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INVITED PAPER

Stability Analysis of Systems With Delay-Dependent Coefficients: An Overview

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ABSTRACT This paper gives an overview of the stability analysis of systems with delay-dependent coefficients. Such systems are frequently encountered in various scientific and engineering applications. Most such analyses are generalization of those on systems with *delay-independent coefficients*. Therefore an introduction on systems with delay-independent coefficients is also given, with an emphasis on the τ-decomposition approach. Methods for two key ingredients of this approach are discussed, namely the identification of imaginary characteristic roots with the corresponding delays, and local behavior analysis of these roots as the delay increases through these critical values. For systems with *delay-dependent coefficients*, we review the methods of analysis for systems with a single delay and commensurate delays, their application to output feedback control and a geometric perspective that establishes a link between systems with and without delay-dependent coefficients. We provide the main ideas of various stability analysis methods and their advantages and limitations. We also present our perspectives on future directions of research on this interesting topic.

INDEX TERMS Delay systems, stability analysis, stability criteria.

I. INTRODUCTION

Systems with time delays are present in a broad spectrum of scientific and engineering disciplines, ranging from biology [44], [69], [92], chemistry [37], economy [119] to physics [129], engineering [31], [98], [117] and control systems [72], [114]. The presence of time delay is often caused by the time needed to transmit material, energy and information between different parts of a system.

The stability analysis of time delay system has been an active field in the control community since time delay can considerably change the performance and stability of a control system. Examples include sampled-data control systems [10], [40], networked control systems, [33], data transfer in high-speed networks [96], design of PID or Proportional-Integral-Retard controllers [100], [107], [108], [124], consensus seeking in multi-agent systems [86], [101], supply chain systems [111], traffic flow [118], and neural networks [25], [91], [133], just to name a few.

The vigorous research efforts have produced a rich collection of literature on the stability analysis of time delay systems. The readers are referred to the [50], [51], [74], [88], [93], [123] for a summary of recent progress. Most stability analyses are based on the Lyapunov approach or the spectrum approach.

The *Lyapunov-based methods* consist in constructing a Lyapunov function or functional to prove stability, by invoking the Razumikhin theorem or the Lyapunov-Krasovskii theorem, respectively [57]. This approach applies to general time delay systems with possibly nonlinear and time varying dynamics and is able to handle uncertainty. However, the construction of Lyapunov functionals is often a challenging task. The methods based on simple Lyapunov-Krasovskii functionals require a low computation load, but the results are in general conservative. For linear systems, computational tools are available leading to computer-aided numerical constructions. By restricting to a piecewise linear kernel, Gu [46] and [47] developed a discretized Lyapunov functional approach for stability analysis, which is guaranteed to find a Lyapunov functional for a large class of asymptotically stable linear systems if the gridding is sufficiently fine [53].

Various techniques for constructing Lyapunov functionals can be found in [39], [50], [73], [78], [112], and [113]. Recently, some different functional approximation approaches have been proposed using polynomial bases, Wirtinger inequalities and/or SOS techniques [102]–[104], [120]–[122]. An explicit inverse of the Lyapunov operators was obtained, and thus opens the way to full-state feedback for LTI delay systems via convex optimization [84]. Nevertheless, the computation burden grows very rapidly as the number of decision variables increases. Moreover, the optimization based approach does not always provide insight into the link between system structures and stability.

The spectrum approach is based on the fact that the stability of LTI time delay systems is determined by the roots of the associated characteristic equation. A system is asymptotically stable if and only if all the roots of the associated characteristic equation are located on the left half complex plane with a finite distance to the imaginary axis. Since the characteristic equation of time delay systems are quasi-polynomials, computing the right most characteristic root is a challenging problem, which is known to be N-P hard [126]. Several approaches for numerically computing the rightmost characteristic roots exist [88]–[90], [125], but the intensive computation involved limits their application. There exist several software packages devoted to delay characteristic equation, as, for example, DDE-BIFTOOL [38], TRACE-DDE [11], QPmR [127].

As in the finite-dimensional case, one of the important problems is the characterization of the *spectrum behavior* as a function of the variation of parameters (including delays). In this context, the continuity property of the spectrum with respect to the parameters is essential (see, for instance, [88] and the references therein). Two particular methods have been largely addressed and discussed in the literature mainly in the case when the number of parameters is small: *D-decomposition method* and *τ-decomposition method*. *The D-decomposition method* is an effective means of determining the number of characteristic roots on the right half complex plane for a given parameter domain without knowing the exact locations of characteristic roots [43], [128]. The main idea of *D*-decomposition method is to separate the parameter space into disjoint regions. In the interior of each region, the number of unstable roots is constant. On the boundary of these regions, the characteristic equation has at least one imaginary root. By analyzing the behaviors of these imaginary characteristic roots with respect to small variation of parameters, the system stability can be determined for different regions in the parameter space.

The *τ-decomposition method* can be viewed as a special case of the *D*-decomposition method when the parameter involved in this case is the delay τ. The underlying idea may be traced back to at least the 1960s (see [76] and the references therein), and a large number of related results can be found in the literature, see [50], [88], [99], [128]. The τ-decomposition methods roughly proceed as follows: starting with one value of delay τ^l for which one knowns the number of characteristic roots in the right half complex plane (usually $\tau^l = 0$), one sweeps through an delay interval of interest (τ^l, τ^u) and identify all delays τ_k, $k = 1, 2, \cdots, N-1$ for which there exist characteristic roots on the imaginary axis. These delay values are referred to as *critical delays* and the frequency of the corresponding imaginary roots are called *crossing frequencies*. By identifying the direction these roots cross the imaginary axis, one may determine the change of the number of right half complex plane roots as τ goes through each τ_k. Thus, one may divide (τ_0, τ_N) into subintervals (τ_{k-1}, τ_k), and the number of right half plane roots within each subinterval is constant and can be explicitly determined. Especially, the subintervals of delay for the systems to be stable can be computed.

Unfortunately, these methods are not sufficient for a complete stability analysis when the system coefficients also depend on the delay. Delay systems of this type are encountered in, for instance, population dynamics with age structure [8], [69], [70], the blowfly model [29], the hematopoietic models [27] as well as the stellar dynamo [129]. Systems of this type may also arise in the analysis of systems that do not contain delay-dependent coefficients. Examples include the analysis of partial differential equations using the method of characteristics [42], stability and convergence analysis of control systems based on delayed feedback [66], [94]. Detailed descriptions of such examples will be given in Section II.

Systems with *delay-dependent coefficients* received much less coverage in the literature. In [70] comprehensive results are obtained for a particular biological model, which cannot be applied to systems in general forms. To the authors' best knowledge, the first general method for stability analysis is proposed [7], which generalizes an earlier work of Cooke and Driessche on systems with delay-independent coefficients [28]. Some restrictive assumptions in [7] are relaxed in [54] to cover a larger set of systems. Both [7] and [54] only consider systems with a single delay of the form

$$D(\lambda, \tau) = P(\lambda, \tau) + Q(\lambda, \tau)e^{-\lambda\tau} = 0. \quad (1)$$

Moreover, the stability analysis methods developed in [7] and [54] do not apply when the characteristic equation has multiple imaginary roots.

Generalization of [54] to address quasi-polynomials with commensurate delays of the form

$$D(\lambda, \tau) = \sum_{i=0}^{N} P_i(\lambda, \tau)e^{-i\lambda\tau} = 0, \quad (2)$$

is pursued in [63]. By using a generalized Schur-Cohn lemma [131], Young proposed a method to detect crossing frequencies and critical delays, and to analyze the local behavior of simple characteristic roots on the imaginary axis. The two-parameter insight provided in [65] not only gives an intuitive interpretation of the results in [54], but also facilitating the development of more general analysis applicable also to the case with multiple imaginary characteristic roots. In [66], it is shown that delayed output signal can be used to approximate the first-order derivative of the output signals for the purpose of feedback control, resulting in a closed-loop system with delay-dependent coefficients. For these systems, the method developed in [54] is applied to find the delay intervals that guarantee a pre-specified exponential convergence rate of closed-loop trajectories. Approximation of higher-order derivatives of the outputs can be made by using commensurate delays and interpolating polynomials as detailed in [62], where the detailed procedure of finding the appropriate values of the delay parameter is also given.

The remaining part of this article is organized as follows. We first review how systems with delay-dependent coefficients arise in Section II. As the analysis of delay-dependent coefficients may be considered as a generalization of one with delay-independent coefficients, we devote Section III to the τ-decomposition related methods for stability analysis of systems with delay-independent coefficients. The analysis of systems with delay-dependent coefficients is presented in Section IV. Section V concludes this paper with our perspectives on this interesting topic of research.

A preliminary version of this paper has appeared in [64]. This article extends [64] in the following two aspects. First, additional examples are presented to show the diversity of applications involving systems with delay-dependent coefficients. Second, more detailed and extensive literature review is contained.

