On the Exact Reliability Enhancements of Small Hammock Networks

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ABSTRACT In this paper, we study small hammock networks, more precisely the 29 hammock networks presented by Moore and Shannon in their prescient paper from 1956, where this type of network was introduced. We will first of all review the concept of the reliability polynomial, and define hammock networks, and emphasize some of their properties. We will then review the state of the art from three different angles: theoretical, algorithmic, and design oriented. These will show that results obtained over many years advocate very strongly for the small hammock networks. Because array-based designs (e.g., FinFETs) seem to be a perfect fit for small hammocks (as are nanoelectromechanical systems or NEMS), we have decided to start investigating them closely. The analyses we pursue here are exact, the coefficients of the reliability polynomials being determined using our own program. The reliability polynomials for non-trivial hammock networks are reported here as functions of the probability (p) that the device is closed. As far as we know, this is the first time they have appeared in this form. Our purpose in determining exact results is a practical one, as we foresee that small hammock networks could play a key role in any array-based design. That is why their reliability enhancements are evaluated thoroughly, using both classical and non-standard cost functions. Finally, conclusions are followed by a longer list of future directions of research, some of them practical while others more theoretical.

INDEX TERMS Hammock network, minimal network, network topology, redundancy, reliability, two-terminal network.

I. INTRODUCTION

In [1] Moore and Shannon studied improvements in reliability obtained by replacing a single unreliable two-contact electromechanical relay (switching device) by a (redundant) network of identical such relays, the network having two distinguished contacts/terminals: an input or source S, and an output or terminus T. Nowadays, the most common switching device is a transistor, namely the complementary metal-oxide-semiconductor (CMOS) transistors used in present-day computers and portable devices. In this paper we shall stick to the original description and notations with a few exceptions (which we will stress), while including in parentheses CMOS interpretations. For the sake of argument, we will assume that each relay in the network is a "break contact" (i.e., with $a < c$). In a "normally closed" relay, a is the

probability of a so-called "stick failure," and $1-c$ is the probability of a so-called "miss failure." By analogy with Science and Medicine, if one regards the openness of the contacts of a normally closed relay as a test for the coil being energized, then $1 - a$ represents the "sensitivity" of the test (the "true positive" rate) and c represents the "specificity" of the test (the "true negative" rate). Note that the examples in [2] are break contacts, whereas the example on page 194 of [1] is a "make contact" (i.e., with $c < a$). In a "normally closed" type relay (corresponding to a pMOS transistor), when the coil (the gate of the transistor) is not energized ("0" on the gate of a pMOS transistor), current is supposed to be able to flow between the two contacts, and when the coil is energized ("1" on the gate of a pMOS transistor), current is supposed not to be able to flow between the two contacts.

In a "normally open" type relay (corresponding to an nMOS transistor), the opposite holds. *Note that—unless they are extremely unreliable—a normally closed relay will be a break contact, and a normally open relay will be a make contact.* We will assume that the coils of all the relays in the network are energized, or not energized, simultaneously (i.e., the gates of all the transistors are tied together, as is common for FinFETs). When the coils are energized, the contacts of each relay remain closed with probability a, independently of each other. When the coils are not energized, the contacts of each relay remain closed with probability c, independently of each other. By symmetry, we will omit the case of a make contact, without loss of generality (i.e., we shall consider only relays working like unreliable pMOS transistors). In the original article, Moore and Shannon called such relays "crummy relays." Their goal was to design a two-terminal network $\mathbb{N}$ of n crummy relays which, with probability greater than c, admits a short-circuit between S and T when the coils are not energized ("0" on all the gates), and which, with probability less than a, admits a short-circuit between S and T when the coils are energized ("1" on all the gates). Such a network $\mathbb{N}$ behaves like a single break contact relay but with better a and c values (i.e., a closer to 0 and c closer to 1 than for the individual crummy relays). In other words, compared with a single relay (pMOS transistor), such a network $\mathbb{N}$ of crummy relays is more reliably closed when it should be closed, and more reliably open when it should be open.

We will represent the n crummy relays in the network $\mathbb{N}$ by a sequence of independent identically distributed (i.i.d.) random variables taking the values "closed" with probability p and "open" with probability $q = 1 - p$. Obviously, the state (either "closed" or "open") of the network $\mathbb{N}$ is a Boolean function of those random variables, and hence is itself a random variable. Moreover, the probability that the network $\mathbb{N}$ is closed is a function of p, which we will denote $h(p)$, and which is known as the reliability polynomial (of degree n).

Moore and Shannon proved that the curve $y = h(p)$ can cross the line $y = p$ at no more than one particular $p = p_0 \in (0, 1)$; see Fig. 1 (a). The ideal $h(p)$ is a Heaviside step function transitioning from 0 to 1 at p_0, with $p_0 \in (a, c)$; see Fig. 1 (b). However, $h(p)$ is a polynomial in p, and can only approximate a step function. As an example, the graphs in Fig. 1 (c) show better and better approximations to the ideal.

Larger networks can be formed from two smaller networks by connecting them in series or in parallel, or by composing them (i.e., using one network as the basic element of the other network). Moore and Shannon showed that the composition of networks improves $h(p)$, and if iterating composition with itself i times we get in the limit

$$\lim_{i \to \infty} h^{(i)}(p) = \begin{cases} 1 & p > p_0 \\ p_0 & p = p_0 \\ 0 & p < p_0, \end{cases} \quad (1)$$

i.e., the ideal Heaviside step function. In fact, the graphs in Fig. 1 (c) include the first three iterated

compositions—i.e., $h^{(1)}(p)$, $h^{(2)}(p)$, and $h^{(3)}(p)$—of $h(p)$, which is also plotted both in Fig 1 (a) and in Fig. 1 (c). The very practical issue is how we should select an $h(p)$, corresponding to a (very) small $\mathbb{N}$ (i.e., having n small), which would need (very) few iterations i to achieve a good approximation to the Heaviside step function. At the outset, one should know $h(p)$ as precisely as possible (ideally, exactly).

In [3] several forms for $h(p)$ are mentioned, but it took till the mid 80's [4] to have them settled, two forms being

$$h(p) = \sum_{k=l}^{n} P_k p^k, \quad (2)$$

$$h(p) = f(p, q) = \sum_{k=l}^{n} N_k p^k q^{n-k} \quad (3)$$

where $q = 1 - p$. N_k is the number of ways one can select a subset of k contacts in the network $\mathbb{N}$ such that if these k contacts are closed and the remaining $n-k$ contacts are open, then the network $\mathbb{N}$ will be closed. Obviously, eq. (3) sums the probabilities of all the different ways in which the network $\mathbb{N}$ can be closed. By replacing $q = 1 - p$ in eq. (3), and expanding, one gets eq. (2). It also follows that the two vectors of coefficients $\vec{P} = (P_0, \ldots, P_n)$ and $\vec{N} = (N_0, \ldots, N_n)$ correspond uniquely, according to the linear system

$$\vec{P} = M\vec{N}, \quad (4)$$

where $P_0 = P_1 = \cdots = P_{l-1} = 0$, $N_0 = N_1 = \cdots = N_{l-1} = 0$, and $P_l = N_l \neq 0$, and M is the invertible $(n+1) \times (n+1)$ lower triangular matrix given by

$$M_{i,j} = (-1)^{i-j} \binom{n+1-j}{i-j}, \quad 1 \leq j \leq i \leq n+1. \quad (5)$$

Another common form is

$$h(p) = g(p, q) = 1 - \sum_{k=w}^{n} C_k q^k p^{n-k}, \quad (6)$$

while, by replacing $p = 1 - q$ in any of eqs. (2), (3) or (6) and expanding, one gets a reliability polynomial in q

$$h(q) = \sum_{k=w}^{n} H_k q^k. \quad (7)$$

We want to stress that these notations for the reliability polynomials $h(p)$ differ from those used in the original article, namely that N_k (eq. (3)) and C_k (eq. (6)), which are forms from [4], are used instead of A_k ([1, eq. (1)]) and, respectively, B_k ([1, eq. (2)]), used in the original Moore and Shannon article [1]. Additionally, it is to be mentioned that

