IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received March 19, 2018, accepted April 18, 2018, date of publication May 7, 2018, date of current version June 5, 2018.

Digital Object Identifier 10.1109/ACCESS.2018.2834231

Optimal Strong Solution of the Weighted Minimax Problem With Fuzzy Relation Equation Constraints

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This work was supported in part by the Natural Science Foundation, Hanshan Normal University, China, under Grant LQ201702 and Grant QD20171001 and in part by the Natural Science Foundation of Guangdong Province under Grant 2016A030307037 and Grant 2017A030307020.

ABSTRACT Max–min fuzzy relation equations could be used to describe three-tier multimedia streaming architecture. To make all the regional servers take part in the system, we define a strong solution concept. The corresponding weighted minimax problem is studied in this paper. A novel resolution is developed for obtaining the optimal strong solution of our proposed fuzzy relation weighted minimax problem.

INDEX TERMS Fuzzy relation equation, strong solution, max-min composition, weighted minimax problem, discrimination matrix.

I. INTRODUCTION

A. FUZZY RELATION EQUATION AND THE CORRESPONDING OPTIMIZATION PROBLEM

Afuzzy relation is a natural extension of the classical 0-1 relation. In such an extension, the relation between two objectives is represented by a real number lying in [0, 1] to describe the relative degree. One of the most commonly used operations between two fuzzy relations is composition, such as *max-min and max-product*. A fuzzy relation equation is usually in the form of

$$A_{m\times n} * x_{n\times 1} = b_{m\times 1}, \quad \text{or} \quad x_{1\times m} * A_{m\times n} = b_{1\times n},$$

where $*$ represents the composition operation. The max-min fuzzy relation equation was first proposed and investigated by Sanchez [1]. He focused on the resolution of this equation, as well as its potential application in medical diagnoses [2]. Searching all of the solutions of this equation produces the most important targets for the fuzzy relation equation [4]–[14]. It is now well known that a consistent system of fuzzy relation equations or inequalities, composed by a t-norm (triangular norm), has a unique maximum solution and a finite number of minimal solutions. Moreover, any vector between the maximum solution and a minimal solution is always a solution [15]–[18]. Hence the resolution of fuzzy relation equations depends on solving all the minimal solutions [19]–[23]. It is easy to verify that the complete solution is non-convex when its minimal solution is not unique. Consequently, an optimization problem subject to fuzzy relation equations or inequalities is often non-convex. However, due to the specific structure of the feasible domain, i.e., the complete solution set of the fuzzy relation system, the resolution of the corresponding optimization problem was studied, which is different from the typical convex optimization problem [18]. Optimizing a linear [26]–[35] or nonlinear [36]–[41] objective function with a fuzzy relation constraint was investigated with application in various management fields.

B. MOTIVATION OF OUR WORK

A system of max-min fuzzy relation equations was introduced to describe the three-tier multimedia streaming architecture in [3]. In such an architecture, *multimedia server*, *regional server* and *client workstation* represent three tiers. The multimedia streaming data are transmitted from the multimedia server to the regional servers. As the relay stations, the regional servers then relay the data to the client workstations. Suppose there are n regional servers, i.e., $RS_1, RS_2, \cdots, RS_n$, and m client workstations in the architecture, i.e., $CW_1, CW_2, \cdots, CW_m$. The multimedia server is the source supplying the multimedia streaming services. The streaming data are transmitted to each regional server on the networks through the virtual circuit c_j,

$j \in J = \{1, 2, \cdots, n\}$. The jth regional server further relays the data to every client workstation with data transmission quality level (measured as *bits per second*) x_j. On the other hand, each client workstation will select at least one regional server fulfilling its quality requirement, denoted by $b_i > 0$, $i \in I = \{1, 2, \cdots, m\}$, to receive the multimedia streaming data. Since RS_j is wire connected to CW_i, there exists a bandwidth limitation a_{ij} between them. Consequently, the ith client workstation CW_i receives streaming data from RS_j with quality level $a_{ij} \wedge x_j$. Therefore, the quantitative relation lying in the three-tier multimedia streaming services can be described by the following max-min fuzzy relation equations:

$$\begin{cases} (a_{11} \wedge x_1) \vee (a_{12} \wedge x_2) \vee \cdots \vee (a_{1n} \wedge x_n) = b_1, \\ (a_{21} \wedge x_1) \vee (a_{22} \wedge x_2) \vee \cdots \vee (a_{2n} \wedge x_n) = b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (a_{m1} \wedge x_1) \vee (a_{m2} \wedge x_2) \vee \cdots \vee (a_{mn} \wedge x_n) = b_m, \end{cases} \tag{1}$$

where $a_{ij}, b_i, x_j \in [0, 1]$ and $b_i > 0$, $i \in I = \{1, 2, \cdots, m\}$, $j \in J = \{1, 2, \cdots, n\}$. The matrix form of (1) is

$$A \circ x^T = b^T,$$

where $A = (a_{ij})_{m \times n}, x = (x_1, x_2, \cdots, x_n)$, $b = (b_1, b_2, \cdots, b_m)$. Scheduling of the quality levels $x_1, x_2, \cdots, x_n$ is an important issue for the multimedia streaming service supplier. The following two distinguishing points of view are commonly considered: utilitarianism versus egalitarianism [24], [25].

In this paper, we denote $X = [0, 1]^n$ and

$$X(A, b) = \{x \in X | A \circ x^T = b^T\}.$$

Hence $X(A, b)$ represents the complete solution set of system (1). A solution of system (1) is in fact a feasible schedule of the quality levels, indicating that all the client workstations are supplied by the multimedia services.

1) **Utilitarianism : considered in the existing work [3]**
Under consideration of utilitarianism, the administrator only pays attention to the efficiency. In the above three-tiered multimedia streaming services, highest efficiency means minimizing the total cost of the multimedia service suppler. Based on such a consideration, Lee and Guu [3] proposed the following linear programming with max-min fuzzy relation equation constraints:

$$\begin{aligned} &\min \ c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ &\text{s.t.}\ A \circ x^T = b^T. \end{aligned} \tag{2}$$

In problem (2), c_jx_j is the service cost per unit time spent by the jth regional server. An optimal solution of problem (2) is an optimal scheduling of the quality levels to reach the minimum total cost

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n. \tag{3}$$

Although the total cost is minimized, the cost spent on each regional server might not be egalitarian. Moreover, it is possible that some of the regional servers are non-occupational. These regional servers are scheduled by $x_j = 0$ and have no client workstation to delay their local streaming data. On the other hand, there might exist some overloaded regional servers, which are scheduled to supply too many client workstations. This leads to the unbalanced scheduling on the regional servers. In order to avoid such unbalanced supply-demand relationship, we introduce the following concept of "Egalitarian".

2) **Egalitarian: considered in our work**
To overcome the above shortage, we try to establish another model in which every regional server plays a part in the system. That is, in our model, the jth regional server RS_j is scheduled by $x_j > 0$, enabling RS_j to delay the streaming of data to at least one client workstation, $\forall j \in J$. Moreover, to balance the cost among all the regional servers, we replace the total cost by the highest cost, i.e.,

$$c_1x_1 \vee c_2x_2 \vee \cdots \vee c_nx_n. \tag{4}$$

Before establishing our model, we should define a concept that enables every regional server to take part in the system.

Definition 1 (Strong Solution): A solution $x^s \in X(A, b)$ is called a strong solution of system (1) if, for every $j \in J$, there exists an $i_j \in I$ such that $a_{i_jj} \wedge x^s_j = b_{i_j}$.

