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Two-Dimensional Direction of Arrival Estimation for Coprime Planar Arrays via Polynomial Root Finding Technique

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ABSTRACT In this paper, the problem of 2-D direction-of-arrival (2-D-DOA) estimation for coprime planar arrays is investigated, and a method based on the polynomial root finding technique with high accuracy and computational efficiency is proposed. In conventional methods, the whole coprime planar array is usually divided into two subarrays to individually estimate 2-D-DOA, and then the true 2-D-DOA can be distinguished from the ambiguous result utilizing the coprime property. Different from the conventional methods, the proposed method utilizes the signal space and noise space of all data to improve the estimation accuracy. In addition, the computational complexity is reduced by converting the 2-D spectral peak searching problem into double polynomial root finding problem. Furthermore, the matching error is eliminated in the scenario of multiple targets. Simulation results demonstrate that the proposed method achieves not only higher estimation accuracy but also lower computational complexity over conventional methods.

INDEX TERMS 2D-DOA, coprime planar array, polynomial root finding, matching error, improved MUSIC algorithm.

I. INTRODUCTION

Direction of arrival (DOA) estimation is an important problem in array signal processing and has been widely used in radar, sonar, wireless communications, navigation, etc [1]–[4]. Due to the Vandermonde structure of steering vector of uniform linear array (ULA) which is easy to deal with mathematically [5], a series of spatial spectrum estimation algorithms are proposed to get the DOA estimation of ULA [6].

Recently, non-uniform linear array has attracted much attention due to its high resolution and large degrees of freedom (DOFs) [7]–[10], among which coprime linear array has become a hot research direction for its closed form solution about array design and good mutual coupling resistance [11], [12]. A coprime linear array consists of two ULAs whose number of array elements are M and N respectively. And the inter-element spacing of the two ULAs are M times and N times half wavelength respectively, where M and N are coprime integers. It is easy to obtain the degree of freedom of O(MN) by using $M + N$ array elements and improve the estimation accuracy [13]. Zhou *et al.* [14] proposed a DECOM method where the entire matrix was first decomposed into two uniform subarrays and then the unique DOA estimation can be reached by combing the total spectral search results of two subarrays. Moreover, a projection-like search-free DOA estimation algorithm for coprime linear array was investigated in [15]. Reference [16] proposed a combined ESPRIT method which can reduce the complexity significantly. Sun *et al.* [17] derived a partial spectral search method which can improve the computational efficiency. Zhang *et al.* [18] put forward an improved DOA estimation method which avoided the matching error problem and improved the estimation accuracy with less complexity. The above work researched the coprime linear array from the theoretical aspect. Some experimental works about coprime linear array are listed as follows: the super resolution of the coprime linear microphone array is confirmed by the experiment in [19]; the broadband implementation of coprime linear microphone arrays is confirmed in [20]; the feasibility of N-Tuple coprime array is verified in [21]. The above experimental works verified the feasibility of the coprime array.

All the aforementioned methods focus on the coprime linear array and can obtain one dimensional angle estimation result. In comparison, 2D DOA has wider applications in practice, which is attracted extensive attention. Next, the research works of coprime planar array are introduced. Shi *et al.* [22] addressed the issue of 2D DOA estimation with coprime planar arrays via sparse representation which can achieve aperture extension. An improved coprime planar matrix was proposed in [23] which obtained a higher degree of freedom than square coprime planar array. Wu *et al.* [24] presented the 2D partial spectral search method (2D-PSS) for coprime planar array, by which the computational complexity can be reduced. Then, Zheng et al. [25] utilized the operation of decreasing dimensions and proposed 1D partial spectral search (1D-PSS) method which further reduces the complexity. However, there are some imperfections in the above works. In [22], the selection of the regulation parameter affects the estimation accuracy of DOA. In [23], the operation of two dimensional search results in a very large amount of computation. In [24] and [25], they both decomposed the whole coprime planar array into two uniform planar arrays to perform DOA estimation respectively, and some information in the data is lost which affects the estimation accuracy. The search-based algorithms in [24] and [25] have high computational complexity, especially with high estimation accuracy requirement. In the situation of multiple targets, the methods in [24] and [25] may produce false targets when eliminating the ambiguity of DOA and extra operation is need to eliminate them. The reason of producing false targets is that the coprime property only ensure the elimination of angle ambiguity in the situation of single target.

