as $card\ (C_i \rightarrow C_i) = 0$, where $i = 1, 2, \ldots, m$.

However, the next two properties do not hold for the cardinality of the soft incomplete discernibility matrix,

(ii) The cardinality does not satisfy the commutative law. $card(C_i \rightarrow C_j) = card(C_j \rightarrow C_i)$, where $i, j = 1, 2, \ldots, m$.

(iii) The cardinality does not satisfy the inclusion relation. $card(C_i \rightarrow C_j) = card(C_i \rightarrow C_k) + card(C_k \rightarrow C_j)$, where $i, j, k = 1, 2, \ldots, m$.

Remark 1: If $card(C_i \rightarrow C_j) = card(C_j \rightarrow C_i)$, then C_i and C_j must have identical classifications. If $card(C_i \rightarrow C_j) \neq card(C_j \rightarrow C_i)$, i.e., either $card(C_i \rightarrow C_j) > card(C_j \rightarrow C_i)$ or $card(C_i \rightarrow C_j) < card(C_j \rightarrow C_i)$, then there must be an order decision-making relation between C_i and C_j. Thus, there is an order decision-making relation that either C_i is superior to C_j or C_j is superior to C_i in the incomplete soft sets, which can be applied to solve decision-making issues.

We find that the cardinalities of $\left|E^{i(1)}\right|$ and $\left|E^{j(1)}\right|$ are easy to calculate, but there is an uncertain value decision parameter in $\left|E^{i(*)}\right|$, $\left|E^{j(*)}\right|$. How can we estimate the unknown cardinality value $\left|E^{i(*)}\right|$ *and* $\left|E^{j(*)}\right|$? We define

$$\left|E^{i(*)}\right| = \begin{cases} 1 - p(1)\left|E^{i(1)}\right| - p(0)\left|E^{i(0)}\right|, & e_l^i = 1, e_l^j = *, \\ p(1)\left|E^{i(1)}\right| + p(0)\left|E^{i(0)}\right|, & e_l^i = *, e_l^j = 0, \end{cases}$$

TABLE 5. Cardinality of the soft parameter dominant incomplete discernibility matrix of Table 1.

C_i	C_1	C_2	C_3	C_4	C_5
C_1	$\emptyset$	1.67	0.75	1	2
C_2	1.25	$\emptyset$	0.25	1.25	2
C_3	2	1.92	$\emptyset$	1.25	1.25
C_4	1.33	2	0.33	$\emptyset$	1
C_5	3.33	3.75	2.33	2	$\emptyset$

$$\left|E^{j(*)}\right| = \begin{cases} 1 - p(1)\left|E^{j(1)}\right| - p(0)\left|E^{j(0)}\right|, & e_l^j = 1, e_l^i = *, \\ p(1)\left|E^{j(1)}\right| + p(0)\left|E^{j(0)}\right|, & e_l^j = *, e_l^i = 0, \end{cases}$$

$l < m$, where $p(1)$ and $p(0)$ represent the probability of an object belonging to and not belonging to $F(e)$, with $\left|E^{i(1)}\right|, \left|E^{j(1)}\right|$ being the cardinality that a known parameter value is in $D(C_i, C_j)$ and $\left|E^{i(0)}\right| = \left|E^{j(0)}\right| = 0$, and are defined [18] by $p(1) = \frac{n_1}{n_1+n_0}$ and $p(0) = \frac{n_0}{n_1+n_0}, e \in E$, where n_1 and n_0 are the numbers of objects that belong to and do not belong to $F(e)$, respectively.