It is obvious that a strong solution of system (1) is a feasible schedule of the quality levels, making the following true: (i) all the client workstations are supplied by the multimedia services; (ii) every regional server is able to supply its own multimedia streaming data to at least one client workstation.

Denote the set of all strong solutions of system (1) by

$$X^s(A, b).$$

Then, our target model could be formulated as

$$\begin{aligned} &\min \ z(x) = c_1x_1 \vee c_2x_2 \vee \cdots \vee c_nx_n \\ &\text{s.t.}\ x \in X^s(A, b). \end{aligned} \tag{5}$$

In our proposed model, all regional servers are treated equally. Each regional server has the chance to delay streaming of its own data, and the highest cost $z(x)$ is minimized. Hence, our model is based on the consideration of egalitarianism.

C. NOVELTY AND STRUCTURE OF OUR WORK

To avoid some of the regional servers becoming unserviceable, we define the concept of a strong solution for the max-min fuzzy relation equations in this paper. The corresponding weighted minimax problem is established and investigated based on practical application. A novel approach is developed to find the optimal solution, carried out via a step-by-step detailed algorithm. The remaining sections of this paper are arranged as follows. In Section 2, we study the strong solution set of system (1), including its properties and structure. In Section 3, we provide a resolution method for our proposed problem (5). Section 4 presents a detailed algorithm and an illustrative example. Advantages of the algorithm are discussed in Section 5, while a simple conclusion is presented in Section 6.

II. STRONG SOLUTION SET OF SYSTEM (1)

A. THE EXISTENCE OF A STRONG SOLUTION

In this subsection, we first introduce some related concepts and results on system (1). Then, we will discuss the existence of the strong solution in Theorems 2-5.

Definition 2: System (1) is said to be consistent if $X(A, b) \neq \emptyset$. Otherwise, it is said to be inconsistent.

Definition 3: For any $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n) \in X$, $x \geq y$ when $x_j \geq y_j$ for all $j \in J$. Furthermore, $x > y$ if $x \geq y$ and there exists $j_0 \in J$ such that $x_{j_0} > y_{j_0}$. The symbols "$\leq$" and "$<$" are defined in a similar way.

Definition 4: A solution $\hat{x} \in X(A, b)$ is said to be the maximum solution of system (1) when $x \leq \hat{x}$ for all $x \in X(A, b)$. A solution $\check{x} \in X(A, b)$ is said to be a minimal solution of system (1) when $x \leq \check{x}$ implies $x = \check{x}$ for any $x \in X(A, b)$.

To determine the maximum of system (1), the classic method is as follows:

Denote $\hat{x} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n)$. For any $j \in J$, $\hat{x}_j = \bigwedge_{i \in I}(a_{ij} \circledast b_i)$, where the operator "$\circledast$" is defined as

$$a_{ij} \circledast b_i = \begin{cases} b_i & a_{ij} > b_i, \\ 1, & a_{ij} \leq b_i. \end{cases} \tag{6}$$

Then, $\hat{x}$ is the maximum solution when system (1) is consistent, as the following theorem describes.

Theorem 1 [2]: System (1) is consistent if and only if $\hat{x} \in X(A, b)$. Furthermore, if $\hat{x} \in X(A, b)$, then $\hat{x}$ is the unique maximum solution of system (1).

Remark 1: i) A strong solution is certainly a solution, but not all of the solutions are a strong solution of system (1). For example, in the fuzzy relation equation $(0.3 \wedge x_1) \vee (0.5 \wedge x_2) = 0.4$, the solution set is $\{(x_1, x_2) | x_1 \in [0, 1], x_2 = 0.4\}$. Any solution in the set is not a strong solution since $0.3 \wedge x_1 \neq 0.4$ for any x_1.

ii) Any strong solution is less than or equal to $\hat{x}$ since $\hat{x}$ is the maximum solution of system (1).

The properties of strong solutions will be further discussed with the aid of two kinds of index sets as follows:

$$I_j(x) = \{i \in I | a_{ij} \wedge x_j = b_i\}, \quad j = 1, 2, \ldots, n,$$
$$J_i(x) = \{j \in J | a_{ij} \wedge x_j = b_i\}, \quad i = 1, 2, \ldots, n$$

for any $x = (x_1, x_2, \ldots, x_n) \in X$.

Proposition 1: A solution $x^s \in X(A, b)$ is a strong solution of system (1) if $I_j(x^s) \neq \emptyset$ for any $j \in J$.

Proof: For any $j \in J$, since $I_j(x^s) \neq \emptyset$, there exists at least an $i_j \in I$ such that $a_{i_j j} \wedge x_j^s = b_{i_j}$, which means x^s is a strong solution.

Corollary 1: If $x^s \in X$ is a strong solution of system (1), then $I_j(x^s) \neq \emptyset$ for any $j \in J$ and $J_i(x^s) \neq \emptyset$ for any $i \in I$.

Proof: For any $i \in I$, since $x^s = (x_j^s)_{1 \times n}$ is a solution of system (1), it satisfies

$$(a_{i1} \wedge x_1^s) \vee (a_{i2} \wedge x_2^s) \vee \cdots \vee (a_{in} \wedge x_n^s) = b_i,$$

which implies that there exists some $j_i \in J$ such that $a_{ij_i} \wedge x_{j_i}^s = b_i$, i.e., $J_i(x^s) \neq \emptyset$.

Furthermore, x^s is a strong solution, so $I_j(x^s) \neq \emptyset$ for any $j \in J$ by Proposition 1.

Proposition 2: If $x^s \in X^s(A, b)$, then $x_j^s > 0$ for any $j \in J$, where $x^s = (x_1^s, x_2^s, \ldots, x_n^s)$.

Proof: If $x_{j_0}^s = 0$ for some $j_0 \in J$, $a_{ij_0} \wedge x_{j_0}^s = 0 \neq b_i$ for any $i \in J$, which implies $I_{j_0}(x) = \emptyset$. Thus, $x^s \notin X^s(A, b)$. This is contradictory to the assumption.

Proposition 3: If $x \in X(A, b)$ and $x^s \in X^s(A, b)$, then $x \vee x^s \in X^s(A, b)$. That is, in system (1), the maximum of a solution and a strong solution is a strong solution.

Proof: First, $x \vee x^s \in X(A, b)$. In fact, since $x, x^s \in X(A, b)$, by the solution structure of system (1), there exists some minimal solution $\check{x}_1, \check{x}_2 \in X(A, b)$ such that $\check{x}_1 \leq x \leq \hat{x}$ and $\check{x}_2 \leq x^s \leq \hat{x}$. Then, $\check{x}_1 \leq \check{x}_1 \vee \check{x}_2 \leq x \vee x^s \leq \hat{x}$, which means $x \vee x^s \in X(A, b)$.

Furthermore, $x \vee x^s \in X^s(A, b)$. Considering that x^s is a strong solution, for any $j \in J$, there exists $i_j \in I$ such that $a_{i_j j} \wedge x_j^s = b_{i_j}$ by definition. Since $x \in X(A, b)$, $a_{i_j j} \wedge x_j \leq \bigvee_{j \in J} a_{i_j j} \wedge x_j = b_{i_j}$. Thus, $a_{i_j j} \wedge (x_j \vee x_j^s) = (a_{i_j j} \wedge x_j) \vee (a_{i_j j} \wedge x_j^s) = b_{i_j}$, which implies $I_j(x \vee x^s) \neq \emptyset$. Hence, $x \vee x^s \in X^s(A, b)$ by Proposition 1.

