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H_∞ Synchronization for Uncertain Time-Delay Chaotic Systems With One-Sided Lipschitz Nonlinearity

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ABSTRACT This paper addresses H_∞ synchronization for uncertain chaotic systems with one-sided Lipschitz nonlinearity under the output and intrinsic state delays. By utilizing the one-sided Lipschitz condition and quadratic inner boundedness, constructing an appropriate Lyapunov–Krasovskii (LKF), robust controller design conditions based on Lyapunov stability theory are derived for synchronization of chaotic systems under disturbances or perturbations bounded by L_2 norm. By introducing the delay-derivative limits and delay-interval bounds into LKF, the intrinsic state time-varying delay can be tackled by the delay-range-dependent strategy. Less conservative stability condition can be obtained by the further improved inequality of Jensen inequality and reciprocally convex approach, which can lead to the tighter upper bound for integral inequality. Numerical simulations are provided to verify the validity of the proposed methodology for synchronization of chaotic systems.

INDEX TERMS Chaos synchronization, one-sided Lipschitz nonlinearity, delay-range-dependency, reciprocally convex approach.

I. INTRODUCTION

Owing to the appealing properties of chaotic system like unpredictability, strange attractors and sensitivity to initial conditions, chaos synchronization has flourished as an appealing research area in various applications, such as image encryption, communication security, biomedical engineering, chemical reaction, brain disorder and so on [1]–[5]. The main purpose of chaos synchronization is to realize the identical behavior between the drive and response systems by means of feedback control. In the several past years, all sorts of nonlinear control methods [6]–[10] have been focused on the synchronization of chaotic systems, such as impulse control [11], adaptive control [12] and fuzzy control [13], etc.

Recently, a novel finite-time synchronization scheme aimed at the hybrid systems with lags has been presented in [14], in which a point of view on synchronization is set forth on the proposed system model.Based on the receding horizon control and Takagi-Sugeno (TCS) fuzzy model,a new H_∞ synchronization method is proposed for chaotic systems with external disturbance in [15]. By applying a sliding mode control approach with the time-varying switching surface, the synchronization control law has been derived in [16], which can synchronize two fractional-order chaotic systems precisely at a pre-specified time without concerning about the initial conditions. In terms of time-delay chaotic systems under the unknown and determined parameters, Ahmad *et al.* [17] put forward an original robust synchronization controller design law to investigate the reduced-order chaos synchronization behaviour. In [18], the synchronization behaviour for the chaotic systems with the coupling and intrinsic state time-varying delays has been investigated.

Although some achievements have been obtained in the above-mentioned works, there are still many problems little studied to be worthy of attention in the field. In practical applications of chaos synchronization theory, there inevitably exist some factors affecting the system performance, such as the time-delay, perturbations, disturbances and external uncertainties [19]–[21]. For instance, time-delay can bring about inaccurate feedback control behaviours, and then result in the instability and non-synchronous phenomena of chaotic systems. In addition, some efforts should be still made to reduce the conservatism in order to synchronize the chaotic systems more accurately and efficiently. Motivated by the above discussions, the paper addresses the robust H_∞ controller design for synchronization of chaotic systems subjected to the time-delay and external uncertainties. Based on the one-sided Lipschitz condition and the quadratic inner boundedness [22], we design a robust feedback controller for synchronization of time-delay chaotic systems against disturbances bounded by the L_2 norm [23]. By introducing the delay-interval bounds and delay-derivative limits into LKF, the intrinsic state time-varying delay is handled by the delay-range-dependent strategy. Rather than the Jensen and Wirtinger inequalities, the tighter upper bound for inequality can be obtained by employing the reciprocally convex approach [24], the improved method based on Jensen inequality [25].

Major contributions of the study lie are summarized in following points. (i) Rather than most studies that use the conventional Lipschitz condition for deriving synchronization controller design condition [18], [26], [27], in the paper, we employ the one-side Lipschitz condition [28] for derivation, which has less conservatism and a wide range of applications with the smaller value than Lipschitz constant. (ii) To implement easily and control precisely in practical engineering, the robust H_∞ mixed output time-delay feedback controller is proposed in contrast to the conventional state feedback controller [2], [5], [17], [18]. That is, the past state information as well as the current is employed, and the suppression control of external uncertainties can be achieved by regulating the disturbance attenuation parameter. (iii) To reduce the conservatism, the reciprocally convex approach, improved method based on Jensen inequality, one-sided Lipschitz condition and quadratic inner boundedness are employed for the derivation of synchronization control condition in comparison with the aforementioned works.

The rest of the paper is as shown as follows: Section II exhibits the problem description and the preliminaries. The proposed robust synchronization controller design condition is provided in Section III. Simulation results are detailed in Section IV, Section V presents conclusions.

Notation: In addition to usual notations, A^+ is the pseudo-inverse matrix of A, $\langle\cdot\rangle$ denotes the inner product, $||\cdot||$ and $||\cdot||_2$ represent the Euclidean norm and the L_2 norm, where $||\cdot||_2 = \sqrt{\int_0^\infty ||\cdot||^2 dt}$. The column vector and block diagonal matrix are expressed by $col\{\cdot\}$ and $diag\{\cdot\}$. $e_i^T = col\{0_{n*(i-1)n}, I_{n*n}, 0_{n*(10-i)n}\}$ shows the block entry matrix for $i = 1, 2, \ldots, 10$. For simplicity, the notation $x(t)$ can be abbreviated as x.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following drive-response systems:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + A_1x(t-\tau(t)) + \vartheta(x(t), t)\\ &\quad +\hbar(x(t-\tau(t)), t) + H\omega_x(t),\\ z_x(t) &= Cx(t), x(t) = \ell_x(t),\end{aligned} \tag{1}$$

$$\begin{aligned}\dot{y}(t) &= Ay(t) + A_1y(t-\tau(t)) + \vartheta(y(t), t)\\ &\quad +\hbar(y(t-\tau(t)), t) + H\omega_y(t) + Bu(t),\\ z_y(t) &= Cy(t), y(t) = \ell_y(t),\end{aligned} \tag{2}$$

where $x, y \in R^n$, $z_x, z_y \in R^m$, $u \in R^p$ and $\omega_x, \omega_y \in R^q$ represent the state vectors, output vectors, control input vector and external disturbances respectively. A, A_1, B, C and H having appropriate dimensions denote the constant matrices. The term τ represents the time-varying delays caused by the inherent modeling of drive-response systems, satisfying

$$0 \le \tau_m \le \tau(t) \le \tau_M, \dot{\tau}(t) \le \mu. \tag{3}$$

In addition, $\vartheta(x, t)$, $\vartheta(y, t)$ and $\hbar(x(t-\tau), t)$, $\hbar(y(t-\tau), t)$ represent the nonlinear dynamics of drive-response systems in the absence and presence of time-delay. The initial conditions are described by ℓ_x and ℓ_y respectively.