Next, we analyze $D(C_1, C_2)$ in Example 1. The soft incomplete discernibility matrix is $D(C_1, C_2) = E^1 \cup E^2 = \left\{e_2^{j(1)}, e_3^{i(*)}, e_5^{j(*)}, e_6^{i(1)}\right\}$. Based on definition 15, the cardinality of $D(C_1, C_2)$ can be expressed in the tabular form shown in Table 5. $card(C_1 \to C_2) = |D(C_1 \to C_2)| = |E^i| = |E^{i(1)}| + |E^{i(*)}| = 1.25$, $card(C_2 \to C_1) = |D(C_2 \to C_1)| = |E^j| = |E^{j(1)}| + |E^{j(*)}| = 1.67$. We can say that C_2 is superior to C_1 in the incomplete soft sets. Furthermore, the cardinality of the soft parameter dominant incomplete discernibility matrix can be computed using the same calculation method and Table 5.

Through the analysis of Table 5, we can obtain the result of the order decision-making relation

$$C_2 \succ C_1 \succ C_4 \succ C_3 \succ C_5$$

The corresponding object relation is $\{p_2, p_6\} \succ \{p_1\} \succ \{p_4\} \succ \{p_3\} \succ \{p_5\}$. Thus, we can select p_2, p_6 as one of the best objects in the sense of incomplete soft sets with the best candidate employee in Example 1.

C. ALGORITHM TO SOLVE DECISION-MAKING PROBLEMS USING THE SOFT INCOMPLETE DISCERNIBILITY MATRIX

Most existing methods regarding incomplete soft sets mainly involve filling in the unknown parameter data and subsequently selecting the optimal selection object according to the sum value of the corresponding parameters. In this paper, we focus on studying the classification ability of the known and unknown corresponding parameters for a given incomplete soft set and solve decision-making problems by constructing the soft incomplete discernibility matrix and soft parameter dominant incomplete discernibility matrix.

In an incomplete soft set over U, for every C_i and C_j from the classification of U, the corresponding parameter is determined by F with known value "1, 0" and unknown value "$*$" in $D(C_i, C_j)$ if we can compare the values of known and unknown corresponding parameters in the soft incomplete discernibility matrix and do not need to directly predict by filling in the unknown data. Meanwhile, by constructing the comparison results of the cardinality in the soft parameter dominant incomplete discernibility matrix, we can easily obtain an order decision-making relation from the soft incomplete discernibility matrix. The result of the order decision-making relation can form an order relationship to select the object from the best choice to the subchoice.

Therefore, based on the above discussion and analysis, a new algorithm of incomplete soft sets is applied to decision-making problems as follows:

Algorithm 1: Soft incomplete discernibility matrix to solve decision-making problems.

Input: Pair of incomplete soft sets (F, E) over U, where $h_i \in U (i = 1, 2, \ldots, m)$.

Output: Form an order relation of the objects, and select the best object(s).

Step 1: Construct the discernibility relation of U and the soft incomplete discernibility matrix $D = D(C_i, C_j), i, j = 1, 2, \ldots, m$.

Step 2: Compute the cardinality of the soft parameter dominant incomplete discernibility matrix $D(C_i \to C_j) = \left\{E^{i(1)} \cup E^{i*} \cup E^{**}\right\}, i > j$, and $D(C_j \to C_i) = \left\{E^{j(1)} \cup E^{j*} \cup E^{**}\right\}, i < j$. Select the corresponding cardinality $\left|E^i\right|$ and $\left|E^j\right|$ from the soft incomplete discernibility matrix.

Step 3: For the elements in the soft parameter dominant incomplete discernibility matrix $D(C_i, C_j)$, we compare the cardinality of $card(C_i \to C_j)$ and $card(C_j \to C_i)$, where $h_i \in C_i$ and $h_j \in C_j$. If the cardinality $card(C_i \to C_j) = card(C_j \to C_i)$, then h_i and h_j must have identical classifications. Otherwise, if $card(C_i \to C_j) \neq card(C_j \to C_i)$, there must be an order decision-making relation between objects h_i and h_j, i.e., we can obtain the result that either h_i is superior to h_j or the object h_j is superior to h_i.