Corollary 2: The maximum of any two strong solutions is a strong solution.

Theorem 2 (Existence Theorem 1): $X^s(A, b) \neq \emptyset$ if and only if $\hat{x} \in X^s(A, b)$, where $\hat{x}$ is the maximum solution of the consistent system (1).

Proof: The sufficiency of the theorem is clear. We just need to prove the necessity of the theorem.

Suppose $X^s(A, b) \neq \emptyset$ and $x^s \in X^s(A, b)$. Since $\hat{x}$ is the maximum of $X(A, b)$ in the consistent system (1), $\hat{x} = \hat{x} \vee x^s \in X^s(A, b)$ by Proposition 3.

Remark 2: The maximum solution of $X(A, b)$ is not always a strong solution. For example, in the fuzzy relation equation $(0.3 \wedge x_1) \vee (0.5 \wedge x_2) = 0.4$, the maximum solution is $(1, 0.4)$, which is not a strong solution. In this case, $X^s(A, b) = \emptyset$ by Theorem 2.

Example 1: Judge the existence of a strong solution of the following fuzzy relation equations.

$$\begin{cases} (0.3 \wedge x_1) \vee (0.2 \wedge x_2) \vee (0.7 \wedge x_3) \vee (0.8 \wedge x_4) = 0.7, \\ (0.5 \wedge x_1) \vee (0.4 \wedge x_2) \vee (0.4 \wedge x_3) \vee (0.9 \wedge x_4) = 0.4, \\ (0.7 \wedge x_1) \vee (0.3 \wedge x_2) \vee (0.2 \wedge x_3) \vee (0.7 \wedge x_4) = 0.4, \\ (0.9 \wedge x_1) \vee (0.6 \wedge x_2) \vee (0.1 \wedge x_3) \vee (0.2 \wedge x_4) = 0.3, \\ (0.8 \wedge x_1) \vee (0.5 \wedge x_2) \vee (0.6 \wedge x_3) \vee (0.4 \wedge x_4) = 0.6. \end{cases} \tag{7}$$

Solve: Let

$$A = \begin{bmatrix} 0.3 & 0.2 & 0.7 & 0.8 \\ 0.5 & 0.4 & 0.4 & 0.9 \\ 0.7 & 0.3 & 0.2 & 0.7 \\ 0.9 & 0.6 & 0.1 & 0.2 \\ 0.8 & 0.5 & 0.6 & 0.4 \end{bmatrix} \text{ and } b^T = \begin{bmatrix} 0.7 \\ 0.4 \\ 0.4 \\ 0.3 \\ 0.6 \end{bmatrix};$$

then,

$$A \circledast b = (a_{ij} \circledast b_i) = \begin{bmatrix} 1 & 1 & 1 & 0.7 \\ 0.4 & 1 & 1 & 0.4 \\ 0.4 & 1 & 1 & 0.4 \\ 0.3 & 0.3 & 1 & 0.7 \\ 0.6 & 1 & 1 & 0.7 \end{bmatrix}.$$

Thus, $\hat{x} = (0.3, 0.3, 1, 0.4)$. It becomes easy to verify that $\hat{x}$ is a solution of system (7), which implies that system (7) is consistent and $\hat{x}$ is the maximum solution by Theorem 1.

Since $I_1(\hat{x}) = \{4\}, I_2(\hat{x}) = \{4\}, I_3(\hat{x}) = \{1, 2, 5\}$, $I_4(\hat{x}) = \{2, 3, 5\}$, i.e., $I_j(\hat{x}) \neq \emptyset$ for every $j \in \{1, 2, 3, 4\}$, $\hat{x}$ is a strong solution of system (7) by Theorem 2 .

Definition 5 (Discrimination Matrix): A matrix $D = (d_{ij})_{m\times n}$ is called the discrimination matrix of system (1) if

$$d_{ij} = \begin{cases} b_i & a_{ij} \wedge \hat{x}_j \geq b_i, \\ 0, & a_{ij} < b_i, \end{cases} \forall i \in I,\ j \in J, \tag{8}$$

where $\hat{x} = (\hat{x}_j)_{1\times n}$ is the the maximum solution of system (1).

Theorem 3: If system (1) is consistent, then each row in discrimination matrix D has at least one nonzero element.

Proof: If system (1) is consistent, then $\hat{x} = (\hat{x}_j)_{1\times n}$ is the maximum solution by Theorem (1). Thus, for each $i \in I$, the following holds:

$$(a_{i1} \wedge \hat{x}_1) \vee (a_{i2} \wedge \hat{x}_2) \vee \cdots \vee (a_{in} \wedge \hat{x}_n) = b_i,$$

which means that at some $j_0 \in J$, $a_{ij_0} \wedge \hat{x}_{j_0} = b_i$. Then, $d_{ij_0} = b_i > 0$ by definition 5, i.e., the ith row of D has nonzero element d_{ij_0}. Based on the arbitrariness of i, each row in discrimination matrix D has at least one nonzero element.

Theorem 4 (Necessary Condition): Let D be the discrimination matrix and $X^s(A, b)$ be the strong solution set of system (1). If $X^s(A, b) \neq \emptyset$, then

i) each row in D has at least one nonzero element;

ii) each column in D has at least one nonzero element.

Proof: i) If $X^s(A, b) \neq \emptyset$, then system (1) is consistent. By Theorem 3, it holds that each row in D has at least one nonzero element.

ii) Let $x^s \in X^s(A, b)$; then, by Proposition 1,

$$I_j(x) = \{i \in I | a_{ij} \wedge x_j = b_i\} \neq \emptyset, \quad j = 1, 2, \ldots, n.$$

Thus, for any $j \in J$, there exists some $i_0 \in I$ such that $a_{i_0j} \wedge x^s_j = b_{i_0}$. Then, $a_{i_0j} \wedge \hat{x}_j \geq a_{i_0j} \wedge x^s_j = b_{i_0} > 0$. Hence, $d_{i_0j} = b_{i_0} > 0$ by definition 5, i.e., the jth column of D has nonzero element d_{i_0j}. Based on the arbitrariness of j, each column in discrimination matrix D has at least one nonzero element.

Theorem 5 (Existence Theorem 2): Let D be the discrimination matrix and $X^s(A, b)$ be the strong solutions set of system (1). Then $X^s(A, b) \neq \emptyset$ if and only if

i) $X(A, b) \neq \emptyset$;

ii) each row in D has at least one nonzero element;

iii) each column in D has at least one nonzero element.

Proof: The necessity is clear by Theorems 2-4. The following proves the sufficiency.

Since $X(A, b) \neq \emptyset$, $\hat{x} = (\hat{x}_j)_{1\times n}$ is the maximum solution of system (1) by Theorem 1. We will prove that $\hat{x}$ is a strong solution.

Since each column in D has at least one nonzero element, $\forall j \in J$, $\exists\, i_0 \in I$ such that $a_{i_0j} \wedge \hat{x}_j \geq b_{i_0}$. On the other hand, as a solution $\hat{x}$ satisfies

$$(a_{i_01} \wedge \hat{x}_1) \vee (a_{i_02} \wedge \hat{x}_2) \vee \cdots \vee (a_{i_0n} \wedge \hat{x}_n) = b_{i_0},$$

which indicates $a_{i_0j} \wedge \hat{x}_j \leq b_{i_0}$, $a_{i_0j} \wedge \hat{x}_j = b_{i_0}$. In this way, we have proved that $I_j(\hat{x}) \neq \emptyset$ for any $j \in J$, or $\hat{x}$ is a strong solution by Proposition 1.

