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A Continuous Finite-Time Output Feedback Control Scheme and Its Application in Quadrotor UAVs

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ABSTRACT A continuous output feedback control scheme is presented for double integrator systems subject to non-vanishing perturbation. In the method, no explicit state observer or disturbance observer is designed. The geometric homogeneity technique and Lyapunov stability theory are utilized to ensure the global finite-time stability of the closed-loop system. The extension of the algorithm to multi-input multi-output is developed, and its application in quadrotor unmanned aerial vehicles is investigated. Finally, the numerical simulation results are provided to validate the efficiency of the proposed method.

INDEX TERMS Finite-time stability, output feedback, quadrotor UAVs, nonlinear control, robust control.

I. INTRODUCTION

The system with double-integrator dynamics is one of the most fundamental systems in control theory and has many applications in practice, such as spacecraft rotation [1], rotary crane motion [2] and manipulator motion [3]. Most of the techniques used for linear or nonlinear feedback stabilization and observation are asymptotic stability, which means the system trajectories settle at the origin as the time approaches infinity. In practice, the finite-time stability is more expected, since it provides faster convergent rate, higher precision and better disturbance rejection properties [4]. Hence, much effort has been devoted to the topic during the last years.

Geometric homogeneity technique, Lyapunov stability theory and Implicit Lyapunov Function (ILF) are commonly used in finite-time control system design. In [5], it was shown that a system is finite-time stability if it is locally asymptotically stable and homogeneous with negative degree, which provides a sufficient condition for the design of controller and observer with finite-time convergence. In [6], a global saturated finite-time controller is developed based on homogeneous method for a rigid spacecraft system. In [7], the homogeneous technique was utilized to the observer design for triple integrator. In addition, a homogeneous Lyapunov function was developed to select gains explicitly to ensure the finite-time convergence of the closed-loop system. In [8], a finite-time control algorithm is proposed using homogeneous technique for arbitrary order integrator. However, the convergence of the algorithm was only proved when the degree of homogeneity was sufficiently close to 0 without more tractable information. Furthermore, in [9] the implicit Lyapunov function method and homogeneous technique were combined together to present a control algorithm for finite-time stabilization of a chain of integrator with arbitrary order. Moreover, the tuning of control parameter was presented using linear matrix inequality technique. In [10], an adaptive continuous twisting algorithm is developed for perturbed double integrator. In the method, finite-time convergence was achieved by a continuous control signal even in the case that upper bound of the perturbation is unknown. Moreover, the control gains are derived explicitly using Polya's Theorem.

A main drawback of these finite-time control methods aforementioned is that the full state information is assumed to be available. However, the velocity information of second order systems may be difficult to obtain in practice [11]. To address the issue, some finite-time output feedback control schemes have been developed. In [12], a class of output feedback finite-time stabilizing control law was proposed for the double integrator system, where the finite-time separation principle was used to ensure the finite-time stability of the closed-loop system under output-feedback framework. In [13], a unified framework for the finite time output feedback stabilization of a double integrator was proposed using a modification of the twisting controller and the supertwisting observer. In [14], a homogeneous controller and a homogeneous observer are designed with different degree of homogeneity to ensure the finite-time astabilization of the double integrator. Furthermore, the robustness and effects of discretization on the closed-loop system were investigated in [15], which shows that an improved robustness is achieved with respect to the result in [13]. However, only the disturbance vanishing in the origin was discussed in [12]–[15]. In [16], a globally finite-time output feedback control law, combining state observer and discontinuous integral controller, was presented for double integrator systems with non-vanishing perturbation. Based on the method in [17], a homogeneous Lyapunov function was constructed to ensure the finite-time stabilization of the closed-loop system. A general idea in the methods aforementioned is that a finite-time state observer is designed, and then the estimate value is incorporated into finite-time controller. As a result, the finite-time output control scheme is constructed by combining the controller and observer. However, the observer relies on the control input, which may affect the transient response of the observer. In [18], a novel finite-time output control law was developed. In the method, a simple nonlinear filter, which does not refer to the control input, is developed to replace the velocity measurement. Nevertheless, the proposed control law is only insensitive to a class of perturbation vanishing at the origin, and it can not reject the non-vanishing perturbation completely. In practice, many systems, such as attitude system of reusable launch vehicle [11] or manipulator [3], are subject to non-vanishing perturbation. Therefore, it is imperative to study this issue.

Inspired by the work in [18] and [16], a new finite-time output feedback control scheme is developed for double integrator with non-vanishing perturbation. The key features of the algorithm are threefold. First, an output feedback control framework is developed for double integrator systems with non-vanishing perturbation, where the velocity measurements doesn't resort to control input. Second, the proof the finite-time stability of the closed-loop system is achieved through geometric homogeneity technique and Lyapunov stability theory. Finally, an extension of the algorithm from scalar to vector is given, and its application in attitude control of quadrotor UAV is investigated.

The outline of this work is as follows. Notation and some useful concepts, lemmas and the problem formulation are presented in Section 2. A continuous finite-time output feedback control algorithm is developed in Section 3. Some results in numerical simulation are presented in Section 4, and the conclusions are summarized in Section 5.