In the sequel, we will refer to $(j\omega^*, \tau^*)$ as a critical pair, where ω^* is a crossing frequency, τ^* a critical delay, if $\lambda = j\omega^*$, $\omega^* \geq 0$ is an imaginary root of the characteristic equation corresponding to τ^*. Due to the symmetry of the characteristic roots about the real axis, the critical pairs only include positive crossing frequencies. We use ∂_x to denote partial differentiation w.r.t argument x. For instance, $\partial_x F(x,y) = \frac{\partial F(x,y)}{\partial x}$. For any complex number c, $\Re(c)$, $\Im(c)$ and $\bar{c}$ denote its real part, imaginary part and complex conjugate, respectively. $\mathbb{R}$ stands for the set of real numbers, $\mathbb{R}_+$ for non-negative reals, and $\mathbb{C}$ for complex numbers. $\mathbb{D}$ and $\partial\mathbb{D}$ stand for the unit disk and the unit circle centered at the origin in the complex plane, respectively.

II. SYSTEMS WITH DELAY-DEPENDENT COEFFICIENTS

In this section we introduce several situations where systems with delay-dependent coefficients arise. In some applications, the mathematical modeling of a physical or biological process directly leads to this type of systems. Take the stellar dynamo given in [129] for example. Its dynamics is described by the following equations

$$\dot{B}_\phi(t) = c_1 e^{-c_2 T_0} A(t - T_0) - c_2 B_\phi(t),$$
$$\dot{A}(t) = c_3 e^{-c_2 T_1} B_\phi(t - T_1) - c_2 A(t),$$

where B_ϕ is the strength of toroidal field, and A is the strength of poloidal field, and c_1, c_2, c_3, T_0, T_1 are positive constants. The delay parameters that appear in the exponential terms in the system coefficients reflect the dissipation of energy over time. The characteristic equation of the above system can be easily obtained as

$$\lambda^2 + 2c_2\lambda + c_2^2 - c_1 c_3 e^{-c_2\tau} e^{-\tau\lambda} = 0, \quad (3)$$

where $\tau = T_0 + T_1$. It is clear that the delay-dependent terms in the coefficients of a differential equation carry over to the associated characteristic equation.

There are various reasons for system coefficients to be delay-dependent. As the information, substance or energy is transmitted, their quantity or magnitude may in general decrease over time due to dissipation, causing their influence to be delay-correlated. As discussed in [7], for population dynamics, the need to incorporate time delay is often the result of the existence of some stage structure [70]. It is easy to conceive that these models will involve some delay-dependent parameters since the through-stage survival rate is often a function of the time delay. Consider the second model with time delay and stage structure introduced by Bence and Nisbet [8] for a population of sessile invertebrates. This model is a two-stage model in which population is divided into adult and juvenile populations. The model takes the form:

$$\begin{cases} \dot{J}(t) = s[F(t) - e^{-m_J\tau}F(t-\tau)] - m_J J(t), \\ \dot{A}(t) = s e^{-m_J\tau} F(t-\tau) - m_A A(t), \\ F(t) = \max\{0, 1 - a_J J(t) - a_A A(t)\}, \end{cases} \quad (4)$$

where $sF(t)$ represents the newly settled juveniles and $se^{-m_J\tau}F(t-\tau)$ the ones that become adults. The corresponding characteristic equation is

$$\lambda^2 + a\lambda + c + (b(\tau)\lambda + d(\tau))e^{-\lambda\tau} = 0 \quad (5)$$

Characteristic equations with delay-dependent coefficients may also arise during the analysis of other type of systems. For instance, the model of hematopoietic stem cell dynamics given in [27] is nonlinear with delay-free coefficients, and possesses two equilibria. The linearized equation in the neighborhood of the nonzero equilibrium has the following characteristic equation

$$\lambda + A(\tau) - B(\tau)e^{-\lambda\tau} = 0,$$

where A, B are continuous functions of τ. In this case, the delay-dependent coefficients are caused by the fact that for a nonlinear systems the positions of the equilibria may depend explicitly on the delay.

time delay systems with persistent repeated roots are typical examples of systems that may be constructed using delay-dependent coefficients. As a matter of fact, consider the simple scalar equation

$$\dot{x}(t) = ax(t) + bx(t-\tau).$$

The corresponding characteristic equation is

$$\lambda - be^{-\tau\lambda} - a = 0. \quad (6)$$

It was shown in [17] that the maximal multiplicity that a spectral value of (6) can have is two. Indeed, let λ_0 be a repeated root of (6) for some given τ, it must hold that

$\lambda_0 = a - \frac{1}{\tau}$ and $b = \frac{1}{\tau}e^{\lambda_0\tau}$. Substitute the expression of b into (6), we arrive at

$$\lambda - \frac{1}{\tau}e^{a\tau-1}e^{-\tau\lambda} - a = 0, \tag{7}$$

which is a system with delay-dependent coefficients. A birkhoff approach in characterizing multiple spectral values is proposed in [17]. The dominancy of multiple spectral values is further explored in [18] and [19] and analytically shown in the case of second-order systems, and a rightmost root assignment based design using delayed state feedback is proposed where its applicability in damping active vibrations for a piezo-actuated beam is demonstrated.

time delay systems with delay-dependent coefficients also find their applications in the analysis of models described by partial differential equations. In [85], the dynamics of age-structured hematopoietic stem cells is described by:

$$\begin{cases} \partial_t\tilde{r}(t,a) + \partial_a\tilde{r}(t,a) = -(\tilde{\delta} + \tilde{\beta}(C(t)))\tilde{r}(t,a), \\ \qquad \text{for } a > 0, t > 0, \\ \partial_t\tilde{p}(t,a) + \partial_a\tilde{p}(t,a) = -\tilde{\gamma}\tilde{p}(t,a), \\ \qquad \text{for } 0 < a < \tilde{\tau}, t > 0, \\ \partial_t r(t,a) + \partial_a r(t,a) = -(\delta + \beta(C(t)))r(t,a), \\ \qquad \text{for } a > 0, t > 0, \\ \partial_t p(t,a) + \partial_a p(t,a) = -\gamma p(t,a), \\ \qquad \text{for } 0 < a < \tau, t > 0, \end{cases}$$

where $r(t, a)$ is the density of resting healthy cells at time t and age a, $\tilde{r}(t, a)$ denotes the density of resting unhealthy cells, $p(t, a)$ the density of proliferating healthy cells and $\tilde{p}(t, a)$ the density of proliferating unhealthy cells. The boundary condition for all $t > 0$ is given by

$$\begin{cases} \tilde{r}(t,0) = 2(1-\tilde{K})\tilde{p}(t,\tilde{r}), \\ \tilde{p}(t,0) = \tilde{\beta}(C(t))\tilde{x}(t) + 2\tilde{K}\tilde{p}(t,\tilde{\tau}), \\ r(t,0) = 2p(t,\tau), \\ p(t,0) = \beta(C(t))x(t). \end{cases}$$

Using the method of characteristics [42] and following similar arguments as those in [4] and [42], the partial differential equation can be reduced to a delay-difference equation with delay-dependent coefficients [36]:

$$\begin{cases} \dot{\tilde{x}}(t) = -[\tilde{\delta} + \tilde{\beta}(x(t)+\tilde{x}(t))]\tilde{x}(t) + 2(1-\tilde{K})e^{-\tilde{\gamma}\tilde{\tau}} \\ \qquad \times\tilde{u}(t-\tau), \\ \tilde{u}(t) = \tilde{\beta}(x(t)+\tilde{x}(t))\tilde{x}(t) + 2\tilde{K}e^{-\tilde{\gamma}\tilde{\tau}}\tilde{u}(t-\tilde{\tau}), \\ \dot{x}(t) = -[\delta + \beta(x(t)+\tilde{x}(t))]x(t) + 2e^{-\gamma\tau}\beta(x(t-\tau) \\ \qquad +\tilde{x}(t-\tau))x(t-\tau). \end{cases} \tag{8}$$

The model of cell density in a generic compartment in [42] is described by a different set of PDEs, and it can also be reduced to a system with delay-dependent coefficients using the method of characteristics.

In control practice, it is often difficult to measure the entire system state for feedback control, therefore output feedback is a common scenario. To improve control performance, in many control schemes, including the popular PID control [3], [55] and the internal model control [71], the time derivative of the output $y(t)$ is used for feedback. In a PID control schem, a delay difference can be used to approximate $\dot{y}(t)$:

$$\dot{y}(t) \approx \frac{y(t) - y(t-\tau)}{\tau}, \tag{9}$$

where τ is a positive delay value. If the plant transfer function is rational, then the characteristic equation of the closed-loop system has delay-dependent coefficients and may be written as:

$$G_0(\lambda) + G_1(\lambda)\frac{1-e^{-\lambda\tau}}{\tau} = 0, \tag{10}$$

where $G_0(\lambda)$ and $G_1(\lambda)$ are polynomials. To estimate higher-order derivatives of the output, it is possible to generalize the finite-difference scheme described above using multiple commensurate delays. This idea is exploited in [94], where the output signal over a past period of time is interpolated with a polynomial function and the higher-order derivatives of the output can be approximated using the derivatives of the interpolation polynomial. Similar to the single delay case, the commensurate-delay based controller in [94] also leads to a closed-loop system with the delay parameter appearing in the coefficients. By exploiting the particular structure of a chain of integrators, it is shown there that a rescaling technique can be applied to transform the system in such a way that the coefficients become delay-independent. For general systems, however, this technique can not be applied.