Considering the drawbacks mentioned above in convention methods, a 2D-DOA estimation method based on polynomial root finding technique in coprime planar arrays is proposed. Firstly, signal subspace of all data is used to estimate the relation between the directional matrices of the two sub planar arrays. Then, by utilizing the estimated relation and noise subspace, the 2D-DOA estimation problem can be converted into two sub problems of double polynomial root finding. Lastly, the pairing of the same target under the two subarrays is realized using the estimated relation between the directional matrices, and the ambiguities of all targets are solved successively. The proposed method make use of the signal subspace and noise subspace of all data to improve the estimation accuracy with low computational complexity. Besides, by using the relation of the subarray direction matrix, the problem of eliminating ambiguity with multiple targets is transformed into a series of single-target ambiguity elimination problems, and the false target is avoided.

The main contribution of this paper can be summarized as follows:(i) a 2D-DOA estimation method with high accuracy is proposed; (ii) the computational complexity is reduced by utilizing two double polynomial root finding technique; (iii) the false targets is avoided using the relation between the direction matrix. The rest of this paper is summarized as follows:

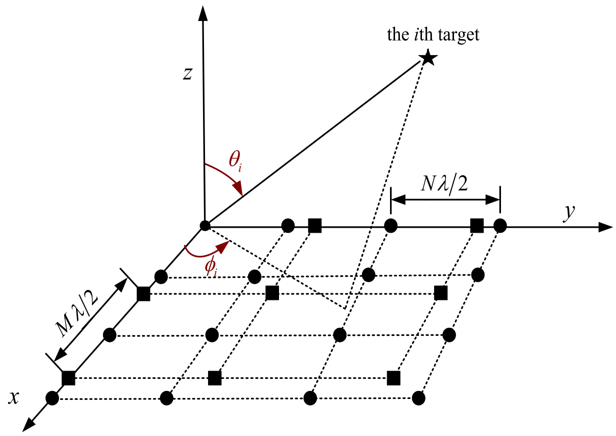

FIGURE 1. Array structure of coprime planar array with $M = 4$ and $N = 3$.

Section 2 describes the signal model of coprime planar array. Section 3 derives the proposed method. Section 4 analyzes the performance of the proposed method. Section 5 provides the simulation results. And Section 6 concludes the work.

Notations: Vectors (matrices) are denoted as lower-case (upper-case) bold characters. $(\cdot)^T$ represents the transpose and $(\cdot)^H$ represents the conjugate transpose. diag $(\cdot)$ represents a diagonal matrix which uses matrix elements as its diagonal elements. $E(\cdot)$ represents statistical expectation. $\otimes$ and $\oplus$ denotes the Kronecker product and Hadamard product respectively. angle $(\cdot)$ represents the extraction of phase operator and $\lceil\cdot\rceil$ denotes the ceiling function. $(\cdot)^+$ represents the pseudo inversion. $\boldsymbol{I}_M$ represents the $M \times M$ identity matrix. angle $(\cdot)$ represents the determinant of square matrix

II. SIGNAL MODEL OF COPRIME PLANAR ARRAY

Assuming there is a coprime planar array which contains two uniform planar subarrays. The first subarray consists of $M \times M$ sensors and the inter-element spacing is $d_1 = N\lambda/2$. The second subarray consists of $N \times N$sensors and the inter-element spacing is $d_2 = M\lambda/2$, where λ is the wavelength, M and N are coprime integers. And the positions of all the elements are in the set $L_S = \{(md_1, nd_1)\,|0 \le m, n \le M\} \cup \{(pd_2, qd_2)\,|0 \le p, q \le N-1\}$.

As shown in Fig. 1, there is only one common sensor between the two subarrays which is located in the position $(0, 0)$. Thus, there are $T = M^2 + N^2 - 1$ sensors in total. The Fig. 1 gives an example of the coprime planar array with $M = 4$ and $N = 3$.

Assume there are K far-field uncorrelated signals are incident on the planar array from $\{(\theta_k, \phi_k)\,|k = 1, 2, \ldots, K\}$, where θ_k is the elevation angle and ϕ_k is the azimuth angle of the kth signal. In order to facilitate processing, the formula can be defined as follows:

$$\begin{cases} u_k = \sin\theta_k \cos\phi_k \\ v_k = \sin\theta_k \sin\phi_k. \end{cases} \tag{1}$$

Because of the non-uniform distribution of element position, it is tedious to express the steering vector of coprime planar array. In brief expression, the whole coprime array can be formulated by two uniform planar subarrays. Then the whole received data can be represented as