Both Theorem 2 and Theorem 5 are used to judge whether a max-min fuzzy relation equation system has a strong solution, which is illustrated for Theorem 5.

Example 2: The existence of a strong solution of system (7) is presented in Example 1.

Solve: In system (7),

$$A = \begin{bmatrix} 0.3 & 0.2 & 0.7 & 0.8 \\ 0.5 & 0.4 & 0.4 & 0.9 \\ 0.7 & 0.3 & 0.2 & 0.7 \\ 0.9 & 0.6 & 0.1 & 0.2 \\ 0.8 & 0.5 & 0.6 & 0.4 \end{bmatrix}, \quad b = (0.7, 0.4, 0.4, 0.3, 0.6)$$

and $\hat{x} = (0.3, 0.3, 1, 0.4)$. By definition, the discrimination matrix is

$$D = (d_{ij}) = \begin{bmatrix} 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0.4 \\ 0.3 & 0.3 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \end{bmatrix}.$$

It is easy to verify that $\hat{x} \in X(A, b)$ and D has a nonzero element in any row and column. Therefore, system (7) has a strong solution by Theorem 5.

B. STRUCTURE OF THE STRONG SOLUTION SET

In this subsection, we discuss the structure of the strong solution set of system (1). First, we prove Proposition 4, which allows us to define the **"Minimal strong solution."** Next, to find the minimal strong solutions, we introduce some concepts: the **strong solution matrix, strong solution corresponding to matrix** and **pseudo minimal strong solution**. Finally, it is shown in Theorem 8 that the structure of the strong solution set of system (1) is completely determined by its minimal strong solution set $\check{X}^s(A, b)$ and the maximum solution.

Proposition 4: If $x^{s_1}, x^{s_2} \in X^s(A, b)$, then $y \in X^s(A, b)$ holds for any y satisfying $x^{s_1} \leq y \leq x^{s_2}$. That is, any vector between two strong solutions is a strong solution.

Proof: As $x^{s_1} \leq y \leq x^{s_2}$ and $x^{s_1}, x^{s_2} \in X^s(A,b) \subset X(A,b)$, $y \in X(A,b)$ by the structure of $X(A,b)$. Thus, $y = y \vee x^{s_1} \in X^s(A,b)$ by Proposition 3.

Corollary 3: If $x^s \in X^s(A,b)$ and, $\forall y$, $x^s \leq y \leq \hat{x}$, then $y \in X^s(A,b)$.

Proof: The proof is trivial.

Definition 6 (Minimal Strong Solution): A strong solution $\check{x}^s \in X^s(A,b)$ is said to be a minimal strong solution of system (1) when $x^s \leq \check{x}^s$ implies $x^s = \check{x}^s$ for any $x^s \in X^s(A,b)$.

Denote $\check{X}^s(A,b)$ as the set of all the minimal strong solutions of system (1).

Definition 7 (Strong Solution Matrix): Let $D = (d_{ij})_{m\times n}$ be the discrimination matrix and $E = (e_{ij})_{m\times n}$. E is called a strong solution matrix belonging to D if

i) $e_{ij} \in \{d_{ij}, 0\}$ for any $i \in I, j \in J$;

ii) each row in E has at least one nonzero element;

iii) each column in E has at least one nonzero element.

Let $E = (e_{ij})_{m\times n}$ be a strong solution matrix belonging to D. We denote $x^E = (x_j^E)_{1\times n}$, where

$$x_j^E = \bigvee_{i\in I} e_{ij}, \quad \forall j \in J. \tag{9}$$

It is proved in the following theorem that x^E is a strong solution of system (1).

Theorem 6 (Strong Solution Corresponding to E): Let $E = (e_{ij})_{m\times n}$ be a strong solution matrix belonging to discrimination matrix D. Then, x^E is a strong solution in system (1) if $X^s(A,b) \neq \emptyset$. We call x^E the strong solution corresponding to E.

Proof: Two steps are used to prove the theorem.

Step 1. We prove that x^E is in $X(A,b)$.

According to the definition of E, each row in E has at least one nonzero element, i.e., for any $i \in I$, there exists $j_0 \in J$ such that $e_{ij_0} = d_{ij_0} = b_i > 0$. Then,

$$\begin{aligned}&(a_{i1}\wedge x_1^E)\vee(a_{i2}\wedge x_2^E)\vee\cdots\vee(a_{in}\wedge x_n^E)\\ &\quad=(a_{i1}\wedge\bigvee_{i\in I}e_{i1})\vee(a_{i2}\wedge\bigvee_{i\in I}e_{i2})\vee\cdots\vee(a_{in}\wedge\bigvee_{i\in I}e_{in})\\ &\quad\geq a_{ij_0}\wedge e_{ij_0}=a_{ij_0}\wedge b_i=b_i.\end{aligned} \tag{10}$$

The last equation holds since $b_i > 0$ when $a_{ij_0} \wedge \hat{x}_j \geq b_i$ by Definition 5.

On the other hand, since $x^E \leq \hat{x}$, by Corollary 3,

$$\begin{aligned}&(a_{i1}\wedge x_1^E)\vee(a_{i2}\wedge x_2^E)\vee\cdots\vee(a_{in}\wedge x_n^E)\\ &\quad\leq(a_{i1}\wedge\hat{x}_1)\vee(a_{i2}\wedge\hat{x}_2)\vee\cdots\vee(a_{in}\wedge\hat{x}_n)=b_i.\end{aligned} \tag{11}$$

Thus, $(a_{i1}\wedge x_1^E)\vee(a_{i2}\wedge x_2^E)\vee\cdots\vee(a_{in}\wedge x_n^E) = b_i$ for any $i \in I$, i.e., x^E is a solution of system (1).

Step 2. We show that x^E is in $X^s(A,b)$.

Since each column in E has at least one nonzero element, for any $j \in J$, there exists $i_0 \in I$ such that $e_{i_0 j} = \bigvee_{i\in I} e_{ij} = x_j^E > 0$. Thus, $a_{i_0 j} \wedge x_j^E = a_{i_0 j} \wedge e_{i_0 j} = a_{i_0 j} \wedge b_{i_0} = b_{i_0}$, which indicates $I_j(x^E) \neq \emptyset$. Therefore, x^E is a strong solution.

Remark 3: A strong solution matrix does not always exist. It depends on the discrimination matrix D. D is a strong solution matrix belonging to itself when D satisfies any row and any column having nonzero elements.

Example 3: Find all the strong matrices belong to discrimination matrix D in Example 2, as well as their strong solutions accordingly.

Solve: In system (7), the discrimination matrix is

$$D=\begin{bmatrix}0&0&0.7&0\\0&0&0.4&0.4\\0&0&0&0.4\\0.3&0.3&0&0\\0&0&0.6&0\end{bmatrix}.$$

All the strong matrices belonging to D are as follows:

$$E_1=\begin{bmatrix}0&0&0.7&0\\0&0&0&0.4\\0&0&0&0.4\\0.3&0.3&0&0\\0&0&0.6&0\end{bmatrix},$$

$$E_2=\begin{bmatrix}0&0&0.7&0\\0&0&0.4&0\\0&0&0&0.4\\0.3&0.3&0&0\\0&0&0.6&0\end{bmatrix}, \quad E_3=D.$$

It is easy to obtain $x^{E_1} = x^{E_2} = x^{E_3} = (0.3, 0.3, 0.7, 0.4)$, which is the only strong solution.