Finally, consider a system with characteristic equation

$$P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0 \tag{11}$$

It is often desirable to make sure the system is not only stable but also converges exponentially to the origin with a rate no less than α, with α being a positive number. This can be achieved by making a substitution $\lambda - \alpha \rightarrow \lambda$ in (11), resulting in

$$P(\lambda-\alpha) + Q(\lambda-\alpha)e^{-\alpha\tau}e^{-\lambda\tau} = 0. \tag{12}$$

Thus the problem is transformed to the stability problem of the system represented by (12). Obviously, the coefficients polynomial $Q(\lambda-\alpha)e^{-\alpha\tau}$ depends on the delay τ even though no coefficient polynomial in (11) depends on the delay.

III. STABILITY ANALYSIS OF SYSTEMS WITH DELAY-INDEPENDENT COEFFICIENTS

In this section, we review some techniques pertaining to the τ-decomposition approach for systems with delay-independent coefficients. We focus on those results that have been, or can potentially be extended to systems with delay-dependent coefficients.

A. CHARACTERISTIC EQUATIONS

Consider linear time-invariant time delay systems with the following characteristic equation:

$$D(\lambda,\tau) = \sum_{i=0}^{N} P_i(\lambda)e^{-i\lambda\tau} = 0, \tag{13}$$

where $P_i(\lambda)$, $i = 0, 1, \cdots, N$ are a polynomials of the Laplace variable λ, and τ is the delay parameter. In the simplest case of $N = 1$, the characteristic equation (13) becomes

$$D(\lambda, \tau) = P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0. \quad (14)$$

It is worth mentioning that the characteristic equation (13) can be derived from functional differential equations of both the retarded and neutral type. The linearized dynamics of the retarded type equations can be written as

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t - i\tau), \quad (15)$$

and that of the neutral type as

$$\frac{d}{dt}\left(x(t) + \sum_{k=1}^{m} H_k x(t - k\tau)\right) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t - i\tau). \quad (16)$$

In (15) and (16), $x(t)$ denotes the state variables at time t and H_k, $k = 1, \cdots, m$, A_i, $i = 0, \cdots, m$ are matrices of appropriate dimensions. Then the characteristic equation is given by:

$$D(\lambda, \tau) = \det\left(\lambda I - A_0 - \sum_{i=1}^{m} A_i e^{-\lambda\tau_i}\right) \quad (17)$$

for the retarded type system (15) and

$$D(\lambda, \tau) = \det\left(\lambda(I + \sum_{k=1}^{m} H_k e^{-\lambda\tau_k}) - A_0 - \sum_{i=1}^{m} A_i e^{-\lambda\tau_i}\right) \quad (18)$$

for the neutral type system (16). An expansion of the determinant in (17) or (18) yields (13). It should be pointed out that even if $m = 1$ in (15) or (16), we may still have $N > 1$ in general in the characteristic equation in (13).

time delay systems may also be alternatively modeled using differential equations on abstract space [6] or over rings of operators [67], although the most common description of time delay systems is functional differential equations. A number of monographs covering different aspects of general time delay systems as well as functional differential equations are available, see e.g. [9], [56], [57], [75].

B. THE τ-DECOMPOSITION APPROACH

The τ-decomposition approach outlined in the introduction section provides a convenient method for determining stability of time delay system by avoiding computing the rightmost characteristic root. A crucial premise of the τ-decomposition approach is that characteristic roots in a vicinity of the imaginary axis and the right half complex plane must vary continuously with respect to τ. While this always holds for retarded type systems, extra conditions are needed to guarantee such continuity for the neutral type systems with multiple delays. Furthermore, it is also possible that a system is unstable even though all the characteristic roots are in the strict left half-plane (see, e.g., [52], [88]), if there exists a sequence of characteristic roots that approach the imaginary axis.

There are two key ingredients of the τ-decomposition approach. The first one is the detection of all *critical pairs*, namely imaginary characteristic roots and the corresponding critical delays. The second is the local behavior analysis of imaginary characteristic roots, that is, to determine whether these critical imaginary characteristic roots will become stable or unstable as τ increases through these critical values. Various methods for solving these two problems will be reviewed in this section. For the sake of brevity, we only discuss systems with a single delay or commensurate delays. For more general time delay systems, such as systems with multiple independent delays [34], [116] or with distributed delays [5], [30], see, e.g., the survey papers [51], [105], the monograph [88] and the references therein.

C. DETECTION OF CRITICAL PAIRS

To illustrate the basic idea, consider first systems with a single delay represented by the characteristic equation (14). When λ lies on the imaginary axis, i.e., $\lambda = j\omega$ for some $\omega \in \mathbb{R}$, $e^{-\lambda\tau}$ lies on the unit circle $\partial\mathbb{D}$ of the complex plane. This is an important property exploited in most techniques that identify imaginary characteristic roots.

The following polynomial is introduced in [28] for this purpose:

$$F(\omega) = P(j\omega)P(-j\omega) - Q(j\omega)Q(-j\omega). \quad (19)$$

Noticing that $e^{-j\omega\tau} \in \partial\mathbb{D}$ in (14) implies $|P(j\omega)| = |Q(j\omega)|$. Therefore, if ω is a real root of $F(\omega)$, then $\lambda = j\omega$ must be an imaginary characteristic root of (14). Let $\omega_1 < \omega_2 < \cdots < \omega_H$ be the non-negative solution of

$$F(\omega) = 0, \quad (20)$$

then each ω_k, $1 \leq k \leq H$ is a crossing frequency. Suppose $(j\omega^*, \tau^*)$ is a critical pair, it follows from (14) that

$$-\frac{P(j\omega_k)}{Q(j\omega_k)} = e^{j\tau_{km}\omega_k}. \quad (21)$$

It can be deduced from the equation above that corresponding to each crossing frequency ω_k, there is a sequence of critical delays τ_{km}, $m = 0, 1, 2, \cdots$, that satisfy

$$\tau_{km} = \frac{1}{\omega_k}\angle\left(-\frac{P(j\omega_k)}{Q(j\omega_k)}\right) + \frac{2\pi m}{\omega_k}, \quad (22)$$

where $\angle(\cdot)$ is the phase angle of a complex number.

As an example, consider the following characteristic equation of neutral type:

$$2\lambda + 1 + (\lambda + 2)e^{-\lambda\tau} = 0. \quad (23)$$

Then

$$F(\omega) = |2j\omega + 1|^2 - |j\omega + 2|^2 = 3\omega^2 - 3.$$

The only positive solution of (20) for the above $F(\omega)$ is $\omega_1 = 1$. According to (22), all the critical delays corresponding to the imaginary characteristic root $j\omega_1$ can be expressed as

$$\tau_{1m} = \tau_0 + 2\pi m,$$

where $\tau_0 \approx 2.4981$. It is easy to see that, in this case, when the delay increases, the characteristic roots will cross the imaginary axis towards the right half complex plane at each $\tau_{1m} > 0$.

Next, for systems with commensurate delays, the simple magnitude condition $|P(j\omega)| = |Q(j\omega)|$ at some crossing frequency ω is no longer available. Nevertheless, the generalized Schur-Cohn lemma given in [131] suggests that it is still possible to define a polynomial $F(\omega)$ for detecting all crossing frequencies. To construct this polynomial, first associate (13) with the following function:

$$\hat{D}(\lambda, x) = \sum_{k=0}^{N} P_k(\lambda)x^k, \tag{24}$$

where x can be a scalar or a matrix. Let $\mathcal{H}$ be the Schur's hermitian form associated with (13) defined as

$$\mathcal{H}(\lambda, X) = \sum_{k=1}^{N} |P_0 x_k + P_1 x_{k+1} + \ldots + P_{N-k} x_N|^2 - \sum_{k=1}^{N} |\overline{P_N} x_k + \overline{P_{N-1}} x_{k+1} + \ldots + \overline{P_k} x_N|^2 \tag{25}$$

where $X = col(x_1, x_2, \ldots, x_N) \in \mathbb{C}^N$. The hermitian form $\mathcal{H}$ can be expressed as

$$\mathcal{H}(\lambda) = X^H \mathrm{H}(\lambda) X \tag{26}$$

where X^H is the conjugate transpose of X, and

$$\begin{aligned} \mathrm{H}(\lambda) &= \hat{Q}(\lambda, S)^H \hat{Q}(\lambda, S) \\ &\quad - \hat{D}(\lambda, S)^H \hat{D}(\lambda, S), \\ \hat{Q}(\lambda, S) &= \sum_{k=0}^{N} \overline{P_k} S^{N-k}, \end{aligned} \tag{27}$$

and

$$S = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

is an $N \times N$ shift matrix. Then the function F for the commensurate-delay systems (13) can be defined as

$$F(\omega) = -\det(\mathrm{H}(j\omega)). \tag{28}$$

The following equality is known as the generalized Schur-Cohn lemma (see for instance [131]):

$$F(\omega) = -|P_N(j\omega)|^{2N} \prod_{i,k=1}^{N} (1 - z_i \overline{z_k}), \tag{29}$$

where z_i, $i = 1, 2, \ldots, N$ are the roots of the polynomial $\hat{D}(j\omega, x)$ in x for fixed ω.