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X_1} \\ \boldsymbol{X_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A_1} \\ \boldsymbol{A_2} \end{bmatrix} \boldsymbol{S} + \boldsymbol{N} \tag{2}$$

where $\boldsymbol{X} = [x(1), x(2), \ldots, x(L)]$ represents the whole time domain data. L represents the number of snapshots. $\boldsymbol{X_1} = [x_1(1), x_1(2), \ldots, x_1(L)]$ and $\boldsymbol{X_2} = [x_2(1), x_2(2), \ldots, x_2(L)]$ represent the receiving data of the two subarrays respectively. $\boldsymbol{S} = [s_1, s_2, \ldots, s_K]^T$ denote the incident signals and $s_k = [s_k(1), s_k(2), \ldots, s_k(L)]\ (k = 1, 2, \ldots, K)$. $\boldsymbol{N}$ denotes white Gaussian noise with mean value zero and variance σ^2. $\boldsymbol{A_i} \in C^{M_i^2 \times K} (i = 1, 2)$ denotes the steering matrix of the ith subarray, where $M_1 = M, M_2 = N$ and

$$\begin{aligned} \boldsymbol{A_i} &= [\boldsymbol{a}_i(u_1, v_1), \boldsymbol{a}_i(u_2, v_2), \ldots, \boldsymbol{a}_i(u_K, v_K)] \\ &= [\boldsymbol{a}_{yi}(v_1) \otimes \boldsymbol{a}_{xi}(u_1), \boldsymbol{a}_{yi}(v_2) \otimes \boldsymbol{a}_{xi}(u_2), \ldots, \\ &\qquad \boldsymbol{a}_{yi}(v_K) \otimes \boldsymbol{a}_{xi}(u_K)] \end{aligned} \tag{3}$$

where $\boldsymbol{a}_{xi}(u_k)\ (k = 1, 2, \ldots, K)$ and $\boldsymbol{a}_{yi}(v_k)\ (k = 1, 2, \ldots, K)$ denote the x-direction and y-direction steering vectors of the ith subarray respectively. And the specific forms is as follows

$$\boldsymbol{a}_{xi}(u_k) = [1, \exp(-j2\pi d_i u_k/\lambda), \ldots, \exp(-j2\pi (M_i - 1) d_i u_k/\lambda)]^T \tag{4}$$

$$\boldsymbol{a}_{yi}(v_k) = [1, \exp(-j2\pi d_i v_k/\lambda), \ldots, \exp(-j2\pi (M_i - 1) d_i v_k/\lambda)]^T. \tag{5}$$

The steering vector of the whole coprime array can be represented as

$$\boldsymbol{a}(u_k, v_k) = \begin{bmatrix} \boldsymbol{a}_1(u_k, v_k) \\ \boldsymbol{a}_2(u_k, v_k) \end{bmatrix} \quad (k = 1, 2, \ldots, K). \tag{6}$$

Both the 2D partial spectral search method and the 1D partial spectral search method divide the all data into two parts to estimate the 2D-DOA respectively. Therefore, the estimation of two covariance matrices can be obtained as follows

$$\hat{\boldsymbol{R}}_i = (1/L) \sum_{l=1}^{L} \boldsymbol{x}_i(l) \boldsymbol{x}_i^H(l) \quad (i = 1, 2), \tag{7}$$

which is used to estimate the 2D-DOA estimation results. Due to the large space between adjacent array elements, the two estimation results are ambiguous which can be eliminated utilizing the coprime property. However, the partition of the whole array restricts the estimation accuracy. Furthermore, the elimination of ambiguity is only guaranteed in the scenario of single target and may result in false target in the situation of multiple targets. In addition, small search step has to be selected to get high precision which leads to a great number of search times and high computational complexity. The quantitative analysis is given in (34)-(36) of Subsection A, Section IV.

Considering the drawbacks mentioned above, the 2D-DOA estimation method based on polynomial root finding technique is proposed. It can make use of the all data to estimate the angles, which has higher estimation accuracy. Besides, the proposed method does not need any spectral search operation and it just needs polynomial root finding operation, so that the complexity is optimized. In addition, the relation between the direction matrices of the two planar subarrays is used to pair the same target, then the ambiguity of all targets can be successively eliminated. Therefore, different targets will not affect each other and the false targets can be avoided.

