IEEE Access
Multidisciplinary ⁝ Rapid Review ⁝ Open Access Journal

Received March 14, 2018, accepted April 4, 2018, date of publication April 10, 2018, date of current version May 9, 2018.

Digital Object Identifier 10.1109/ACCESS.2018.2825338

The Influence of Attitude Dilution of Precision on the Observable Degree and Observability Analysis With Different Numbers of Visible Satellites in a Multi-Antenna GNSS/INS Attitude Determination System

YONG WANG, XIUBIN ZHAO, CHUNLEI PANG, BO FENG, AND LIANG ZHANG
Information and Navigation College, Air Force Engineering University, Xi'an 710077, China
Corresponding author: Yong Wang (wangyongnav@163.com)

This work was supported by the National Natural Science Foundation of China under Grant 61601506.

ABSTRACT The observability and observable degree of attitude errors in tightly coupled global navigation satellite system/inertial navigation system attitude determination system are studied in this paper. The mathematical relationship between the observable degree and the attitude dilution of precision (ATDOP) is determined by the singular value decomposition of the observability Gramian. The influence of the ATDOP on the observable degree is analyzed. The observability analysis of the attitude errors under different numbers of visible satellites is performed via a rank test of the observability Gramian. The results of a field test demonstrate that, a higher observable degree of attitude error will be obtained for the pitch, roll and yaw at a given time under a lower ATDOP. When the number of visible satellites is equivalent, a lower ATDOP leads to a higher observable degree of error for any one of the three attitudes at different times. Moreover, the results show that the three attitude errors are all externally observable under the condition of at least three visible satellites with three antennas.

INDEX TERMS Observability, observable degree, attitude dilution of precision (ATDOP), singular value decomposition (SVD), global navigation satellite system/inertial navigation system (GNSS/INS), attitude determination.

I. INTRODUCTION

Observability is an important property of a dynamical system, and whether the states can be estimated with a Kalman filter is mainly determined by the system observability [1], [2]. For an unobservable system, a successful estimation cannot be achieved, even if the measurement is sufficiently accurate. The system states with high observability usually have a high accuracy estimated values, whereas the states with low observability will have a degraded estimation accuracy, and these system states may even become divergent. In certain cases, the degree of observability of the system states must be measured when all states are observable. The degree of observability is quantitatively expressed by the observable degree [3], [4]. Thus, the observability and observable degree of the states must be evaluated for the design of a dynamical system.

For the observability analysis, the rank test and null space test of the observability matrix have been widely studied. The rank test of the observability matrix method was introduced in many works [5]–[9]. To avoid the complexity of the observability matrix, time-varying systems were modeled as piece-wise constant systems, and the observability of error states in the time-varying systems was then determined by testing the rank of the stripped observability matrix [5], [6]. The null space test of the observability matrix was mainly investigated in [10]–[12]. By testing the null space of the observability matrix, the observability properties of the error states in the navigation system with a low-grade inertial measurement unit (IMU) and an accurate single-antenna Global Positioning System (GPS) were studied [11]. For observable degree analyses, the eigenvalue decomposition of the error covariance matrix and the singular value decomposition (SVD) of the observability matrix are typically investigated. To provide insights into the degree of observability for a higher order system, the eigenvalues and eigenvectors of the error covariance matrix have been used to determine the observable degree of a linear combination of the state variables [13]. The SVD of the observability matrix was proposed in [14] to analyze the observable degree of complete and incomplete observable systems without the prior use of a Kalman filter [15], [16]. However, the error covariance behavior is sensitive to the initial error covariance; therefore, the relationship between the observability and the error covariance can be misleading [17]. Moreover, analytically determining the rank of an observability matrix is usually difficult except for simple system models [18]. Thus, observability measures based on the observability Gramian were proposed in [19] to analyze the observability and observable degree of multi-input/multi-output time-varying systems. In this work, we use these observability measures to study the influence of dilution of precision (DOP) on the observable degree of the state variable because the observability Gramian is intuitively related to the DOP.

System observability is improved during maneuvers, and there has been a number of pieces of literature talking about the effect of vehicle maneuvers on the system observability [1], [7], [9], [11], [20]. In [7], linear acceleration did not change the number of observable modes but did affect the structure of the observable space, and the observability of attitude angles is enhanced via nonconstant axial accelerating maneuvers in an integrated GPS/inertial navigation system (INS). In [1] and [9], acceleration changes were shown to enhance the observability of attitude angles and gyro bias, and angular velocity changes were shown to enhance the lever arm observability for GPS/INS integration. Almost all types of translational and angular maneuvers could make a three-channel tightly coupled GPS/strapdown inertial navigation system (SINS) instantaneous observability [9]. Roll and pitch errors, accelerometer biases in the east and north directions and gyro biases will become observable states in the absence of IMU continuous rotation around the X, Y and Z axes in the inertial system [20]. However, with the exception of vehicle maneuvers, the DOP may also affect the observable degree of the system state in a tightly coupled Global Navigation Satellite System (GNSS)/INS. The work reported in [21] demonstrated via numerical simulations that the observability worsens as the geometry dilution of precision (GDOP) increases. To date, research has not been reported regarding how the DOP influences the observable degree of the system state. In this paper, we analyze the observable degrees of three attitude errors, namely, the pitch, roll and yaw errors, using the SVD of the observability Gramian and provide the explicit mathematical relationship between the attitude dilution of precision (ATDOP) and the observable degree of attitude error in a tightly coupled GNSS/INS attitude determination system.

GNSS attitude determination is driven by precise GNSS carrier phase measurements. To fully exploit the high precision, the unknown integer ambiguities of the carrier phase measurements must be resolved. Teunissen developed a new integer least-squares (ILS) theory for the GNSS compass model together with efficient integer search strategies. This work extends the unconstrained ILS theory to the nonlinearly constrained case, which is particularly suited for precise attitude determination [22]. Based on this theory, instantaneous integer ambiguity resolution performance was demonstrated in [23] and [24].

The number of GNSS antennas could impact the observability of the error states in GNSS/INS integration. In [10], the error states were all observable with at least three antennas and at least one state was unobservable in the case of two antennas. However, the observability of the error states may also be affected by the number of visible satellites in a tightly coupled GNSS/INS attitude determination system. Thus, in this paper, we analyze the observability of attitude errors with different numbers of visible satellites via the rank test of the observability Gramian for the integrated GNSS/INS attitude determination system.

The paper is organized as follows. Section II demonstrates the mathematical relationship between the ATDOP and observable degree of attitude error by the SVD of the observability Gramian. Section III discusses the observability of a tightly coupled GNSS/INS attitude determination system under the condition of different numbers of visible satellites. Section IV presents the results of a field test. Section V discusses the conclusions.