II. PRELIMINARIES AND PROBLEM STATEMENT

A. NOTATIONS

Throughout the paper, the following notations will be used. $\mathbb{R}$ is the set of real numbers and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. For any non-negative real number α, the function $x \mapsto \lceil x \rfloor^\alpha$ is defined as $\lceil x \rfloor^\alpha = |x|^\alpha \text{sign}(x)$ for any $x \in \mathbb{R}$. It follows from the definition that $\frac{d\lceil x \rfloor^\alpha}{dx} = \alpha |x|^{\alpha-1}$, $\lceil x \rfloor^0 = \text{sign}(x)$, $\lceil x \rfloor = x$ and $\lceil x \rfloor^2 = x|x|$. For a given vector $\boldsymbol{x} = [x_1, .., x_n]^T \in \mathbb{R}^n$, let $\| \boldsymbol{x} \| = \sqrt{\boldsymbol{x}^T \boldsymbol{x}}$ be the Euclidean norm of vector $\boldsymbol{x}$. For any $\boldsymbol{x} \in \mathbb{R}^n$, define the multi-variable sign function $\text{sign}(\boldsymbol{x}) = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$. The transpose of multivariable sign function is defined by $\text{sign}^T(\boldsymbol{x}) = \frac{\boldsymbol{x}^T}{\|\boldsymbol{x}\|}$. For any non-negative real number α, the function $\boldsymbol{x} \mapsto \lceil \boldsymbol{x} \rfloor^\alpha$ is defined as $\lceil \boldsymbol{x} \rfloor^\alpha = \|\boldsymbol{x}\|^\alpha \text{sign}^T(\boldsymbol{x})$ for any $x \in \mathbb{R}^n$. When one says that vector function $\boldsymbol{f}(\boldsymbol{x}) \in \mathbb{R}^n$ is Lipschitz continuous, it means that each element of $\boldsymbol{f}(\boldsymbol{x})$, i.e., $f_i(\boldsymbol{x})$, is Lipschitz continuous.

B. DEFINITIONS AND LEMMAS

Consider the nonlinear dynamical system

$$\dot{x}(t) = f(x(t)), \quad t > t_0, \ x(t_0) = x_0 \tag{1}$$

where $x = [x_1, .., x_n]^T \in \mathbb{R}^n$ is the state vector, $f(x) : \mathbb{R}^n \to \mathbb{R}^n$ is a possibly discontinuous vector field. In this case, the solutions of (2) are understood in the sense of Filippov [19]. It is assumed that the origin is an equilibrium point of system (1). Throughout the paper, it is assumed that the solution of (1) starts at $t_0 = 0$, denoted by $X(t, x_0)$ with x_0 as the initial condition.

Let $r = [r_1, ..., r_n] \in \mathbb{R}^n$ be the weight vector with $r_i > 0, (i = 1, ..., n)$. The dilation mapping is defined as $\Lambda_\lambda^r(x) = [\lambda^{r_1} x_1, ..., \lambda^{r_n} x_n]^T$ for any $\lambda > 0$.

Definition 1 [20]: A function $g(x) : \mathbb{R}^n \to \mathbb{R}$ is said to be r-homogeneous with degree $k \in \mathbb{R}$ if for all $x \in \mathbb{R}^n$ and all $\lambda > 0$ we have $g(\Lambda_\lambda^r(x)) = \lambda^k g(x)$.

Definition 2 [20]: A vector field $f(x) : \mathbb{R}^n \to \mathbb{R}^n$ is said to be r-homogeneous with degree k if for each $i \in \{1, ..., n\}$, the element $f_i(x)$ is r-homogeneous of degree of $k + r_i$; that is $f_i(\Lambda_\lambda^r(x)) = \lambda^{k+r_i} f_i(x)$ for any $\lambda > 0$ and $x \in \mathbb{R}^n$.

Definition 3 [5]: The origin of the system (1) is said to be globally uniformly finite-time stable if it is uniformly Lyapunov stable and finite-time attractive, i.e., there exists $0 \leq T < +\infty$ such that $X(t, x_0) = 0$ for all $t \geq T$. The function $T_0(x_0) = \inf\{T \geq 0 : X(t, x_0) = 0, \forall t \geq T\}$ is called the settling-time function of the system (1).

Lemma 1 [5]: The origin of the system (1) is globally finite-time stable if it is locally asymptotically stable and homogeneous with negative degree.

Lemma 2 [21]: For any positive real numbers $a > 0, b > 0, c > 0, p > 1$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, the inequality $\frac{c^p}{p} a^p + \frac{c^{-q}}{q} b^q - ab \geq 0$ is always satisfied.

Lemma 3 [16]: Let $\eta(x) : \mathbb{R}^n \to \mathbb{R}$ and $\gamma(x) : \mathbb{R}^n \to \mathbb{R}_+$ be two homogeneious with degree m with respect to weight vector $r = [r_1, ..., r_n]$ such that $\{x \in \mathbb{R}^n \setminus \{0\} : \gamma(x) = 0\} \subset \{x \in \mathbb{R}^n \setminus \{0\} : \eta(x) < 0\}$ holds. Then, there exists a real number λ^* such that, for all $\lambda \geq \lambda^*$ and for all $x \in \mathbb{R}^n \setminus \{0\}$, and some $c > 0$, $\eta(x) - \lambda\gamma(x) < -c \| x \|_{r,p}^m$.

C. PROBLEM FORMULATION

Consider the following double integrator system described by

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u + \Delta(t) \tag{2}$$

where $x_1, x_2 \in \mathbb{R}$ are the states, $u \in \mathbb{R}$ is the control and $\Delta(t)$ represents the perturbation, satisfying the following assumption

Assumption 1: Suppose that the perturbation $\Delta(t)$ in (2) is a Lipschitz continuous time signal with a known Lipschitz constant L_1, i.e. $|\dot{\Delta}(t)| \leq L_1$.

The problem is to design a continuous control law using only the system output x_1 such that the states of system (2) converge to zero in finite time.