III. PROPOSED 2D-DOA ESTIMATION METHOD

In the proposed method, the covariance matrix of the whole array is estimated as

$$\hat{\boldsymbol{R}} = (1/L) \sum_{l=1}^{L} \begin{bmatrix} \boldsymbol{x}_1(l) \\ \boldsymbol{x}_2(l) \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1(l) \\ \boldsymbol{x}_2(l) \end{bmatrix}^H. \tag{8}$$

Then the eigenvalue decomposition is carried out

$$\hat{\boldsymbol{R}} = \hat{\boldsymbol{E}}_s \hat{\boldsymbol{D}}_s \hat{\boldsymbol{E}}_s^H + \hat{\boldsymbol{E}}_n \hat{\boldsymbol{D}}_n \hat{\boldsymbol{E}}_n^H \tag{9}$$

where $\hat{\boldsymbol{D}}_s$ denotes the $K \times K$ diagonal matrix which contains the largest K eigenvalues of $\hat{\boldsymbol{R}}$, where $\hat{\boldsymbol{D}}_n$ denotes the diagonal matrix which contains the other eigenvalues. $\hat{\boldsymbol{E}}_s$ stands for the signal subspace which is made up of the eigenvectors of the largest K eigenvalues, while $\hat{\boldsymbol{E}}_n$ stands for the noise subspace which consists of the other eigenvectors.

A. RELATION DERIVATION

The signal subspace and the direction matrix span the same space

$$\hat{\boldsymbol{E}}_s = \begin{bmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \end{bmatrix} \boldsymbol{T} \tag{10}$$

where $\boldsymbol{T} \in C^{K \times K}$ denotes a non-singular matrix. And the signal subspace is decomposed into two parts

$$\hat{\boldsymbol{E}}_s = \begin{bmatrix} \boldsymbol{E}_{s1} \\ \boldsymbol{E}_{s2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \end{bmatrix} \boldsymbol{T}. \tag{11}$$

Then, $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ are obtained as follows:

$$\boldsymbol{H}_1 = \boldsymbol{E}_{s2} \boldsymbol{E}_{s1}^{+} = \boldsymbol{A}_2 \boldsymbol{T} \boldsymbol{T}^{-1} \boldsymbol{A}_1^{+} = \boldsymbol{A}_2 \boldsymbol{A}_1^{+} \tag{12}$$

$$\boldsymbol{H}_2 = \boldsymbol{E}_{s1} \boldsymbol{E}_{s2}^{+} = \boldsymbol{A}_1 \boldsymbol{T} \boldsymbol{T}^{-1} \boldsymbol{A}_2^{+} = \boldsymbol{A}_1 \boldsymbol{A}_2^{+} \tag{13}$$

which means

$$\boldsymbol{A}_2 = \boldsymbol{H}_1 \boldsymbol{A}_1 \tag{14}$$

$$\boldsymbol{A}_1 = \boldsymbol{H}_2 \boldsymbol{A}_2. \tag{15}$$

In this way, the relation between the two direction matrices is obtained by using the signal subspace. How to use these relation to pair the same target under different subarrays is shown in Subsection D, Section III. The double polynomial rooting algorithm is derived in the following two subsections.

B. IMPROVED MUSIC SPECTRUM

According to [26], the MUSIC spectrum of the all data is

$$f(u,v) = \frac{1}{\mathbf{a}^H(u,v)\mathbf{E}_n\mathbf{E}_n^H\mathbf{a}(u,v)} = \frac{1}{\left[\mathbf{a}_1^H(u,v)\,\mathbf{a}_2^H(u,v)\right]\mathbf{E}_n\mathbf{E}_n^H\begin{bmatrix}\mathbf{a}_1(u,v)\\ \mathbf{a}_2(u,v)\end{bmatrix}}. \quad (16)$$

Due to the specific formulation of the coprime array steering vector, a direct application of the polynomial root finding technique has to be adopted involving (14) and (15) so as to transform the MUSIC spectrum of the all data into

$$f_1(u,v) = \frac{1}{\left[\mathbf{a}_1^H(u,v)\,\mathbf{a}_1^H(u,v)\mathbf{H}_1^H\right]\mathbf{E}_n\mathbf{E}_n^H\begin{bmatrix}\mathbf{a}_1(u,v)\\ \mathbf{H}_1\mathbf{a}_1(u,v)\end{bmatrix}} = \frac{1}{\mathbf{a}_1^H(u,v)\mathbf{Q}_1\mathbf{Q}_1^H\mathbf{a}_1(u,v)} \quad (17)$$

$$f_2(u,v) = \frac{1}{\left[\mathbf{a}_2^H(u,v)\mathbf{H}_2^H\,\mathbf{a}_2^H(u,v)\right]\mathbf{E}_n\mathbf{E}_n^H\begin{bmatrix}\mathbf{H}_2\mathbf{a}_2(u,v)\\ \mathbf{a}_2(u,v)\end{bmatrix}} = \frac{1}{\mathbf{a}_2^H(u,v)\mathbf{Q}_2\mathbf{Q}_2^H\mathbf{a}_2(u,v)} \quad (18)$$

where $\mathbf{Q}_1 = \left[\mathbf{I}_{M^2}\mathbf{H}_1^H\right]\mathbf{E}_n$ and $\mathbf{Q}_2 = \left[\mathbf{H}_2^H\mathbf{I}_{N^2}\right]\mathbf{E}_n$. (17) and (18) are the deformations of (16). That is to say, (17) and (18) both contain all the data of the whole plane array. So the above two MUSIC spectral use the all output to estimate the 2D-DOA. In [24] and [25], the whole plane array is divided into two parts to be processed respectively. Therefore, higher estimation accuracy can be expected than that of algorithms in [24] and [25].