Define $e = x - y$, and the error system is derived as

$$\begin{aligned}\dot{e} &= Ae + A_1e(t-\tau) + \vartheta(x, t) - \vartheta(y, t)\\ &\quad +\hbar(x(t-\tau_2), t) - \hbar(y(t-\tau_2), t)\\ &\quad +H(\omega_x - \omega_y) - Bu.\end{aligned} \tag{4}$$

Construct a mixed output time-delay feedback controller as

$$\begin{aligned}u = K\big((z_x - z_y) &- \varepsilon_1\big(z_x(t-d) - z_y(t-d)\big)\\ &-\varepsilon_2\int_{t-d}^{t}\big(z_x(\varphi) - z_y(\varphi)\big)d\varphi\big),\end{aligned} \tag{5}$$

where K represents the gain matrix of appropriate dimensions, ε_1 and ε_2 are regulatory factors which can regulate the strength of delayed feedback, τ_1 denotes the constant output delay.

Combining (4) and (5), we can obtain

$$\begin{aligned}\dot{e} &= (A - BKC)e + A_1e(t-\tau) + \varepsilon_1 BKCe(t-d)\\ &\quad +\Phi(x, y, t) + \Upsilon(x, y, \tau, t) + H\omega\\ &\quad +\varepsilon_2 BKC\int_{t-d}^{t} e(\varphi)d\varphi,\end{aligned} \tag{6}$$

where

$$\begin{aligned}\Phi(x, y, t) &= \vartheta(x, t) - \vartheta(y, t),\\ \Upsilon(x, y, \tau, t) &= \hbar(x(t-\tau), t) - \hbar(y(t-\tau), t).\end{aligned}$$

To derive the robust H_∞ synchronization control condition, following assumptions and lemmas need be satisfied.

Assumption 1: Nonlinear terms ϑ and $\hbar$ are one-sided Lipschitz if there exist

$$\begin{aligned}&\langle\vartheta(x, t) - \vartheta(y, t), x - y\rangle \le \nu_1||(x-y)||^2,\\ &\quad\langle\hbar(y(t-\tau), t) - \hbar(x(t-\tau), t), y(t-\tau) - x(t-\tau)\rangle\\ &\quad\le \nu_2||(y(t-\tau) - x(t-\tau))||^2,\end{aligned}$$

where $\nu_1, \nu_2 \in R$ are the one-sided Lipschitz constants.

Assumption 2: For ϑ and $\hbar$, there exist the quadratic inner boundedness conditions given by

$$\begin{aligned}
&(\vartheta(x,t)-\vartheta(y,t))^T(\vartheta(x,t)-\vartheta(y,t))\\
&\quad\leq \kappa_1||(x-y)||^2+\varpi_1\langle x-y,\vartheta(x,t)-\vartheta(y,t)\rangle,\\
&(\hbar(y(t-\tau),t)-\hbar(x(t-\tau),t))^T\\
&(\hbar(y(t-\tau),t)-\hbar(x(t-\tau),t))\\
&\leq \kappa_2||(y(t-\tau)-x(t-\tau))||^2\\
&\quad+\varpi_2\langle y(t-\tau)-x(t-\tau),\hbar(y(t-\tau),t)-\hbar(x(t-\tau),t)\rangle,
\end{aligned}$$

for scalars $\kappa_1,\kappa_2\in R$ and $\varpi_1,\varpi_2\in R$.

Assumption 3: The error system satisfies the H_∞ performance index with the zero initial condition, if there exists [18]

$$\int_0^\infty e^T(t)e(t)dt\leq\gamma^2\int_0^\infty\omega^T(t)\omega(t)dt,$$

where γ represents the disturbance attenuation rate.

Lemma 1: For the positive-definite matrices $Z_1=Z_1^T\in R^{m\times m}$ and $Z_2=Z_2^T\in R^{n\times n}$, there exists the matrix $S\in R^{m\times n}$ satisfying $\begin{bmatrix} Z_1 & S\\ * & Z_2\end{bmatrix}\geq 0$. If $\beta\in(0,1)$, the following inequality holds [24]

$$\begin{bmatrix}\frac{1}{\beta}Z_1 & 0\\ * & \frac{1}{1-\beta}Z_2\end{bmatrix}\geq\begin{bmatrix} Z_1 & S\\ * & Z_2\end{bmatrix}.$$

Lemma 2: For the continuously differentiable function $x(t)$, assume that $a<b$ and $Y=Y^T>0$. Then, there exists [25]

$$\begin{aligned}
-\int_a^b\dot{x}^T(\varphi)Y\dot{x}(\varphi)d\varphi&\leq-\frac{1}{b-a}\Im_0^T(a,b)Y\Im_0(a,b)\\
&-\frac{3}{b-a}\Im_1^T(a,b)Y\Im_1(a,b)-\frac{5}{b-a}\Im_2^T(a,b)Y\Im_2(a,b).
\end{aligned}$$

where

$$\begin{aligned}
\Im_0(a,b)&=x(b)-x(a),\\
\Im_1(a,b)&=x(b)+x(a)-\frac{2}{b-a}\int_a^b x(\varphi)d\varphi,\\
\Im_2(a,b)&=x(b)-x(a)-\frac{12}{(b-a)^2}\int_a^b(\varphi-\frac{a+b}{2})x(\varphi)d\varphi.
\end{aligned}$$

III. ROBUST H_∞ SYNCHRONIZATION CONTROLLER DESIGN

In this section, the robust H_∞ control conditions are provided for synchronization of chaotic systems. Define a vector as

$$\begin{aligned}
\zeta^T=\big[&e^T,e^T(t-\tau),e^T(t-\tau_m),e^T(t-\tau_M),\\
&e^T(t-d),\Phi^T,\Upsilon^T,\omega^T,\frac{1}{\tau_m}\int_{t-\tau_m}^t e^T(\varphi)d\varphi,\\
&\frac{1}{d}\int_{t-d}^t e^T(\varphi)d\varphi,\frac{1}{\tau_m^2}\int_{t-\tau_m}^t(\varphi-\frac{2t-\tau_m}{2})e^T(\varphi)d\varphi,\\
&\frac{1}{d^2}\int_{t-d}^t(\varphi-\frac{2t-d}{2})e^T(\varphi)d\varphi.\big].
\end{aligned}\tag{7}$$