III. MAIN RESULT

In fact, various methods exist to address the problem through state observer, such as [13] and [16], to name just a few. However, the state observer is generally dependent on control input, which may affect the transient response of the closed-loop system. Inspired from [16] and [18], an output feedback control law without resorting to a state observer is proposed for system (2) such that $x_1, x_2 \to 0$ in a finite time, i.e., T, and the perturbation $\Delta(t)$ can also be estimated after finite time. To this end, the following control law is proposed

$$u = -k_1\lceil x_1\rfloor^{\frac{1}{3}} - k_2\lceil x_3\rfloor^{\frac{1}{3}} - \int_0^t (k_{I_1}\lceil x_1 + k_{I_2}x_3\rfloor^0)dt \quad (3)$$

where x_3 is designed by

$$x_3 = q + x_1 \quad (4)$$

with

$$\dot{q} = -k_3\lceil q + x_1\rfloor^{\frac{2}{3}} \quad (5)$$

By defining

$$x_4 = -\int_0^t (k_{I_1}\lceil x_1 + k_{I_2}x_3\rfloor^0)dt + \Delta(t) \quad (6)$$

the closed-loop dynamics of (2) results in the following differential equation

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -k_1\lceil x_1\rfloor^{\frac{1}{3}} - k_2\lceil x_3\rfloor^{\frac{1}{3}} + x_4 \\ \dot{x}_3 = -k_3\lceil x_3\rfloor^{\frac{2}{3}} + x_2 \\ \dot{x}_4 = -k_{I_1}\lceil x_1 + k_{I_2}x_3\rfloor^0 + \dot{\Delta}(t) \end{cases} \quad (7)$$

Since the right hand side of (7) is discontinuous, its solutions will be understood in the sense of Filippov. A direct verification shows that the system (7) is homogeneous with degree $k = -1$ with respect to weight vector $r = [3, 2, 3, 1]$. Next, the following theorem is developed to ensure that the system (7) is finite-time stable.

Theorem 1: Consider the closed-loop system (7) under the Assumption 1, Then, there exists some positive constants k_1, k_2, k_3, k_{I_1} and k_{I_2} such that the states of system (7) converge to zero in finite-time.

Proof: Consider the following continuously differentiable Lyapunov function candidate

$$V(x) = r_1|\epsilon_1|^{\frac{5}{3}} + r_{12}\epsilon_1 x_2 + |x_2|^{\frac{5}{2}} + |x_3|^{\frac{5}{3}} + |x_4|^5 \quad (8)$$

with $\epsilon_1 = x_1 - \frac{\lceil x_4\rfloor^3}{k_1^3}$. It follows from Lemma 2 that the Lyapunov function is positive definite, which can be achieved by selecting large enough r_1 for any given r_{12}. Moreover, a direct verification shows that the Lyapunov function (8) is homogeneous with degree 5 with respect to weight vector $r = [3, 2, 3, 1]$. Furthermore, taking the derivative of the Lyapunov function (8) along the states of (7) results in

$$\dot{V}(x) = \left(\frac{5}{3}r_1\lceil\epsilon_1\rfloor^{\frac{2}{3}} + r_{12}x_2\right)\dot{\epsilon}_1 + \left(\frac{5}{2}\lceil x_2\rfloor^{\frac{3}{2}} + r_{12}\epsilon_1\right)\dot{x}_2 + \frac{5}{3}\lceil x_3\rfloor^{\frac{2}{3}}\dot{x}_3 + 5\lceil x_4\rfloor^4\dot{x}_4 \quad (9)$$

Substituting $\dot{\epsilon}_1 = x_2 - \frac{3x_4^2}{k_1^3}\dot{x}_4$ into (9) results in

$$\dot{V}(x) = \left(\frac{5}{3}r_1\lceil\epsilon_1\rfloor^{\frac{2}{3}} + r_{12}x_2\right)x_2 + \left(r_{12}\epsilon_1 + \frac{5}{2}\lceil x_2\rfloor^{\frac{3}{2}}\right)\dot{x}_2 + \frac{5}{3}\lceil x_3\rfloor^{\frac{2}{3}}\dot{x}_3 + \left(5\lceil x_4\rfloor^4 - \frac{3x_4^2}{k_1^3}(\frac{5}{3}r_1\lceil\epsilon_1\rfloor^{\frac{2}{3}} + r_{12}x_2)\right)\dot{x}_4 \quad (10)$$

Substituting the derivatives of x_2, x_3 and x_4 in (7) into (10) yields

$$\begin{aligned}\dot{V}(x) &= \left(\frac{5}{3}r_1\lceil\epsilon_1\rfloor^{\frac{2}{3}} + r_{12}x_2\right)x_2 + \left(r_{12}\epsilon_1 + \frac{5}{2}\lceil x_2\rfloor^{\frac{3}{2}}\right) \\ &\times \left(x_4 - k_1\lceil x_1\rfloor^{\frac{1}{3}} - k_2\lceil x_3\rfloor^{\frac{1}{3}}\right) + \frac{5}{3}\lceil x_3\rfloor^{\frac{2}{3}}\left(x_2 - k_3\lceil x_3\rfloor^{\frac{2}{3}}\right) \\ &+ \left(5\lceil x_4\rfloor^4 - \frac{3x_4^2}{k_1^3}(\frac{5}{3}r_1\lceil\epsilon_1\rfloor^{\frac{2}{3}} + r_{12}x_2)\right) \\ &\times \left(-k_{I_1}\lceil x_1 + k_{I_2}x_3\rfloor^0 + \dot{\Delta}(t)\right)\end{aligned} \quad (11)$$