Step 4: Output the results of the order decision-making relation with all objects.The maximum cardinality of objects should be selected after the comparison of the cardinality of the soft parameter dominant incomplete discernibility matrix.

We will continue to use Example 1 to explain the algorithm and output the order decision-making relation to select the best object(s).

Step 1: From Table 1, we can obtain that the covering classes of U is $\{\{p_1\}, \{p_2, p_6\}, \{p_3\}, \{p_4\}, \{p_5\}\}$. We will denote $\{h_1\}$ by $\{C_1\}$, $\{p_2, p_6\}$ by $\{C_2\}$, $\{p_3\}$ by $\{C_3\}$, $\{p_4\}$ by $\{C_4\}$, $\{p_5\}$ by $\{C_5\}$, and the constructed soft incomplete discernibility matrix is shown in Table 3.

Step 2: By constructing the cardinality of the soft parameter dominant incomplete discernibility matrix from

TABLE 6. A Comparative study of solutions in decision making based on incomplete soft set.

comparison	*Datafilling*	*Description*	*complexity*	*result*
[20] *method*	*yes*	*Data filling with strong association and maybe no association between parameters and parameters*	$O(n^2)$	*certain*
[21] *method*	*yes*	*Data filling with all possible value with higher computational complexity*	$O(2^n)$	*certain*
OUR method	*no*	*Construct Soft incomplete discernibility matrix and Soft parameter dominant incomplete discernibility matrix*	$O(n^2)$	*certain*
Remark	[20, 21] *method filled with unknown data, our method not direct to unknown data filling*	[20, 21] *method filled with data from different aspects, Our method construct incomplete discernibility matrix and solve decision − making problems*	*The* [20] *and our method have lower computational compared to* [21] *method*	*All three method have certain result*

Table 4, we can obtain the cardinality between the objects $card(C_1 \rightarrow C_2) = 1.75$, $card(C_2 \rightarrow C_1) = 1.67$, $card(C_1 \rightarrow C_3) = 2$, $card(C_3 \rightarrow C_1) = 0.75, \ldots, card(C_4 \rightarrow C_5) = 2$, $card(C_5 \rightarrow C_4) = 1$. The constructed soft parameter dominant incomplete discernibility matrix is shown in Table 4.

Step 3: Comparing the results of the cardinality in the soft parameter dominant incomplete discernibility matrix in step 2, we have $card(C_1 \rightarrow C_2) < card(C_2 \rightarrow C_1)$, $card(C_1 \rightarrow C_3) > d(C_3 \rightarrow C_1), \ldots, card(C_4 \rightarrow C_5) > card(C_5 \rightarrow C_4)$. An order decision-making relation can be obtained: $C_2 \succ C_1, C_1 \succ C_3, C_1 \succ C_4, \ldots, C_4 \succ C_5$.

Thus, we can select p_2 or p_6 as one of the best objects for the decision-making problems.

D. COMPARISON WITH THE EXISTING SOLUTION FOR INCOMPLETE SOFT SET

In this section, we compare our algorithm in decision making with the other existing solution [20], [21] provided by the literature. The summary of the comparison of three method shown by table 6 from the ascepts of description, limitation, complexity and result.

IV. WEIGHTED SOFT INCOMPLETE DISCERNIBILITY MATRIX AND DECISION MAKING

"Should a membership function be regarded as the only characteristic function of a fuzzy set?" Lin [30] asked and answered this fundamental problems. The author introduced a new mathematical theory, which is called the weighted soft sets, i.e., the "theory of W-soft sets". Maji *et al.* [10] used Lin's definition to introduce the weighted table of soft sets. Not limited to only 0 and 1, the weighted table of soft sets has entries $d_{ij} = w_{ij} \times h_{ij}$, where w_{ij} is the weights of the soft set parameter e_j, and h_{ij} is the value for the ith object that corresponds to the jth parameter e_j of the soft set.

In association with the decision-making project, an incomplete soft set can now be easily expanded to the parameters that have the form with an important weight.