Theorem 1: Consider the error system (6) under $\omega\neq 0$ satisfying Assumption 1-3 and condition (3). By application of the controller (5), the error system asymptotically converges to zero if there exist the positive-definite and symmetric matrices P, Q_i, Z_j, matrix V of appropriate dimensions satisfying $\Theta=\begin{bmatrix} Z_2 & V\\ * & Z_2\end{bmatrix}\geq 0$, and positive scalars ε_k, ϵ_i, and for $i=1,2,3,4$, $j=1,2,3$, $k=1,2$, such that the inequality

$$\begin{bmatrix}\Pi & \tau_m\Xi^TZ_1 & \tau_M\Xi^TZ_2 & d\Xi^TZ_3\\ * & -Z_1 & 0 & 0\\ * & * & -Z_2 & 0\\ * & * & * & -Z_3\end{bmatrix}<0,\tag{8}$$

is satisfied, where

$$\begin{aligned}
\Pi&=e_1^TP\Xi+\Xi^TPe_1+\Lambda+e_1^Te_1-\gamma^2e_8^Te_8\\
&\quad-\Gamma_1^T\Theta\Gamma_1-\Gamma_2^TW_1\Gamma_2-\Gamma_3^TW_2\Gamma_3\\
&\quad-e_6^T\epsilon_1Ie_1+e_1^T\epsilon_2\varpi_1Ie_6-e_6^T\epsilon_2Ie_6\\
&\quad-e_7^T\epsilon_3Ie_2+e_2^T\epsilon_4\varpi_2Ie_7-e_7^T\epsilon_4Ie_7\\
&\quad+e_1^T(\epsilon_1\nu_1I+\epsilon_2\kappa_1I)e_1+e_2^T(\epsilon_3\nu_2I+\epsilon_4\kappa_2I)e_2,\\
\Xi&=(A-BKC)e_1+A_1e_2+\varepsilon_1BKCe_5\\
&\quad+Ie_6+Ie_7+He_8+\varepsilon_2dBKCe_{10},\\
\Lambda&=diag\big\{Q_1+Q_2+Q_3+Q_4,-(1-\mu)Q_3,\\
&\quad-Q_1,-Q_2,-Q_4,0,0,0,0,0\big\},\\
\Gamma_1&=col\big\{e_2-e_4,e_3-e_2\big\},\ \tau_{Mm}=\tau_M-\tau_m,\\
\Gamma_2&=col\big\{e_1-e_3,e_1+e_3-2e_9,e_1-e_3-12e_{11}\big\},\\
\Gamma_3&=col\big\{e_1-e_5,e_1+e_5-2e_{10},e_1-e_5-12e_{12}\big\},\\
W_1&=diag\left\{Z_1,3Z_1,5Z_1\right\},\ W_2=diag\left\{Z_3,3Z_3,5Z_3\right\}.
\end{aligned}$$

Proof: Consider the following LKF candidate:

$$V(e,t)=V_1(e,t)+V_2(e,t)+V_3(e,t),\tag{9}$$

where

$$\begin{aligned}
V_1(e,t)&=e^T(t)Pe(t),\\
V_2(e,t)&=\int_{t-\tau_m}^t e^T(\varphi)Q_1e(\varphi)d\varphi+\int_{t-\tau_M}^t e^T(\varphi)Q_2e(\varphi)d\varphi\\
&\quad+\int_{t-\tau}^t e^T(\varphi)Q_3e(\varphi)d\varphi+\int_{t-d}^t e^T(\varphi)Q_4e(\varphi)d\varphi,\\
V_3(e,t)&=\int_{-\tau_m}^0\int_{t+s}^t\tau_m\dot{e}^T(\varphi)Z_1\dot{e}(\varphi)d\varphi ds\\
&\quad+\int_{-\tau_M}^{-\tau_m}\int_{t+s}^t\tau_{Mm}\dot{e}^T(\varphi)Z_2\dot{e}(\varphi)d\varphi ds\\
&\quad+\int_{-d}^0\int_{t+s}^t d\dot{e}^T(\varphi)Z_3\dot{e}(\varphi)d\varphi ds.
\end{aligned}$$

Taking the derivative of (9) along the trajectory of (3) and (6) yields

$$\begin{aligned}
\dot{V}_1(e,t)&=2e^TP\big[(A-BKC)e+A_1e(t-\tau)\\
&\quad+\varepsilon_1BKCe(t-d)+\Phi(x,y,t)+\Upsilon(x,y,\tau,t)\\
&\quad+H\omega+\varepsilon_2BKC\int_{t-d}^t e(\varphi)d\varphi\big],\\
&=\zeta^T\left\{e_1^TP\Xi+\Xi^TPe_1\right\}\zeta,
\end{aligned}\tag{10}$$

$$\begin{aligned}\dot{V}_2(e,t) \leq\ & e^T Q_1 e - e^T(t-\tau_m)Q_1 e(t-\tau_m) + e^T Q_2 e\\ & - e^T(t-\tau_M)Q_2 e(t-\tau_M) + e^T Q_3 e\\ & -(1-\mu)e^T(t-\tau)Q_3 e(t-\tau) + e^T Q_4 e\\ & - e^T(t-d)Q_4 e(t-d)\\ =\ & \zeta^T \Lambda \zeta,\end{aligned} \tag{11}$$

$$\begin{aligned}\dot{V}_3(e,t) =\ & \zeta^T \Xi^T \left(\tau_m^2 Z_1 + \tau_{Mm}^2 Z_2 + d^2 Z_3\right)\Xi\zeta\\ & - \int_{t-\tau_m}^{t} \tau_m \dot{e}^T(\varphi) Z_1 \dot{e}(\varphi)d\varphi\\ & - \int_{t-\tau_M}^{t-\tau_m} \tau_{Mm}\dot{e}^T(\varphi)Z_2\dot{e}(\varphi)d\varphi\\ & - \int_{t-d}^{t} d\dot{e}^T(\varphi)Z_3\dot{e}(\varphi)d\varphi.\end{aligned} \tag{12}$$

According to Assumption 1-2, we have

$$\begin{aligned}&\epsilon_1\nu_1 e^T e - \epsilon_1 \Phi^T\\ &\quad e = \zeta^T\left(e_1^T\epsilon_1\nu_1 I e_1 - e_6^T\epsilon_1 I e_1\right)\zeta \geq 0,\\ &\epsilon_2\kappa_1 e^T e + \epsilon_2\varpi_1 e^T\Phi - \epsilon_2\Phi^T\Phi\\ &\quad = \zeta^T\left(e_1^T\epsilon_2\kappa_1 I e_1 + e_1^T\epsilon_2\varpi_1 I e_6 - e_6^T\epsilon_2 I e_6\right)\zeta \geq 0,\end{aligned} \tag{13}$$

and

$$\begin{aligned}&\epsilon_3\nu_2 e^T(t-\tau)e(t-\tau) - \epsilon_3\Upsilon^T e(t-\tau)\\ &\quad = \zeta^T\left(e_2^T\epsilon_3\nu_2 I e_2 - e_7^T\epsilon_3 I e_2\right)\zeta \geq 0,\\ &\epsilon_4\kappa_2 e^T(t-\tau)e(t-\tau) + \epsilon_4\varpi_2 e^T(t-\tau)\Upsilon - \epsilon_4\Upsilon^T\Upsilon\\ &\quad = \zeta^T\left(e_2^T\epsilon_4\kappa_2 I e_2 + e_2^T\epsilon_4\varpi_2 I e_7 - e_7^T\epsilon_4 I e_7\right)\zeta \geq 0,\end{aligned} \tag{14}$$

where ϵ_i represents the positive parameter for adjusting the range.