Taking into account the identities $x_4 - k_1\lceil x_1\rfloor^{\frac{1}{3}} - k_2\lceil x_3\rfloor^{\frac{1}{3}} = x_4 - k_1\lceil x_1\rfloor^{\frac{1}{3}} - k_2\lceil x_3\rfloor^{\frac{1}{3}} + k_3(\epsilon_1 + \lceil x_2\rfloor^{\frac{3}{2}}) - k_3(\epsilon_1 + \lceil x_2\rfloor^{\frac{3}{2}})$ and $\lceil x_3\rfloor^{\frac{2}{3}} = \lceil x_3\rfloor^{\frac{2}{3}} - \frac{1}{k_3}x_2 + \frac{1}{k_3}x_2$ and let $r_{12} = \frac{5}{2}$, the (11) can be rewritten as

$$V(x) = V_1(x) + V_2(x) + V_3(x) \quad (12)$$

with $V_1(x)$, $V_2(x)$ and $V_2(x)$ being defined by

$$\begin{aligned}V_1(x) &= -k_3\left(\frac{5}{2}\left(\epsilon_1 + \lceil x_2\rfloor^{\frac{3}{2}}\right)^2 + \frac{5}{3}\left(\lceil x_3\rfloor^{\frac{2}{3}} - \frac{1}{k_3}x_2\right)^2\right) \\ V_2(x) &= \left(\frac{5}{3}r_1\lceil\epsilon_1\rfloor^{\frac{2}{3}} + \frac{5}{2}x_2\right)x_2 - \frac{5}{2}\left(\epsilon_1 + \lceil x_2\rfloor^{\frac{3}{2}}\right) \\ &\times \left(k_1\lceil x_1\rfloor^{\frac{1}{3}} + k_2\lceil x_3\rfloor^{\frac{1}{3}} - k_3(\epsilon_1 + \lceil x_2\rfloor^{\frac{3}{2}}) - x_4\right) \\ &+ \frac{5}{3k_3}x_2\left(x_2 - k_3\lceil x_3\rfloor^{\frac{2}{3}}\right) \\ V_3(x) &= \left(5\lceil x_4\rfloor^4 - \frac{3x_4^2}{k_1^3}(\frac{5}{3}r_1\lceil\epsilon_1\rfloor^{\frac{2}{3}} + r_{12}x_2)\right) \\ &\times \left(-k_{I_1}\lceil x_1 + k_{I_2}x_3\rfloor^0 + \dot{\Delta}(t)\right)\end{aligned} \quad (13)$$

It is easily to be verified that the functions $V_1(x)$, $V_2(x)$ and $V_3(x)$ are homogeneous with degree 4 with respect to weight vector $r = [3, 2, 3, 1]$. Furthermore, it follows form the definition (13) that $V_1(x)$ is negative semidefinite, and it becomes to zero only at the set $\mathcal{S}_1 = \{\lceil x_2 \rfloor^{\frac{1}{2}} + \lceil \epsilon_1 \rfloor^{\frac{1}{3}} = 0\} \bigcap \{\lceil x_3 \rfloor^{\frac{2}{3}} - \frac{1}{k_3} x_2 = 0\}$. On set $\mathcal{S}_1$, the value of $V_2(x)$ becomes $V_2(x) = -(\frac{5}{3} r_1 - \frac{5}{2}) |\epsilon_1|^{\frac{4}{3}}$ which is negative when $r_1 > \frac{3}{2}$. Application of Lemma 3 shows that $V_1(x) + V_2(x) < -c \parallel x \parallel_{r,p}^{m}$ for k_3 sufficiently large. Note the fact that $V_1(x) + V_2(x) = 0$ on the set $\mathcal{S}_2 = \{\epsilon_1 = x_2 = x_3 = 0\}$. Obviously, $V_3(x)$ reduces to the following identity

$$V_3(x) = 5 \left(-k_I \lceil x_1 \rfloor^0 + \dot{\Delta}(t) \right) \lceil x_4 \rfloor^4 \tag{14}$$

when the system states evolve on the set $\mathcal{S}_2$. Recall the relationship $\epsilon_1 = x_1 - \frac{\lceil x_4 \rfloor^3}{k_1^3}$, and $\epsilon_1 = 0$ implies that $sign(x_1) = sign(x_4)$. Hence, $V_3(x)$ in (14) satisfies $V_3(x) \leq -(k_{I1} - L)|x_4|^4$ which is negative when $k_{I1} > L$. It follows from Lemma 3 that there exist appropriate k_{I1} such that $\dot{V}(x) = V_1(x) + V_2(x) + V_3(x) < 0$, implying the asymptotic stability of the system (7). Taking into account the homogeneity of the system (7) with negative homogeneous degree -1, it follows from Lemma 1 that the system (7) is finite-time stability.

Remark 1: It is worth noting that an analytic calculation of the gains k_1, k_2, k_3, k_{I1} and k_{I2} may be found using the Polyas Theorem [17]. However, the derivation is intractable for high-order system, such as the system (7)*. An alternative way to determine these parameters is trial and error.*

Remark 2: It should be noted that the parameter k_{I_2} is independent on the stability of the system (7)*, which can be observed from the proof in Theorem 1. Hence, k_{I_2} can be selected arbitrarily at least in theory.*

Remark 3: When the states of system (7) *reach zero in a finite time T, it follows from* (6) *that the perturbation $\Delta(t)$ can be estimated through $\int_0^t (k_{I_1} \lceil x_1 + k_{I_2} x_3 \rfloor^0) dt$, i.e., $\int_0^t (k_{I_1} \lceil x_1 + k_{I_2} x_3 \rfloor^0) dt = \Delta(t)$ for any $t \geq T$.*

IV. APPLICATION IN QUADROTOR UAV

When applying the scalar algorithm in attitude control of UAV, the attitude model has to be decoupled and divided into multi SISO system. In practice, a multivariable algorithm is expected when it is used in MIMO systems. Hence, an extension of the control law from scalar to vector is given, and then its application in attitude control of quadrotor UAV is discussed.