$$N_k + C_{n-k} = \binom{n}{k} = \frac{n!}{k!(n-k)!}. \quad (8)$$

Intuitively, the more parallel $\mathbb{N}$ is, the more reliable it is in closed-circuit mode, and the more serial $\mathbb{N}$ is, the more reliable it is in open circuit mode. As one would expect, the quality of "parallel-ness" is associated with the width w, which is the least number of relays which must be opened in order for the network to be opened (more concisely, w is the size of a "minimal cut" separating S from T). Similarly,
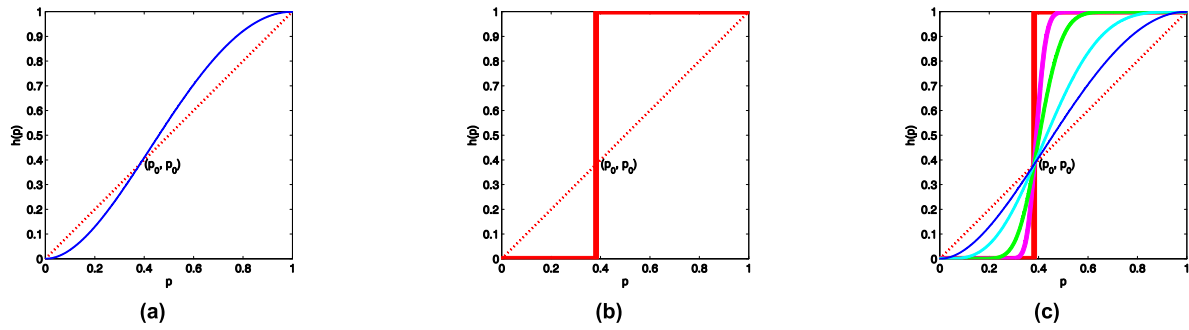
FIGURE 1. Reliability polynomial $h(p)$: (a) smooth; (b) Heaviside; (c) different sigmoids.
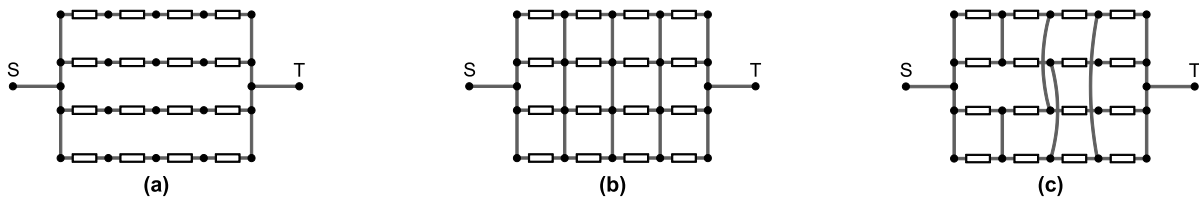

FIGURE 2. Minimal $N_{4,4}$: (a) parallel-of-series (planar); (b) series-of-parallel (planar); (c) a non-planar network.

the quality of "serial-ness" is associated with the length l, which is the least number of relays which must be closed in order for the network to be closed (concisely, l is the length of a "minimal path" from S to T). This explains why $N_0 = N_1 = \cdots = N_{l-1} = 0$, the first non-zero coefficient being N_l, the number of minimal paths from S to T. Moore and Shannon [1] make these two intuitions more precise by using eqs. (3) and (6) to show that for p close to 0, $h(p)$ behaves as $P_l p^l$ (as $P_l = N_l$). Similarly, for p close to 1, $h(p)$ behaves as $1 - C_w q^w$, C_w being the number of minimal cuts separating S from T. Therefore, a large value of l tends to produce a more rapid decay of $h(p)$ towards 0 as p tends to 0, while a large value of w tends to produce a more rapid approach of $h(p)$ towards 1 as p tends to 1. In this sense, larger values of both l and w will yield an $h(p)$ curve closer to the ideal step function of Fig. 1 (b).

However, there is a cost represented by the number of relays $n \geq wl$ (see [1, Th. 3]). This motivates the definition of a minimal network $\mathcal{N}$, which is one having $n = wl$ exactly. In [1, Th. 4] the minimal networks $\mathcal{N}$ of width w and length l are characterized as those obtained from a parallel combination of w series combinations of l relays (Fig. 2 (a)) by adding connections between specific points in the network, in a particular way. Namely, considering the column of w points at which a given column of relays is connected to an adjacent column of relays, connections between those points are to be made arbitrarily, and the same goes for all $l - 1$ such columns. Therefore, without accounting for symmetries, there are $(\mathcal{B}_w)^{l-1}$ minimal networks of width w and length l, where $\mathcal{B}_w$ is the w-th Bell number (i.e., the number of partitions of a set of w elements). At the two extremes, no vertical connections are made (leading to a

parallel-of-series or PoS as in Fig. 2 (a)), and respectively all vertical connections are made (leading to a series-of-parallel or SoP as in Fig. 2 (b)). By allowing the vertical connections to "skip over" horizontal rails, non-planar minimal networks $\mathcal{N}$ can be obtained, i.e., minimal networks $\mathcal{N}$ which cannot be realized in 2 dimensions (2D) without an edge crossing (see Fig. 2 (c) for an example).

In **Section II** we will describe a special class of minimal networks $H_{w,l}$ introduced in [1] as "hammock networks". **Section III** will follow, with a review of the current state-of-the-art on network reliability. **Section IV** will present the exact reliability polynomials $h(p)$ for the 29 hammock networks $H_{w,l}$ detailed in [1] (for $w \leq 5$ and $l \leq 5$). These exact $h(p)$ reliability polynomials will be used for comparative simulations. The simulations will first of all show the absolute improvements that the hammocks bring to reliability, but will also reveal comparisons based on a figure-of-merit (*FoM*) accounting for both *reliability improvements* and *costs* (defined as the redundancy factors). Afterwards, we will also analyze $H_{w,l}$ networks by the steepness of $h(p)$ (by computing $h'(p)$), as well as by Shannon's information capacity C (by considering $H_{w,l}$ networks as binary channels). These analyses will reveal where and by how much $H_{w,l}$ can augment steepness and C. Finally, we will present a straightforward extension of $H_{w,l}$ in which we map the network onto a cylinder. We call the resulting networks cylindrical (2.5D) hammock networks. In so doing, we are taking steps towards a practical view of the benefits of small hammock networks, which seem to be a perfect fit for array-based design (including FinFETs), and obviously for nano-electromechanical systems (NEMS). These will be followed by conclusions and further directions of research.

II. THE SOMEHOW FORGOTTEN HAMMOCKS

During the evolution of computer technology, the electromechanical relays discussed by Moore and Shannon were replaced first by vacuum tubes, then by discrete transistors, and afterwards by integrated CMOS transistors.

The planar minimal networks known as $H_{w,l}$, on which we have focused our study, can all be realized starting from networks like the ones in Fig. 2 (a) by connecting pairs of nodes which are vertically nearby neighbors (i.e., adjacent) by short (of length equal to the gap between the horizontal rails) vertical wires. Informally, we shall call such short vertical wires 'matchsticks'. It seems that Moore and Shannon considered only those minimal networks which can be built using such matchsticks. Of these, without accounting for symmetries, there are $2^{(w-1)(l-1)}$ of width w and length l. This is many fewer than $(\mathcal{B}_w)^{l-1}$, the number of minimal networks of width w and length l. It is not clear whether Moore and Shannon realized that they were neglecting the majority of the networks characterized by their Theorem 4. Besides the two extreme cases of no matchsticks (Fig. 2 (a)) and all the $(w - 1) \times (l - 1)$ matchsticks (Fig. 2 (b)), Moore and Shannon [1] consider an intermediate particular case. In this particular case, of all the possible $(w - 1) \times (l - 1)$ matchsticks, half are present and the other half are absent. The $\frac{(w-1)(l-1)}{2}$ matchsticks are arranged regularly in an alternate way which gives rise to the well-known 'brick-wall' pattern shown in Fig. 3. Moore and Shannon named these special cases of minimal planar networks 'hammocks' ($H_{w,l}$) from their appearance when nodes S and T are pulled apart and every matchstick collapses into a node—as rectangles deform into rhombs.
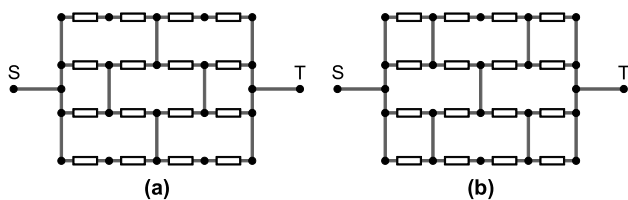

FIGURE 3. The two hammocks for $w = 4$ and $l = 4$: (a) $H_{4,4}$; and (b) $H_{4,4}^{+}$.

In the case when w and l are both even (*i.e.*, $w = 2i$ and $l = 2j$) two distinct cases are possible. This is because both $w - 1$ and $l - 1$ are odd so $\frac{(w-1)(l-1)}{2}$ is not an integer. In this paper, the hammock with $\left\lfloor \frac{(w-1)(l-1)}{2} \right\rfloor$ matchsticks will be denoted $H_{w,l}$, while the one having $\left\lceil \frac{(w-1)(l-1)}{2} \right\rceil$ matchsticks will be denoted $H_{w,l}^{+}$ (having exactly one matchstick more than $H_{w,l}$). Fig. 3 (a) shows $H_{4,4}$ having 4 matchsticks, while Fig. 3 (b) shows $H_{4,4}^{+}$ which has 5 matchsticks.

III. TWO-TERMINAL NETWORKS

A brief review of the field of reliability starting from the mid-50's (when Moore and Shannon introduced $H_{w,l}$ [1]) was presented by Ushakov in 2000 [5] (followed in 2004 by [6]). Among other problems, Ushakov's review examines the two-terminal network reliability problem, emphasizing the importance of the so-called S-shaped dependence of $h(p)$ discussed by Moore and Shannon [1]. It has been established more than once (see later under **III. C. DESIGN**) that using very small $H_{w,l}$ as a device-level redundancy scheme leads to much more reliable systems than: triple modular redundancy, von Neumann multiplexing, and even reconfiguration—at similar redundancy factors (i.e., the multiplicative increase in the number of identical elements/devices used to replace one element/device).

The main research directions following on the prescient [1] have been: (i) theoretical, (ii) algorithmic, and (iii) design oriented.