II. INFLUENCE OF THE ATDOP ON THE OBSERVABLE DEGREE

In this section, the observable degree based on the SVD of the observability Gramian is introduced for a linear discrete system. The mathematical relationship between the observable degree of the system state and the DOP is demonstrated. The influence of the ATDOP on the observable degree of attitude error is discussed for a tightly coupled GNSS/INS attitude determination system.

Consider the following linear discrete system:

$$\begin{aligned} \mathbf{x}_k &= \mathbf{\Phi}_{k,k-1}\mathbf{x}_{k-1} \\ \mathbf{y}_k &= \mathbf{H}_k\mathbf{x}_k + \mathbf{v}_k \end{aligned} \tag{1}$$

where $\mathbf{x}_k \in \mathbf{R}^n$ is the state vector at time step k, $\mathbf{\Phi}_{k,k-1} \in \mathbf{R}^{n\times n}$ is the state transition matrix from time step $k-1$ to time step k, $\mathbf{y}_k \in \mathbf{R}^m$ is the measurement vector at time step k, $\mathbf{H}_k \in \mathbf{R}^{m\times n}$ is the measurement matrix at time step k, and $\mathbf{v}_k \in \mathbf{R}^m$ is the measurement noise vector at time step k.

Assume that $\mathbf{R}_k$ is the covariance matrix of $\mathbf{v}_k$ and $\mathbf{P}_k = E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T]$ is the error covariance matrix of $\mathbf{x}_k$. $\mathbf{P}_k$ represents the uncertainty in the state estimate. If $\mathbf{P}_k$ is "large", then a high uncertainty is present in the state estimate. Thus, the observable degree of the system state can be determined from the error covariance behavior according to the matrix perturbation theory. If the error covariance of a state undergoes a large decrease from the initial error covariance, the observable degree for the corresponding state is usually considered high. However, the behavior of the error covariance can be strongly influenced by the choice of the initial error covariance matrix, and the detailed proof has been given in [18]. Therefore, an observable degree analysis method based on the SVD of the observability Gramian was proposed by Hong to eliminate the influence of the initial error covariance on the observable degree [18].

For the linear discrete system (1), the information matrix is defined as follows:

$$\mathbf{I}_k = \mathbf{P}_k^{-1} \tag{2}$$

where $\mathbf{I}_k$ represents the certainty in the state matrix. If $\mathbf{I}_k$ is "large", then the confidence in the state estimate is high [25]. Analogous to the error covariance matrix, if the information matrix of a state undergoes a large increase from the initial information matrix, then the observable degree for the corresponding state is also considered high.

From the time step 0 to the time step k, the following is derived:

$$\mathbf{I}_k = \mathbf{I}_0 + \sum_{i=0}^{k} \mathbf{\Phi}_{i,0}^T \mathbf{H}_i^T \mathbf{R}_i^{-1} \mathbf{H}_i \mathbf{\Phi}_{i,0} \tag{3}$$

where $\mathbf{L}_{k,0} = \sum_{i=0}^{k} \mathbf{\Phi}_{i,0}^T \mathbf{H}_i^T \mathbf{R}_i^{-1} \mathbf{H}_i \mathbf{\Phi}_{i,0}$ is the observability Gramian on $[0, k]$ for the linear discrete system (1). If the observability Gramian is a symmetric matrix, then the SVD of $\mathbf{L}_{k,0}$ can be expressed as follows:

$$\mathbf{L}_{k,0} = \mathbf{U}_k \mathbf{\Sigma}_k \mathbf{U}_k^T \tag{4}$$

where $\mathbf{U}_k = [\, \mathbf{u}_1 \; \mathbf{u}_2 \; \cdots \; \mathbf{u}_n \,]$ is an orthogonal matrix consisting of singular vectors and $\mathbf{\Sigma}_k = diag(\, \sigma_1 \; \sigma_2 \; \cdots \; \sigma_n \,)$ is a diagonal matrix that consists of singular values. The singular value of the observability Gramian can be considered a measure of the observability for the subspace spanned by the corresponding singular vector. A large singular value implies that a large change must occur in the information matrix is necessary for the subspace spanned by the corresponding singular vector to be unobservable. Let the ith singular vector of the orthogonal matrix be represented as $\mathbf{u}_i = [\, u_{i1} \; u_{i2} \; \cdots \; u_{in} \,]^T$. Then, the observable degree of the state is as follows [18]:

$$\mu_i = \sqrt{\sum_{j=1}^{n} \sigma_j^2 u_{ij}^2} \tag{5}$$

Equation (5) shows that the observable degree of the system state is determined by the singular values and the elements of their corresponding singular vectors. Moreover, the observable degree will improve as the singular value and the elements of the singular vector increase.

In the observability Gramian $\mathbf{L}$, the state transition matrix $\mathbf{\Phi}$ reflects the influence of vehicle maneuvers on the observable degree, whereas the measurement matrix $\mathbf{H}$ reflects the influence of the DOP on the observable degree. The effect of the DOP on the observable degree is studied using the observability Gramian at time step k. To avoid the effect of vehicle maneuvers, the state transition matrix is not considered in the observability Gramian; thus, (3) can be expressed as follows [25]:

$$\mathbf{I}_k^+ = \mathbf{I}_k^- + \mathbf{L}_k \tag{6}$$

$$\mathbf{L}_k = \mathbf{H}_k^T \mathbf{R}_k^{-1} \mathbf{H}_k \tag{7}$$

where $\mathbf{I}_k^+$ and $\mathbf{I}_k^-$ are the information matrices after and before the processing of the measurement at time step k, respectively, and $\mathbf{L}_k$ is the observability Gramian at time step k. The externally observable degree, namely, the observable degree of the system state that has an externally measurement, can be obtained by the SVD of $\mathbf{L}_k$ according to (5).

In a tightly coupled GNSS/INS system, the DOP is usually defined as the square root of the diagonal element of $\mathbf{L}_k^{-1}$. In addition, the diagonal element of $\mathbf{L}_k^{-1}$ can be expressed by the singular value and singular vector of $\mathbf{L}_k$ as described in the following theorem.

Theorem 1: The ith diagonal element of $\mathbf{L}_k^{-1}$ is

$$d_i = \sum_{j=1}^{n} \frac{u_{ij}^2}{\sigma_j} \tag{8}$$

Proof: Because the SVD of $\mathbf{L}_k$ is shown in (4), and $\mathbf{U}_k$ is an orthogonal matrix, the SVD of $\mathbf{L}_k^{-1}$ can be expressed as follows:

$$\begin{aligned} \mathbf{L}_k^{-1} &= \left(\mathbf{U}_k \mathbf{\Sigma}_k \mathbf{U}_k^T \right)^{-1} \\ &= \mathbf{U}_k \mathbf{\Sigma}_k^{-1} \mathbf{U}_k^T \\ &= \sum_{j=1}^{n} \frac{\mathbf{u}_j \mathbf{u}_j^T}{\sigma_j} \end{aligned} \tag{9}$$

Therefore, Theorem 1 is proved.