A. MULTIVARIABLE ALGORITHM

The main result in this section is summarized by the following theorem.

Theorem 2: Consider the multivariable double integrator systems

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{x}_2, \quad \dot{\boldsymbol{x}}_2 = \boldsymbol{u} + \boldsymbol{\Delta}(t) \tag{15}$$

where $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$ are the states, $\boldsymbol{u} \in \mathbb{R}^n$ is the control and $\boldsymbol{\Delta}(t) \in \mathbb{R}^n$ is a sufficiently smooth uncertain vector, satisfying $\|\dot{\boldsymbol{\Delta}}(t)\| \leq L$. Then, there exists some positive constants k_1, k_2, k_3, k_{I_1} and k_{I_2} such that the states of (15) *is finite-time stable if the control law is designed by*

$$\boldsymbol{u} = -k_1 \lceil \boldsymbol{x}_1 \rfloor^{\frac{1}{3}} - k_2 \lceil \boldsymbol{x}_3 \rfloor^{\frac{1}{3}} - \int_0^t (k_{I_1} \lceil \boldsymbol{x}_1 + k_{I_2} \boldsymbol{x}_3 \rfloor^0) dt \tag{16}$$

where $\boldsymbol{x}_3$ is designed by

$$\boldsymbol{x}_3 = \boldsymbol{q} + \boldsymbol{x}_1 \tag{17}$$

with

$$\dot{\boldsymbol{q}} = -k_3 \|\boldsymbol{q} + \boldsymbol{x}_1\|^{\frac{2}{3}} \operatorname{sign}(\boldsymbol{q} + \boldsymbol{x}_1) \tag{18}$$

Proof: The proof is similar with that provided in the previous. Hence, just a sketch of the proof is given here. The closed-loop dynamics of (15)-(18) is governed by

$$\begin{cases} \dot{\boldsymbol{x}}_1 = \boldsymbol{x}_2 \\ \dot{\boldsymbol{x}}_2 = -k_1 \lceil \boldsymbol{x}_1 \rfloor^{\frac{1}{3}} - k_2 \lceil \boldsymbol{x}_3 \rfloor^{\frac{1}{3}} + \boldsymbol{x}_4 \\ \dot{\boldsymbol{x}}_3 = -k_3 \lceil \boldsymbol{x}_3 \rfloor^{\frac{2}{3}} + \boldsymbol{x}_2 \\ \dot{\boldsymbol{x}}_4 = -k_{I_1} \lceil \boldsymbol{x}_1 + k_{I_2} \boldsymbol{x}_3 \rfloor^0 + \dot{\boldsymbol{\Delta}}(t) \end{cases} \tag{19}$$

where $\boldsymbol{x}_4 = -\int_0^t (k_{I_1} \lceil \boldsymbol{x}_1 + k_{I_2} \boldsymbol{x}_3 \rfloor^0) dt + \boldsymbol{\Delta}(t)$. To examine the stability of (19), consider the following Lyapunov function candidate

$$V(\boldsymbol{x}) = r_1 \|\boldsymbol{\epsilon}_1\|^{\frac{5}{3}} + r_{12} \boldsymbol{\epsilon}_1^T \boldsymbol{x}_2 + \|\boldsymbol{x}_2\|^{\frac{5}{2}} + \|\boldsymbol{x}_3\|^{\frac{5}{3}} + \|\boldsymbol{x}_4\|^5 \tag{20}$$

with $\boldsymbol{\epsilon}_1 = \boldsymbol{x}_1 - \frac{\|\boldsymbol{x}_4\|^3 \operatorname{sign}(\boldsymbol{x}_4)}{k_1^3}$. The Lyapunov function is positive definite by selecting appropriate r_1 for any given r_{12}, and homogeneous with degree 5 with respect to weight vector $r = [3, 2, 3, 1]$. Noting the fact that $[\|\boldsymbol{x}\|^p]' = \frac{p \boldsymbol{x}^T \dot{\boldsymbol{x}}}{\|\boldsymbol{x}\|^{2-p}}$ for any $p \in \mathbb{R}_+$. Furthermore, the derivative of the Lyapunov function (20) is given by

$$\begin{aligned} \dot{V}(\boldsymbol{x}) = & \left(\frac{5}{3} r_1 \lceil \boldsymbol{\epsilon}_1 \rfloor^{\frac{2}{3}} + r_{12} \boldsymbol{x}_2^T \right) \boldsymbol{x}_2 + \left(r_{12} \boldsymbol{\epsilon}_1^T + \frac{5}{2} \lceil \boldsymbol{x}_2 \rfloor^{\frac{3}{2}} \right) \\ & \times \left(\boldsymbol{x}_4^T - k_1 \lceil \boldsymbol{x}_1 \rfloor^{\frac{1}{3}} - k_2 \lceil \boldsymbol{x}_3 \rfloor^{\frac{1}{3}} \right)^T \\ & + \frac{5}{3} \lceil \boldsymbol{x}_3 \rfloor^{\frac{2}{3}} \left(\boldsymbol{x}_2^T - k_3 \lceil \boldsymbol{x}_3 \rfloor^{\frac{2}{3}} \right)^T \\ & + \left(5 \lceil \boldsymbol{x}_4 \rfloor^4 - \left(\frac{5}{3} r_1 \lceil \boldsymbol{\epsilon}_1 \rfloor^{\frac{2}{3}} + r_{12} \boldsymbol{x}_2^T \right) \frac{4 \|\boldsymbol{x}_4\|^2 \boldsymbol{I}_n - \boldsymbol{x}_4 \boldsymbol{x}_4^T}{k_1^3} \right) \\ & \times \left(-k_{I_1} \lceil \boldsymbol{x}_1 + k_{I_2} \boldsymbol{x}_3 \rfloor^0 + \dot{\Delta}^T(t) \right)^T \end{aligned} \tag{21}$$