Theorem 7: For any $x^* \in X^s(A,b)$, there exists some strong solution matrix E^* belonging to D such that $x^{E^*} \leq x^*$, where D is the discrimination matrix of system (1) and x^{E^*} is the strong solution corresponding to E^*.

Proof: For any strong solution $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ in $X^s(A,b)$, we are going to find a strong matrix E^* belonging to D such that $x^* \leq x^{E^*}$.

Let $E^* = (e_{ij}^*)$, where

$$e_{ij}^*=\begin{cases}d_{ij}, & a_{ij}\wedge x_j^*=b_i,\\ 0, & others.\end{cases} \tag{12}$$

Since x^* is a strong solution, $J_i(x^*) \neq \emptyset$ for $i \in I$ and $I_j(x^*) \neq \emptyset$ for $j \in J$ by Corollary 1. Thus, E^* has nonzero elements for any row and column. Furthermore, according to the definition, $e_{ij}^* = d_{ij}$ or $e_{ij}^* = 0$. Therefore, E^* is a strong matrix belonging to D.

The rest is proving $x^* \leq x^{E^*}$. In fact, for any $j \in J$,

$$x_j^{E^*}=\bigvee_{i\in I}e_{ij}^*=\bigvee_{\{i\in I|a_{ij}\wedge x_j^*=b_i\}}d_{ij}=\bigvee_{\{i\in I|a_{ij}\wedge x_j^*=b_i\}}b_i\leq x_j^*, \tag{13}$$

which means $x^{E^*} \leq x^*$ in the sense of a fuzzy relation.

Definition 8 (Pseudo Minimal Strong Solution): For any strong solution matrix E belonging to discrimination matrix D, x^E is the strong solution corresponding to E. We call x^E a pseudo minimal strong solution of system (1).

Denote $\bar{X}^s(A, b)$ as all the pseudo minimal strong solutions of system (1).

Proposition 5: i) $\bar{X}^s(A, b)$ is a finite set;

ii) $\check{X}^s(A, b) \subseteq \bar{X}^s(A, b) \subseteq X^s(A, b) \subseteq X(A, b)$.

Proof: i) Note that all the matrices belonging to D are finite. Then, the pseudo minimal strong solutions are finite as well.

ii) This is clear by definition.

Corollary 4 $\check{X}^s(A, b)$ is a finite set.

Theorem 8 (Structure Theorem): If $X^s(A, b) \neq \emptyset$, then the strong solution set of system (1) is

$$X^s(A, b) = \bigcup_{\check{x}^s \in \check{X}^s(A,b)} \{x^s | \check{x}^s \leq x \leq \hat{x}\}, \tag{14}$$

where $\check{X}^s(A, b)$ is a finite set.

Proof: Based on Propositions 4-5, Corollaries 3-4, and Theorem 7, the proof is trivial.

III. RESOLUTION OF PROGRAMMING (5)

In this section, we focus on the resolution of programming (5). A novel method is developed to find the optimal solution of (5).

Suppose $c_j > 0$ for any $j \in J$, which was explained in Subsection 1.2–Motivation of our work. We will find the optimal solution by a matrix approach. First, we define an operator $C \odot D$; then, "the minimum matrix" is found, which is helpful for achieving the optimal value.

Let $D = (d_{ij})_{m\times n}$ be the discrimination matrix and $C = (c_1, c_2, \ldots c_n)$ be the coefficient vector. Define an operator of C and D as follows:

$$C \odot D = (c_j d_{ij})_{m\times n} = \begin{bmatrix} c_1 d_{11} & c_2 d_{12} & \ldots & c_n d_{1n} \\ c_1 d_{21} & c_2 d_{22} & \ldots & c_n d_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ c_1 d_{m1} & c_2 d_{m2} & \ldots & c_n d_{mn} \end{bmatrix}. \tag{15}$$

By (8), we have

$$c_j d_{ij} = \begin{cases} b_i c_j & a_{ij} \wedge \hat{x}_j \geq b_i, \\ 0, & a_{ij} < b_i, \end{cases} \forall i \in I, j \in J, \tag{16}$$

where $\hat{x} = (\hat{x}_j)_{1\times n}$ is the the maximum solution of system (1).

Proposition 6: For any fixed $i_0 \in I, j_0 \in J$, the sign of element $c_{j_0} d_{i_0 j_0}$ and the sign of element $d_{i_0 j_0}$ are the same. That is, the signs of elements of the same subscript are exactly the same in the two matrices $C \odot D$ and D.

Proof: Note that $c_j > 0$ for any $j \in J$; the proof is trivial.

In the remainder of this section, we assume that $D = (d_{ij})_{m\times n}$ is the discrimination matrix that satisfies Theorem 5, i.e., each row and each column of D have at least one nonzero element. By Proposition 6, we have the following:

i) each row in $C \odot D$ has at least one nonzero element;

ii) each column in $C \odot D$ has at least one nonzero element.

We define some sparse matrices $S^{(1)}$, $S^{(2)}$ and S.

Since each row in $C \odot D$ has at least one nonzero element, we construct $S^{(1)}$ by taking the minimum of the nonzero element of each row in $C \odot D$ as the element of the corresponding coordinates in $S^{(1)}$ and zero for the others. Similarly, we use the columns for $S^{(2)}$, i.e.,

$$S^{(1)} = (s^{(1)}_{ij})_{m\times n}, S^{(2)} = (s^{(2)}_{ij})_{m\times n},$$

where

$$s^{(1)}_{ij} = \begin{cases} c_j d_{ij}, & c_j d_{ij} = min\{c_k d_{ik} > 0 | k \in J\}, \\ 0, & others. \end{cases} \tag{17}$$

$$s^{(2)}_{ij} = \begin{cases} c_j d_{ij}, & c_j d_{ij} = min\{c_j d_{lj} > 0 | l \in I\}, \\ 0, & others. \end{cases} \tag{18}$$

$i = 1, 2, \ldots, m, j = 1, 2, \ldots, n.$

We also denote $S = (s_{ij})_{m\times n}$, where

$$s_{ij} = s^{(1)}_{ij} \vee s^{(2)}_{ij}, \quad \forall i \in I, j \in J. \tag{19}$$

Proposition 7: Let S be the sparse matrix defined above. Then,

i) each row in S has at least one nonzero element;

ii) each column in S has at least one nonzero element.

Proof: Note that each row in $S^{(1)} = (s^{(1)}_{ij})_{m\times n}$ has at least one nonzero element; each column in $S^{(2)} = (s^{(2)}_{ij})_{m\times n}$ has at least one nonzero element; and $S = (s_{ij})_{m\times n}$, where $s_{ij} = s^{(1)}_{ij} \vee s^{(2)}_{ij}$. Then, each row and each column in S have at least one nonzero element.

Proposition 8: Denote $C^{-1} = (c_1^{-1}, c_2^{-1}, \ldots, c_n^{-1})$. Then, the matrix $C^{-1} \odot S$ is a strong solution matrix belonging to D, where D is the discriminant matrix, and S is the sparse matrix defined above.

Proof: Note that $C^{-1} \odot (C \odot D) = D$ and S is a matrix belonging to $C \odot D$. Thus, any $t_{ij} \in C^{-1} \odot S$ is equal to d_{ij} or 0. Moreover, S has a nonzero element for both any row and any column, which means $C^{-1} \odot S$ has the same nature. Thus, $C^{-1} \odot S$ is a strong matrix belonging to D by Definition 7.

Corollary 5: i) $x^{C^{-1}\odot S}$ is in $X^s(A, b)$;

ii) $z(x^{C^{-1}\odot S}) = \bigvee_{i\in I} \bigvee_{j\in J} s_{ij}$.