Let $\beta_1 = (\tau_M - \tau)/\tau_{Mm}$ and $\beta_2 = (\tau - \tau_m)/\tau_{Mm}$. It is obvious that $\beta_k > 0$ and $\beta_1 + \beta_2 = 1$. Introduce a matrix V such that $\begin{bmatrix} Z_2 & V \\ * & Z_2\end{bmatrix}$. For the integral terms in (12), applying Lemma 1-2 can reveal

$$\begin{aligned}&-\int_{t-\tau_M}^{t-\tau_m} \tau_{Mm}\dot{e}^T(\varphi)Z_2\dot{e}(\varphi)d\varphi\\ &\quad = -\int_{t-\tau_M}^{t-\tau} \tau_{Mm}\dot{e}^T(\varphi)Z_2\dot{e}(\varphi)d\varphi\\ &\qquad - \int_{t-\tau}^{t-\tau_m} \tau_{Mm}\dot{e}^T(\varphi)Z_2\dot{e}(\varphi)d\varphi\\ &\quad \leq -\frac{1}{\beta_1}\left[e(t-\tau) - e(t-\tau_M)\right]^T Z_2 \left[e(t-\tau) - e(t-\tau_M)\right]\\ &\qquad -\frac{1}{\beta_2}\left[e(t-\tau_m) - e(t-\tau)\right]^T Z_2\left[e(t-\tau_m) - e(t-\tau)\right]\\ &\quad \leq -\zeta^T\Gamma_1^T\Theta\Gamma_1\zeta,\end{aligned} \tag{15}$$

and

$$\begin{aligned}&-\int_{t-\tau_m}^{t}\tau_m\dot{e}^T(\varphi)Z_1\dot{e}(\varphi)d\varphi - \int_{t-d}^{t} d\dot{e}^T(\varphi)Z_3\dot{e}(\varphi)d\varphi\\ &\qquad \leq -\zeta^T\Gamma_2^T W_1\Gamma_2\zeta - \zeta^T\Gamma_3^T W_2\Gamma_3\zeta.\end{aligned} \tag{16}$$

Combining (10) − (16), we have

$$\begin{aligned}\dot{V}(e,t) \leq \zeta^T\Big\{& e_1^T P\Xi + \Xi^T P e_1 + \Lambda - \Gamma_1^T\Theta\Gamma_1 - \Gamma_2^T W_1\Gamma_2\\ & - \Gamma_3^T W_2\Gamma_3 + \Xi^T\left(\tau_m^2 Z_1 + \tau_{Mm}^2 Z_2 + d^2 Z_3\right)\Xi\\ & - e_6^T\epsilon_1 I e_1 + e_1^T\epsilon_2\varpi_1 I e_6 - e_6^T\epsilon_2 I e_6\\ & - e_7^T\epsilon_3 I e_2 + e_2^T\epsilon_4\varpi_2 I e_7 - e_7^T\epsilon_4 I e_7\\ & + e_1^T\left(\epsilon_1\nu_1 I + \epsilon_2\kappa_1 I\right)e_1 + e_2^T\left(\epsilon_3\nu_2 I + \epsilon_4\kappa_2 I\right)e_2\Big\}\zeta,\end{aligned} \tag{17}$$

According to Assumption 3, in order to ensure H_∞ performance, the quadratic performance index $J(e,\omega)$ is defined as

$$J(e,\omega) = \int_0^\infty \left[e^T e - \gamma^2\omega^T\omega\right]d\varphi, \tag{18}$$

and under the zero initial condition, we can derive

$$\begin{aligned}J(e,\omega) &= \int_0^\infty \left[e^T e - \gamma^2\omega^T\omega + \dot{V}(e,t)\right]d\varphi - V(e,t)\big|_{t\to\infty}\\ &\leq \int_0^\infty \left[e^T e - \gamma^2\omega^T\omega + \dot{V}(e,t)\right]d\varphi\\ &\leq \int_0^\infty \zeta^T\Delta\zeta d\varphi,\end{aligned} \tag{19}$$

where

$$\begin{aligned}\Delta =\ & e_1^T P\Xi + \Xi^T P e_1 + \Lambda - \Gamma_1^T\Theta\Gamma_1 - \Gamma_2^T W_1\Gamma_2 - \Gamma_3^T W_2\Gamma_3\\ & + \Xi^T\left(\tau_m^2 Z_1 + \tau_{Mm}^2 Z_2 + d^2 Z_3\right)\Xi + e_1^T e_1 - \gamma^2 e_8^T e_8\\ & - e_6^T\epsilon_1 I e_1 + e_1^T\epsilon_2\varpi_1 I e_6 - e_6^T\epsilon_2 I e_6\\ & - e_7^T\epsilon_3 I e_2 + e_2^T\epsilon_4\varpi_2 I e_7 - e_7^T\epsilon_4 I e_7\\ & + e_1^T\left(\epsilon_1\nu_1 I + \epsilon_2\kappa_1 I\right)e_1 + e_2^T\left(\epsilon_3\nu_2 I + \epsilon_4\kappa_2 I\right)e_2.\end{aligned}$$

When $\Delta < 0$ holds, $J(e,\omega) < V(e,t)\big|_{t\to 0} = 0$ shows the H_∞ performance. By application of Schur complement to $\Delta < 0$, the inequality (8) can be produced, which is sufficient for stability analysis. This completes the proof of Theorem 1.

By employing a known mixed output time-delay feedback controller (5), the condition in Theorem 1 can be employed for stability analysis of the error system (4). Now a sufficient condition is derived for obtaining the controller gain K.

Theorem 2: A sufficient condition for the solution in Theorem 1 is that there exist the positive-definite and symmetric matrices X, $\bar{Q}_i$, $\bar{Z}_j$, matrices $\bar{V}$ and M of appropriate dimensions satisfying $\bar{\Theta} = \begin{bmatrix}\bar{Z}_2 & \bar{V}\\ * & \bar{Z}_2\end{bmatrix} \geq 0$, positive scalars ε_k, ϵ_i for $i = 1,2,3,4$, $j = 1,2,3$, $k = 1,2$, such that the following condition

$$\begin{bmatrix}\bar{\Pi} & \Psi & e_1^T X & e_1^T\sqrt{|\eta_1|}X & e_2^T\sqrt{|\eta_2|}X\\ * & -\Omega & 0 & 0 & 0\\ * & * & -I & 0 & 0\\ * & * & * & -I & 0\\ * & * & * & * & -I\end{bmatrix} < 0, \tag{20}$$