A. THEORETICAL

A large body of work has been dedicated to computing two-terminal connection probabilities or two-terminal network reliabilities $h(p)$. It was established early on that in general there are no simple solutions [7]–[9], the two-terminal reliability problem being a #P-complete problem [7]. This means it is at least as hard as the NP-complete problems [7]–[9]. Still, exact analytical solutions for the reliabilities $h(p)$ of particular ladder networks of arbitrary size, affected by faulty nodes and/or edges, have been detailed [4], [10]–[13]. The methods used to obtain such results work by simplifying the network through pivotal decomposition or deletion-contraction. Along with classical series-parallel and triangle-star transformations, simplifications are repeated backwards (from T to S) taking advantage of the recursive and regular structure of the network. For particular ladder networks (e.g., simple, symmetric, Brecht-Colbourn, fan, double fan, K_4, K_3 cylinders, Manhattan $3 \times n$ grid) the associated two-terminal network reliability $h(p)$ was calculated exactly, using a product of transfer matrices M_i, the elements of M_i being polynomials in the individual edge and node reliabilities:

$$R_n = v_L \times M_n M_{n-1} \ldots M_1 M_0 \times v_R. \quad (9)$$

In [13] it was suggested how to extend this approach to arbitrarily wide networks. In fact, for networks represented by recursive families of graphs, the two-terminal reliability may be expressed exactly as a product of transfer matrices, where individual edge and node reliabilities are taken into account.

It is to be mentioned that similar formulas were determined in the late 1980's in the study of the reliability of consecutive-k-out-of-$n : F$ systems [14]–[16] using the Markov chain method:

$$R(k, n; q) = (1, 0, \ldots, 0) \times M^n \times (1, \ldots, 1, 0)^t. \quad (10)$$

Publications such as [4], [10]–[13] also analyzed in depth the resulting reliability polynomials $h(p)$. A few years later, Graver and Sobel [17] reiterated the importance of the reliability polynomials $h(p)$, which provide a rich theoretical basis, for example, for a study of the effects of structural properties on diffusion dynamics [18].

Trying to reduce the intrinsic computational burden of the problem, a particular theoretical approach has focused on determining bounds. Their accuracy is obviously of critical importance [19].

Finally, turning our attention toward algorithmic approaches, Jane *et al.* [20] have presented an exact algorithm and modified it for obtaining a practical bounding algorithm. This has led to both lower and upper bounds on two-terminal network reliability, and allows a trade-off between execution time and accuracy (if the algorithm runs to the end the results are exact).

B. ALGORITHMIC

The early papers on exactly computing $h(p)$, such as, e.g., [21], relied on the inclusion-exclusion principle applied to the union of all paths from S to T, and considered both the reliability of vertices as well as edges. Similarly, [22] considered the reliability of vertices and edges (when the failure probabilities of the elements need not be equal, but the elements are still independent), as well as both directed and undirected graphs. The authors pointed out that direct use of the inclusion-exclusion principle requires checking of a number of cases exponential in the number of paths from S to T, which is obviously unfeasible for large $\mathcal{N}$. However, by discarding many irrelevant cases, they were able to reduce the necessary amount of computation to the point where $\mathcal{N}$ with up to 20 paths between S and T could be solved. Continuing on the theme of reducing algorithmic complexity, [23] emphasized that prior factoring methods were exponentially more complex than similar methods for the edge-only case. They introduced a new method in which the expense of including unreliable vertices is less than exponential, albeit by assuming that elements are independent. Similarly, after classifying exact reliability algorithms as decomposition or factoring, inclusion-exclusion or sum of disjoint products, [24] credited [7] for showing how to solve the two-terminal case in time quadratic in the number of minimal cuts. A few years later, Suh and Chang [25] presented two algorithms for enumerating the minimal cuts of $\mathcal{N}$. The one that uses the branch addition method achieves a time complexity linear in the product between the number of edges and the number of vertices. A new algorithm for trimming irrelevant "culs de sac" from $\mathcal{N}$ was introduced in [26]. It was shown that pre-processing $\mathcal{N}$ in this way can save considerable time. The algorithm for undirected graphs (such as are considered in this paper) was shown to be linear in the sum of the number of edges and the number of vertices.

After it was established that computing $h(p)$ exactly can be quite taxing [7]–[9], approximate methods began to attract attention. One example is [27] where the authors considered a network $\mathcal{N}$ of unreliable elements with two failure modes, and advocated for randomly seeded genetic algorithms to find approximate optimal solutions in reasonable time. Other examples are [28] and [29], which also classified exact reliability computation algorithms for $\mathcal{N}$ as factoring/decomposition and minimal path/cut enumeration.

They reviewed many of the methods for exactly computing $h(p)$ including: brute-force enumeration, sum of disjoint products, inclusion-exclusion, reductions and decompositions, and binary tree methods. The conclusion was that for large networks none of these methods is feasible. This has stimulated research on approximate methods (e.g., Monte Carlo) and studies on upper and lower bounds. In [28], the Boolean function giving the state of the network was represented as a function of the states of the devices by a so-called "Propositional Directed Acyclic Graph," leading to a 'structure function' for the reliability, which is agnostic as to the choice of reliabilities for the individual devices. In [29] it was suggested to optimize Monte Carlo by using methods from machine learning. The main contribution of [29] was an efficient binary tree algorithm (based on a heuristic search for the most important minimal cuts) for upper and lower bounds on $h(p)$, which allowed the user to specify the execution time, and thus the accuracy of the bounds. Further progress was made in [30], which used an ordered subset of the minimal cuts and another ordered subset of the minimal paths to calculate an all-terminal reliability upper and lower bound, respectively.

Finally, a fresh publication [31] has proposed an efficient strategy for calculating the two-terminal reliability of a binary-state network (based on logical-probabilistic calculus). It is probably the first parallel implementation on multi-core processors for estimating two-terminal reliability. With the advent of many cores, graphical processing units (GPUs), and parallel software we expect to see more progress on these lines in the future.

C. DESIGN

The reviews [5] and [6] were theoretically-oriented and did not delve into design-oriented reliability concerns such as variations (already large and expected to grow), aging (devices are aging faster), temperature (accelerates aging), and soft errors (their rates are increasing). These factors are exactly the ones looked at by Henkel *et al.* [32]. Such factors would need to be tackled accurately through an integrated CAD (computer aided design) flow for design, analyses, and optimizations. In particular, the interdependence/correlations of various reliability factors should be properly accounted for, to avoid pessimistic overdesign. In fact, reliability has been recognized by ITRS (International Technology Roadmap for Semiconductors) as a major design challenge [33] since about a decade ago. On one side, most hardware redundant design solutions have focused on the Boolean gate, circuit and system levels (on error sensing, detection, and masking) [32]. On the other side, for practical design purposes Moore and Shannon articles [1] and [2] have made a very strong case for using small hammock $H_{w,l}$ networks at the device-level (i.e., the lowest level), as being by far the most effective solution. Hammock reliability polynomials were computed exactly for series, parallel, $H_{2,2}$ and $H_{2,2}^+$ in [1], and $H_{3,3}$ in [2], but it is Creveling [34] who *made the case for 'quad'* ($H_{2,2}$) *even before (by a few months) hammocks were formally*

introduced by Moore and Shannon. Creveling considered $H_{2,2}$ networks of diodes, and analyzed the probabilities of two different failures: open circuit and short circuit.

An extension of the results of Moore and Shannon to the case of permanent failures was presented in [35] and [36]. The analysis was done both for $H_{2,2}^+$ (2×2 SoP) in [35] and [36] as well as for $H_{2,2}$ (2×2 PoS) in [36]. These have considered both a and c, and also the fact that these are varying in time. *The mean time to failure (MTTF) of such structures was evaluated for the first time.*

Sorensen [37] is most certainly the first paper taking the concepts of Moore and Shannon at heart and working out detailed designs for several device-level redundant circuits: a flip-flop, a reversible counter, and an AND-3 gate. All of these were solid-state switching networks using discrete diodes, bipolar transistors, resistances and capacitors. The same concepts were used to design a NAND gate in 1964 [38]. Different probabilities for ''good,'' ''short,'' and ''open'' were considered as well as the fact that various devices fail in different ways. That is why three different redundancy schemes have been used: series ($H_{1,2}$), parallel ($H_{2,1}$), and quad as 2×2 SoP ($H_{2,2}^+$). The ensuing analyses were very detailed, taking time into consideration. For one year of operation the device-level redundant circuits are more reliable than the non-redundant ones: the flip-flop $300\times$, the reversible counter $3300\times$, and the AND-3 gate $620\times$. Similar analyses were performed for circuit-level triple modular redundancy (TMR). *TMR also improved reliability, but about $5\times$ less than when using hammocks, while needing about 25% more devices and dissipating $2.3\times$ more energy.* The conclusion was clearly that *using hammocks at the device-level is superior to circuit-level redundancy as far as reliability improvement, weight, size, number of parts and power are concerned.* These have made the case for hammocks very strong. Still, hammocks have two drawbacks: firstly that it is far from trivial to design with them (even when using just $H_{2,2}$ and $H_{2,2}^+$), and secondly that the resulting circuits are very difficult to test. Later, Bolchini *et al.* [39] used the same $H_{2,2}^+$ quad (2×2 SoP transistor structure) in the design of CMOS gates, showing (once again) that *the $H_{2,2}^+$ quad always achieves better reliability than TMR structures, even if the voter is assumed to be perfect (i.e., fault-free).*

Anghel and Nicolaidis [40] revisited $H_{2,2}$ and $H_{2,2}^+$ at the device-level (CMOS transistors) for INV and NAND-2, while also showing how such concepts could be intermingled at the gate-level by quadruplicating the wires and the gates (see also [41]). This blends device-level with gate-level reliability schemes, and was used for INV, NAND-2, flip-flops, and memory cells. Each CMOS transistor was replaced by 4 transistors (either $H_{2,2}$ or $H_{2,2}^+$), and each gate was replaced by 4 gates, so the overall redundancy factor ended up being $4 \times 4 = 16$. Power and delay have been obtained from simulations of the layout in $0.18\,\mu m$ CMOS. The fault tolerance was obtained analytically at the logic level. For transistor defect densities of 1% it was shown that this quad-quad scheme (at a redundancy significantly

less than modular redundancy, von Neumann multiplexing, and even reconfiguration) improves reliability by $5\times$ for NAND-2 and by $30\times$ for MUX-2. The idea of combining hammocks at the device-level (quadded transistors) with a gate-level redundancy scheme (quadded logic) was revisited in [42] and [43]. Both articles showed that better reliability than TMR or triple interwoven redundancy (or both of them combined), is obtained on the ISCAS'85 benchmark circuits.