Theorem 1 shows that the DOP is also determined by the singular values and the elements of their corresponding singular vectors. However, compared with the observable degree, the DOP will decrease as the singular value increases.

According to (5) and (8), the mathematical relationships between the externally observable degree of the system state and the corresponding DOP can be obtained with the singular value and singular vector of the SVD of the observability Gramian.

For the tightly coupled multi-antenna GNSS/INS attitude determination system, we can obtain the ATDOPs, namely, the pitch dilution of precision (PIDOP), roll dilution of precision (RDOP) and yaw dilution of precision (YDOP), according to their definitions in [26] and (8) as follows:

$$PIDOP = \sqrt{\sum_{j=1}^{n} \frac{u_{1j}^2}{\sigma_j}} \tag{10}$$

$$RDOP = \sqrt{\sum_{j=1}^{n} \frac{u_{2j}^2}{\sigma_j}} \tag{11}$$

$$YDOP = \sqrt{\sum_{j=1}^{n} \frac{u_{3j}^2}{\sigma_j}} \tag{12}$$

According to (5), the externally observable degree of the pitch, roll and yaw errors are respectively calculated as:

$$\mu_1 = \sqrt{\sum_{j=1}^{n} \sigma_j^2 u_{1j}^2} \tag{13}$$

$$\mu_2 = \sqrt{\sum_{j=1}^{n} \sigma_j^2 u_{2j}^2} \tag{14}$$

$$\mu_3 = \sqrt{\sum_{j=1}^{n} \sigma_j^2 u_{3j}^2} \tag{15}$$

Thus, the mathematical relationship between the externally observable degree of attitude error and the corresponding ATDOP can be obtained as shown in (10)-(15) for a tightly coupled multi-antenna GNSS/INS attitude determination system. In addition, the attitude error state with a lower DOP has a higher externally observable degree.

III. OBSERVABILITY ANALYSIS WITH DIFFERENT NUMBERS OF VISIBLE SATELLITES

In this section, the observability of attitude errors for a tightly coupled multi-antenna GNSS/INS attitude determination system using a two visible satellites scheme, three visible satellites scheme and more than three visible satellites scheme is analyzed.

The rank test of the observability Gramian $\mathbf{L}_k$ is usually used for observability analyses [27]. If $\mathbf{L}_k$ is of full rank, then the system states are considered to be observable. However, if $\mathbf{L}_k$ is rank deficient, then the rank of $\mathbf{L}_k$ is the number of the system states that are observable [28]. For $\mathbf{L}_k$ in (7), the observability analysis can be simplified by the following theorem.

Theorem 2: If $\mathbf{R}_k$ is nonsingular, then $rank(\mathbf{H}_k^T\mathbf{R}_k^{-1}\mathbf{H}_k) = rank(\mathbf{H}_k^T\mathbf{H}_k)$.

Proof: Because $\mathbf{R}_k$ is nonsingular, $rank(\mathbf{H}_k^T\mathbf{R}_k^{-1}) = rank(\mathbf{H}_k^T)$ [29]. Therefore, $rank(\mathbf{H}_k^T\mathbf{R}_k^{-1}\mathbf{H}_k) = rank(\mathbf{H}_k^T\mathbf{H}_k)$.

The covariance matrix $\mathbf{R}_k$ is generally nonsingular in the measurement equation of a tightly coupled multi-antenna GNSS/INS attitude determination system. Thus, Theorem 2 shows that only the measurement matrix $\mathbf{H}_k$ needs to be considered for the rank test of $\mathbf{L}_k$.

In a tightly coupled multi-antenna GNSS/INS attitude determination system, the measurement equation is usually formed by the double-differential (DD) carrier phase equations. Thus, the measurement matrix with two baselines formed by three antennas is expressed as follows [30]:

$$\mathbf{H}_k = \begin{pmatrix} \mathbf{S}_b^T\,[\mathbf{b}_1\times] \\ \mathbf{S}_b^T\,[\mathbf{b}_2\times] \end{pmatrix} \tag{16}$$

where $\mathbf{S}_b = [\,\mathbf{s}_b^1\ \mathbf{s}_b^2\ \cdots\ \mathbf{s}_b^n\,]$ is the DD unit line-of-sight matrix in the body frame, $\mathbf{s}_b^i = [\,s_{bx}^i\ s_{by}^i\ s_{bz}^i\,]^T$, $i = 1, 2, \ldots, n$ is the DD unit line-of-sight vector in the body frame, $\mathbf{b}_1 = [\,0\ l_1\ 0\,]^T$ and $\mathbf{b}_2 = [\,l_2\ 0\ 0\,]^T$ are two baseline vectors in the body frame, and $[\mathbf{b}\times]$ is the skew symmetric matrix of $\mathbf{b}$. In certain cases, only one baseline formed by two antennas is used to determine the two attitudes of pitch, roll and yaw. In this instance, the measurement matrix can be simplified as follows:

$$\mathbf{H}_k = \mathbf{S}_b^T\,[\mathbf{b}_1\times] \tag{17}$$

A. TWO VISIBLE SATELLITES SCHEME

Only one DD carrier phase equation is available for the case in which two satellites are visible. Thus, n is equal to 1 in (16). Then, by substituting (16) into (7), we obtain the following:

$$\begin{aligned}\mathbf{L}_k &= \begin{pmatrix} \mathbf{S}_b^T\,[\mathbf{b}_1\times] \\ \mathbf{S}_b^T\,[\mathbf{b}_2\times] \end{pmatrix}^T \begin{pmatrix} \mathbf{S}_b^T\,[\mathbf{b}_1\times] \\ \mathbf{S}_b^T\,[\mathbf{b}_2\times] \end{pmatrix} \\ &= \begin{pmatrix} l_1^2\left(s_{bz}^1\right)^2 & 0 & -l_1^2 s_{bz}^1 s_{bx}^1 \\ 0 & l_2^2\left(s_{bz}^1\right)^2 & -l_2^2 s_{bz}^1 s_{by}^1 \\ -l_1^2 s_{bz}^1 s_{bx}^1 & -l_2^2 s_{bz}^1 s_{by}^1 & l_1^2\left(s_{bx}^1\right)^2 + l_2^2\left(s_{by}^1\right)^2 \end{pmatrix}\end{aligned} \tag{18}$$

From (18), we can easily conclude that $rank(\mathbf{L}_k) = 2$ and $\mathbf{L}_k$ is rank deficient; therefore, the tightly coupled three-antenna GNSS/INS attitude determination is externally unobservable, in the case of two visible satellites. Moreover, only two of the three attitude errors are externally observable.