is satisfied. where

$$
\begin{aligned}
\bar{\Pi} &= e_1^T \bar{\Xi} X + X\bar{\Xi}^T e_1 + \bar{\Lambda} - \Gamma_1^T \bar{\Theta}\Gamma_1 - \Gamma_2^T \bar{W}_1 \Gamma_2 \\
&\quad -\Gamma_3^T \bar{W}_2 \Gamma_3 - \gamma^2 e_8^T e_8 \\
&\quad -e_6^T \epsilon_1 I e_1 + e_1^T \epsilon_2 \varpi_1 I e_6 - e_6^T \epsilon_2 I e_6 \\
&\quad -e_7^T \epsilon_3 I e_2 + e_2^T \epsilon_4 \varpi_2 I e_7 - e_7^T \epsilon_4 I e_7, \\
\bar{\Xi} &= (AX - BM)e_1 + A_1 X e_2 + \varepsilon_1 BMe_5 \\
&\quad + Ie_6 + Ie_7 + He_8 + \varepsilon_2 dBMe_{10}, \\
\bar{\Lambda} &= diag\{\bar{Q}_1 + \bar{Q}_2 + \bar{Q}_3 + \bar{Q}_4, -(1-\mu)\bar{Q}_3, \\
&\quad -\bar{Q}_1, -\bar{Q}_2, -\bar{Q}_4, 0, 0, 0, 0, 0\}, \\
\Psi &= \left[\tau_m \bar{\Xi}^T, \tau_M \bar{\Xi}^T, d\bar{\Xi}^T\right], \\
\Omega &= diag\left\{X\bar{Z}_1^{-1}X, X\bar{Z}_2^{-1}X, X\bar{Z}_3^{-1}X\right\}, \\
\bar{W}_1 &= diag\left\{\bar{Z}_1, 3\bar{Z}_1, 5\bar{Z}_1\right\}, \bar{W}_2 = diag\left\{\bar{Z}_3, 3\bar{Z}_3, 5\bar{Z}_3\right\} \\
\eta_1 &= \epsilon_1 \nu_1 + \epsilon_2 \kappa_1, \eta_2 = \epsilon_3 \nu_2 + \epsilon_4 \kappa_2.
\end{aligned}
$$

Then, the controller gain can be computed by $K = MX^{-1}C^+$.

Proof: For the terms $\epsilon_1\nu_1 I + \epsilon_2\kappa_1 I$ and $\epsilon_3\nu_2 I + \epsilon_4\kappa_2 I$ of Π in Theorem 1, the inequalities $\epsilon_1\nu_1 + \epsilon_2\kappa_1 \le |\epsilon_1\nu_1 + \epsilon_2\kappa_1|$ and $\epsilon_3\nu_2 + \epsilon_4\kappa_2 \le |\epsilon_3\nu_2 + \epsilon_4\kappa_2|$ hold. Based on the Schur complements and congruence transform, by pre- and post-multiplying opposite sides of the $\left[\begin{smallmatrix} Z_2 & V \\ * & Z_2 \end{smallmatrix}\right]$ by the matrix $diag\{X, X\}$, and (8) by the matrix

$$
diag\{\overbrace{X, ..., X}^{5}, I, I, I, X, X, Z_1^{-1}, Z_2^{-1}, Z_3^{-1}\} \tag{21}
$$

where $X = P^{-1}$, $M = KCX$, $\bar{Q}_i = P^{-1}Q_iP^{-1}$, $\bar{Z}_j = P^{-1}Z_jP^{-1}$, $\bar{V} = P^{-1}VP^{-1}$, we can obtain $\left[\begin{smallmatrix} \bar{Z}_2 & \bar{V} \\ * & \bar{Z}_2 \end{smallmatrix}\right] \ge 0$ and (20). This proves the Theorem 2.

Note that the upper bound on $\epsilon_1\nu_1 + \epsilon_2\kappa_1$ and $\epsilon_3\nu_2 + \epsilon_4\kappa_2$ are considered in Theorem 3, which can lead to a relatively simple condition. However, owing to the utilization of the upper bound, conservatism inevitably exists in (20). If $\epsilon_1\nu_1 + \epsilon_2\kappa_1 \le 0$ or $\epsilon_3\nu_2 + \epsilon_4\kappa_2 \le 0$, the condition in (20) can be infeasible for the larger values $-(\epsilon_1\nu_1 + \epsilon_2\kappa_1)$ or $-(\epsilon_3\nu_2 + \epsilon_4\kappa_2)$. To reduce the conservatism, Theorem 3 is provided based on the sign of $\epsilon_1\nu_1 + \epsilon_2\kappa_1$ and $\epsilon_3\nu_2 + \epsilon_4\kappa_2$.

Theorem 3: A sufficient and necessary condition for the solution in Theorem 1 is that there exist the positive-definite and symmetric matrices X, $\bar{Q}_i$, $\bar{Z}_j$, matrices $\bar{V}$ and M of appropriate dimensions satisfying $\bar{\Theta} = \left[\begin{smallmatrix} \bar{Z}_2 & \bar{V} \\ * & \bar{Z}_2 \end{smallmatrix}\right] \ge 0$, positive scalars ε_k, ϵ_i for $i = 1, 2, 3, 4, j = 1, 2, 3, k = 1, 2$, such that either of the following conditions:

(i) If $\epsilon_1\nu_1 + \epsilon_2\kappa_1 > 0$ and $\epsilon_3\nu_2 + \epsilon_4\kappa_2 > 0$, then

$$
\begin{bmatrix}
\bar{\Pi} & \Psi & e_1^T X & e_1^T \sqrt{\eta_1} X & e_2^T \sqrt{\eta_2} X \\
* & -\Omega & 0 & 0 & 0 \\
* & * & -I & 0 & 0 \\
* & * & * & -I & 0 \\
* & * & * & * & -I
\end{bmatrix} < 0, \tag{22}
$$

(ii) If $\epsilon_1\nu_1 + \epsilon_2\kappa_1 > 0$ and $\epsilon_3\nu_2 + \epsilon_4\kappa_2 \le 0$, then

$$
\begin{bmatrix}
\bar{\Pi} - G_2 & \Psi & e_1^T X & e_1^T \sqrt{\eta_1} X \\
* & -\Omega & 0 & 0 \\
* & * & -I & 0 \\
* & * & * & -I
\end{bmatrix} < 0, \tag{23}
$$

(iii) If $\epsilon_1\nu_1 + \epsilon_2\kappa_1 \le 0$ and $\epsilon_3\nu_2 + \epsilon_4\kappa_2 > 0$, then

$$
\begin{bmatrix}
\bar{\Pi} - G_1 & \Psi & e_1^T X & e_1^T \sqrt{\eta_2} X \\
* & -\Omega & 0 & 0 \\
* & * & -I & 0 \\
* & * & * & -I
\end{bmatrix} < 0, \tag{24}
$$

(iv) If $\epsilon_1\nu_1 + \epsilon_2\kappa_1 \le 0$ and $\epsilon_3\nu_2 + \epsilon_4\kappa_2 \le 0$, then

$$
\begin{bmatrix}
\bar{\Pi} - G_1 - G_2 & \Psi & e_1^T X \\
* & -\Omega & 0 \\
* & * & -I
\end{bmatrix} < 0, \tag{25}
$$

is satisfied for $G_1 = -X(\epsilon_1\nu_1 + \epsilon_2\kappa_1)X$ and $G_2 = -X(\epsilon_3\nu_2 + \epsilon_4\kappa_2)X$. Then, the controller gain can be computed by $K = MX^{-1}C^+$.