Proof: i) Note that $C^{-1} \odot S$ is a strong matrix belong to D. The proof is trivial.

ii) $z(x^{C^{-1}\odot S}) = \bigvee_{j\in J} c_j x_j^{C^{-1}\odot S} = \bigvee_{j\in J} x_j^S = \bigvee_{j\in J} \bigvee_{i\in I} s_{ij}$.

Theorem 9: If $X^s(A, b) \neq \emptyset$, then the optimal value of Programming (5) is $\bigvee_{i\in I} \bigvee_{j\in J} s_{ij}$, and $x^{C^{-1}\odot S}$ is an optimal strong solution, where $S = (s_{ij})_{m\times n}$ is the sparse matrix defined by (19) and $x^{C^{-1}\odot S}$ is the strong solution corresponding to $C^{-1} \odot S$.

Proof: For any $x^* = (x_1^*, x_2^*, \ldots, x_n^*) \in X^s(A, b)$, according to Theorem 7, there exists some strong solution matrix E^* belonging to D such that $x^{E^*} \leq x^*$, where D is the discrimination matrix of system (1) and x^{E^*} is the strong solution corresponding to E^*. Hence, we have

$$z(x^*) \geq z(x^{E^*}).$$

Let $E^* = (e^*_{ij})_{m\times n}$. Then

$$z(x^{E^*}) = \bigvee_{j\in J} c_j x_j^{E^*} = \bigvee_{j\in J}\bigvee_{i\in I} c_j e_{ij}.$$

For any $i \in I$, $\bigvee_{j\in J} c_j e_{ij} \geq \bigvee_{j\in J} s^{(1)}_{ij}$ by definition of $S^{(1)}$. Similarly, for any $j \in J$, $\bigvee_{i\in I} c_j e_{ij} \geq \bigvee_{i\in I} s^{(1)}_{ij}$ by definition of $S^{(2)}$. Hence,

$$\bigvee_{j\in J}\bigvee_{i\in I} c_j e_{ij} \geq \bigvee_{j\in J}\bigvee_{i\in I}(s^{(1)}_{ij} \vee s^{(2)}_{ij}) = \bigvee_{j\in J}\bigvee_{i\in I} s_{ij}.$$

Therefore,

$$z(x^*) \geq z(x^{E^*}) \geq \bigvee_{j\in J}\bigvee_{i\in I} s_{ij}.$$

Especially, according to Corollary 5, $x^{C^{-1}\odot S}$ is a strong solution in system (1) and $z(x^{C^{-1}\odot S}) = \bigvee_{i\in I}\bigvee_{j\in J} s_{ij}$. Thus, we have found an optimal strong solution of Programming (5).

IV. ALGORITHM FOR PROGRAMMING (5)

Step 1: Check the consistency of system (1) by Theorem 1. If inconsistent, then Programming (5) has no solution, so stop. Otherwise, go to Step 2.

Step 2: Calculate the characteristic matrix D, and judge the existence of a strong solution by Theorem 5, i.e., if D has a nonzero element for both any row and any column, then $X^s(A, b) \neq \emptyset$.

Step 3: Calculate $C \odot D$ by (15).

Step 4: Compute the sparse matrices $S^{(1)}$ and $S^{(2)}$, and then, obtain $S = (s_{ij})_{m\times n}$ by $S^{(1)} \vee S^{(2)}$.

Step 5: Find the optimal value, which is $\bigvee_{i\in I}\bigvee_{j\in J} s_{ij}$ of Programming (5) by Theorem 9 .

Step 6: Obtain an optimal strong solution $x^{C^{-1}\odot S}$ of Programming (5) by Theorem 10.

Example 4: The fuzzy relation equations with the max-min operator are

$$A \circ x = b, \tag{20}$$

where

$$A = \begin{bmatrix} 0.25 & 0.18 & 0.48 & 0.25 & 0.19 & 0.22 \\ 0.72 & 0.80 & 0.74 & 0.64 & 0.92 & 0.56 \\ 0.48 & 0.32 & 0.82 & 0.48 & 0.43 & 0.38 \\ 0.89 & 0.94 & 0.87 & 0.45 & 0.57 & 0.49 \\ 0.64 & 0.75 & 0.95 & 0.80 & 0.90 & 0.90 \end{bmatrix},$$

$$b = \begin{bmatrix} 0.25 \\ 0.64 \\ 0.48 \\ 0.57 \\ 0.80 \end{bmatrix}.$$

Denote by $X^s(A, b)$ the strong solution set of system (20).
i) Prove $X^s(A, b) \neq \emptyset$.
ii) Solve the optimization problem

$$\begin{aligned} &\min\ z(x) = x_1 \vee x_2 \vee x_3 \vee x_4 \vee x_5 \vee x_6 \\ &\text{s.t.} x \in X^s(A, b). \end{aligned} \tag{21}$$

iii) Solve the optimization problem

$$\begin{aligned} &\min\ z(x) = 0.9x_1 \vee 0.6x_2 \vee 0.8x_3 \vee 0.7x_4 \vee 0.5x_5 \vee 0.4x_5 \\ &\text{s.t. } x \in X^s(A, b). \end{aligned} \tag{22}$$

Solve:

i) Step 1. Check the consistency of system (20).
Since

$$(a_{ij} \circledast b_i) = \begin{bmatrix} 1 & 1 & 0.25 & 1 & 1 & 1 \\ 0.64 & 0.64 & 0.64 & 1 & 0.64 & 1 \\ 1 & 1 & 0.48 & 1 & 1 & 1 \\ 0.57 & 0.57 & 0.57 & 1 & 1 & 1 \\ 1 & 1 & 0.80 & 1 & 0.80 & 0.80 \end{bmatrix},$$

$\hat{x} = (0.57, 0.57, 0.25, 1, 0.64, 0.80)$. It is easy to check that $\hat{x}$ is the solution of system (20).

Step 2. Calculate the characteristic matrix D by Definition 5.

$$D = \begin{bmatrix} 0.25 & 0 & 0.25 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0.64 & 0.64 & 0 \\ 0.48 & 0 & 0 & 0.48 & 0 & 0 \\ 0.57 & 0.57 & 0 & 0 & 0.57 & 0 \\ 0 & 0 & 0 & 0.8 & 0 & 0.8 \end{bmatrix}.$$

Each row and each column have nonzero elements in D; thus, the strong solution set $X^s(A, b) \neq \emptyset$.

ii) In Steps 1-2 of Question i), we checked the existence of system (20).

Step 3. $C \odot D = D$, as $C = (1, 1, 1, 1, 1, 1)$ in the situation.

Step 4. Compute the sparse matrices $S^{(1)}$ and $S^{(2)}$ as follows.

$$S^{(1)} = \begin{bmatrix} 0.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.64 & 0 & 0 \\ 0.48 & 0 & 0 & 0 & 0 & 0 \\ 0.57 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0 & 0 \end{bmatrix},$$

$$S^{(2)} = \begin{bmatrix} 0.25 & 0 & 0.25 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.57 & 0 & 0 & 0.57 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.8 \end{bmatrix}.$$

Hence,

$$S = (s_{ij}) = \begin{bmatrix} 0.25 & 0 & 0.25 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0.64 & 0 & 0 \\ 0.48 & 0 & 0 & 0 & 0 & 0 \\ 0.57 & 0.57 & 0 & 0 & 0.57 & 0 \\ 0 & 0 & 0 & 0.8 & 0 & 0.8 \end{bmatrix}.$$

Step 5. Obtain the optimal value, which is $\bigvee_{i\in I}\bigvee_{j\in J} s_{ij} = 0.8$.