where $\boldsymbol{I}_n$ represents a n-dimension identity matrix. Following the method in Theorem 1, (21) can be rewritten as

$$\dot{V}(\boldsymbol{x}) = V_1(\boldsymbol{x}) + V_2(\boldsymbol{x}) + V_3(\boldsymbol{x}) \tag{22}$$

with $V_1(\boldsymbol{x})$, $V_2(\boldsymbol{x})$ and $V_2(\boldsymbol{x})$ being defined by

$$V_1(\boldsymbol{x}) = -k_3 \left(\frac{5}{2} \|\boldsymbol{\epsilon}_1^T + \lceil \boldsymbol{x}_2 \rfloor^{\frac{3}{2}}\|^2 + \frac{5}{3} \|\lceil \boldsymbol{x}_3 \rfloor^{\frac{2}{3}} - \frac{1}{k_3} \boldsymbol{x}_2^T\|^2 \right)$$

$$\begin{aligned} V_2(\boldsymbol{x}) = & \left(\frac{5}{3} r_1 \lceil \boldsymbol{\epsilon}_1 \rfloor^{\frac{2}{3}} + \frac{5}{2} \boldsymbol{x}_2^T \right) \boldsymbol{x}_2 - \frac{5}{2} \left(\boldsymbol{\epsilon}_1^T + \lceil \boldsymbol{x}_2 \rfloor^{\frac{3}{2}} \right) \\ & \times \left(k_1 \lceil \boldsymbol{x}_1 \rfloor^{\frac{1}{3}} + k_2 \lceil \boldsymbol{x}_3 \rfloor^{\frac{1}{3}} - k_3 (\boldsymbol{\epsilon}_1^T + \lceil \boldsymbol{x}_2 \rfloor^{\frac{3}{2}}) - \boldsymbol{x}_4^T \right)^T \\ & + \frac{5}{3k_3} \boldsymbol{x}_2^T \left(\boldsymbol{x}_2^T - k_3 \lceil \boldsymbol{x}_3 \rfloor^{\frac{2}{3}} \right)^T \end{aligned}$$

$$V_3(\boldsymbol{x}) = \left(5\lceil \boldsymbol{x}_4 \rfloor^4 - (\frac{5}{3} r_1 \lceil \boldsymbol{\epsilon}_1 \rfloor^{\frac{2}{3}} + r_{12} \boldsymbol{x}_2^T)\frac{4\|\boldsymbol{x}_4\|^2 \boldsymbol{I}_n - \boldsymbol{x}_4 \boldsymbol{x}_4^T}{k_1^3}\right) \times \left(-k_{I_1}\lceil \boldsymbol{x}_1 + k_{I_2} \boldsymbol{x}_3\rfloor^0 + \dot{\boldsymbol{\Delta}}^T(t)\right)^T \quad (23)$$

It follows form the definition of homogeneity in **Definition 1** that $V_1(\boldsymbol{x})$, $V_2(\boldsymbol{x})$ and $V_3(\boldsymbol{x})$ are homogeneous with respect to weight vector $r = [3, 2, 3, 1]$. Hence, the selection of the parameters k_1, k_2, k_3, k_{I1} and k_{I1} provided in Theorem 1 can be directly utilized to ensure $\dot{V}(\boldsymbol{x})$ in (22) is negative. This completes the proof.

Remark 4: It should be noted that the extension of the algorithm from scalar to vector is useful in mulivariable systems, and it can bring some improvements, such as chattering suppression and non-decoupled design [11], [22].

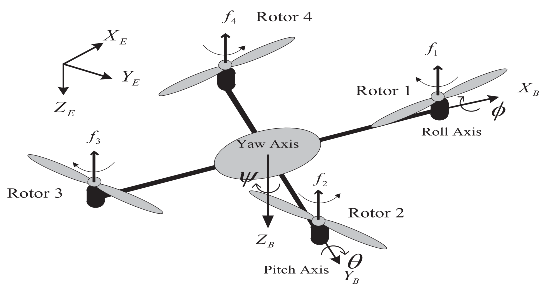

FIGURE 1. The configuration of quadrotor.

B. ATTITUDE CONTROL OF QUADROTOR UAV

The configuration of quadrotor is given in Fig. 1. It is composed by a rigid cross frame and four rotors. The position (x, y and z direction) and attitude (pitch, roll and yaw) motion can be achieved through appropriate combination of the rotors 1 to 4 [23], [24].

The rotational motions of quadrotor UAV is described by

$$\dot{\boldsymbol{\Theta}} = \boldsymbol{W\Omega} \quad (24)$$

$$\boldsymbol{I}\dot{\boldsymbol{\Omega}} = -\boldsymbol{\Omega} \times \boldsymbol{I}\Omega + \boldsymbol{\tau} + \boldsymbol{\Delta} \quad (25)$$

where $\boldsymbol{\Theta} = [\phi, \theta, \psi]^T$ is the Euler angle, $\boldsymbol{\Omega} = [\omega_x, \omega_y, \omega_z]^T$ is attitude angular velocity, and $\boldsymbol{\Delta}$ is disturbance. $\boldsymbol{\tau} = [\tau_1, \tau_2, \tau_3]^T \in \mathbb{R}^3$ is the control torque vector. $\boldsymbol{I} = \mathrm{diag}[I_x, I_y, I_z]$ is a symmetric positive definite constant matrix and the matrix $\boldsymbol{W}$ is calculated by

$$\boldsymbol{W} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \quad (26)$$