Hence, the weighted parameter forms can be utilized to replace the ordinary parameter forms related to the application of the decision-making problems in the incomplete soft set. Next, we will discuss the problems of decision making with the weighted parameters using the weighted soft incomplete discernibility matrix in incomplete soft sets.

When the weights are applied to the soft incomplete discernibility matrix in definition 13, we can obtain the weighted soft incomplete discernibility matrix. Therefore, the weighted parameters of the soft incomplete discernibility matrix are defined as follows:

$$\begin{aligned} D\left(C_i, C_j\right) &= \{E^i \cup E^j : i, j \leq |U|.\} \\ &= \{E^{i(1)} \cup E^{i(*)} \cup E^{j(1)} \cup E^{j(*)} \cup E^{**} : i, j \leq |U|.\} \end{aligned}$$

where $E^{i(*)} = \{e_l^{i(*)\times w_i} : F(h_i, e_l) = 1 \text{ and } F(h_j, e_l) = *, \text{or } F(h_i, e_l) = * \text{ and } F(h_j, e_l) = 0, \forall h_i \in C_i, \forall h_j \in C_j, e_l \in A\}$ and $F(h_i, e_l)$ denote the objects value in C_i associated with e_l, and $e_l^{i\times w_i}$ denotes the value of parameters with weight w_i in C_i of the objects.

According to the discussion and analysis, a new algorithm of incomplete soft set decision-making problems with weighted parameters projected on the soft incomplete discernibility matrix is obtained as follows:

Algorithm 2: Weighted soft incomplete discernibility matrix for decision-making problems.

Input: Pair of incomplete soft sets (F, E), where $h_i \in U(i = 1, 2, \ldots, m)$, and the set of weights is $W = \{w_1, w_2, w_3, \ldots, w_{|E|}\}$.

Output: For all objects in incomplete soft sets, output the order decision-making relation and select the best object(s).

Step 1: Construct the classification of U and the soft incomplete discernibility matrix $D = D(C_i, C_j), i, j \leq m$.

Step 2: Compute the cardinality of the soft parameter dominant incomplete discernibility matrix $D(C_i \rightarrow C_j) = \left\{E^{i(1)} \cup E^{i*} \cup E^{**}\right\}, i > j$, and $D(C_j \rightarrow C_i) = \left\{E^{j(1)} \cup E^{j*} \cup E^{**}\right\}, i < j$. Select the items $\left|E^i\right| \times w_i$ and $\left|E^j\right| \times w_j$ from the soft incomplete discernibility matrix, where w_i and w_j are the weights.

Step 3: Compare the results of the weighted cardinality in the soft parameter dominant incomplete discernibility matrix

TABLE 7. Weighted soft parameter dominant incomplete discernibility matrix.