The use of $H_{2,2}^+$ in output-wired (ganged logic) MIN-3 gates was analyzed in [44]. These MIN-3 gates were simulated in 22nm PTM (predictive technology model, http://ptm.asu.edu/). This was probably *the first paper presenting Monte Carlo simulations* of gates using quad-connected CMOS transistors (instead of analytical analyses). Almost simultaneously [45] used $H_{2,2}$ (calling it S2× P2 as having 2 transistors in series and 2 in parallel, i.e., SoP as in Fig. 2 (a)) in an AND gate with pull-up resistor (either with nMOS transistors and negative-logic, or pMOS transistors and positive-logic). These AND gates were used to realize k-to-2^k DEMUX (k-bit inputs and 2^k output lines). Robinett *et al.* [45] relied on a detailed analytical analysis to compare different implementations of a 1-to-64 DEMUX, which showed that *at transistor defect probabilities of 1% the reliability is increased from 2.2% to 94.5% when using $H_{2,2}$.* They also suggested extending $H_{2,2}$ to larger SoP networks (like Fig. 2 (a)). Shortly afterwards, El-Maleh *et al.* [46] designed both classical and quad (both $H_{2,2}$ and $H_{2,2}^+$) INV, NAND-2, and NOR-2 gates in a $0.5\,\mu m$ CMOS process. Delay and power estimates were obtained from simulations, while circuit failure probabilities were obtained using both Monte Carlo and analytic approaches. Several of the classical ISCAS'85 benchmark circuits were compared, showing (once more) that *$H_{2,2}$ and $H_{2,2}^+$ structure are the best for enhancing reliability (much better than TMR and triple interwoven redundancy).* The paper also suggested replacing $H_{2,2}$ and $H_{2,2}^+$ by an n^2-transistor structure (an $n \times n$ SoP network, like Fig. 2 (b)), for achieving even better reliabilities. Recently, this approach has been reevaluated, for a combination of duplication with asymmetric transistor sizing [47].

Finally, [48], [49] presented new design approaches in standard CMOS for variability hardened digital CMOS cells. These targeted sub-threshold operation (for reducing power) in 40nm CMOS. The impact of various nanometer effects was quantified using dedicated analyses for random doping fluctuations, inverse narrow-width effects, and short-channel effects. These suggested that *many narrow channel devices are more efficient than a single wide device*, implicitly advocating for fingered devices. The concept of using arrays of minimum size devices is in fact older [50] and the benefits of segmented bulk MOSFET (SegFET) were appreciated even earlier [51]. Obviously, the proposed vertical FET (VFET, see Fig. 4 (a)), vertical slit FET (VeSFET) [52], as well as FinFETs [53], [54] and arrays of beyond CMOS devices [55] (e.g., various types of nanotubes [56]), and also
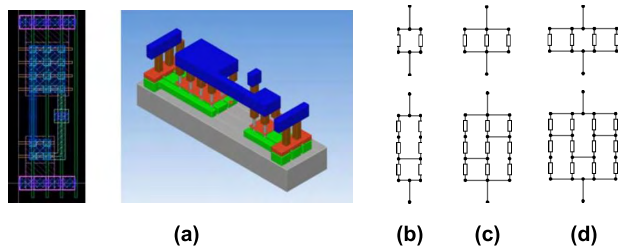
FIGURE 4. (a) Inverter using VFETs (©imec, Scientific Report 2010/ 1159089); three possible solutions (with $R_{pMOS} \cong 3R_{nMOS}$ from Fig. 4(a)): (b) $H_{2,1}$ and $H_{2,3}$; (c) $H_{3,1}$ and $H_{3,3}$; (d) $H_{4,1}$ and $H_{4,3}$.

NEMS, could very easily take advantage of small hammock networks.

The preceding cases make it clear that an accurate evaluation of the reliability enhancements conferred by small hammock networks is timely, and is potentially beneficial to advanced FinFETs as well as to beyond CMOS array-based designs. However, the present paper is not immediately intended for practical design purposes, but rather for theoretical and mathematical insights into the problem. That is why we show results for idealized devices (relays/transistors) whose failures can be described by probabilities (modeling intermittent faults) which do not vary with time (hence which ignore ageing). On the other hand, the results reported here are for small hammock networks and are exact, which should be of practical interest. In this regard, we plan to follow up by analyzing simple hammock-based designs for FinFETs (see Fig. 4 (b)). Such an approach would allow for an accurate analytical estimate of the probability of an inverter P_{INV} as a function of p (see [57]). This should be evaluated against:

- approximations like $P_{INV} = exp\left[-2V_{DD}/\sigma\left(V_{th}\right)\right]$ from [58] (which would link p to the supply voltage V_{DD} and the standard deviation of the threshold voltage V_{th});
- detailed Monte Carlo simulations.

IV. EXACT RESULTS FOR SMALL HAMMOCKS

All the computations were performed using Mathematica ver. 9.0.0.0 running on an HP Z840 workstation, sometimes in parallel on some or all of the 24 kernels available. Preliminary results were reported in [59], while more details as well as results concerning channel capacity will be presented here.

Our first algorithm was our simplest. We considered each of the 2^n possible states of the network, corresponding to the possible choices of "open" or "closed" for each relay (i.e., a brute force approach). We tested each state for connectedness of S and T, and obtained $h(p) = f(p,q)$ as the sum of $p^k q^{n-k}$ over all states passing this test. The test for connectedness of S and T was as follows. Terminals S and T are connected if and only if $((I_m + J)^n)_{1,m} > 0$. This is so because $(I_m + J)^n = \sum_{k=0}^{n}\binom{n}{k}J^k$ and $(J^k)_{1,m}$ is equal to the number of paths of length k from S to T. In practice the performance of this algorithm for some cases was far from impressive. In particular, the computation of $h(p)$ for those

hammocks for which $w+l \leq 8$ was practically instantaneous, but for $H_{4,5}$ and $H_{5,4}$ this took around 18 minutes, while for $H_{5,5}$ it went on for about 4.5 days. Still, this naïve algorithm has helped us to verify the correctness of several enhanced algorithms.

Our latest algorithm is based on a recursive depth-first traversal of a binary tree. The two children of a parent node in the tree are realized by conditioning on an arbitrary edge of the graph $\mathcal{G}$. That edge is either deleted (assuming that the corresponding relay is "open") or contracted (assuming that the corresponding relay is "closed"). The basic idea of conditioning on an edge of the graph was suggested by Moore and Shannon [1] (see eq. (3) on page 196) and [60], as well as by recursive definitions of certain graph polynomials, e.g., the Tutte polynomial. The recursion terminates when either S and T coincide, or else when S and T are found to belong in separate connected components of $\mathcal{G}$. At each node of the tree, some pre-processing is performed, before conditioning on the chosen edge. First, any components of the graph not containing S and T are discarded. Then, any paths having no other connection to the rest of the graph than at their start and end vertices are replaced by single edges labeled with the appropriate probabilities. After this, any loops are removed and any multiple edges are replaced by single edges with the appropriate probabilities. Then an arbitrary edge e of the resulting graph $\mathcal{G}$ is chosen, and two graph minors $\mathcal{G}/e$ and $\mathcal{G} - e$ are generated, by contraction and deletion of that edge, respectively. The module then calls itself recursively, to compute the reliabilities of these two minors, and finally returns the combined result given by

$$h_{\mathcal{G}}(p) = P(e)\, h_{\mathcal{G}/e}(p) + [1 - P(e)]\, h_{\mathcal{G}-e}(p). \quad (11)$$

Here $P(e)$ is the probability associated with the edge e. In this manner, the module traverses an incomplete binary tree of graph minors depth-first. This algorithm takes around 0.6 seconds to compute $h(p)$ for $H_{5,5}$, an improvement by roughly 640,000 times over the naïve approach.

For each of the hammock networks $H_{w,l}$ (shown in Fig. 5), we have computed $h(p)$ exactly as a reliability polynomial in p (i.e., as given by eq. (2)). This complements the results reported in [59] where we have shown $h(p,q)$ (i.e., as given by eq. (3)). That is, rather than using numerical methods to approximate $h(p)$ for some large but finite sequence of values of p, we have computed here all the integer coefficients P_k of the $h(p)$ polynomials. All these $h(p)$ reliability polynomials are presented in a compact form in Table 1. As far as we know, these exact $h(p)$ functions for these small hammock networks have not been published before in this form, except obviously for series ($H_{1,2}$, $H_{1,3}$, $H_{1,4}$, $H_{1,5}$), parallel ($H_{2,1}$, $H_{3,1}$, $H_{4,1}$, $H_{5,1}$), $H_{2,2}$, $H_{2,2}^+$ (in [1]) and $H_{3,3}$ (in [2]). It is to be mentioned that N_k (eq. (3)) were recently reported in [59] and are linked to P_k by $\vec{N} = M^{-1}\vec{P}$ (see eqs. (4) and (5)).