For a dual-antenna GNSS/INS attitude determination, $\mathbf{L}_k$ can be obtained by substituting (17) into (7) as follows:

$$\begin{aligned}\mathbf{L}_k &= \left(\mathbf{S}_b^T\,[\mathbf{b}_1\times]\right)^T \left(\mathbf{S}_b^T\,[\mathbf{b}_1\times]\right) \\ &= \begin{pmatrix} l_1^2\left(s_{bz}^1\right)^2 & 0 & -l_1^2 s_{bz}^1 s_{bx}^1 \\ 0 & 0 & 0 \\ -l_1^2 s_{bz}^1 s_{bx}^1 & 0 & l_1^2\left(s_{bx}^1\right)^2 \end{pmatrix}\end{aligned} \tag{19}$$

The rank of (19) is 1, therefore, the tightly coupled dual-antenna GNSS/INS attitude determination system is also externally unobservable with two visible satellites. In this situation, only one of the three attitude errors is externally observable.

B. THREE VISIBLE SATELLITES SCHEME

The case of three satellites presents two DD carrier phase equations. Similar to the two visible satellites scheme, we can also obtain the observability Gramian by substituting (16) into (7) as follows:

$$\begin{aligned}\mathbf{L}_k &= \begin{pmatrix} l_1^2\left(s_{bz}^1\right)^2 & 0 & -l_1^2 s_{bz}^1 s_{bx}^1 \\ 0 & l_2^2\left(s_{bz}^1\right)^2 & -l_2^2 s_{bz}^1 s_{by}^1 \\ -l_1^2 s_{bz}^1 s_{bx}^1 & -l_2^2 s_{bz}^1 s_{by}^1 & l_1^2\left(s_{bx}^1\right)^2 + l_2^2\left(s_{by}^1\right)^2 \end{pmatrix} \\ &+ \begin{pmatrix} l_1^2\left(s_{bz}^2\right)^2 & 0 & -l_1^2 s_{bz}^2 s_{bx}^2 \\ 0 & l_2^2\left(s_{bz}^2\right)^2 & -l_2^2 s_{bz}^2 s_{by}^2 \\ -l_1^2 s_{bz}^2 s_{bx}^2 & -l_2^2 s_{bz}^2 s_{by}^2 & l_1^2\left(s_{bx}^2\right)^2 + l_2^2\left(s_{by}^2\right)^2 \end{pmatrix}\end{aligned} \tag{20}$$

From (20), $rank(\mathbf{L}_k) = 3$ and $\mathbf{L}_k$ is of full rank, therefore, the attitude determination system is externally observable.

The observability Gramian for the dual-antenna system is expressed as follows:

$$\mathbf{L}_k = \begin{pmatrix} l_1^2\left[\left(s_{bz}^1\right)^2+\left(s_{bz}^2\right)^2\right] & 0 & -l_1^2\left(s_{bz}^1 s_{bx}^1+s_{bz}^2 s_{bx}^2\right) \\ 0 & 0 & 0 \\ -l_1^2\left(s_{bz}^1 s_{bx}^1+s_{bz}^2 s_{bx}^2\right) & 0 & l_1^2\left[\left(s_{bz}^1\right)^2+\left(s_{bz}^2\right)^2\right] \end{pmatrix} \tag{21}$$

Because the rank of (21) is 2, two of the three attitude errors are externally observable for the three visible satellites scheme.

C. MORE THAN THREE VISIBLE SATELLITES SCHEME

The case of more than three visible satellites presents n DD carrier phase equations. The observability Gramian of three-antenna system is expressed (22), as shown at the bottom of this page.

According to (22), $rank(\mathbf{L}_k) = 3$ and $\mathbf{L}_k$ is of full rank, therefore, the attitude determination system is also externally observable in the scheme for more than three visible satellites.

The observability Gramian of the dual-antenna system is expressed as follows:

$$\mathbf{L}_k = \begin{pmatrix} l_1^2 \sum_{i=1}^{n}\left(s_{bz}^i\right)^2 & 0 & -l_1^2 \sum_{i=1}^{n} s_{bz}^i s_{bx}^i \\ 0 & 0 & 0 \\ -l_1^2 \sum_{i=1}^{n} s_{bz}^i s_{bx}^i & 0 & l_1^2 \sum_{i=1}^{n}\left(s_{bz}^i\right)^2 \end{pmatrix} \tag{23}$$

In (23), the rank is 2, therefore, the system is also externally unobservable and only two of the three attitude errors are externally observable.

In conclusion, for the tightly coupled three-antenna GNSS/INS attitude determination, the system is externally observable with more than two visible satellites. Moreover, one of three attitude errors will be externally unobservable with only two visible satellites. However, for the dual-antenna GNSS/INS attitude determination, one attitude error is always externally unobservable. In addition, only one of the three attitude errors is externally observable when the number of visible satellites is two.

IV. FIELD TEST RESULTS

To verify the observability and observable degree of the tightly coupled GNSS/INS attitude determination, a field test is performed under stationary conditions at Xi'an, China. Three ComNav K501 receivers, three NovAtel 702-GGG antennas and a MicroStrain 3DM-GX3-25 IMU are used in the field test. The configuration of the antennas and IMU in the body frame of the field test is shown in Fig. 1. The lengths of the two baselines are 3.965 m and 9.445 m.

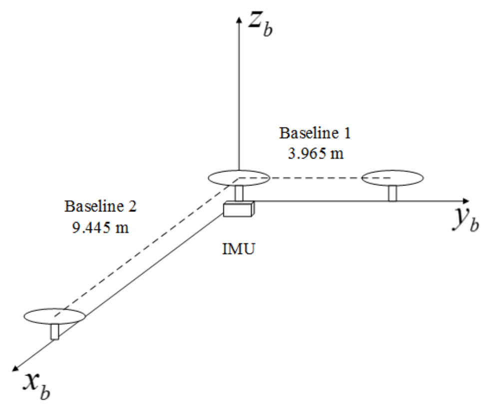


FIGURE 1. Configuration of the antennas and IMU in the body frame of the field test

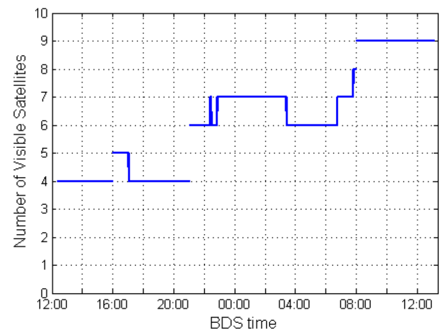


FIGURE 2. The number of BDS visible satellites in the field test.