The equations (13) and (24) imply that a necessary and sufficient condition for $j\omega^*$ to be an imaginary characteristic root is that $\hat{D}(j\omega^*, x)$ has a root in x on the unit circle $\partial\mathbb{D}$. Therefore, $j\omega^*$ being an imaginary characteristic root implies that $F(\omega^*) = 0$ with $F(\omega)$ given in (29). However, the converse is not necessarily true. It is possible that none of the z_i's lies on $\partial\mathbb{D}$ when $\omega = \omega^*$ but we still have $F(\omega^*) = 0$ because there may exist some z_k, z_l such that $z_1\overline{z_k} = 1$. In other words, if $\lambda = j\omega$ is an imaginary characteristic root of (13), then the following must hold:

$$F(\omega) = 0. \tag{30}$$

The polynomial $F(\omega)$ defined in (28) was first introduced in [21] for identifying crossing frequencies. Once the imaginary characteristic roots are known, the critical delay values can be easily computed.

The critical pairs can also be identified using bilinear transformations [50]. This method is known as the *pseudo delay technique* or the *Rekasius substitution* [106]. The idea is to replace the term $e^{j\omega\tau}$ in (13) with $\frac{1-j\omega T}{1+j\omega T}$, which leads to

$$D_T(\omega) = \sum_{i=0}^{N} p_i(j\omega) \left(\frac{1 - j\omega T}{1 + j\omega T}\right)^i = 0. \tag{31}$$

Then $\omega^* \geq 0$ is a crossing frequency if and only if the above equation is satisfied with $\omega = \omega^*$. The equation above can be written as

$$\begin{cases} \Re(D_T(\omega)) = 0, \\ \Im(D_T(\omega)) = 0. \end{cases}$$

To obtain a polynomial equation of ω only, the variable T can be eliminated from the above equations using the resultant theory (see, e.g., [32], [109]), or the other elimination techniques.

The aforementioned methods consist in first identifying all crossing frequencies and then computing the corresponding critical delays. Finally, an alternative approach is first to detect all $z \in \partial\mathbb{D}$ such that

$$\sum_{i=0}^{N} P_i(j\omega) z^i = 0. \tag{32}$$

Suppose $z = z^*$ is such a solution, then after replacing z with z^* in (32) one obtains a polynomial equation in ω. By solving this polynomial equation for real ω, all crossing frequencies corresponding to z^* can be obtained. The *matrix pencil methods* embody this idea and directly work on the state-space formulation. Consider for example the retarded type system (15), let n be the number of the state variables, define the matrix pencil as:

$$\Lambda(z) = zW + U,$$

where $z \in \mathbb{C}$ and $W, U \in \mathbb{R}^{(2mn^2)\times(2mn^2)}$ are given by:

$$W = \begin{pmatrix} I_{n^2} & 0 & \cdots & 0 & 0 \\ 0 & I_{n^2} & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & I_{n^2} & 0 \\ 0 & 0 & \cdots & 0 & B_m \end{pmatrix},$$

$$U = \begin{pmatrix} 0 & -I_{n^2} & 0 & \cdots & 0 \\ 0 & 0 & -I_{n^2} & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & I_{n^2} \\ B_{-m} & B_{-m+1} & \cdots & B_{m-2} & B_{m-1} \end{pmatrix}.$$

and $B_{-k}, k = 1, \cdots, m$ are defined as:

$$B_{-k} = I_n \otimes A_k^T, \quad B_i = A_i \otimes I_n,\ B_0 = A_0 \oplus A_0^T. \quad (33)$$

The operators $\otimes$ and $\oplus$ are the Kronecker product and sum, (see, e.g., [77]).

It is shown in [95] that $\lambda = j\omega^*$ is an imaginary characteristic root if and only if there exists some complex number $z^* \in \partial\mathbb{D}$ such that

$$\det(\Lambda(z^*)) = 0, \quad (34)$$

and

$$\det(A_0 + \sum_{i=1}^{m} A_i z^{*i} - j\omega^* I) = 0 \quad (35)$$

are both satisfied. One first solve (34) to obtain all solutions in z^* on the unit circle, then solve (35) for all the crossing frequencies ω^* corresponding to each given z^*. It is worth mentioning that not all $z^* \in \partial D$ generate crossing frequencies (see, for instances, [97]). Once z^* and ω^* are obtained, the critical delays can be easily computed from the equality $e^{-j\omega^*\tau} = z^*$.

D. BEHAVIOR ANALYSIS OF IMAGINARY ROOTS

Given a critical pair $(j\omega^*, \tau^*)$. If $\lambda = j\omega^*$ is a simple root of $D(\lambda, \tau)$, or equivalently $\partial_\lambda D(\lambda, \tau)|_{(j\omega^*,\tau^*)} \neq 0$, then by the implicit function theorem [2], [68], the characteristic root λ is a function of τ denoted here as $\lambda(\tau)$ in a neighborhood of $(j\omega^*, \tau^*)$. The first-order derivative of the characteristic roots with respect to the delay can be computed as

$$\frac{d\lambda(\tau^*)}{d\tau} = -\frac{\partial_\tau D(\lambda, \tau)}{\partial_\lambda D(\lambda, \tau)}\bigg|_{(j\omega^*,\tau^*)}. \quad (36)$$

If $\text{sgn}(\Re(\frac{d}{d\tau}\lambda(\tau^*))) = 1$, the characteristic root crosses the imaginary axis from $\mathbb{C}_-$ to $\mathbb{C}_+$ as τ increases through τ^*. If it is -1 instead, the characteristic root moves from $\mathbb{C}_+$ to $\mathbb{C}_-$ and becomes stable. Return to the example (23). At the critical delay $\tau^* \approx 2.4981$, the characteristic equation has a pair of imaginary roots $\lambda = \pm j$. Consider the root with positive frequency, straightforward computation yields

$$\frac{d\lambda(\tau^*)}{d\tau} \approx 0.0905 - 0.3790j. \quad (37)$$

Since the real part of this derivative is positive, this pair of characteristic roots move into the right half complex plane and thus become unstable as τ increases from τ^*.

It is possible to obtain the crossing direction of simple imaginary characteristic roots by working directly on the state-space formulation of the system. In the monograph [88], the following class of delay systems with parameterized coefficients and delays are discussed:

$$\dot{x} = A_0(p)x(t) + \sum_{i=1}^{m} A_i(p)x(t - \tau_i(p)). \quad (38)$$

Suppose $\lambda = j\omega^*$ is a simple characteristic root of $D(\lambda, p^*)$, and let $\lambda(p)$ be the trajectory of the characteristic root that passes through $j\omega^*$ as p passes through p^*. Then using the Jacobi's formula and some properties of left and right eigenvalues of rank one matrices, the following equation is derived

$$\partial_{p_i}\lambda(p) = -\frac{v_0^T \cdot \partial_{p_i} M \cdot u_0}{v_0^T \cdot \partial_\lambda M \cdot u_0}, \quad (39)$$

where

$$M(\lambda, p) = \lambda I - A_0(p) - \sum_{i=1}^{m} A_i e^{-\lambda\tau_i(p)}, \quad (40)$$

and v_0^T and u_0 are the left and right eigenvectors of $M(j\omega^*, p^*)$, respectively.

It is worth mentioning that the equation (38) can actually represent a large class of systems with delay-dependent coefficients, although this fact was not mentioned in [88]. For instance, if one sets $p = \tau$, $\tau_i = ip$, then (38) represents systems with commensurate delays. On the other hand, suppose $p = col\{p_1, \cdots, p_m\}$ and $\tau_i(p) = p_i$, then (38) represents systems with m independent delays. Therefore the formula (39) can indeed be applied to systems with delay-dependent coefficients.

When the right hand side of (36) or (39) is zero, higher-order analysis is necessary, which is reported in [41] and briefly introduced in [88] using the eigenvalue perturbation technique. The method in [41] apply to systems represented both by the state-space equations and characteristic equations of quasi-polynomials.