Proof: Before two successive Schur complements to (20) in the proof process of Theorem 2, the original inequality is shown as

$$
\begin{bmatrix}
\bar{\Pi} + X\eta_1 X + X\eta_2 X & \Psi & e_1^T X \\
* & -\Omega & 0 \\
* & * & -I
\end{bmatrix} < 0. \tag{26}
$$

On the one hand, if $\epsilon_1\nu_1 + \epsilon_2\kappa_1 > 0$ and $\epsilon_3\nu_2 + \epsilon_4\kappa_2 > 0$, (22) and (26) are equivalent after two successive Schur complements. For the inequality (26), if $\epsilon_1\nu_1 + \epsilon_2\kappa_1 > 0$, $\epsilon_3\nu_2 + \epsilon_4\kappa_2 \le 0$ or $\epsilon_1\nu_1 + \epsilon_2\kappa_1 \le 0$, $\epsilon_3\nu_2 + \epsilon_4\kappa_2 > 0$, by substituting G_1 and G_2 and using the Schur complement, we can derive (23) and (24). If $\epsilon_1\nu_1 + \epsilon_2\kappa_1 \le 0$ and $\epsilon_3\nu_2 + \epsilon_4\kappa_2 \le 0$, (25) and (26) are equivalent by substituting G_1 and G_2. On the other hand, suppose that the solution in Theorem 2 exists. For the inequality (20), using two successive Schur complements can yield

$$
\begin{bmatrix}
\bar{\Pi} + X|\eta_1|X + X|\eta_2|X & \Psi & e_1^T X \\
* & -\Omega & 0 \\
* & * & -I
\end{bmatrix} < 0. \tag{27}
$$

If $\epsilon_1\nu_1 + \epsilon_2\kappa_1 > 0$ and $\epsilon_3\nu_2 + \epsilon_4\kappa_2 > 0$, (20) and (22) are equivalent. Because $\epsilon_1\nu_1 + \epsilon_2\kappa_1 \le |\epsilon_1\nu_1 + \epsilon_2\kappa_1|$ and $\epsilon_3\nu_2 + \epsilon_4\kappa_2 \le |\epsilon_3\nu_2 + \epsilon_4\kappa_2|$ hold, we can derive $-G_1 \le |X(\epsilon_1\nu_1 + \epsilon_2\kappa_1)X|$ and $-G_2 \le |X(\epsilon_3\nu_2 + \epsilon_4\kappa_2)X|$. If (20) is satisfied, (27) must hold. when $\epsilon_1\nu_1 + \epsilon_2\kappa_1 > 0$, $\epsilon_3\nu_2 + \epsilon_4\kappa_2 \le 0$ or $\epsilon_1\nu_1 + \epsilon_2\kappa_1 \le 0$, $\epsilon_3\nu_2 + \epsilon_4\kappa_2 > 0$, by using the Schur complement, we can obtain (23) and (24). If $\epsilon_1\nu_1 + \epsilon_2\kappa_1 \le 0$ and $\epsilon_3\nu_2 + \epsilon_4\kappa_2 \le 0$, (27) can result into (25). This completes the proof.

Remark 1: Different from the Theorem 2, Theorem 3 is a sufficient and necessary condition for the solution. By considering the sign of $\epsilon_1\nu_1 + \epsilon_2\kappa_1$ and $\epsilon_3\nu_2 + \epsilon_4\kappa_2$ instead of their upper bounds, we can obtain a less conservative condition. Furthermore, if the condition in Theorem 2 holds, the solution in Theorem 3 will always exist. That is, if Theorem 2 is satisfied for the real number $\eta_1 = \epsilon_1\nu_1 + \epsilon_2\kappa_1$ and

$\eta_2 = \epsilon_3\nu_2 + \epsilon_4\kappa_2$, the condition of Theorem 3 will always be valid for the interval $-\infty < \epsilon_1\nu_1 + \epsilon_2\kappa_1 \leq \eta_1$ and $-\infty < \epsilon_3\nu_2 + \epsilon_4\kappa_2 \leq \eta_2$.

Remark 2: Because nonlinear terms $N_j = X\bar{Z}_j^{-1}X$ for $j = 1, 2, 3$ exist in Theorems 2-3, we don't employ the conventional convex feasibility approach to seek the solution. To address this, we can convert the problem into a nonlinear optimal one. According to the presence or absence of the nonlinear terms $G_1 = -X(\epsilon_1\nu_1 + \epsilon_2\kappa_1)X$ and $G_2 = -X(\epsilon_3\nu_2 + \epsilon_4\kappa_2)X$ in (22) $-$ (25), different optimization schemes can be provided for solution by using the cone complementary linearization.

(i) If $\epsilon_1\nu_1 + \epsilon_2\kappa_1 > 0$ and $\epsilon_3\nu_2 + \epsilon_4\kappa_2 > 0$, then

$$\begin{cases} \text{Minimize Trace} \\ PX + \sum_{j=1}^{3}(\bar{Z}_jU_j + N_jF_j + PU_jPF_j), \\ \text{subject to (22)} \\ \begin{bmatrix} P & I \\ * & X \end{bmatrix} \geq 0, \begin{bmatrix} \bar{Z}_j & I \\ * & U_j \end{bmatrix} \geq 0, \begin{bmatrix} N_j & I \\ * & F_j \end{bmatrix} \geq 0, \begin{bmatrix} U_j & X \\ * & F_j \end{bmatrix} \geq 0. \end{cases} \tag{28}$$

(ii) If $\epsilon_1\nu_1 + \epsilon_2\kappa_1 > 0$ and $\epsilon_3\nu_2 + \epsilon_4\kappa_2 \leq 0$, then

$$\begin{cases} \text{Minimize Trace} \\ 0.5\bar{G}_2G_2 + 0.5X\eta_2X\bar{G}_2 + PX \\ + \sum_{j=1}^{3}(\bar{Z}_jU_j + N_jF_j + PU_jPF_j), \\ \text{subject to (23)} \\ \begin{bmatrix} G_2 & I \\ * & \bar{G}_2 \end{bmatrix} \geq 0, \begin{bmatrix} -\eta_2 & P \\ * & \bar{G}_2 \end{bmatrix} \geq 0, \begin{bmatrix} P & I \\ * & X \end{bmatrix} \geq 0, \\ \begin{bmatrix} \bar{Z}_j & I \\ * & U_j \end{bmatrix} \geq 0, \begin{bmatrix} N_j & I \\ * & F_j \end{bmatrix} \geq 0, \begin{bmatrix} U_j & X \\ * & F_j \end{bmatrix} \geq 0. \end{cases} \tag{29}$$

(iii) If $\epsilon_1\nu_1 + \epsilon_2\kappa_1 \leq 0$ and $\epsilon_3\nu_2 + \epsilon_4\kappa_2 > 0$, then

$$\begin{cases} \text{Minimize Trace} \\ 0.5\bar{G}_1G_1 + 0.5X\eta_1X\bar{G}_1 + PX \\ + \sum_{j=1}^{3}(\bar{Z}_jU_j + N_jF_j + PU_jPF_j), \\ \text{subject to (24)} \\ \begin{bmatrix} G_1 & I \\ * & \bar{G}_1 \end{bmatrix} \geq 0, \begin{bmatrix} -\eta_1 & P \\ * & \bar{G}_1 \end{bmatrix} \geq 0, \begin{bmatrix} P & I \\ * & X \end{bmatrix} \geq 0, \\ \begin{bmatrix} \bar{Z}_j & I \\ * & U_j \end{bmatrix} \geq 0, \begin{bmatrix} N_j & I \\ * & F_j \end{bmatrix} \geq 0, \begin{bmatrix} U_j & X \\ * & F_j \end{bmatrix} \geq 0. \end{cases} \tag{30}$$