Receivers were used to collect the BeiDou Navigation Satellite System (BDS) data at 1 Hz from Nov. 8, 2016, to Nov. 9, 2016, for approximately one day of data. The number of BDS satellites visible in the field test is shown in Fig. 2. As illustrated in Fig. 2, the number of visible BDS satellites during the field test is 4-9. The pseudo-ranges and carrier phase measurements for BDS B1 are used in the field test. The simplified equally weighted model is the stochastic model used in attitude determination. The integer ambiguities are resolved with the C-LAMBDA method [23]. The attitude estimation is based on the integrated three-antenna solution [31].

Figs. 3-5 show the observable degrees of the pitch, roll and yaw errors and their corresponding ATDOPs, namely, the PIDOP, RDOP and YDOP, in the field test. The observable

$$\mathbf{L}_k = \begin{pmatrix} l_1^2 \sum_{i=1}^{n}\left(s_{bz}^i\right)^2 & 0 & -l_1^2 \sum_{i=1}^{n} s_{bz}^i s_{bx}^i \\ 0 & l_2^2 \sum_{i=1}^{n}\left(s_{bz}^i\right)^2 & -l_2^2 \sum_{i=1}^{n} s_{bz}^i s_{by}^i \\ -l_1^2 \sum_{i=1}^{n} s_{bz}^i s_{bx}^i & -l_2^2 \sum_{i=1}^{n} s_{bz}^i s_{by}^i & l_1^2 \sum_{i=1}^{n}\left(s_{bx}^i\right)^2 + l_2^2 \sum_{i=1}^{n}\left(s_{by}^i\right)^2 \end{pmatrix} \tag{22}$$

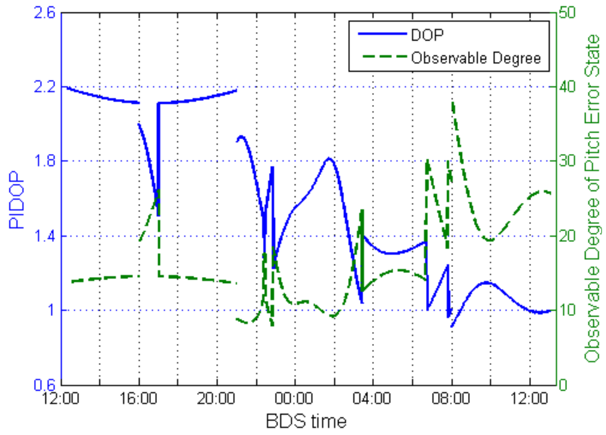


FIGURE 3. PIDOP and observable degree of the pitch error in the field test.

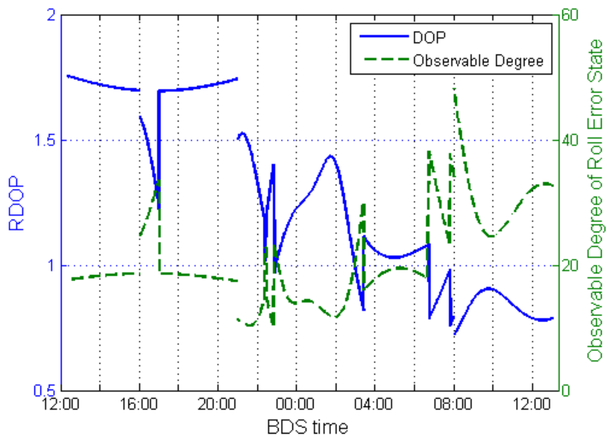


FIGURE 4. RDOP and observable degree of the roll error in the field test.

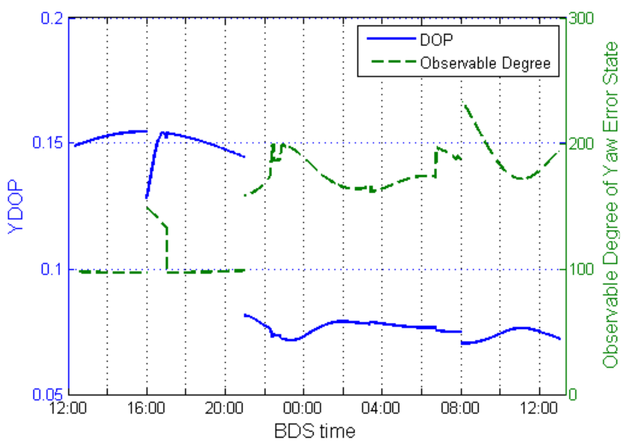


FIGURE 5. YDOP and observable degree of the yaw error in the field test.

degrees are calculated according to (13)-(15). The ATDOP values are calculated according to the definitions in [26] and [30]. As shown in Figs. 3-5, among the three elements of ATDOP, the YDOP value is the smallest and the PIDOP value is the largest. However, the observable degree of yaw error is the highest and the observable degree of pitch error is the lowest. Thus, for the pitch, roll and yaw at a given time, a higher observable degree is obtained for lower ATDOP values. Moreover, an examination of Figs. 2-5 shows that when the number of visible satellites is equivalent, for any one attitude among the pitch, roll and yaw at different times, a lower ATDOP will lead to a higher observable degree.

Table 1 shows the visible satellite number, singular values and orthogonal matrix of the SVD of the observability Gramian, ATDOPs, observable degrees of attitude errors, and the square root of d_i in (8) at BDS times of 14:00:00, 15:00:00, 16:00:00, 00:00:00, 01:00:00, 02:00:00, 04:00:00, 05:00:00, 06:00:00, 10:00:00, 11:00:00, and 12:00:00. For simplicity of notation, exponentials with 10 as the base are expressed by E, such that 1.0E-4 means 1.0×10^{-4}, and the unit of the elements of the orthogonal matrix in Table 1 is E-4. An examination of the data in Table 1 shows the following results.

1) At BDS time 14:00:00, the YDOP value is 0.1530, which is the smallest among the three ATDOPs, and the PIDOP value is 2.14, which is the largest. The observable degree of yaw error is 97.01, which is the highest among the three observable degrees, and the observable degree of pitch error is 14.26, which is the lowest. A further comparison of the results of the ATDOP and observable degree at other BDS times showed that for simultaneously obtained pitch, roll and yaw values, a lower ATDOP corresponds to a higher observable degree.
2) At BDS times 14:00:00, 15:00:00, and 16:00:00, the number of visible satellites is 4; the PIDOP values are 2.14, 2.12, and 2.11, respectively and the observable degrees are 14.26, 14.45, and 14.55, respectively; the RDOP values are 1.72, 1.71, and 1.70, respectively, and the observable degrees are 18.27, 18.52, and 18.65, respectively; and the YDOP values are 0.1530, 0.1543, and 0.1547, respectively, and the observable degrees are 97.01, 96.75, and 96.67, respectively. These results show that for the pitch, roll and yaw obtained at different times, a lower ATDOP corresponds to a higher observable degree when the number of visible satellites is equivalent. This conclusion is supported by the results obtained for BDS times 00:00:00, 01:00:00, and 02:00:00; 04:00:00, 05:00:00, and 06:00:00; 10:00:00, 11:00:00, and 12:00:00.
3) The orthogonal matrices of the SVD at BDS times 00:00:00, 01:00:00, and 02:00:00 show slight differences, which may be related to the equivalent number of visible satellites and the slight changes in the satellite geometry in the body frame. Therefore, the SVD of the observability Gramian may be related to the satellite geometry in the body frame.
4) In the orthogonal matrices, the second and third elements of the first and second rows are much larger than the first element, and the first element of the third row is much larger than the second and third elements. This difference is particularly pronounced

TABLE 1. SVD of the observability Gramian, ATDOP and observable degree.