Denote $\boldsymbol{x}_1 = \Theta$ and $\boldsymbol{x}_2 = \boldsymbol{W}\Omega$, equations (24) and (25) can be rewritten as

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{x}_2, \dot{\boldsymbol{x}}_2 = \boldsymbol{F} + \boldsymbol{\Delta}_\delta + \tau' \quad (27)$$

where $\boldsymbol{F} = -\boldsymbol{WI}^{-1}\Omega \times \boldsymbol{I}\Omega$, $\boldsymbol{\Delta}_\delta = \boldsymbol{WI}^{-1}\Delta + \dot{\boldsymbol{W}}\Omega$ and $\tau' = \boldsymbol{WI}^{-1}\tau$. Define the attitude tracking error $\boldsymbol{e}_1 = \Theta - \Theta_{ref}$ and $\boldsymbol{e}_2 = \boldsymbol{x}_2 - \dot{\Theta}_{ref}$, the error dynamics of system (27) is governed by

$$\dot{\boldsymbol{e}}_1 = \boldsymbol{e}_2, \dot{\boldsymbol{e}}_2 = \boldsymbol{F} + \boldsymbol{\Delta}_\delta + \tau' - \ddot{\boldsymbol{\Theta}}_{ref} \quad (28)$$

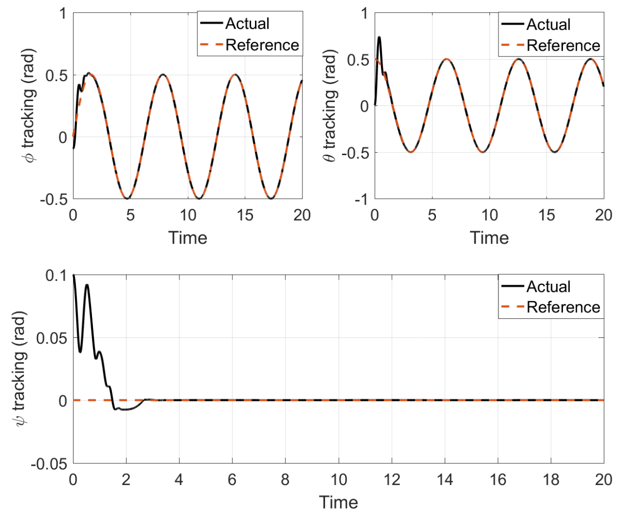

FIGURE 2. Attitude tracking curves.

Now the aim of the work is to design a continuous control input τ' using only $\boldsymbol{e}_1$ such that $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ converge to zero in finite time. Since the attitude angular velocity $\boldsymbol{\Omega}$ is not available, $\boldsymbol{F}$ in (27) can not be obtained directly. Hence, the following assumption is required.

Assumption 2: It is assumed that $\boldsymbol{\Delta}_\delta + \boldsymbol{F}$ *in* (28) *is Lipschitz continuous with a Lipschitz constant* $\widetilde{L}$ *satisfying* $\|\dot{\boldsymbol{\Delta}}_\delta + \dot{\boldsymbol{F}}\| \le \widetilde{L}$.

Assumption 3 [23]: Suppose that pitch and raw angles satisfy the conditions $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ *and* $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$.

Applying Theorem 2 to system (27), we can obtain the following lemma to ensure the finite-time stability of system (28).

Lemma 4: Consider the error dynamics (28) *with Assumptions 2 and 3, there exist some appropriate positive real numbers* k_1, k_2, k_3, k_{I1} *and* k_{I2} *such that* $\boldsymbol{e}_1$ *and* $\boldsymbol{e}_2$ *converge to zero in a finite time, if the control law is designed by*

$$\boldsymbol{\tau}' = \ddot{\boldsymbol{\Theta}}_{ref} - k_1\|\boldsymbol{e}_1\|^{\frac{1}{3}}\mathrm{sign}(\boldsymbol{e}_1) - k_2\|\boldsymbol{e}_3\|^{\frac{1}{3}}\mathrm{sign}(\boldsymbol{e}_3) - m\int_0^t k_{I_1}\mathrm{sign}(\boldsymbol{e}_1 + k_{I_2}\boldsymbol{e}_3)dt \quad (29)$$

where $\boldsymbol{e}_3$ is an augmented variable which can be obtained according to Theorem 2 with a similar form in (17) and (18). For brevity, it is omitted here. The proof of the lemma follows directly from Theorem 2. It should be noted that the real control torque can be calculated by $\tau = \boldsymbol{IW}^{-1}\tau'$ when τ' is available due to the non-singularity of the matrix $\boldsymbol{W}$ under Assumption 3.

V. SIMULATION AND DISCUSSION

In this section, the numerical simulation is provided to validate the efficiency of the developed algorithm. The physical parameters for a typical quadrotor UAV are summarized as: $I_x = 2.3 \times 10^{-3}\mathrm{kgm}^2$, $I_y = 2.4 \times 10^{-3}\mathrm{kgm}^2$ and $I_z = 2.6 \times 10^{-3}\mathrm{kgm}^2$. The controller parameters are selected as: $k_1 = 5, k_2 = 25, k_3 = 3, k_{I_1} = 8, k_{I_2} = 0.5$. To verify the robustness of the proposed method, the disturbances

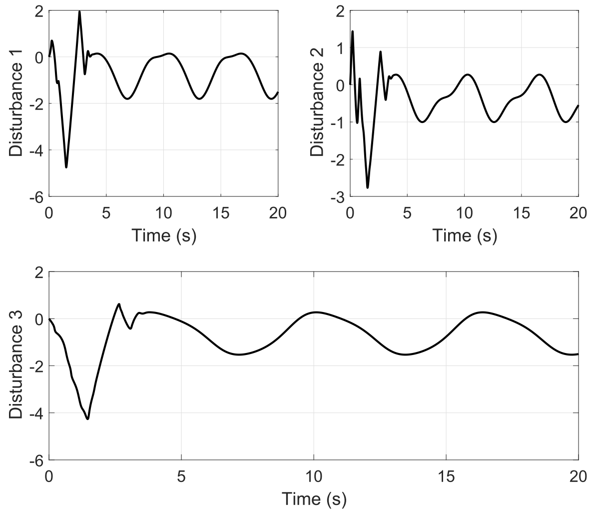

FIGURE 3. Disturbance estimation curves.