Step 6. An optimal strong solution is $x^S = (0.57, 0.57, 0.25, 0.8, 0.57, 0.8)$.

iii) Compared with Question ii), the objective function of the model has changed while the constraint conditions remain unchanged. We start it from Step 3.

Step 3.

$$C \odot D = \begin{bmatrix} 0.225 & 0 & 0.2 & 0.175 & 0 & 0 \\ 0 & 0 & 0 & 0.448 & 0.32 & 0 \\ 0.432 & 0 & 0 & 0.336 & 0 & 0 \\ 0.513 & 0.342 & 0 & 0 & 0.285 & 0 \\ 0 & 0 & 0 & 0.56 & 0 & 0.32 \end{bmatrix}.$$

Compute the sparse matrices $S^{(1)}$ and $S^{(2)}$ as follows.

$$S^{(1)} = \begin{bmatrix} 0 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.32 & 0 \\ 0 & 0 & 0 & 0.336 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.285 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.32 \end{bmatrix},$$

$$S^{(2)} = \begin{bmatrix} 0.225 & 0 & 0.2 & 0.175 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.342 & 0 & 0 & 0.285 & 0 \\ 0 & 0 & 0 & & 0 & 0.32 \end{bmatrix}.$$

Hence,

$$S = (s_{ij}) = \begin{bmatrix} 0.225 & 0 & 0.2 & 0.175 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.32 & 0 \\ 0 & 0 & 0 & 0.336 & 0 & 0 \\ 0 & 0.342 & 0 & 0 & 0.285 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.32 \end{bmatrix}.$$

Step 5. Obtain the optimal value, which is $\bigvee_{i \in I} \bigvee_{j \in J} s_{ij} = 0.342$.

Step 6. Since

$$C^{-1} \odot S = \begin{bmatrix} 0.25 & 0 & 0.25 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.64 & 0 \\ 0 & 0 & 0 & 0.48 & 0 & 0 \\ 0 & 0.57 & 0 & 0 & 0.57 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.8 \end{bmatrix},$$

an optimal strong solution is $x^{C^{-1} \odot S} = (0.25, 0.57, 0.25, 0.48, 0.64, 0.8)$.

Example 5: In this example we consider a Three-tier Multimedia Streaming Architecture with three regional servers and five client workstations (See Fig. 1).

The bandwidth limitation that the ith client workstation CW_i receives multimedia streaming from the jth regional server RS_j is a_{ij}, shown in the following Table 1, where $j \in \{1, 2, 3\}$, $i \in \{1, 2, 3, 4, 5\}$. The quality requirements of the five client workstations are 300, 250, 200, 250, 200, respectively, with measure unit *Mbps*. The jth regional server RS_j relays the multimedia streaming data to every client workstation with data transmission quality level x_j *Mbps*, $j = 1, 2, 3$.

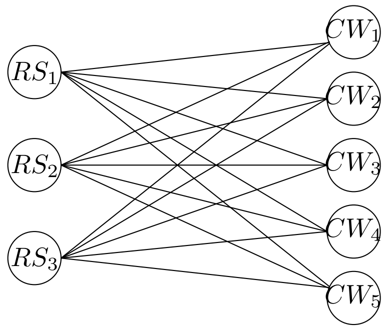

FIGURE 1. The Three-tier Multimedia Streaming Architecture.

TABLE 1. Value of a_{ij} (measure unit: Mbps).

i \ j	1	2	3
1	200	300	250
2	200	250	200
3	200	200	150
4	150	200	250
5	200	200	150

According to the above-given information, we are able to obtain the equations describing the Three-tier Multimedia Streaming Architecture as follows,

$$\begin{cases} (200 \wedge x_1) \vee (300 \wedge x_2) \vee (250 \wedge x_3) = 300, \\ (200 \wedge x_1) \vee (250 \wedge x_2) \vee (200 \wedge x_3) = 250, \\ (200 \wedge x_1) \vee (200 \wedge x_2) \vee (150 \wedge x_3) = 200, \\ (150 \wedge x_1) \vee (200 \wedge x_2) \vee (250 \wedge x_3) = 250, \\ (200 \wedge x_1) \vee (200 \wedge x_2) \vee (150 \wedge x_3) = 200. \end{cases} \quad (23)$$

The target is to search a strong solution of Eq. (23) minimizing the highest cost $3.5x_1 \vee 2x_2 \vee 2.5x_3$. Furthermore, we equivalently change Eq. (23) into a system of fuzzy relation equations by setting $a'_{ij} = \frac{a_{ij}}{500}$, $x'_j = \frac{x_j}{500}$, $b'_i = \frac{b_i}{500}$. Then we have

$$\begin{cases} (0.4 \wedge x_1) \vee (0.6 \wedge x_2) \vee (0.5 \wedge x_3) = 0.6, \\ (0.4 \wedge x_1) \vee (0.5 \wedge x_2) \vee (0.4 \wedge x_3) = 0.5, \\ (0.4 \wedge x_1) \vee (0.4 \wedge x_2) \vee (0.3 \wedge x_3) = 0.4, \\ (0.3 \wedge x_1) \vee (0.4 \wedge x_2) \vee (0.5 \wedge x_3) = 0.5, \\ (0.4 \wedge x_1) \vee (0.4 \wedge x_2) \vee (0.3 \wedge x_3) = 0.4. \end{cases} \quad (24)$$

Denote the strong solution set of system (24) by $X^S(A', b')$. Then we get the corresponding weighted minimax programming

$$\begin{aligned} &\min\ z'(x') = 3.5x'_1 \vee 2x'_2 \vee 2.5x'_3 \\ &\text{s.t. } x' \in X^S(A', b'). \end{aligned} \quad (25)$$

Following the resolution algorithm provided in this section, we find an optimal strong solution of programming (25) that

$$x'^* = (x'^*_1, x'^*_2, x'^*_3) = (0.4, 0.6, 0.5).$$

Considering $x_j' = \frac{x_j}{500}, j = 1, 2, 3$, we get an optimal strong solution of the original problem that

$$x^* = (x_1^*, x_2^*, x_3^*) = (200, 300, 250),$$

with measure unit *Mbps*. Under this optimal strong solution, All the three regional servers are active. One possible service scheduling is as follows: RS_1 services CW_3 and CW_5, RS_2 services CW_1 and CW_2, while RS_3 services CW_4. There is no non-occupational regional server.

Optimal service scheduling, searched from the strong solution set, is compared to that searched from the classical solution set, in the following Example 6. The advantage of our proposed optimization model in this paper also lies in this example.

Example 6: In the above Example 5, if we do not consider the strong solution set, but just search the optimal solution minimizing objective function from the solution set of Eq. (23), then the unbalanced service scheduling could not be avoid. When searching the optimal solution from the solution set of (23), we could establish the optimization problem as

$$\begin{aligned} &\min\ z(x) = 3.5x_1 \vee 2x_2 \vee 2.5x_3 \\ &\text{s.t. } x \in X(A, b), \end{aligned} \tag{26}$$

or as

$$\begin{aligned} &\min\ z(x) = 3.5x_1 + 2x_2 + 2.5x_3 \\ &\text{s.t. } x \in X(A, b), \end{aligned} \tag{27}$$

where $X(A, b)$ denotes the solution set of (23). Both problems 26 and 27 own the same optimal solution, i.e.

$$\bar{x}^* = (0, 300, 250).$$

Under the optimal solution $\bar{x}^* = (0, 300, 250)$, the service scheduling is that RS_1 services no client workstation, while RS_2 services CW_1, CW_2, CW_3, CW_5, and RS_3 services CW_4. In this situation, the second regional server RS_2 is overloaded, since it need to service four client workstations. But however, the second regional server RS_1 is non-occupational, since it services no client workstation. This leads to an unbalanced service scheduling.