BDS time	Number of visible satellites	Singular value	Orthogonal matrix (E-4)	ATDOP			Observable degree of attitude error			$\sqrt{d_i}$		
				PIDOP	RDOP	YDOP	Pitch	Roll	Yaw	i=1	i=2	i=3
14:00:00	4	99.67 3.79 0.137	$\begin{bmatrix} -1412 & -6034 & 7849 \\ -1810 & -7637 & -6197 \\ 9733 & -2295 & -13.77 \end{bmatrix}$	2.14	1.72	0.1530	14.26	18.27	97.01	2.14	1.72	0.1530
15:00:00	4	99.49 3.80 0.140	$\begin{bmatrix} -1434 & -6030 & 7847 \\ -1839 & -7629 & -6198 \\ 9724 & -2332 & -14.73 \end{bmatrix}$	2.12	1.71	0.1543	14.45	18.52	96.75	2.12	1.71	0.1543
16:00:00	4	99.44 3.830 0.141	$\begin{bmatrix} -1444 & -6029 & 7847 \\ -1852 & -7625 & -6199 \\ 9720 & -2349 & -15.16 \end{bmatrix}$	2.11	1.70	0.1547	14.55	18.65	96.67	2.11	1.70	0.1547
00:00:00	7	188.76 16.66 0.261	$\begin{bmatrix} -188.0 & -6168 & 7869 \\ -247.1 & -7865 & -6171 \\ 9995 & -310.4 & -4.576 \end{bmatrix}$	1.55	1.22	0.0732	10.88	13.91	188.67	1.55	1.22	0.0732
01:00:00	7	174.43 13.64 0.221	$\begin{bmatrix} -374.3 & -6164 & 7865 \\ -491.7 & -7850 & -6175 \\ 9981 & -617.8 & -9.268 \end{bmatrix}$	1.68	1.33	0.0774	10.65	13.72	174.10	1.68	1.33	0.0774
02:00:00	7	165.55 12.31 0.197	$\begin{bmatrix} -320.1 & -6160 & 7871 \\ -400.9 & -7861 & -6168 \\ 9987 & -513.0 & -4.677 \end{bmatrix}$	1.78	1.41	0.0790	9.25	11.74	165.33	1.78	1.41	0.0790
04:00:00	6	164.04 22.47 0.349	$\begin{bmatrix} -86.17 & -6170 & -7870 \\ -108.6 & -7869 & 6170 \\ -9999 & 138.7 & 0.7818 \end{bmatrix}$	1.34	1.06	0.0781	13.93	17.77	164.02	1.34	1.06	0.0781
05:00:00	6	170.36 23.47 0.369	$\begin{bmatrix} -291.9 & -6162 & -7871 \\ -361.0 & -7862 & 6169 \\ -9989 & 464.2 & 7.069 \end{bmatrix}$	1.30	1.03	0.0771	15.29	19.45	170.18	1.30	1.03	0.0771
06:00:00	6	174.10 22.30 0.352	$\begin{bmatrix} -339.8 & -6157 & -7872 \\ -414.1 & -7861 & 6167 \\ -9986 & 535.5 & 12.15 \end{bmatrix}$	1.33	1.05	0.0766	14.96	18.96	173.86	1.33	1.05	0.0766
10:00:00	9	182.76 30.95 0.479	$\begin{bmatrix} -182.5 & -6164 & -7872 \\ -179.3 & -7870 & 6167 \\ -9997 & 253.7 & 33.14 \end{bmatrix}$	1.14	0.90	0.0742	19.37	24.58	182.71	1.14	0.90	0.0742
11:00:00	9	171.77 35.67 0.551	$\begin{bmatrix} -19.20 & -6170 & -7869 \\ -21.07 & -7869 & 6170 \\ -10000 & 4.737 & 28.11 \end{bmatrix}$	1.07	0.84	0.0764	22.01	28.07	171.77	1.07	0.84	0.0764
12:00:00	9	178.20 40.60 0.629	$\begin{bmatrix} -155.5 & -6167 & -7871 \\ -162.3 & -7869 & 6169 \\ -9997 & 223.7 & 22.27 \end{bmatrix}$	1.00	0.79	0.0750	25.19	32.08	178.16	1.00	0.79	0.0750

for the orthogonal matrices for BDS times 00:00:00, 01:00:00, 02:00:00, 04:00:00, 05:00:00, 06:00:00, 10:00:00, 11:00:00, and 12:00:00. Thus, according to the observable degree of pitch, roll and yaw errors in (13)-(15), the observable degree of pitch and roll errors is mainly determined by the second and third singular values, and the observable degree of yaw is mainly determined by the first singular value.

5) The square root of d_i calculated with the singular values and orthogonal matrices is equal to the ATDOP. Thus, (7) is verified by the field test.

Table 2 shows the observability Gramians and their ranks for the case with different numbers of visible satellites at BDS time 00:00:00. As shown in Table 2, for the dual-antenna attitude determination system, the rank of the observability Gramian is 1 when the number of visible satellites is two, whereas the rank of the observability Gramian is 2 when the number of visible satellites is greater than two. These findings show that only one attitude error is externally observable when two satellites are visible and that two attitude errors are externally observable when more than two satellites are visible. For the three-antenna attitude determination system, the rank of the observability Gramian is 2 when the number

TABLE 2. Observability Gramian and its rank.