The $h(p)$ presented in Table 1 have been used to plot the graphs in Fig. 5. On all of them the yellow/shaded area shows where no improvements on closure reliability are
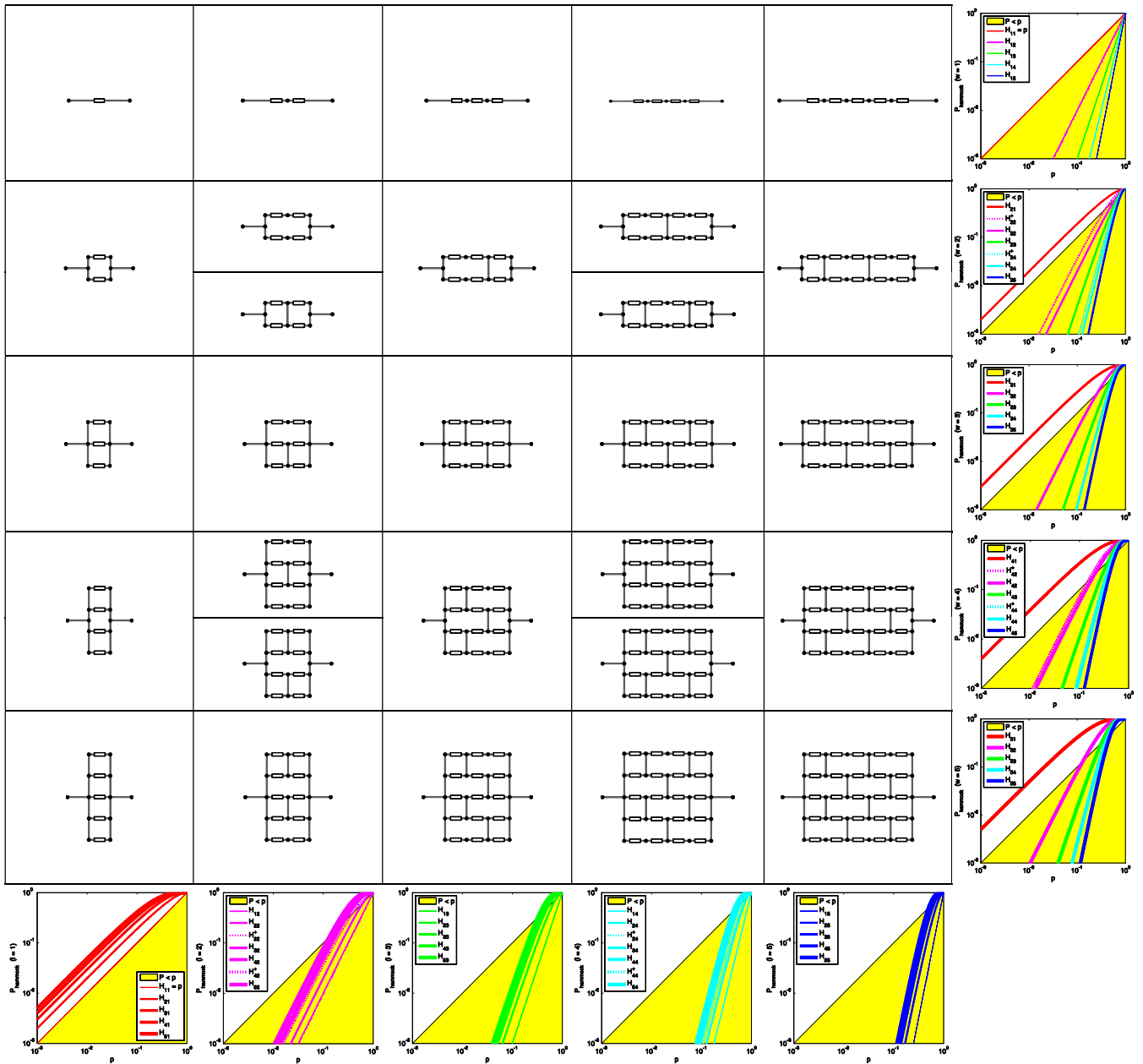
FIGURE 5. All the hammocks $H_{w,l}$ for $w \leq 5$, $l \leq 5$ and exact simulation results (log scale).

made, i.e., where the $H_{w,l}$ network of crummy relays is less likely to be closed than is a single crummy relay.

V. EVALUATING ENHANCEMENTS
A. RELIABILITY IMPROVEMENTS
A well-established measure for estimating reliability enhancements was introduced by Klaschka [61] in 1967, namely the *reliability improvement index (RII)*, defined as

$$RII = \frac{log(p)}{log[h(p)]}. \qquad (12)$$

A thorough explanation of the method for quantifying the performance of a redundancy scheme by *RII* and *a cost factor (cost)* was detailed in 1971 [62]. In practice, finding

an optimal redundancy scheme corresponds to either maximizing *RII* at a given *cost*, or minimizing *cost* for a given *RII*. Here we have defined the cost as the redundancy factor lw ($cost = lw$), or equivalently $cost = n$ for any minimal network, including hammocks. It follows that *a figure-of-merit (FoM)* defined as the ratio of the reliability improvement index (*RII*) to the redundancy factor (lw) captures both parameters:

$$FoM = \frac{RII}{cost} = \frac{RII}{wl}. \qquad (13)$$

Both the *RII* and the *FoM* were calculated exactly for all the $H_{w,l}$ networks presented in Fig. 5. They can be seen in Fig. 6. In the yellow/shaded area, no improvements

TABLE 1. Reliability polynomials $h_n(p)$ for hammocks $H_{w,l}$ ($w \leq 5, l \leq 5$).

l	$w = 1$
1	p
2	p^2
3	p^3
4	p^4
5	p^5

l	$w = 2$	
1	$\mathbf{2}\,p - p^2$	
2	$\mathbf{2}\,p^2 - p^4$	$\mathbf{4}\,p^2 - 4p^3 + p^4$
3	$\mathbf{4}\,p^3 - 2p^4 - 2p^5 + p^6$	
4	$\mathbf{4}\,p^4 - 4p^6 + p^8$	$\mathbf{8}\,p^4 - 8p^5 - 2p^6 + 4p^7 - p^8$
5	$\mathbf{8}\,p^5 - 4p^6 - 8p^7 + 4p^8 + 2p^9 - p^{10}$	

l	$w = 3$
1	$\mathbf{3}\,p - 3p^2 + p^3$
2	$\mathbf{5}\,p^2 - 4p^3 - 3p^4 + 4p^5 - p^6$
3	$\mathbf{8}\,p^3 - 6p^4 - 6p^5 + 12p^7 - 9p^8 + 2p^9$
4	$\mathbf{13}\,p^4 - 10p^5 - 10p^6 - 8p^7 + 27p^8 + 2p^9 - 26p^{10} + 16p^{11} - 3p^{12}$
5	$\mathbf{21}\,p^5 - 16p^6 - 19p^7 - 20p^8 + 49p^9 + 28p^{10} - 52p^{11} - 25p^{12} + 60p^{13} - 30p^{14} + 5p^{15}$

l	$w = 4$	
1	$\mathbf{4}\,p - 6p^2 + 4p^3 - p^4$	
2	$\mathbf{6}\,p^2 - 4p^3 - 8p^4 + 8p^5 + 2p^6 - 4p^7 + p^8$	$\mathbf{8}\,p^2 - 8p^3 - 14p^4 + 32p^5 - 24p^6 + 8p^7 - p^8$
3	$\mathbf{12}\,p^3 - 10p^4 - 10p^5 - 18p^6 + 64p^7 - 30p^8 - 38p^9 + 48p^{10} - 20p^{11} + 3p^{12}$	
4	$\mathbf{18}\,p^4 - 12p^5 - 14p^6 - 32p^7 + 10p^8 + 156p^9 - 126p^{10} - 128p^{11}$ $+ 188p^{12} - 24p^{13} - 68p^{14} + 40p^{15} - 7p^{16}$	$\mathbf{24}\,p^4 - 24p^5 - 18p^6 - 40p^7 + 98p^8 + 40p^9 - 6p^{10} - 472p^{11}$ $+ 852p^{12} - 696p^{13} + 308p^{14} - 72p^{15} + 7p^{16}$
5	$\mathbf{36}\,p^5 - 30p^6 - 26p^7 - 92p^8 + 122p^9 + 62p^{10} + 328p^{11} - 760p^{12} - 306p^{13} + 1708p^{14}$ $- 1234p^{15} - 312p^{16} + 932p^{17} - 566p^{18} + 156p^{19} - 17p^{20}$	