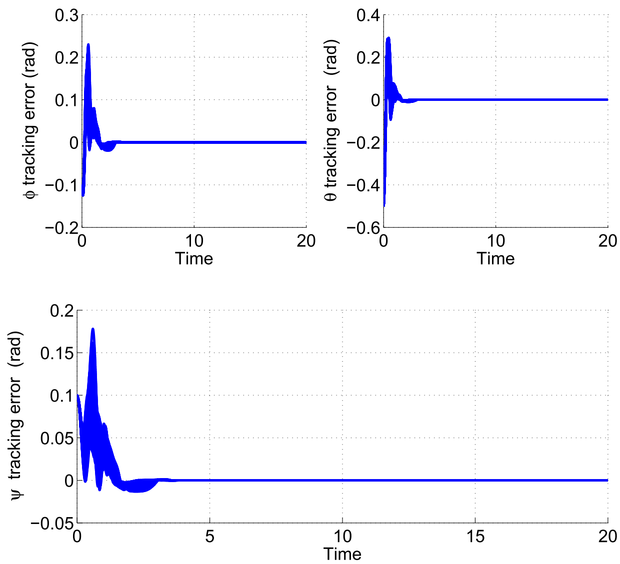

FIGURE 4. Attitude tracking errors with Monte Carlo simulation.

$\boldsymbol{\Delta} = 0.5(1 + \sin(t) + \cos(t))[1\ 1\ 1]^T \boldsymbol{I}$ are added in the simulation. The initial values of the attitude and attitude angular velocity are set to $\Theta = [-0.1, 0, 0.1]$rad and $\Omega = [0, 0, 0]$rad/s. The tracking curves of attitude are shown in Fig. 2, from which it can be observed that the desired attitude can be tracked effectively by the proposed control scheme even only the attitude angle is used. Fig. 3 shows the disturbance estimation of $\boldsymbol{F} + \boldsymbol{\Delta}_\delta$, where "Disturbance estimation i",$i = 1, 2, 3$ represents the i-th element of $\boldsymbol{F} + \boldsymbol{\Delta}_\delta$. The good tracking performance illustrates the effectiveness of the disturbance estimator. To further demonstrate the robustness of the proposed algorithm, a Monte Carlo with 200 times runs is carried out. The model parameter uncertainty $\Delta_{\boldsymbol{I}}$ determined randomly in $[-30\%, +30\%]\boldsymbol{I}$ is added in this case, which means that the real inertia matrix $\boldsymbol{I} + \Delta_{\boldsymbol{I}}$ is used in the simulation. In this case, the simulation results

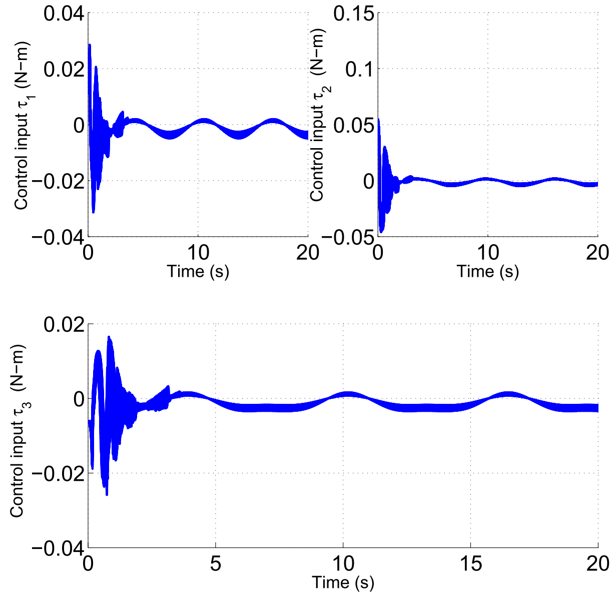

FIGURE 5. Control torques with Monte Carlo simulation.

are plotted in Figs. 4 and 5. Fig. 4 demonstrates the attitude tracking errors of pitch, roll and yaw angles, from that it can be seen that the tracking errors converge to zero quickly even in the presence of external disturbances, model parameter uncertainties and unmeasurable attitude angular velocity. The control torques including pitch, roll and yaw are plotted in Fig. 5.

VI. CONCLUSION

A scalar and vector finite-time stabilizing output feedback control law is developed for double integrator systems with non-vanishing disturbances. The remarkable features of the algorithm is that it avoids the dependence of state observer, which is generally used in output feedback framework. To reject the non-vanishing disturbance, a discontinuous integral control signal is included, which also can be used to estimate the non-vanishing disturbance after a finite time. The application of the method in quadrotor UAV is investigated. Finally, the efficiency of the proposed method is verified through numerical simulation.

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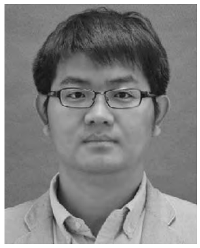

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