(iv) If $\epsilon_1\nu_1 + \epsilon_2\kappa_1 \leq 0$ and $\epsilon_3\nu_2 + \epsilon_4\kappa_2 \leq 0$, then

$$\begin{cases} \text{Minimize Trace} \\ 0.5\bar{G}_kG_k + 0.5X\eta_kX\bar{G}_k + PX \\ + \sum_{j=1}^{3}(\bar{Z}_jU_j + N_jF_j + PU_jPF_j), \\ \text{subject to (25)} \\ \begin{bmatrix} G_k & I \\ * & \bar{G}_k \end{bmatrix} \geq 0, \begin{bmatrix} -\eta_k & P \\ * & \bar{G}_k \end{bmatrix} \geq 0, \begin{bmatrix} P & I \\ * & X \end{bmatrix} \geq 0, \\ \begin{bmatrix} \bar{Z}_j & I \\ * & U_j \end{bmatrix} \geq 0, \begin{bmatrix} N_j & I \\ * & F_j \end{bmatrix} \geq 0, \begin{bmatrix} U_j & X \\ * & F_j \end{bmatrix} \geq 0. \end{cases} \tag{31}$$

Algorithm:

1) If $\epsilon_1\nu_1 + \epsilon_2\kappa_1 > 0$ and $\epsilon_3\nu_2 + \epsilon_4\kappa_2 > 0$,solve the cone complementary linearization algorithm with $(P, X, \bar{Z}_j, U_j, N_j, F_j, M)$ subject to (28).Set $P^{c+1} = P, X^{c+1} = X, \bar{Z}_j^{c+1} = \bar{Z}_j, U_j^{c+1} = U_j, N_j^{c+1} = N_j, N_j^{c+1} = N_j, F_j^{c+1} = F_j, M^{c+1} = M$.
   If $\epsilon_1\nu_1 + \epsilon_2\kappa_1 > 0$ and $\epsilon_3\nu_2 + \epsilon_4\kappa_2 \leq 0$,solve the cone complementary linearization algorithm with $(G_2, P, X, \bar{Z}_j, U_j, N_j, F_j, M, \eta_1)$ subject to (29).Set $G_2^{c+1} = G_2, P^{c+1} = P, X^{c+1} = X, \bar{Z}_j^{c+1} = \bar{Z}_j, U_j^{c+1} = U_j, N_j^{c+1} = N_j, N_j^{c+1} = N_j, F_j^{c+1} = F_j, M^{c+1} = M$.
   If $\epsilon_1\nu_1 + \epsilon_2\kappa_1 \leq 0$ and $\epsilon_3\nu_2 + \epsilon_4\kappa_2 > 0$,solve the cone complementary linearization algorithm with $(G_1, P, X, \bar{Z}_j, U_j, N_j, F_j, M, \eta_2)$ subject to (30).Set $G_1^{c+1} = G_1, P^{c+1} = P, X^{c+1} = X, \bar{Z}_j^{c+1} = \bar{Z}_j, U_j^{c+1} = U_j, N_j^{c+1} = N_j, N_j^{c+1} = N_j, F_j^{c+1} = F_j, M^{c+1} = M$.
   If $\epsilon_1\nu_1 + \epsilon_2\kappa_1 \leq 0$ and $\epsilon_3\nu_2 + \epsilon_4\kappa_2 \leq 0$,solve the cone complementary linearization algorithm with $(G_k, P, X, \bar{Z}_j, U_j, N_j, F_j, M, \eta_k)$ subject to (31).Set $G_k^{c+1} = G_k, P^{c+1} = P, X^{c+1} = X, \bar{Z}_j^{c+1} = \bar{Z}_j, U_j^{c+1} = U_j, N_j^{c+1} = N_j, N_j^{c+1} = N_j, F_j^{c+1} = F_j, M^{c+1} = M$.
2) If (22),(23),(24),(25) is feasible for $K = MX^{-1}C^+$, then exit. Or else, set c=c+1, then go 1).

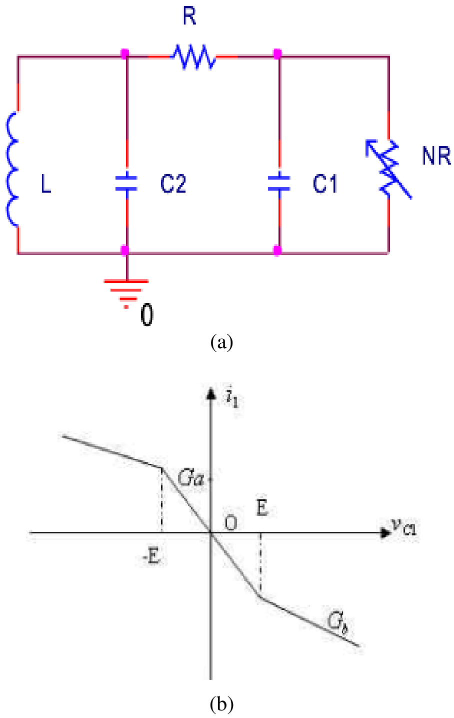

FIGURE 1. The structure plot and model analysis of Chua's circuit. (a) The topological graph of Chua's circuit. (b) Volt ampere characteristics of Chua's diode.

IV. SIMULATION

To exhibit the effectiveness of the aforementioned control approach, in the section, we introduce the Chua's circuit

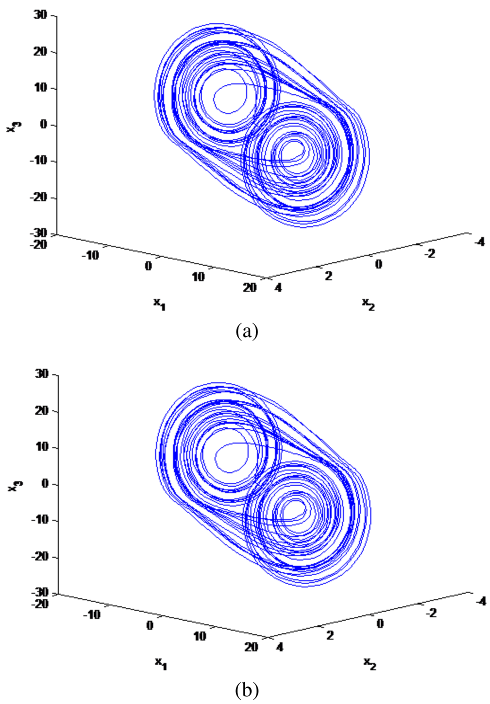

FIGURE 2. The phase-space plots of the drive system subjected to the external uncertainty. (a) Phase plane trajectory without the control. (b) Phase plane trajectory with the control.