C_i	C_1	C_2	C_3	C_4	C_5
C_1	$\emptyset$	$\{e_2^{j(1)\times 0.2}, e_5^{j(*)\times 0.8}\}$	$\{e_2^{j(*)\times 0.2}, e_5^{**\times 0.8}\}$	$\{e_2^{j(1)\times 0.2}\}$	$\{e_2^{j(1)\times 0.2}, e_7^{j(1)\times 0.9}\}$
C_2	$\{e_3^{i(*)\times 0.4}, e_6^{i(1)\times 0.9}\}$	$\emptyset$	$\{e_3^{j(*)\times 0.4}\}$	$\{e_3^{j(*)\times 0.4}, e_6^{j(1)\times 0.9}\}$	$\{e_6^{j(1)\times 0.9}, e_7^{j(1)\times 0.9}\}$
C_3	$\{e_1^{i(1)\times 0.1}, e_5^{**\times 0.8}, e_6^{i(1)\times 0.9}\}$	$\{e_1^{i(1)\times 0.1}, e_2^{i(*)\times 0.2}, e_5^{i(*)\times 0.8}\}$	$\emptyset$	$\{e_2^{j(*)\times 0.2}, e_6^{j(1)\times 0.9}\}$	$\{e_2^{j(*)\times 0.2}, e_6^{j(1)\times 0.9}, e_7^{j(1)\times 0.9}\}$
C_4	$\{e_1^{i(1)\times 0.1}, e_5^{i(*)\times 0.8}\}$	$\{e_1^{i(1)\times 0.1}, e_5^{i(1)\times 0.8}\}$	$\{e_5^{i(*)\times 0.8}\}$	$\emptyset$	$\{e_7^{j(1)\times 0.9}\}$
C_5	$\{e_1^{i(1)\times 0.1}, e_3^{i(1)\times 0.4}, e_4^{i(1)\times 0.7}, e_5^{i(*)\times 0.8}\}$	$\{e_1^{i(1)\times 0.1}, e_3^{i(*)\times 0.4}, e_4^{i(1)\times 0.7}, e_5^{i(1)\times 0.8}\}$	$\{e_3^{i(1)\times 0.4}, e_4^{i(1)\times 0.7}, e_5^{i(*)\times 0.8}\}$	$\{e_3^{i(1)\times 0.4}, e_4^{i(1)\times 0.7}\}$	$\emptyset$

obtained in step 2, i.e., compare $|E^i| \times w_i$ with $|E^j| \times w_j$, where w_i and w_j are the weights.

Step 4: Combine the results of step 3. Output the order decision-making relation of all objects.

By implementing weights on the parameter forms, the decision maker can use the above modified algorithm to select his optimal object. As mention in Example 1 of incomplete soft sets, if weights are assigned to the parameter forms, the decision maker selects the objects considering his preference in all conditions. Suppose that the improved weights of the choice parameters are as follows: $\omega_1 = 0.1, \omega_2 = 0.2, \omega_3 = 0.4, \omega_4 = 0.7, \omega_5 = 0.8, \omega_6 = 0.9, \omega_7 = 0.9$. Then, the weighted soft parameter dominant incomplete discernibility matrix is as shown in Table 7.

Table 7 shows that in $D(C_1, C_2)$, $card(C_1 \to C_2) = |D(C_1 \to C_2)| = |E^{i(1)\times 0.9}| + |E^{i(*)\times 0.4}| = 1$ and $card(C_2 \to C_1) = |D(C_2 \to C_1)| = |E^{j(1)\times 0.2}| + |E^{j(*)\times 0.8}| = 0.736$. Thus, the comparison result is that C_1 superior to C_2, i.e., the order decision-making relation h_1 is superior to h_2 and h_6. Similarly, the new order decision-making relation among all objects is $C_1 \succ C_4 \succ C_2 \succ C_5 \succ C_3$. Therefore, the optimal choice object is h_1. The new order decision-making relation and the new best choice change after improving the weight of the parameters. In fact, people will have different preference views when facing decision-making problems. Thus, this weighted-based approach to the selection of parameters will be useful for solving real-life problems.

V. CONCLUSION

The incomplete soft set theory has made some progress from theory to practical applications, particularly in decision support analysis problems. In our paper, the definitions of the soft incomplete discernibility matrix and soft parameter dominant incomplete discernibility matrix are introduced. We focus on the classification ability of the corresponding parameter for a given incomplete soft set and solve the decision-making problems. An order decision-making relation among all objects can be output by constructing the soft incomplete discernibility matrix with this new algorithm. Finally, we define the weighted soft incomplete discernibility matrix for application in decision making.

In future research, we intend to study different function-mapping forms of incomplete soft sets and the soft incomplete discernibility matrix to reduce the parameters and maintain the order of the relation.

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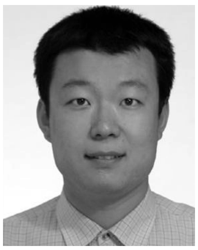

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