Number of satellites for attitude determination	Dual-antenna		Three-antenna	
	Observability Gramian	Rank	Observability Gramian	Rank
2	$\begin{bmatrix} 0.0049 & 0 & 0.0225 \\ 0 & 0 & 0 \\ 0.0225 & 0 & 0.1036 \end{bmatrix}$	1	$\begin{bmatrix} 0.0770 & 0.0944 & -1.117 \\ 0.0944 & 0.1234 & -1.490 \\ -1.117 & -1.490 & 18.10 \end{bmatrix}$	2
3	$\begin{bmatrix} 0.0331 & 0 & 0.0379 \\ 0 & 0 & 0 \\ 0.0379 & 0 & 0.1120 \end{bmatrix}$	2	$\begin{bmatrix} 0.5205 & 0.6375 & -0.8686 \\ 0.6375 & 0.8337 & -1.186 \\ -0.8686 & -1.186 & 18.24 \end{bmatrix}$	3
4	$\begin{bmatrix} 0.1989 & 0 & 0.2470 \\ 0 & 0 & 0 \\ 0.2470 & 0 & 0.3758 \end{bmatrix}$	2	$\begin{bmatrix} 3.132 & 3.837 & -5.050 \\ 3.837 & 5.018 & -6.928 \\ -5.050 & -6.928 & 26.38 \end{bmatrix}$	3
5	$\begin{bmatrix} 0.2774 & 0 & 0.1926 \\ 0 & 0 & 0 \\ 0.1926 & 0 & 0.4135 \end{bmatrix}$	2	$\begin{bmatrix} 4.368 & 5.350 & 2.056 \\ 5.350 & 6.996 & 2.437 \\ 2.056 & 2.437 & 70.73 \end{bmatrix}$	3
6	$\begin{bmatrix} 0.3613 & 0 & 0.0708 \\ 0 & 0 & 0 \\ 0.0708 & 0 & 0.5905 \end{bmatrix}$	2	$\begin{bmatrix} 5.689 & 6.968 & -2.941 \\ 6.968 & 9.113 & -3.939 \\ -2.941 & -3.939 & 90.11 \end{bmatrix}$	3
7	$\begin{bmatrix} 0.4033 & 0 & -0.0476 \\ 0 & 0 & 0 \\ -0.0476 & 0 & 0.9237 \end{bmatrix}$	2	$\begin{bmatrix} 6.352 & 7.779 & -2.104 \\ 7.779 & 10.17 & -2.690 \\ -2.104 & -2.690 & 91.92 \end{bmatrix}$	3

of visible satellites is 2, whereas the rank of the observability Gramian is 3 when the number of visible satellites is greater than two. These findings show that the system is externally observable when more than two satellites are visible. These conclusions are consistent with the observability analysis in Section III.

V. CONCLUSIONS

In this paper, the observability and observable degree of attitude errors in a tightly coupled multi-antenna GNSS/INS attitude determination system are analyzed. The mathematical relationship between the observable degree of attitude errors and the corresponding ATDOP is determined by the SVD of the observability Gramian. The influence of the PIDOP, RDOP and YDOP on the observable degrees of pitch, roll and yaw errors is studied. The observability of attitude errors with different numbers of visible satellites is analyzed by the rank test of the observability Gramian.

A field test is conducted to verify the observable degree and observability analysis. The results show that for the pitch, roll and yaw at a given time, the attitude with the lower DOP has a higher observability degree. For any one of the three attitudes at different times, the lower DOP also leads to the higher observable degree when the number of visible satellites is the same. The rank test of the observability Gramian shows that only one of the attitude errors is externally observable for the two visible satellites scheme and that two states are externally observable for the more than two visible satellites scheme in the dual-antenna GNSS/INS attitude determination. Two of the attitude errors are externally observable for the two visible satellites scheme, and all states are externally observable for the more than two visible satellites scheme in the three-antenna GNSS/INS attitude determination.

In addition to the observability of attitude errors discussed in this paper, other error states, such as the position error, velocity error, gyros bias and accelerometer bias, can be observed in a tightly coupled GNSS/INS integrated attitude determination system. A considerable amount of additional work is required to thoroughly answer the questions related to these errors, which represent potential research subjects for further studies.

REFERENCES

[1] Y. G. Tang, Y. Wu, M. Wu, W. Wu, X. Hu, and L. Shen, "INS/GPS integration: Global observability analysis," *IEEE Trans. Veh. Technol.*, vol. 58, no. 3, pp. 1129–1142, Mar. 2009.

[2] V. L. Bageshwar, D. Gebre-Egziabher, W. L. Garrard, and T. T. Georgiou, "A stochastic observability test for discrete-time Kalman filters," *J. Guid., Control, Dyn.*, vol. 32, no. 4, pp. 1356–1370, 2009.

[3] Y. Ma, J. Fang, W. Wang, and J. Li, "Decoupled observability analyses of error states in INS/GPS integration," *J. Navigat.*, vol. 67, no. 3, pp. 473–494, Jan. 2014.

[4] J.-H. Yoon and H. Peng, "A cost-effective sideslip estimation method using velocity measurements from two GPS receivers," *IEEE Trans. Veh. Technol.*, vol. 63, no. 6, pp. 2589–2599, Jul. 2014.

[5] D. Goshen-Meskin and I. Y. Bar-Itzhack, "Observability analysis of piece-wise constant systems—Part I: theory," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 28, no. 4, pp. 1056–1067, Oct. 1992.

[6] D. Goshen-Meskin and I. Y. Bar-Itzhack, "Observability analysis of piece-wise constant systems—Part II: Application to inertial navigation in-flight alignment," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 28, no. 4, pp. 1068–1075, Oct. 1992.

[7] I. Rhee, M. F. Abdel-Hafez, and J. L. Speyer, "Observability of an integrated GPS/INS during maneuvers," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 40, no. 2, pp. 526–535, Apr. 2004.

[8] C. Fan, X. Hu, X. He, K. Tang, and B. Luo, "Observability analysis of a MEMS INS/GPS integration system with gyroscope G-sensitivity errors," *Sensors*, vol. 14, no. 9, pp. 16003–16016, Aug. 2014.