l	$w = 5$
1	$\mathbf{5}\,p - 10p^2 + 10p^3 - 5p^4 + p^5$
2	$\mathbf{9}\,p^2 - 8p^3 - 22p^4 + 40p^5 - 10p^6 - 24p^7 + 23p^8 - 8p^9 + p^{10}$
3	$\mathbf{16}\,p^3 - 14p^4 - 13p^5 - 58p^6 + 160p^7 - 31p^8 - 166p^9 + 49p^{10} + 233p^{11} - 300p^{12} + 165p^{13} - 45p^{14} + 5p^{15}$
4	$\mathbf{29}\,p^4 - 26p^5 - 18p^6 - 80p^7 + 18p^8 + 406p^9 - 128p^{10} - 654p^{11} - 410p^{12} + 2740p^{13} - 2606p^{14}$ $- 242p^{15} + 2267p^{16} - 1960p^{17} + 832p^{18} - 184p^{19} + 17p^{20}$
5	$\mathbf{52}\,p^5 - 46p^6 - 23p^7 - 204p^8 + 180p^9 + 22p^{10} + 1288p^{11} - 1228p^{12} - 2120p^{13} - 1720p^{14} + 13614p^{15} - 9622p^{16}$ $- 15645p^{17} + 28443p^{18} - 10045p^{19} - 14861p^{20} + 21065p^{21} - 12719p^{22} + 4306p^{23} - 800p^{24} + 64p^{25}$

are made. It can be seen that the benefits are impressive (by several orders of magnitude) for p close to 1, and that they vanish for p below 0.5. This means that the analysis should be followed by a similar one for a "normally open" relay type (corresponding to an nMOS transistor), which would complement the results reported here. In practice the approach has to be more nuanced as an nMOS transistor is not the exact 'opposite' of a pMOS transistor. However, reliability improvements could be obtained for both of them, albeit that the particular $H_{w,l}$ network for a pMOS might end up being different from that for an nMOS (see Fig. 4 (b)).

B. STEEPNESS IMPROVEMENTS

The intention stated in [1] (and [2]) was to replace a single relay with a two-terminal network of relays (having reliability polynomial $h(p)$). The step/Heaviside function is the ideal function (see Fig. 1 (b)). Therefore, the steepness $h'(p)$ has to have importance. Steepness also relates to measures such as Birnbaum's reliability importance [63]. It follows that another reliability related cost function could be the maximum of $h'(p)$ on [0, 1]. This is because, at its point of discontinuity, the rate of change of the ideal step/Heaviside function is infinite.

We have computed all the $h'(p)$ exactly using the $h(p)$ presented in Table 1. The results are plotted in Fig. 7. It is worth looking closely at Figs. 6 and 7 once again as they point to different "winners." *The networks which are optimal in terms of RII are the more parallel ones* (Fig. 6), while *the networks with the steepest $h(p)$ are those for which $l = w$* (Fig. 7).

C. CHANNEL CAPACITY IMPROVEMENTS

Another well-established measure of the efficacy of a two-terminal network $\mathbb{N}$ of unreliable relays is its channel capacity C [64], [65]. As suggested in [1] one could consider the common state of the coils (i.e., gates) to be the input to the channel, and the connectedness of S and T to be the output.

Shannon did not obtain an explicit formula for C in terms of a and c, except in some special cases such as the symmetric case $c = 1 - a$. Nor did Moore and Shannon compute $h(p)$ exactly except for a few of the $H_{w,l}$ networks they considered. We believe that, but for these obstacles, Moore and Shannon probably would have computed the channel capacities C of the 29 hammocks presented in [1] (at least for some particular choices of a and c). They did observe that $C(a, c) = 0$ precisely when $a = c$, but it is Muroga [66] who first showed
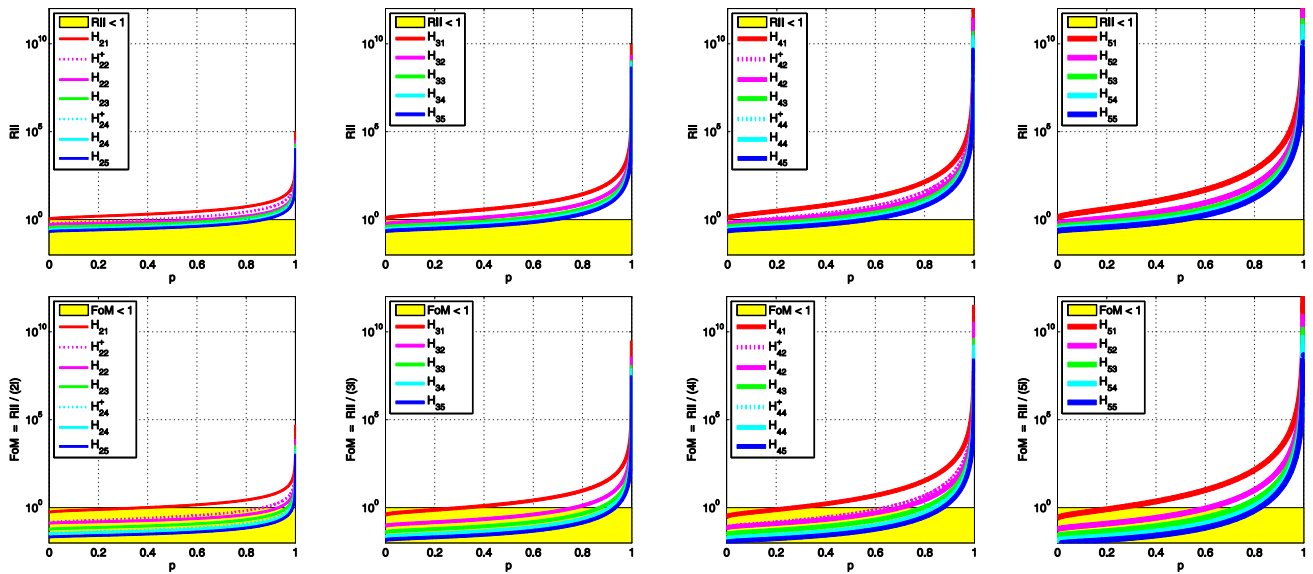
FIGURE 6. The reliability improvement index (*RII*) versus *p* is shown in the top row, while the *FoM* versus *p* is presented in the bottom one.
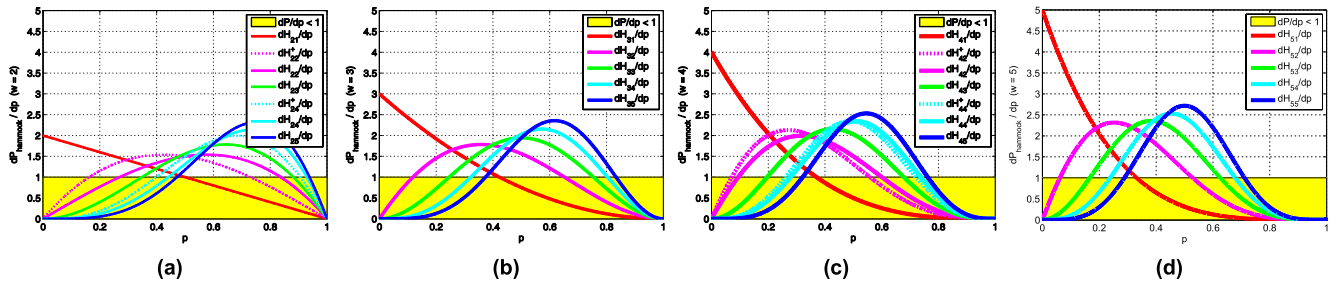

FIGURE 7. The derivative $h'(p)$ for: (a) $w = 2$; (b) $w = 3$; (c) $w = 4$; and (d) $w = 5$.

how to obtain general, explicit formulae for C in terms of a and c, and Silverman [67] who carried out this work in the special case of a binary channel. Our study [68] yielded the formula presented at the bottom of the next page (eq. (14)), where C is measured in bits per symbol transmitted (the symbols being 0 and 1).

A perfect binary channel has capacity 1 bit/symbol, but an unreliable one has capacity between 0 and 1. A two-terminal network $\mathcal{N}$ characterized by a certain $h(p)$, such as the $H_{w,l}$ we have considered here, has channel capacity $C(h(a), h(c))$.

For understanding how $H_{w,l}$ affects C we have plotted in Fig. 8 a unit cube which shows the capacities C of $H_{1,1}$ (i.e., no redundancy), $H_{3,3}$, $H_{5,5}$, and $H_{7,7}$ when varying both a and c in [0, 1]. The reasons we have used $H_{7,7}$ will become clear shortly. Its reliability polynomial is:

$$
\begin{aligned}
h_{7,7}(p) = \ &296p^7 - 276p^8 + 217p^9 - 2520p^{10} \\
&+ 1673p^{11} - 2427p^{12} + 10210p^{13} - 5130p^{14} \\
&+ 47982p^{15} - 56503p^{16} - 600p^{17} - 610686p^{18} \\
&+ 838991p^{19} + 168252p^{20} + 4056898p^{21} \\
&- 7756901p^{22} - 4985504p^{23} - 9073832p^{24}
\end{aligned}
$$

$$
\begin{aligned}
&+ 58119079p^{25} + 41418180p^{26} \\
&- 211199196p^{27} - 143809927p^{28} \\
&+ 633758144p^{29} + 551213356p^{30} \\
&- 2501334428p^{31} - 377539736p^{32} \\
&+ 9856740648p^{33} - 15868399515p^{34} \\
&+ 6450475602p^{35} + 14340458076p^{36} \\
&- 27918283370p^{37} + 22303967257p^{38} \\
&- 4345985716p^{39} - 10663030843p^{40} \\
&+ 14908944586p^{41} - 11235463077p^{42} \\
&+ 5854112676p^{43} - 2235353013p^{44} \\
&+ 631014226p^{45} - 129146412p^{46} \\
&+ 18218736p^{47} - 1590393p^{48} + 64914p^{49}. \quad (15)
\end{aligned}
$$