Although it is quite straightforward to use the formula (36) or (39) to determine the crossing direction of simple characteristic roots, these formula do not provide deep insight into this problem. The right hand side of (36) or (39) rely on λ and τ in a complicated way, which does not reveal how the crossing direction may vary for different critical pairs.

In [28], an interesting relationship between the function $F(\omega)$ defined in (19) and the crossing direction of characteristic roots are derived. Let $(j\omega^*, \tau^*)$ be a critical pair of the characteristic equation (14), $\lambda(\tau)$ be the roots of (14) in a neighborhood of $(j\omega^*, \tau^*)$, then

$$\text{sgn}\left(\Re(\lambda'(\tau^*))\right) = \text{sgn}(F'(\omega^*)). \quad (41)$$

According to the last equation, as τ sweeps through τ^* from left, a pair of imaginary roots $\pm j\omega^*$ cross the imaginary axis toward the right half complex plane if $F'(\omega^*) > 0$. They move toward the left half complex plane if $F'(\omega^*) < 0$.

Two important *invariance properties* now follow from (22) and (41). The former shows that the crossing frequency ω_k is invariant with respect to a shift of $2\pi/\omega_k$ in the delay. The crossing direction of each characteristic root at $\lambda = j\omega_k$ is independent of the corresponding delay, as indicated in (41).

This further implies a simple root crossing pattern in the way characteristic roots with different frequencies cross the imaginary axis. It is easy to see

$$\mathrm{sgn}(F'(\omega_k)) = -\mathrm{sgn}(F'(\omega_{k+1})). \quad (42)$$

Therefore the crossing direction of each two neighboring imaginary roots $j\omega_k$ and $j\omega_{k+1}$ always have opposite crossing directions. We note that this pattern of alternating crossing directions does not necessarily hold for system with commensurate delays. Using this property, it is easy to see that the roots crossing toward one side of the imaginary axis more often than toward the other side. If the characteristic equation (14) admit at most a finite number of roots on the right half plane, then it can be deduces that the imaginary roots must cross toward the right more frequently, otherwise for large delays, the number of characteristic roots lying to the right of the imaginary axis would fall below zero. It is then claimed that there exits some positive number T^* such that the system (14) remains unstable for all $\tau > T^*$ and no stability switches will occur if τ increases beyond T^*.

E. REPEATED CHARACTERISTIC ROOTS

For the characterization and analysis of crossing roots with multiplicity larger than one, several different approaches are available in the literature, including those based on the perturbation theory and Newton-Puiseux series [81], [82], as well as the geometrical approach [45], [59].

It is possible that the real part of $\lambda'(\tau)$ is zero at some critical pair. In this situation, higher-order derivatives of the characteristic root with respect to τ needs to be computed in order to determine the root crossing direction. This type of analysis has been reported in [41].

The frequency-sweeping framework developed in [82] provides a general method for comprehensive stability analysis of multiple roots on the imaginary axis. Recall the characteristic equation for systems with commensurate delays

$$D(\lambda, \tau) = \sum_{i=0}^{N} p_i(\lambda) e^{-i\lambda\tau} = 0, \quad (43)$$

where each $p_i(\lambda)$ is a polynomial. Corresponding to the characteristic equation, the following function is also defined:

$$\hat{D}(\lambda, z) = \sum_{i=0}^{N} p_i(\lambda) z^i = 0.$$

Sweeping through $\omega \geq 0$, for each $\lambda = j\omega$, suppose the equation above admits N solutions in z. Denote these solutions as $z_i(j\omega)$, $i = 1, \cdots, N$. Then the graph of $\Gamma_i(\omega) = |z_i(j\omega)|$ is referred to as a *frequency-sweeping curve*(FSC).

As τ increases through some critical delay $\tau_{\alpha k}$, where $\tau_{\alpha k}$ is given in (22), the increase of number of characteristic roots on the right half complex plane in a small neighborhood of λ_α is equal to $NF_{z_\alpha}(\tau_{\alpha k})$ defined as follows

$$NF_{z_\alpha}(\tau_{\alpha k}) = N_{z_\alpha}(\tau + \epsilon) - N_{z_\alpha}(\tau - \epsilon), \quad (44)$$

where $N_{z_\alpha}(\tau)$ is the number of the FSCs: $\Gamma_i(\omega)$, $i = 1, \cdots, N$ that satisfy 1) $z_i(\omega) = \exp(-\lambda_\alpha \tau_\alpha)$, 2) $\Gamma_i(\omega) > 1$, and ϵ is a sufficiently small positive number. In other words, the characteristic roots crossing the imaginary axis is associated with the corresponding frequency-sweeping curves crossing the horizontal line 1. Using this property, it is shown that the crossing of characteristic roots on the imaginary axis with multiplicity has similar invariance properties as the systems with a single delay and simple imaginary characteristic roots. Furthermore, the system stability can be analyzed completely, in the sense that the eventual number of unstable characteristic roots as $\tau \to +\infty$ can be easily determined.

As noted in [81], the idea of *frequency-sweeping*, consisting in first detecting all potential crossing frequencies when implementing the τ-decomposition methods, is not new, and have indeed appeared in earlier work such as [21], [24], [25], [28], [48], [49], [76], and [115]. However, the frequency-sweeping idea is used only to detect critical delay pairs in [25] and [50] for the commensurate delays case and in [35], [48], [49], and [115] for the incommensurate delay case, without determining the crossing directions of the imaginary characteristic roots. In [28] and [76], frequency-sweeping tests were used for studying the local behavior of imaginary characteristic roots, but characteristic equations considered therein are confined to a class of simple quasi-polynomials.

As mentioned in [82], when the critical pair is not regular, which includes the case of multiple characteristic roots on the imaginary axis, it is necessary to use the *Puiseux series* to analyze the asymptotic behaviors of these roots. Here we give a very rough idea about how Puiseux series can come into play. The characteristic equation (13) can be expanded at each critical pair (λ_0, τ_0) as

$$F(\Delta\lambda, \Delta\tau) = 0,$$

where $\Delta\lambda = \lambda - \lambda_0$, $\Delta\tau = \tau - \tau_0$ and $F(\Delta\lambda, \Delta\tau)$ is a series obtained through the Taylor expansion. The critical pair (λ_0, τ_0) may not be regular, in the sense that $\partial_\lambda D(\lambda_0, \tau_0) = 0$ or $\partial_\tau D(\lambda_0, \tau_0) = 0$. Let n be the number such that $\partial_\lambda^i D(\lambda_0, \tau_0) = 0$, for $i = 1, \ldots, n-1$ and $\partial_\lambda^n D(\lambda_0, \tau_0) \neq 0$. Also let g be such a number that $\partial_\tau^i D(\lambda_0, \tau_0) = 0$, for $i = 1, \ldots, g-1$ and $\partial_\tau^g D(\lambda_0, \tau_0) \neq 0$. Then there exits a positive number v such that the sequence $F(\Delta\lambda, \Delta\tau)$ determines the following v Puiseux series:

$$\Delta\lambda = \sum_{i=g_i}^{\infty} C_{ki} (\Delta\tau)^{\frac{i}{n_k}}, \quad k = 1, \cdots, v,$$

and $n_1 + \cdots n_v = n$. Conversely, we can also express the increase of τ from τ_0 as a series of $\Delta\lambda$, known as the duel

Puiseux series:

$$\Delta\tau = \sum_{i=v_i}^{\infty} D_{ki}(\Delta\lambda)^{\frac{i}{g_k}}, \quad k = 1, \cdots, v,$$

where $g_1 + \cdots + g_v = g$. Then it is easy to see that the curves of $(\lambda(\tau), \tau)$ in a small neighbourhood of (λ_0, τ_0) may have several branches. The local behaviors of these branches are fully characterized by these Puiseux series. The Puiseux series can be obtained based on the Newton polygon. A constructive algorithm for computing the Puiseux series can be found in [81]. Further discussion of the invariance properties can be found in [80]. Finally, strongly related to the newton diagram, some algorithmic procedure to construct the Weierstrass polynomial that captures all information corresponding to multiple roots has been proposed in [83] (see also [20]). We also note that when there are multiple parameters in the characteristic equation, it is not possible to locally expand the trajectory of a repeated characteristic root in the form of Puiseux series of multiple variables. See [1] for detailed analysis.

As mentioned in [22] and [23], the interest in characterizing the algebraic/geometric multiplicities corresponding to characteristic roots on the imaginary axis is emphasized, since such multiplicities characterize the local behavior of imaginary characteristic roots. A constructive approach to the multiplicity of crossing imaginary roots is proposed in [13] through a class of functional confluent Vandermonde matrices and a sharper bound on the multiplicity of imaginary characteristic roots is established. For analysis of multiple roots at the orign, see also [14] and [15].