[9] J. Jiang, F. Yu, H. Lan, and Q. Dong, "Instantaneous observability of tightly coupled SINS/GPS during maneuvers," *Sensors*, vol. 16, no. 6, pp. 765–787, May 2016.
[10] S. Hong, M. H. Lee, J. A. Rios, and J. L. Speyer, "Observability analysis of INS with a GPS multi-antenna system," *Korean Soc. Mech. Eng. Int. J.*, vol. 16, no. 11, pp. 1367–1378, 2002.
[11] S. Hong, M. H. Lee, H.-H. Chun, S.-H. Kwon, and J. L. Speyer, "Observability of error States in GPS/INS integration," *IEEE Trans. Veh. Technol.*, vol. 54, no. 2, pp. 731–743, Mar. 2005.
[12] M. K. Lee, S. Hong, M. H. Lee, S.-H. Kwon, and H.-H. Chun, "Observability analysis of alignment errors in GPS/INS," *J. Mech. Sci. Technol.*, vol. 19, no. 6, pp. 1253–1267, Jun. 2005.
[13] F. M. Ham and R. G. Brown, "Observability, eigenvalues, and Kalman filtering," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-19, no. 2, pp. 269–273, Mar. 1983.
[14] X. Cheng, D. Wan, and X. Zhong, "Study on observability and its degree of strapdown inertial navigation system," *J. Southeast Univ.*, vol. 27, no. 6, pp. 6–11, 1997.
[15] Y. Li, Y. Li, C. Rizos, and X. Xu, "Observability analysis of SINS/GPS during in-motion alignment using singular value decomposition," *Adv. Mater. Res.*, vols. 433–440, pp. 5918–5923, Jan. 2012.
[16] M. Wang, J. Chen, C. Song, Y. Han, and C. Qin, "Observable degree analysis of DGPS/SINS calibration based on singular value decomposition," in *Proc. 35th Chin. Control Conf.*, Chengdu, China, Jul. 2016, pp. 5648–5653.
[17] S. Hong, M. H. Lee, H.-H. Chun, S.-H. Kwon, and J. L. Speyer, "Experimental study on the estimation of lever arm in GPS/INS," *IEEE Trans. Veh. Technol.*, vol. 55, no. 2, pp. 431–448, Mar. 2006.
[18] S. Hong, H.-H. Chun, S.-H. Kwon, and M. H. Lee, "Observability measures and their application to GPS/INS," *IEEE Trans. Veh. Technol.*, vol. 57, no. 1, pp. 97–106, Jan. 2008.
[19] Y. F. Jiang and Y. P. Lin, "Error estimation of INS ground alignment through observability analysis," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 28, no. 1, pp. 92–97, Jan. 1992.
[20] S. Du, W. Sun, and Y. Gao, "Improving observability of an inertial system by rotary motions of an IMU," *Sensors*, vol. 17, no. 4, pp. 698–717, Apr. 2017.
[21] H. He, Y. Zheng, D. Yang, J. Zhang, and S. Wang, "GDOP's influence on observable degree of multi-antenna GPS/SINS integrated attitude measuring system," *World J. Eng. Technol.*, vol. 3, no. 3C, pp. 354–359, Oct. 2015.
[22] P. J. G. Teunissen, "Integer least squares theory for the GNSS compass," *J. Geodesy*, vol. 84, no. 7, pp. 433–447, 2010.
[23] P. J. G. Teunissen, "Testing of a new single-frequency GNSS carrier phase attitude determination method: Land, ship and aircraft experiments," *GPS Solutions*, vol. 15, no. 1, pp. 15–28, 2011.
[24] G. Giorgi, P. J. G. Teunissen, S. Verhagen, and P. J. Buist, "Instantaneous ambiguity resolution in global-navigation-satellite-system-based attitude determination applications: A multivariate constrained approach," *J. Guid., Control, Dyn.*, vol. 35, no. 1, pp. 51–67, 2012.
[25] D. Simon, *Optimal State Estimation: Kalman, H_∞, and Nonlinear Approaches*. Hoboken, NJ, USA: Wiley, 2006.
[26] S. F. Gomez, "Attitude determination and attitude dilution of precision (ADOP) results for international space station global positioning system (GPS) receiver," in *Proc. 13th Int. Tech. Meeting Satellite Division Inst. Navigat. (ION GPS)*, Salt Lake City, UT, USA, Sep. 2000, pp. 1995–2002.
[27] D. Sun and J. L. Crassidis, "Observability analysis of six-degree-of-freedom configuration determination using vector observations," *J. Guid. Control Dyn.*, vol. 25, no. 6, pp. 1149–1157, 2002.
[28] F. M. Callier and C. A. Desoer, *Linear System Theory*. New York, NY, USA: Springer-Verlag, 1991.
[29] X. Zhang, *Matrix Analysis and Applications*, 2nd ed. Beijing, China: Tsinghua Univ. Press, 2013.
[30] Y. Wang, X. Zhao, C. Pang, B. Feng, H. Tong, and L. Zhang, "BDS and GPS stand-alone and integrated attitude dilution of precision definition and comparison," *Adv. Space Res.*, to be published, doi: 10.1016/j.asr.207.11.032.
[31] G. Giorgi, P. J. G. Teunissen, S. Verhagen, and P. J. Buist, "Testing a new multivariate GNSS carrier phase attitude determination method for remote sensing platforms," *Adv. Space Res.*, vol. 46, no. 2, pp. 118–129, 2010.

YONG WANG received the B.Sc. degree in navigation engineering and the M.Sc. degree in communication and information system from the Information and Navigation College, Air Force Engineering University, Xi'an, China, in 2012 and 2014, respectively.

He is currently pursuing the Ph.D. degree with the Information and Navigation College, Air Force Engineering University. His research interests include GNSS/INS integrated attitude determination and instantaneous ambiguity resolution.

XIUBIN ZHAO was born in Hubei, China, in 1965. He received the B.Sc. degree from the Telecommunication Engineering Institute, Air Force Engineering University, Xi'an, China, in 1988, the M.Sc. degree from the School of Electronic Information, Nanjing University of Aeronautics and Astronautics, Nanjing, China, in 1991, and the Ph.D. degree from the School of Electronic and Information, Northwestern Polytechnical University, Xi'an, in 2006.

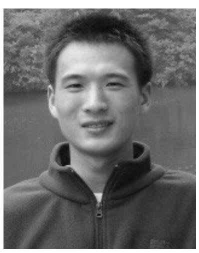
He is currently a Full Professor with the Information and Navigation College, Air Force Engineering University. His research interests include GNSS and its integrated navigation system, aircraft navigation and control.

CHUNLEI PANG received the B.Sc. degree in navigation engineering and the M.Sc. degree in communication and information system from the Telecommunication Engineering Institute, Air Force Engineering University, Xi'an, China, in 2007 and 2009, respectively, and the Ph.D. degree from the Information and Navigation College, Air Force Engineering University, in 2013.

He is currently a Lecturer with the Information and Navigation College, Air Force Engineering University. His research interests include GNSS high-precision positioning and GNSS/INS integrated navigation.

BO FENG received the B.S. degree in automation from Beijing Information Science and Technology University, Beijing, China, in 2009, and the Ph.D. degree from the School of Automation, Beijing Institute of Technology, Beijing, in 2015.

He is currently a Lecturer with the Information and Navigation College, Air Force Engineering University. His research interests include linear filtering algorithm, stability analysis, and inertial navigation technology and its applications.



LIANG ZHANG received the B.Sc. degree in surveying engineering from the School of Geodesy and Geomatics, Wuhan University, Wuhan, China, in 2010, and the Ph.D. degree in aeronautical and astronautical science and technology from the College of Aerospace and Engineering, National University of Defense Technology, Changsha, China, in 2013.

He is currently a Lecturer with the Information and Navigation College, Air Force Engineering University. His research interests include GNSS/SINS integrated navigation and its high-precision data processing.