For a better understanding we introduce here a *capacity improvement index (CII)* as

$$
CII = \frac{C(h(a), h(c))}{C(a, c)} \quad (16)
$$

which is defined analogously to *RII* (eq. (12)), the channel capacity being already logarithmic (see eq. (14)). Fig. 9 shows both 3D views and contour plots of *CII* for $H_{3,3}$ versus
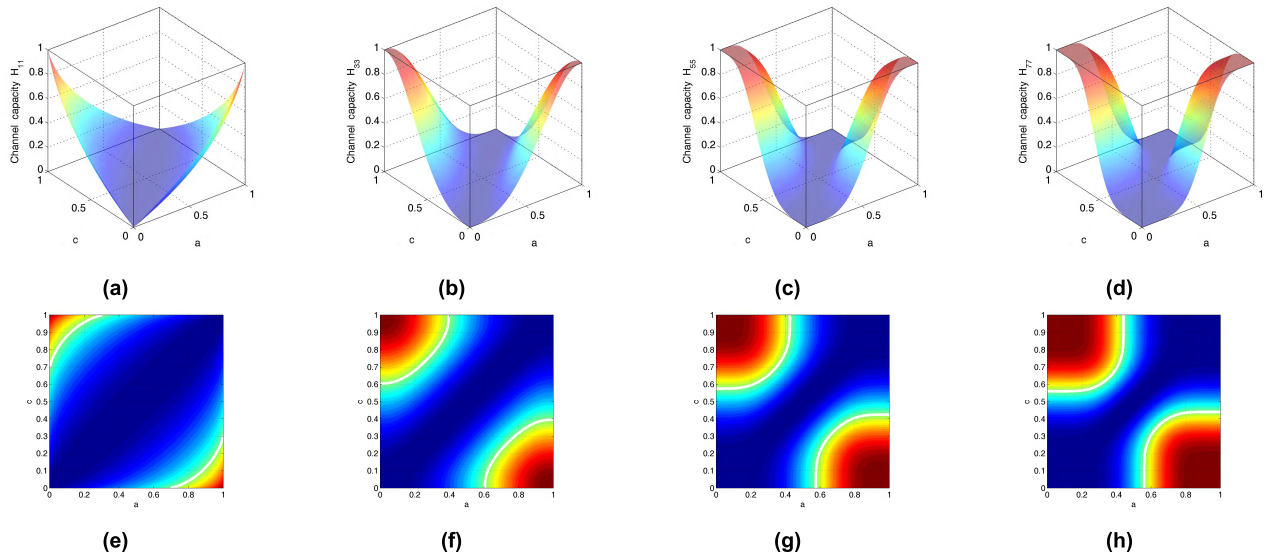
FIGURE 8. The channel capacity $C(a, c)$ for: (a) $H_{1,1}$; (b) $H_{3,3}$; (c) $H_{5,5}$; (d) $H_{7,7}$; (e), (f), (g), (h) associated contour plots.

$H_{1,1}$, $H_{5,5}$ versus $H_{1,1}$, and $H_{7,7}$ versus $H_{1,1}$. The contour plot for $H_{7,7}$ versus $H_{1,1}$ (Fig. 9 (f)) shows the emergence of two "separating lines" which are in fact optimal for Boolean logic (see [69], [70]). That is why we have added $H_{7,7}$, as from Figs. 9 (d) and 9 (e) the fact that the separating curves are closing towards two lines is not yet obvious. As expected, for the most common values of a and c, $H_{3,3}$, $H_{5,5}$, and $H_{7,7}$ offer successive improvements on channel capacity over that of a single relay $H_{1,1}$. Still, in case both a and c are simultaneously skewed towards (0, 0) or (1, 1) adding redundancy does not help (in fact on the contrary). This suggests (once again) that a two-terminal network needs to be customized for the particular values of a and c, for both enhancing channel capacity and improving reliability (see also [71] and [72].

It becomes clear that *increased redundancy leads not only to boosting reliability but also to greater steepness and to augmenting the channel capacity*. Further studies should try to establish correlations among such cost functions.

VI. EXTENSIONS TO CYLINDRICAL HAMMOCKS

In biology, cylindrical structures like pipelines are ubiquitous. Unexpectedly, fresh publications have identified very regular arrays reinforcing neurons' membranes [73]–[77]. These results support the expectation that ion channel distributions on the membrane might follow suit. Lately it has been argued that such regular spacing might facilitate the propagation of action potentials [78].

Axons (and dendrites) are used for communication, and therefore several reliability investigations have modeled them as cylindrical consecutive-k-out-of-n:F systems [79]–[81]. Hammock-inspired alternatives were suggested earlier [82], [83], being evaluated with respect to power/energy consumption, while their reliabilities were only approximated. Inspired by the periodic actin-spectrin-based membrane skeleton organization we followed [82] and [83] by extending the classical (planar or 2D) hammocks $H_{w,l}$ [1] to cylindrical hammocks $H_{w,l}^{2.5D}$ [84] (2.5D, as such

$$
C(a, c) = \begin{cases}
\log_2 \left\{ \left[\dfrac{(1-a)^{(1-a)c} a^{ac}}{(1-c)^{a(1-c)} c^{ac}} \right]^{\frac{1}{c-a}} + \left[\dfrac{(1-c)^{(1-a)(1-c)} c^{(1-a)c}}{(1-a)^{(1-a)(1-c)} a^{a(1-c)}} \right]^{\frac{1}{c-a}} \right\} & 0 < a, c < 1, \quad a \neq c \\[16pt]
0 & 0 \leq a \leq 1, \quad a = c \\[8pt]
\log_2 \left[1 + a(1-a)^{\frac{1-a}{a}} \right] & 0 < a < 1, \quad c = 0 \\[8pt]
\log_2 \left[1 + c(1-c)^{\frac{1-c}{c}} \right] & a = 0, \quad 0 < c < 1 \\[8pt]
\log_2 \left[1 + (1-a)a^{\frac{a}{1-a}} \right] & 0 < a < 1, \quad c = 1 \\[8pt]
\log_2 \left[1 + (1-c)c^{\frac{c}{1-c}} \right] & a = 1, \quad 0 < c < 1 \\[8pt]
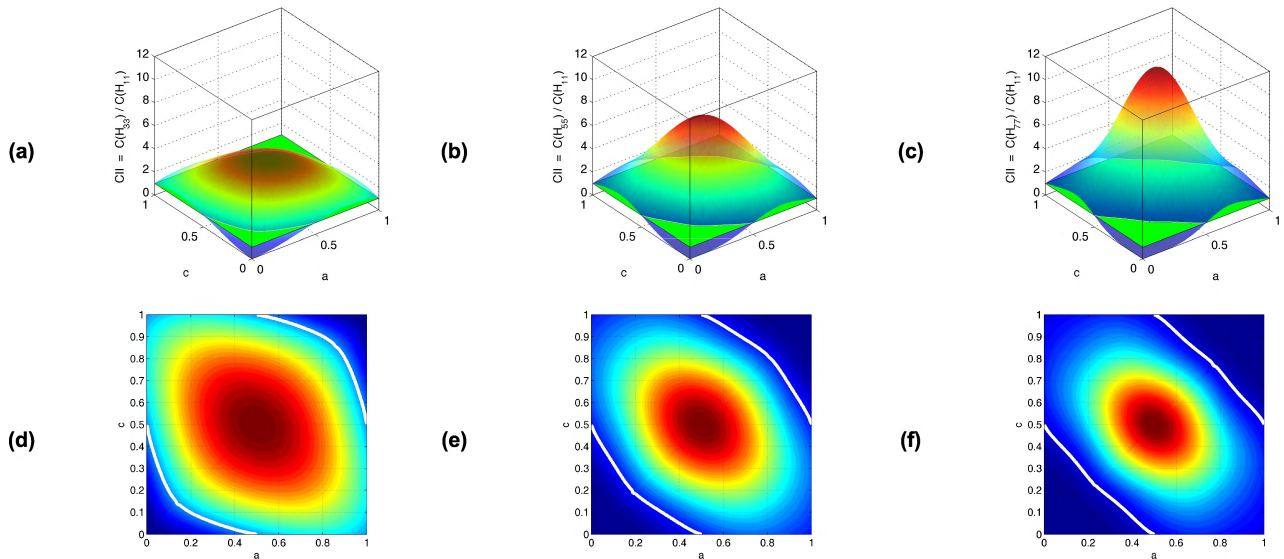1 & (a, c) \in \{(1, 0), (0, 1)\}
\end{cases}
\tag{14}
$$

FIGURE 9. The capacity improvement index ($CII(a, c)$): (a) $H_{3,3}$ versus $H_{1,1}$; (b) $H_{5,5}$ versus $H_{1,1}$; (c) $H_{7,7}$ versuss $H_{1,1}$; (d), (e), (f) associated contour plots.