IV. STABILITY ANALYSIS FOR SYSTEMS WITH DELAY DEPENDENT COEFFICIENTS

Despite the rich literature on time delay systems, very few results are available on general stability analysis of systems with delay-dependent coefficients. In articles focused on the modeling aspects of this type of systems, stability is usually investigated through numerical experiments. In [70] comprehensive study is conducted on a particular biological model. To the best knowledge of the authors, the first general method for stability analysis was proposed in [7], which generalizes an earlier work of Cooke and van den Driessche on systems with delay-independent coefficients [28].

In [7], Berreta and Kuang analyzed characteristic equations of the form (1). The same definition of F in (19) is used there, except that now $F(\omega, \tau)$ in general depends on τ:

$$F(\omega, \tau) = |P(j\omega, \tau)|^2 - |Q(j\omega, \tau)|^2. \tag{45}$$

The delay is restricted to some delay interval of interest denoted as $\mathcal{I} = [\tau^l, \tau^u]$. It is assumed that the number of unstable characteristic roots are known for $\tau = \tau^l$. Each function $P(\lambda, \tau)$ and $Q(\lambda, \tau)$ is polynomial in λ and analytic in τ. In the sequel, we may use $D_\tau(\lambda)$ and $F_\tau(\lambda)$ instead of $D(\lambda, \tau)$, $F(\lambda, \tau)$ to emphasis them as functions of λ for each fixed τ. The same convention apply to other functions parameterized by τ. It is assumed that each positive root of

$$F_\tau(\omega) = 0, \tag{46}$$

denoted as $\omega_k(\tau)$, $k = 1, \cdots, L$ is defined and differentiable in $\mathcal{I}$. If for some $\omega^* > 0$, $j\omega^*$ is an imaginary root of $D_\tau(\lambda)$ for some $\tau = \tau^*$, then there exists some k such that $\omega_k(\tau^*) = \omega^*$. Moreover, the following condition must hold at $\omega = \omega^*$, $\tau = \tau^*$:

$$\angle P(j\omega, \tau) - \angle Q(j\omega, \tau) = -\omega\tau + \pi + 2l\pi \tag{47}$$

where l is an integer. The last condition is transformed into the following equation:

$$S_{k,l}(\tau) := \tau - \frac{\theta_k(\tau) + 2l\pi}{\omega_k(\tau)} = 0, \tag{48}$$

where

$$\theta_k(\tau) = \angle\left(-\frac{P(j\omega_k(\tau), \tau)}{Q(j\omega_k(\tau), \tau)}\right) \tag{49}$$

is a differentiable function under some assumptions. Then the root crossing direction criteria (41) has to be modified to

$$\Re(\lambda'(\tau))|_{\tau=\tau^*} = \mathrm{sgn}(\partial_\omega F(\omega_k(\tau^*), \tau^*))\mathrm{sgn}(S'_{k,n}(\tau^*)). \tag{50}$$

The last equation shows that the invariance properties indicated in [28] no longer hold when the system coefficients depend on the delay. At any given critical delay τ^*, the roots of $D_{\tau^*}(\lambda)$ crosses the imaginary axis at $\pm j\omega_k(\tau^*)$, which is in general different for different τ^*. Since a critical delay τ^* must satisfy (48), a series of constant shifts from τ^* in general does not produce a series of critical delays. The invariance of crossing direction of critical delay does not hold either. Indeed, comparing (50) with (41), it is clear that the root crossing direction depends also on an extra factor, namely $\mathrm{sgn}(S'_{k,n}(\tau^*))$. Therefore it is possible that the crossing directions of imaginary roots associated with the frequency function $\omega_k(\tau)$ may switch at various critical delays.

The computation required by the method developed in [7] is fairly modest. Indeed, sweeping τ through the delay interval, one solves the polynomial equation (46) for τ at some discrete delay points to obtain a series of frequency functions $\omega_k(\tau)$, $k = 1, \cdots, L$, evaluated at these discrete delay points. Critical delays can be identified according to (48) by tracking the graph of each $S_{k,l}(\tau)$ function. Finally, the crossing direction of each characteristic root on the imaginary axis can be easily determined using the condition (50). This method has been applied to several hematopoietic dynamics model in [27] and the hopf bifurcation of blood cell production dynamics in [26].

In [54], some restrictive assumptions in [7] are relaxed and the stability analysis method is presented in a more systematic and general way. The most important relaxation in [54] is that each positive root of $F_\tau(\omega)$ is no longer assumed to exist in the entire interval $\mathcal{I}$. Instead, it is suggested to solve the following equations

$$\begin{cases} F(\omega, \tau) = 0, \\ \partial_\omega F(\omega, \tau) = 0, \end{cases} \tag{51}$$

for $(\omega, \tau) \in \mathbb{R}_+ \times \mathcal{I}$. Let $\tau^{(1)} < \cdots < \tau^{(L-1)}$ be all the τ contained in the solution of the last equation in τ. Define $\tau^{(0)} = \tau^l$ and $\tau^{(L)} = \tau^u$, where τ^l and τ^u are the upper and lower bounds of $\mathcal{I}$, respectively, i.e., $\mathcal{I} = [\tau^l, \tau^u]$. Then $\mathcal{I}$ can be decomposed into subintervals:

$$\mathcal{I} = \bigcup_{i=1}^{L} \mathcal{I}^{(i)}, \tag{52}$$

where $\mathcal{I}^{(i)} = [\tau^{(i-1)}, \tau^{(i)}]$. It is then proved in [54] that the number of positive roots of $F_\tau(\omega)$ is constant in $(\tau^{(i-1)}, \tau^i)$ and these roots are all simple. These functions in any given interval $\mathcal{I}^{(i)}$ are denoted as $\omega_k^{(i)}(\tau)$, $k = 1, \cdots, m(i)$. It is implicitly assumed in [7] that the integer L in (52) is just 1, or equivalently, (51) has no solution in $(\omega, \tau) \in (\mathbb{R}_+, \mathcal{I})$. Such an assumption can easily be violated in practice, as shown in [54] and [65]. We note that when the system coefficients are also polynomials of some function $g(\tau)$, then (51) is a set of polynomial equation in ω, $g(\tau)$. As a result, all solution pairs $(\omega, g(\tau))$ can be detected using the resultant theory.

The phase angle condition in [54] for finding critical delays and the crossing frequency also takes a different form than in [7]. In each given subinterval $\mathcal{I}^{(i)}$, the phase angle functions are defined as

$$\theta_k^{(i)}(\tau) = \angle P(j\omega_k^{(i)}(\tau), \tau) - \angle Q(j\omega_k^{(i)}(\tau), \tau) + \omega_k^{(i)}(\tau)\tau + \pi. \tag{53}$$

Here $\angle(\cdot)$ is a function that measures the angle of a complex number and is required to be continuous in $\mathcal{I}^{(i)}$. Consequently, its range is not restricted within any 2π interval. Then ω^* is a crossing frequency of any imaginary root of $D_\tau(\lambda)$ for some $\tau \in \mathcal{I}^{(i)}$ if and only if $\omega^* = \omega_k^{(i)}(\tau)$ and the following holds:

$$\theta_k^{(i)}(\tau) = 2l\pi, \quad l \text{ integer}. \tag{54}$$

The root crossing criterion is derived as

$$\operatorname{sgn}\left(\Re\left(\frac{d\lambda(\tau^*)}{d\tau}\right)\right) = \operatorname{sgn}\left(\partial_\omega F(\omega_k^{(i)}(\tau^*), \tau^*)\right) \times \operatorname{sgn}\left(\frac{d\theta_k^{(i)}(\tau^*)}{d\tau}\right). \tag{55}$$

Comparing the $S_{k,l}$ function defined in (48) with the phase functions defined in (53), several differences can be identified as follows: in each interval $\mathcal{I}^{(i)}$, only one phase function $\theta_k^{(i)}(\tau)$ is associated with each frequency function $\omega_k^{(i)}(\tau)$. On the other hand, a sequence of functions $S_{k,l}(\tau)$, $l = 0, 1, \cdots$ are associated with each frequency function. Moreover, on the boundary of $\mathcal{I}^{(i)}$, the functions $S_{k,1}(\tau)$ may grow unbounded if $\omega_k(\tau)$ approaches zero, while the functions $\theta_k^{(i)}(\tau)$ are always bounded in $\mathcal{I}^{(i)}$. In [7], to ensure $S_{k,l}(\tau)$ to be well defined, it is assumed that

$$P(j\omega, \tau) + Q(j\omega, \tau) \neq 0, \tag{56}$$

for $(\omega, \tau) \in \mathbb{R} \times \mathcal{I}$. By using the phase angle functions $\theta_k^{(i)}(\tau)$ instead, this assumption is relaxed in [54] to

$$|P(j\omega, \tau)| + |Q(j\omega, \tau)| \neq 0, \tag{57}$$

for $(\omega, \tau) \in \mathbb{R} \times \mathcal{I}$.