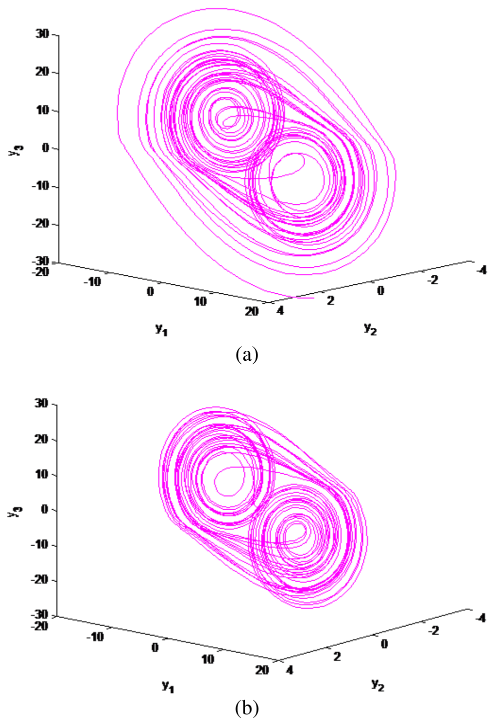

FIGURE 3. The phase-space plots of the response system subjected to the external uncertainty. (a) Phase plane trajectory without the control. (b) Phase plane trajectory with the control.

chaotic systems as the simulation example given by

$$A = \begin{bmatrix} -2.548 & 9.1 & 0 \\ 1 & -1 & 1 \\ 0 & -14.2 & 0 \end{bmatrix}, \quad \hbar(\upsilon, t) = \begin{bmatrix} \delta\upsilon_1 \\ 0 \\ 0 \end{bmatrix},$$

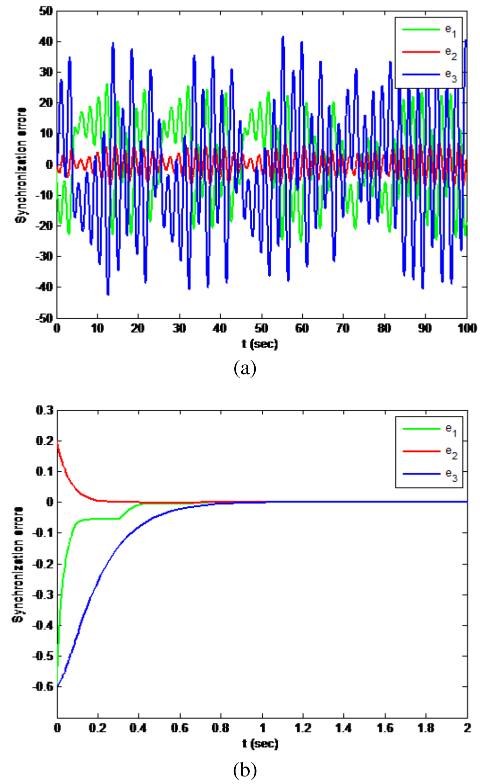

FIGURE 4. The time responses of error system subjected to the external uncertainty. (a) The time response of error system without the control. (b) The time response of error system with the control.

$$\begin{aligned}
\vartheta(\upsilon, t) &= 9.11 \begin{bmatrix} |\upsilon_1 + 1| - |\upsilon_1 - 1| \\ 0 \\ 0 \end{bmatrix}, \\
A_1 &= B = C = H = diag\{1, 1, 1\}, \\
\tau(t) &= 0.1 + 0.02\sin(5t), \ \delta = 0.1 + 0.01\sin(0.1t), \\
\ell_x(t) &= [-0.2, 0.1, -0.4]^T, \ \ell_y(t) = [0.4, -0.1, 0.2]^T, \\
\omega_x(t) &= [0.10\sin(100t), 0.10\sin(120t), 0.10\sin(110t)]^T, \\
\omega_y(t) &= [0.12\sin(120t), 0.13\sin(115t), 0.14\sin(110t)]^T.
\end{aligned} \tag{32}$$

As can be seen from the Fig. 1(a), Chua's circuit is a third-order autonomous circuit composed of resistors, capacitors and inductors and Chua's diode.It can produce chaos phenomenon when it satisfies one of the following conditions: (i) Nonlinear resistance is not less than one. (ii) Linear effective resistance of not less than one.(iii) Not less than three energy storage components.The volt ampere characteristic curve of Chua's diode is showed in Fig. 1(b).

The phase-space and error trajectories plots for the chaotic systems without any control are presented in Fig. 2(a) – 4(a), which indicate that the response system can't asymptotically track the drive system under different initial conditions. To make the systems behave in a synchronous way, we design a mixed output time-delay feedback controller under the conditions

$$\nu_1 = 0,\ \nu_2 = 0,\ \kappa_1 = 45,\ \kappa_2 = 0.25,\ \varpi_1 = 0,\ \varpi_2 = 0,$$
$$\varepsilon_1 = 0.1,\ \varepsilon_2 = 0.045,\ \mu = 0.3,\ d = 0.3,\ \gamma = 0.31. \tag{33}$$

Because $\epsilon_1\nu_1 + \epsilon_2\kappa_1 > 0$ and $\epsilon_3\nu_2 + \epsilon_4\kappa_2 > 0$ are satisfied for any positive scalars ϵ_i, $i = 1, 2, 3, 4$, the controller gain matrix K can be solved through (22) and (28), given by

$$K = \begin{bmatrix} 19.7785 & 9.1254 & 0.0002 \\ 1.0035 & 21.3312 & 1.0017 \\ 0.0002 & -14.2394 & 22.3337 \end{bmatrix}. \tag{34}$$

By application of controller (34), the response system can asymptotically synchronize to the drive system, as illustrated in Fig. $2(b) - 4(b)$. Therefore, the proposed scheme in the paper has certain practical and theoretical significance for synchronization of chaotic systems subjected to the time-delays, disturbances, perturbations and uncertainties.

V. CONCLUSION

This paper designs a mixed output time-delay feedback controller for addressing synchronization of time-delay chaotic systems with one-sided Lipschitz nonlinearity under external uncertainties. Based on the reciprocally convex approach, improved method for Jensen inequality, one-sided Lipschitz condition and the quadratic inner boundedness, taking the output lag and intrinsic state time-varying delays with non-zero lower bound into consideration, an appropriate LKF is proposed to derive the controller design condition for synchronization of chaotic systems subjected to external disturbances bounded by L_2 norm. Because of the existence of nonlinear terms, we can handle the non-linear optimal problems converted from the original problem to obtain the solution by using the cone complementary linearization method. Numerical simulations of the Chua's circuit chaotic systems are demonstrated to verify the effectiveness of the proposed methodology.

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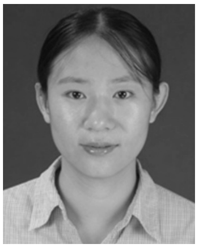

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