TABLE 2. Reliability polynomials $h_n(p)$ for cylindrical hammocks $H_{wl}^{2.5D}$ ($w \leq 6, l \leq 6$).

l	$w = 2$
3	$8\,p^3 - 12p^4 + 6p^5 - p^6$
4	$16\,p^4 - 32p^5 + 24p^6 - 8p^7 + p^8$
5	$32\,p^5 - 80p^6 + 80p^7 - 40p^8 + 10p^9 - p^{10}$
6	$64\,p^6 - 192p^7 + 240p^8 - 160p^9 + 60p^{10} - 12p^{11} + p^{12}$

l	$w = 4$
3	$16\,p^3 - 16p^4 - 12p^5 - 80p^6 + 348p^7 - 528p^8 + 424p^9 - 194p^{10} + 48p^{11} - 5p^{12}$
4	$32\,p^4 - 32p^5 - 8p^6 - 272p^7 + 572p^8 + 696p^9 - 3792p^{10} + 6080p^{11} - 5438p^{12} + 3008p^{13} - 1028p^{14} + 200p^{15} - 7p^{16}$
5	$64\,p^5 - 64p^6 + 16p^7 - 800p^8 + 1376p^9 + 1416p^{10} - 1744p^{11} - 16140p^{12} + 52188p^{13} - 791452p^{14} + 74244p^{15} - 46320p^{16}$ $+ 19416p^{17} - 5286p^{18} + 848p^{19} - 61p^{20}$
6	$128\,p^6 - 128p^7 + 96p^8 - 2112p^9 + 3184p^{10} + 3552p^{11} + 120p^{12} - 33232p^{13} + 8232p^{14} + 235632p^{15} - 674944p^{16}$ $+ 1014096p^{17} - 995240p^{18} + 681096p^{19} - 330486p^{20} + 112208p^{21} - 25464p^{22} + 3480p^{23} - 217p^{24}$

l	$w = 6$
3	$24\,p^3 - 24p^4 - 18p^5 - 192p^6 + 522p^7 - 102p^8 + 182p^9 - 4758p^{10} + 13980p^{11} - 20874p^{12} + 19494p^{13}$ $- 12138p^{14} + 5082p^{15} - 1383p^{16} + 222p^{17} - 16p^{18}$
4	$48\,p^4 - 48p^5 - 324p^7 - 123p^8 + 1572p^9 + 1806p^{10} + 1020p^{11} - 52676p^{12} + 131184p^{13} - 57288p^{14} - 325892p^{15}$ $+ 860283p^{16} - 1159716p^{17} + 1030746p^{18} - 645372p^{19} + 288900p^{20} - 91088p^{21} + 19296p^{22} - 2472p^{23} + 145p^{24}$
5	$96\,p^5 - 96p^6 + 72p^7 - 852p^8 + 336p^9 - 2673p^{10} + 16608p^{11} + 4002p^{12} - 46920p^{13} - 207195p^{14} + 670022p^{15} + 269094p^{16}$ $- 3246282p^{17} + 2608801p^{18} + 11332854p^{19} - 39428604p^{20} + 66878822p^{21} - 75673020p^{22} + 62178714p^{23}$ $- 38274481p^{24} + 17776326p^{25} - 6169128p^{26} + 1556218p^{27} - 270198p^{28} + 28926p^{29} - 1441p^{30}$
6	$192\,p^6 - 192p^7 + 288p^8 - 2112p^9 + 972p^{10} - 7764p^{11} + 24408p^{12} + 56808p^{13} - 70350p^{14} - 291216p^{15} - 913332p^{16}$ $+ 5021496p^{17} - 2279360p^{18} - 9910356p^{19} - 24687150p^{20} + 169664736p^{21} - 266457582p^{22} - 153468720p^{23}$ $+ 1443104058p^{24} - 3251509716p^{25} + 4603238958p^{26} - 4726351516p^{27} + 3703401594p^{28} - 2260750656p^{29}$ $+ 1080747708p^{30} - 402130896p^{31} + 114447411p^{32} - 24115980p^{33} + 3550566p^{34} - 326412p^{35} + 14116p^{36}$

networks, being hollow, are not properly 3D). These are obtained by wrapping classical hammocks $H_{w,l}$ around a cylinder and adding $(l - 1)/2$ matchsticks (equivalent to connecting the two sides of the classical hammock using a zipper). For keeping the brick-wall pattern appearance these *cylindrical hammocks can be defined only for even values of* w (due to the need of having alternating matchsticks). On one side, by [1, Th. 4], $H_{w,l}^{2.5D}$ still belong to the same class of minimal 2-terminal networks as $H_{w,l}$. On the other side, they have $w(l-1)/2$ matchsticks instead of $(w-1)(l-1)/2$,

which means that $H_{w,l}^{2.5D}$ have slightly more paths between S and T than $H_{w,l}$ (i.e., they are more parallel). Using the same algorithm, we have determined exactly the reliability polynomials $h(p)$ associated to small cylindrical hammocks $H_{w,l}^{2.5D}$. The results are reported in Table 2 as reliability polynomials in p (given by eq. (2)). These complement the results reported in [84] which have presented $h(p, q)$ given by eq. (3).

We could take the same route and plot these reliability polynomials and compare them with those of equivalent $H_{w,l}$ (see Fig. 10 comparing $H_{4,l}^{2.5D}$ with $H_{4,l}^{2D}$). Recent results allow
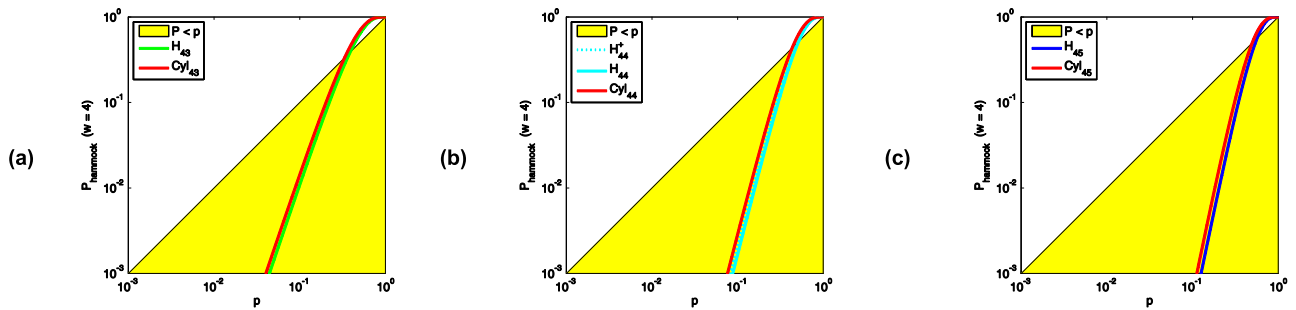
FIGURE 10. Reliability of cylindrical (2.5D) hammocks versus classical (2D) hammocks for $w = 4$: (a) $h_{4,3}^{2D}$ (green) and $h_{4,3}^{2.5D}$ (red); (b) $h_{4,4}^{2D}$ (cyan), $h_{4,4}^{2D+}$ (cyan dotted) and $h_{4,4}^{2.5D}$ (red); (c) $h_{4,5}^{2D}$ (blue) and $h_{4,5}^{2.5D}$ (red).

for a different approach. In [85] we have shown that it is possible to determine exactly the first non-zero coefficient of the reliability polynomials associated to any hammock network. For the particular case of $H_{w,l}^{2.5D}$ the first coefficient is $P_l = N_l = w \times 2^{l-1}$ for any w and l [84] (red and bolded in Table 2). When taken together with the values of the first non-zero coefficients for $H_{w,l}^{2D}$ [85] it becomes clear that $h^{2.5D}(p) > h^{2D}(p)$, hence cylindrical hammocks are slightly more reliable than classical ones for the same w and l when p is close to 0 (see Fig. 10).

VII. CONCLUSION

This paper has looked at small hammock networks, focusing on exactly determining their associated reliability polynomials $h(p)$. We have also extended classical hammocks to cylindrical ones (by mapping them onto cylinders). The state-of-the-art has revealed that array-based designs are a very good fit for (very) small $H_{w,l}$ networks, as are NEMS, hence accurate analyses should prove useful. The coefficients of the reliability polynomials have been determined exactly, while we have also shown that cylindrical hammocks exhibit only a marginal reliability advantage.

Our short-term plan falls on the practical side aiming to use these exact reliability polynomials for evaluating the reliability enhancements (very) small hammocks can bring to Boolean logic gates (e.g., when implemented using FinFETs connected as $H_{w,l}$, as suggested in Fig. 4 (b)). These should be analyzed both analytically as well as through detailed Monte Carlo simulations.

In the long-term we aim to study 3D non-planar networks akin to the cytoskeleton inside axons and dendrites [86]–[89] (e.g., for understanding the reliability/availability of cellular transport networks), with an introductory example presented in [90], where the results reported are approximations (obtained through bounding).

While working on this paper we have been able to identify a few interesting open problems:

- analyze upper/lower bounds on $h(p)$ with respect to accuracy (e.g., start from $N_l p^l q^{n-l}$ and $1 - C_w q^w p^{n-w}$, and afterwards include N_{l+1} and C_{w+1}, etc.);

- obtain formulas for N_l and C_w as functions of w and l (for N_l the formula is reported in [85]);
- express $h(p)$ exactly as a product of transfer matrices, considering individual node/edge reliabilities;
- analyze non-hammock matchstick minimal networks $\mathbb{N}$ with the same $\frac{(w-1)(l-1)}{2}$ number of matchsticks;
- analyze non-matchstick planar minimal networks $\mathbb{N}$;
- determine if there exist planar minimal networks $\mathbb{N}$ not realizable as in [1, Th. 4] without skipping horizontal rails;
- analyze non-planar minimal networks $\mathbb{N}$, and in particular whether there are such networks which are more reliable than any planar minimal network $\mathbb{N}$.

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