We illustrate the analysis procedure above with the stellar dynamos model mentioned in the Introduction. The system characteristic equation is given in (3). Therefore

$$P(\lambda, \tau) = \lambda^2 + 2c_2\lambda + c_2^2,$$
$$Q(\lambda, \tau) = -c_1c_3e^{-c_2\tau}.$$

The parameters are set as: $c_1 = -10$, $c_2 = 2$, $c_3 = 3$. Suppose we are concerned with the stability of the system for some delay interval $\mathcal{I}$ including the origin. Assume $\tau \in \mathcal{I} = [0, 2]$. The function $F(\omega, \tau)$ in this case is

$$F(\omega, \tau) = \omega^4 + 2c_2^2\omega^2 + c_2^4 - c_1^2c_3^2e^{-2c_2\tau}. \tag{58}$$

Only one pair of parameters $(\omega, \tau) = (0, \tau^{(1)})$ simultaneously satisfies (51), where

$$\tau^{(1)} = -\frac{1}{2c_2}\ln\left(\frac{c_2^4}{c_1^2c_3^2}\right) \approx 1.006.$$

The interval $\mathcal{I}$ is thus partitioned into two subintervals $\mathcal{I}^{(1)} = [\tau^{(0)}, \tau^{(1)}]$, $\mathcal{I}^{(2)} = [\tau^{(1)}, \tau^{(2)}]$, where $\tau^{(0)} = 0$, $\tau^{(2)} = 2$. There exits one positive real root $\omega_1^{(1)}(\tau)$ of $F_\tau(\omega)$ for $\tau \in (0, \tau^{(1)})$. As τ reaches $\tau^{(1)}$, this solution merges with the negative solution $-\omega_1^{(1)}(\tau)$, and they become complex as τ enters $\mathcal{I}^{(2)}$, and $F_\tau(\omega)$ does not have any real solution for τ in $\mathcal{I}^{(2)}$. In this case, we have

$$\omega_1^{(1)}(\tau) = \sqrt{|c_1c_3|e^{-c_2\tau} - c_2^2}.$$

Corresponding to $\omega = \omega_1^{(1)}(\tau)$, $\theta_1^{(1)}(\tau)$ is plotted against τ in the top diagram of Figure 1. It can be seen that the curve intersects the horizontal line 2π at $\tau_1 \approx 0.2748$ and $\tau_2 \approx 0.5314$. Therefore, $\omega_{11} = \omega_1^{(1)}(\tau_1) \approx 3.6490$, and $\omega_{21} = \omega_1^{(1)}(\tau_2) \approx 2.5228$.

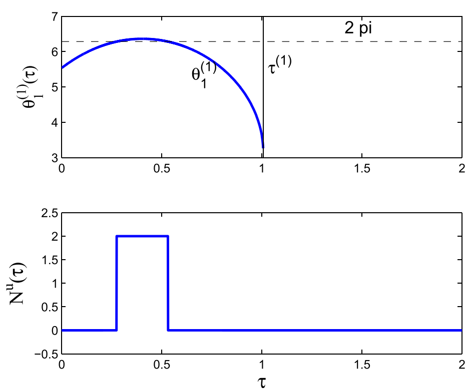

FIGURE 1. **The stability analysis of the stellar dynamos. The two intersections between the graph of $\theta_1^{(1)}$ and the black-dashed line located at 2π corresponds to the two delay values for which the system has a pair of imaginary roots. Here N^u is the number of unstable roots of the stellar dynamos.**

It can be easily calculated that

$$\frac{d}{d\tau}\theta_1^{(1)}(\tau_1) > 0, \quad \frac{d}{d\tau}\theta_1^{(1)}(\tau_2) < 0,$$

which are also obvious from the top diagram in Figure 1. Therefore, we conclude from (55) that a pair of characteristic roots crosses the imaginary axis from the left half-plane to the right half-plane as τ increases through τ_1, and this pair of characteristic roots returns to the left half-plane as τ further increases through τ_2. In other words, $\mathrm{Inc}(\omega_{11}, \tau_1) = 1$, and $\mathrm{Inc}(\omega_{21}, \tau_2) = -1$. Some simple calculation shows that the system is asymptotically stable for $\tau = 0$. A plot of $N^u(\tau)$ is shown in the bottom diagram of Figure 1, from which we conclude that the system is stable for $\tau \in [0, \tau_1) \cup (\tau_2, \tau^{(2)}]$; it is unstable for $\tau \in (\tau_1, \tau_2)$.

There are some limitations of the analysis in both [7] and [54]. The formula (55) does not give any information on the crossing direction if $\omega_k^{(i)}(\tau^*)$ is a multiple root of $F_\tau(\omega)$ or if $\frac{d\theta_k^{(i)}(\tau^*)}{d\tau} = 0$. It is shown in [54] that if $j\omega^*$ is a repeated imaginary root of $D_\tau(\omega)$, then ω^* must also be a repeated real root of $F_\tau(\omega)$. Therefore, the root crossing criterion (55) as well as (50) does not apply to imaginary characteristic roots with multiplicity larger than one. The analysis in [7] and [54] relies on the differentiability of $P(\lambda, \tau)$ and $Q(\lambda, \tau)$ w.r.t τ. However, in the light of these root crossing criteria, one may naturally ask whether the derivative of the phase angle functions can be replaced by its monotonicity at critical delays.

The analysis in [54] is further extended by the same authors in [65] to address some of the aforementioned limitation. It also gives an affirmative answer to the question above. The *two-parameter framework* proposed in [65] significantly simplifies the argument of the proofs and allows for an intuitive geometric interpretation of the root crossing problem. The geometric perspective is that the characteristic equation (1) with a single parameter τ can also be parameterized with two variables subject to some appropriate constraints. Stated more precisely, it is equivalent to

$$D_{rq}(\lambda) = P(\lambda, r) + Q(\lambda, r)e^{-\lambda q} = 0, \qquad (59)$$

where $r, q \in \mathcal{I}$, and r, q are restricted to the 45 degree line $r = q$ on the r-q plane. Given any $r \in \mathcal{I}$, there exists a sequence of critical delay values denoted by $\hat{\tau}_i(r) \geq 0$, $i = 1, 2, \cdots, \infty$ such that when $q = \hat{\tau}_i(r)$, (59) admits imaginary characteristic roots. The graph of each $\hat{\tau}_i(r)$ is said to be a critical delay curve. The r-q parameter space is thus separated into subregions r-q curves as the boundaries. The number of unstable characteristic roots is constant as long as (r, q) remains in the interior of a certain subregion.

This idea can be illustrated with the population dynamics (5). With the following set of parameters: $a = 0.8$, $b = 2.5$, $c = 0.12e^{-m_j\tau}$, $d = 0.2e^{-m_j\tau}$, $m_j = 0.1192$, the corresponding critical delay curves are plotted in Figure 2. The restriction $r = q = \tau$ means that the parameter $(r, q) = (\tau, \tau)$ moves along the 45 degree line marked by the green dashed lines. The green dashed line intersects the critical delay curves at points D and E. The characteristic equation admits imaginary roots for delay at these two points.

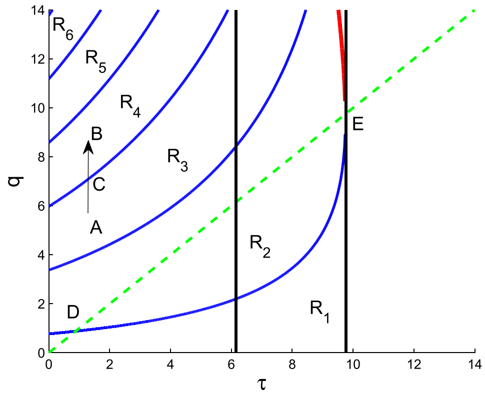

FIGURE 2. The critical delay curves of the population model (5). The characteristic equation admits imaginary roots if and only if the parameter point (r, q) is on these curves. Within each region enclosed by these curves, the number of unstable characteristic roots is a constant.

To determine the number of unstable characteristic roots, it is sufficient to know the number of unstable roots of (59) for (r, q) in different stability regions. This can be done as follows: suppose the number of unstable roots is known for the region R_3, and one needs to determine it in the region R_4. First pick a parameter point $A = (r_A, q_A) \in R_3$, fix $r = r_A$ and increases q from q_A until the parameter reaches point $B = (r_A, q_B) \in R_4$. At point $C = (r_A, q_C)$ on the critical delay curve separating R_3 and R_4, imaginary characteristic roots appear. The crossing directions of these roots can be determined using methods for systems with delay-free coefficients as q further increases from q_c since the parameter r inside system coefficients is fixed at r_A.

This geometric perspective leads to an intuitive interpretation of (55), which is elaborated in [54]. Roughly speaking, the term $\mathrm{sgn}(\partial_\omega F(j\omega_D, \tau_D))$ tells the crossing direction of this characteristic root if (r, q) moves vertically up entering subregions R_2 from R_1; the derivative of the phase angle function reflects whether (r, q) enters R_2 from R_1 or vise versa as it sweeps along the 45 degree line. Now it is easy to see that the crossing direction of a simple imaginary characteristic root is indeed determined by the product of two factors as shown in (55).