V. ADVANTAGES OF THE ALGORITHM

A traditional method for solving programming (5) is to find all the pseudo minimal strong solutions of system (1) and choose the optimal strong solution(s) by comparing their objective function values, which will be proved in the following theorem.

Theorem 10: If $X^s(A, b) \neq \emptyset$, there exists some pseudo minimal strong solutions $\bar{x}$ such that $\bar{x}$ is an optimal strong solution of Programming (5).

Proof: Let x^* be an optimal strong solution. According to Theorem 7, there exists some strong solution matrix E^* belonging to D such that $x^{E^*} \leq x^*$, where D is the discrimination matrix and x^{E^*} is the strong solution corresponding to E^*, i.e., $x^{E^*} \in \bar{X}(A, b)$.

Let $\bar{x} = x^{E^*}$. On the one hand, $z(\bar{x}) \geq z(x^*)$ since x^* is an optimal strong solution. On the other hand $z(\bar{x}) \leq z(x^*)$ since $\bar{x} \leq x^*$. Thus, $z(\bar{x}) = z(x^*)$, which implies that one of the pseudo minimal strong solutions is an optimal strong solution.

Corollary 6: If $X^s(A, b) \neq \emptyset$, there exists $x^s \in X^s(A, b)$ such that x^s is an optimal strong solution of Programming (5).

Proof: By the structure of the strong solution set in Theorem 8, there exists some minimal strong solution x^s of system (1) such that $x^s \leq \bar{x}$, where $\bar{x}$ is the pseudo minimal strong solution in the proof of Theorem 10. It is clear that $z(x^s) = z(\bar{x})$. Thus x^s is an optimal strong solution of Programming (5).

However, it is more complex to solve the Programming (5) by Theorem 10, as we will see in the next example.

Example 7: Solve the optimization programming problem

$$\begin{aligned} &\min\ z(x) = 0.9x_1 \vee 0.6x_2 \vee 0.8x_3 \vee 0.7x_4 \vee 0.5x_5 \vee 0.4x_5 \\ &\text{s.t. } x \in X^s(A, b), \end{aligned} \tag{28}$$

where A, b is the matrices in Example 4.

Solve: The characteristic matrix is

$$D = \begin{bmatrix} 0.25 & 0 & 0.25 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0.64 & 0.64 & 0 \\ 0.48 & 0 & 0 & 0.48 & 0 & 0 \\ 0.57 & 0.57 & 0 & 0 & 0.57 & 0 \\ 0 & 0 & 0 & 0.8 & 0 & 0.8 \end{bmatrix}.$$

By searching all over the matrix D with the aid of the software "MATLAB", we will obtain 10 strong matrices belonging to D, which we denotes as $E_1, E_2, \ldots E_{10}$. Accordingly, there are 10 pseudo minimal strong solutions according to $E_1, E_2, \ldots E_{10}$, which we denote as $x^{E_1}, x^{E_2}, \ldots x^{E_{10}}$. By calculation, the pseudo minimal strong solutions are as follows:

$$\begin{aligned} x^{E_1} &= (0.25, 0.57, 0.25, 0.64, 0.57, 0.8), \\ x^{E_2} &= (0.25, 0.57, 0.25, 0.64, 0.64, 0.8), \\ x^{E_3} &= (0.25, 0.57, 0.25, 0.80, 0.57, 0.8), \\ x^{E_4} &= (0.25, 0.57, 0.25, 0.80, 0.64, 0.8), \\ x^{E_5} &= (0.48, 0.57, 0.25, 0.25, 0.64, 0.8), \\ x^{E_6} &= (0.48, 0.57, 0.25, 0.48, 0.64, 0.8), \\ x^{E_7} &= (0.48, 0.57, 0.25, 0.80, 0.64, 0.8), \\ x^{E_8} &= (0.57, 0.57, 0.25, 0.48, 0.64, 0.8), \\ x^{E_9} &= (0.57, 0.57, 0.25, 0.64, 0.64, 0.8), \\ x^{E_{10}} &= (0.57, 0.57, 0.25, 0.80, 0.64, 0.8). \end{aligned}$$

Accordingly, the objective function values are

$$\begin{aligned} z(x^{E_1}) = 0.342, z(x^{E_2}) = 0.448, \\ z(x^{E_3}) = 0.560, z(x^{E_4}) = 0.560, \\ z(x^{E_5}) = 0.432, z(x^{E_6}) = 0.432, \\ z(x^{E_7}) = 0.560, z(x^{E_8}) = 0.513, \\ z(x^{E_9}) = 0.513, z(x^{E_{10}}) = 0.513. \end{aligned}$$

Among them, the only optimal strong solution is $x^{E_1} = (0.25, 0.57, 0.25, 0.48, 0.64, 0.8)$, and the optimal value of the Programming (28) is 0.342.

Comparing the algorithm proposed in Example 4 with that in Example 7, we find that the former is more convenient and effective. In the former way, we are able to find an optimal solution of the programming (5) without solving all the pseudo strong minimal solutions. However, in the latter approach, we have to search all the potential matrices belonging to D all over the discrimination matrix D.

A comparison of the complexities between the proposed algorithm and the algorithm in Example 7 was conducted. Consider that the number of variables is n and the number of equations is m. Then, the complexity of the proposed algorithm $T_1(m, n) = O(mn)$, while the complexity of the algorithm in Example 7 is $T_2(m, n) = O(m^2n^2)$. The algorithm based on matrices can significantly reduce the computational complexity.

VI. CONCLUSION

In this paper, we introduced the concept of strong solutions for max-min fuzzy relation equations with the application background of three-tier multimedia streaming architecture. A matrix approach is developed to discuss the sufficient and necessary conditions of the existence of a strong solution in Theorems 2-5. We used the concept of the discrimination matrix of system (1) to describe the properties of the strong solution set. As shown in Theorem 8, we found the structure of the strong solution set, which is completely determined by its minimal strong solution set $\check{X}^s(A, b)$ and the maximum solution. Furthermore, we discussed the optimization programming problem (5) and developed a matrix algorithm to obtain the optimal value and an optimal strong solution in Theorem 9. Numerical examples are given to support the algorithm. Finally, we compared the matrix algorithm with the traditional method in Section 5. As proved in Theorem 10, we can find an optimal strong solution from the pseudo strong solution set $\bar{X}(X, a)$. However, a great effort was required to do so, as finding all the pseudo minimal strong solutions is usually an NP hard problem. Comparing Example 4 with Example 7, we find the feasibility and efficiency of the algorithm proposed in Section 5.

Reliability based design optimization (RBDO) for complex systems with fuzzy sets is a promising research region in recent years. Many important works have been published. As far as we know, RBDO with the background mentioned in this paper has not been reported yet. In the future we will consider the RBDO problem in the three-tier multimedia streaming services.

ACKNOWLEDGEMENTS

The authors would like to thank the editor and the anonymous referees for their valuable comments and suggestions, which helped us improve the earlier version of this paper.

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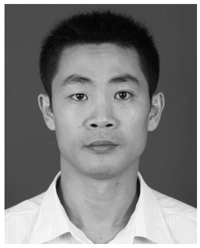

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