The difficulty encountered when analyzing (1) may come from the fact that if the parameter point (r, q) sweeps exactly along the line $r = q$, it can pass through some particular points where the analysis becomes challenging. With the two-parameter perspective, to determine the stability of (1) given any delay τ_0, one can find another path in the r-q plane that connects $(0, 0)$ and (τ_0, τ_0) while bypassing some "unfavorable" parameter points and thus facilitating root crossing analysis. In other words, it can be expected that with appropriately constructed paths, classical D-decomposition or τ-decomposition technique may be applied to systems with delay-dependent coefficients.

By exploiting the idea described above, the fundamental result developed in [65] only requires a continuous dependence of the system coefficients on the delay. The differentiability w.r.t the delay is no longer required. Several version of root crossing criteria are proposed there. The most general version also applies to imaginary characteristic roots with multiplicity larger than one. For a repeated imaginary root $j\omega^*$ and the corresponding delay $\tau^* \in \mathcal{I}$, several frequency functions $\omega_k^{(i)}(\tau)$ with different index k approach ω^* as τ approaches τ^*. Then the phase angle functions $\theta_\tau^{(i)}$ approach some $2l\pi$ horizontal line for some integer l as τ tends to τ^*. Then it is shown that the crossing directions are related to the position of these phase angle functions in relation to $2l\pi$. We do not state the precise result here since it requires some additional notations. Interested readers are referred to [65] for details. By applying such a criterion to a simple characteristic root, it can be confirmed that the term $\mathrm{sgn}(\theta_k^{(i)}(\tau^*))$ in (55) can indeed be replaced by a term that indicate the monotonicity of $\theta_k^{(i)}(\tau)$ at $\tau = \tau^*$. Furthermore, the following higher-order version of (55) is derived for (1) when the coefficients are sufficiently differentiable:

$$\mathrm{sgn}\left(\Re\left(\frac{d^l\lambda(\tau^*)}{d\tau^l}\right)\right) = \mathrm{sgn}\left(\partial_\omega F(\omega_k^{(i)}(\tau^*), \tau^*)\right) \times \mathrm{sgn}\left(\frac{d^l\theta_k^{(i)}(\tau^*)}{d\tau^l}\right), \quad (60)$$

for $l = 1, 2, \cdots, n_d$, where τ^* is a critical delay in the interior of $\mathcal{I}^{(i)}$, $j\omega_k^{(i)}(\tau^*)$ is an imaginary root of $D_{\tau^*}(\lambda)$, and n_d is the lowest order non-vanishing derivative of $\theta_k^{(i)}(\tau)$ at $\tau = \tau^*$. This high-order root crossing criterion is consistent with the version that makes use of the monotonicity of $\theta_k^{(i)}(\tau)$ at $\tau = \tau^*$.

The method developed in [54] can be generalized for systems with both commensurate delays and delay dependent-coefficients. In [63] (see also [62]), the characteristic equation with commensurate delays (13) is studied. The definition of the $F(\omega, \tau)$ function in this case is given by

$$F(\omega, \tau) = -\det(\mathrm{H}(j\omega, \tau)), \quad (61)$$

where the matrix H is given by (27) except that it now depends on τ since the system coefficients are delay-dependent. Using the new definition of $F(\omega, \tau)$, the frequency curves $\omega_k^{(i)}(\tau)$ and phase curves $\theta_k^{(i)}(\tau)$ can be defined in the same way as they are defined for systems with a single delay.

Let $(j\omega^*, \tau^*)$ be a critical pair. Let i, k be such numbers that $\tau^* \in \mathcal{I}^{(i)}$ and $\omega_k^{(i)}(\tau^*) = \omega^*$. Then (2) defines a simple characteristic root λ as a differentiable function of τ in a sufficiently small neighborhood of $(j\omega^*, \tau^*)$. Let n_d be such a number that $(\frac{d}{d\tau})^l\theta_k^{(i)}(\tau^*) = 0$, for $l = 1, 2, \cdots, n_d - 1$, then, under some assumptions realistic from an application point of view (see [65] for details), the following holds:

$$\mathrm{sgn}\left(\Re\left((\tfrac{d}{d\tau})^l\lambda(\tau^*)\right)\right) = (-1)^{N_x(j\omega^*,\tau^*)}\mathrm{sgn}\left(\partial_\omega F(\omega^*, \tau^*)\right) \times \mathrm{sgn}\left((\tfrac{d}{d\tau})^l\theta_k^{(i)}(\tau^*)\right), \quad (62)$$

for $l = 1, 2, \cdots, n_d$, where $N_x(j\omega^*, \tau^*)$ is the number of roots of $\hat{D}_{j\omega^*,\tau^*}(x)$ that are outside the unit disk $\mathbb{D}$.

The stability analysis methods in [54] are extended for control systems subject to *delay-difference approximation* in [66]. Let τ^u be the maximal possible delay value used in the delay-difference approximation scheme given by (9), it is of interest to know all the subintervals contained in $(0, \tau^u]$ such that the closed-loop system is exponentially stable with decay rate α, which is equivalent to the asymptotic stability of

$$P_\alpha(\lambda, \tau) + Q_\alpha(\lambda, \tau)e^{-\lambda\tau} = 0, \quad (63)$$

where

$$P_\alpha(\lambda, \tau) = G_0(\lambda - \alpha) + G_1(\lambda - \alpha)\tau^{-1},$$
$$Q_\alpha(\lambda, \tau) = G_1(\lambda - \alpha)e^{\alpha\tau}\tau^{-1}.$$

The results in [54] and [65] can not be directly applied to (63) since the system coefficients become unbounded as $\tau \to 0^+$. However, suppose the closed-loop system has no characteristic root with real part exactly equal to $-\alpha$, then it is shown in [66] that there exists $\tau^l > 0$ such that no root of (63) crosses the imaginary axis as τ varies in $(0, \tau^l)$. Therefore if τ^l is known, one can simply set $\mathcal{I} = [\tau^l, \tau^u]$ and apply the results from [54] and [65]. Such a τ^l value can be acquired by solving polynomial equations as shown in [66]. In some applications, it is necessary to use higher-order derivatives of the output to construct the feedback. Therefore, it is reasonable to consider higher-order derivatives approximated by multiple delays. Even for the first-order derivative, approximation using multiple delays and linear regression can be desirable for its capability of ameliorating the noise in the measurement. The detailed scheme for the approximation of higher-order derivatives are given in [62, Ch. 6] using the technique proposed in [94]. The idea is to interpolate the time history of the output signal with a polynomial function, then the derivatives of the output can be replaced by the derivatives of the interpolation polynomial as an approximation. This scheme leads to a closed-loop system of the form (2), which can be analyzed with the method proposed in [65].

V. CONCLUSIONS AND PERSPECTIVES

Systems with delay dependent coefficients have been encountered in various scientific and engineering disciplines, including biology, physics, social sciences, as well as control systems. An effective method for stability analysis of such systems is given in [7], which, following a generalized τ-decomposition approach, extends earlier work on systems with delay-independent coefficients. The analysis in [54] relaxes some restrictive assumptions of [7], and produces a stability analysis method that can be applied to a larger class of systems. In [65], some general results concerning the crossing direction of characteristic roots are presented for systems with a single delay. When applied to simple characteristic roots, this result leads to a crossing direction criterion that utilizes higher-order derivatives as well as monotonicity of phase angle functions as opposed to [7] and [54], where only the first-order derivative is taken into account. The two-parameter framework proposed in [65] suggests that by choosing an appropriate path in the two-parameter plane, "classical" *D*-decomposition methods may be applicable to systems with delay-dependent coefficients. The stability analysis methods developed in these papers can be used to determine the number of unstable characteristic roots based on the graph of some functions. Such a geometric correlation might be exploitable to solve other challenging problems, such as the robust stability analysis of systems in which the coefficients do not only depend on the delay, but also contain uncertainties.

A lot of work remains to be done for systems with delay-dependent coefficients. The reported methods in [7], [54], [62], [63], and [65] are only concerned with systems with commensurate delays. Extensions to systems with multiple independent delay parameters and incommensurate delays remain to be done. Robust stability analysis for systems with also parameter uncertainties will be useful in practice. The asymptotic behavior analysis of multiple imaginary characteristic roots have been solved using perturbation methods [22], [23], [95] and the Puiseux series along with the frequency-sweeping framework [82] for systems with fixed coefficients. The extension of these methods to systems with delay-dependent coefficients would also be an interesting research topic in the future.

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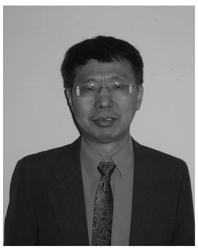

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