C. DOUBLE POLYNOMIAL ROOTING ALGORITHM

2D-PSS or 1D-PSS can be applied to (17) and (18) to estimate the 2D-DOA, but it has large complexity which need to be further improved. To consider this problem, the polynomial root finding procedure is used to estimate the 2D-DOA [27]. Now the specific operation is derived as follows:

Define the function $V_i(u,v)\ (i=1,2)$ as

$$V_i(u,v) = \mathbf{a}_i^H(u,v)\mathbf{Q}_i\mathbf{Q}_i^H\mathbf{a}_i(u,v) \quad (19)$$

where $V_i(u,v)$ is the denominator of $f_i(u,v)$, thus the spectral peak search can be converted to finding the null point of (19). According to (3), (19) can be represented as

$$\begin{aligned} V_i(u,v) &= \left(\mathbf{a}_{yi}(v)\otimes\mathbf{a}_{xi}(u)\right)^H\mathbf{Q}_i\mathbf{Q}_i^H\left(\mathbf{a}_{yi}(v)\otimes\mathbf{a}_{xi}(u)\right)\\ &= \mathbf{a}_{xi}^H(u)\left(\mathbf{a}_{yi}(v)\otimes\mathbf{I}_{M_i}\right)^H\mathbf{Q}_i\mathbf{Q}_i^H\left(\mathbf{a}_{yi}(v)\otimes\mathbf{I}_{M_i}\right)\mathbf{a}_{xi}(u)\\ &= \mathbf{a}_{xi}^H(u)\mathbf{G}_i(v)\mathbf{a}_{xi}(u) \end{aligned} \quad (20)$$

where $\mathbf{G}_i(v) = \left(\mathbf{a}_{yi}(v)\otimes\mathbf{I}_{M_i}\right)^H\mathbf{Q}_i\mathbf{Q}_i^H\left(\mathbf{a}_{yi}(v)\otimes\mathbf{I}_{M_i}\right)$.

Denote $z_{yi} = e^{-j2\pi d_i v/\lambda}$, $z_{xi} = e^{-j2\pi d_i u/\lambda}\ (i=1,2)$. The steering vector $\mathbf{a}_{yi}(v)$ and $\mathbf{a}_{xi}(u)$ can be rewritten as follows:

$$\mathbf{a}_{yi}(v) = \mathbf{a}(z_{yi}) = \left[1, z_{yi}, z_{yi}^2, \ldots, z_{yi}^{M_i-1}\right]^T \quad (21)$$

$$\mathbf{a}_{xi}(u) = \mathbf{a}(z_{xi}) = \left[1, z_{xi}, z_{xi}^2, \ldots, z_{xi}^{M_i-1}\right]^T. \quad (22)$$

Then, (19) can be written as

$$V_i(z_{xi}, z_{yi}) = \mathbf{a}^H(z_{xi})\mathbf{G}(z_{yi})\mathbf{a}(z_{xi}) \quad (23)$$

where $\mathbf{G}(z_{yi}) = \left(\mathbf{a}(z_{yi})\otimes\mathbf{I}_{M_i}\right)^H\mathbf{Q}_i\mathbf{Q}_i^H\left(\mathbf{a}(z_{yi})\otimes\mathbf{I}_{M_i}\right)$.

Then the spectrum peak of (17) and (18) is turned into the root of the following equation:

$$\mathbf{a}^H(z_{xi})\mathbf{G}(z_{yi})\mathbf{a}(z_{xi}) = 0\ (i=1,2). \quad (24)$$

Essentially, the above states derivations convert the estimation of the 2D-DOA into finding the roots of polynomial (24). If z_{yi} does not match with one target and if

$$\text{Rank}\left(\mathbf{Q}_i\right) \geq M_i, \quad (25)$$

then the matrix $\mathbf{G}(z_{yi})$ is invertible, and the determinant is not equal to zero.

To solve the polynomial (24), we can find that z_{yi} satisfies

$$D(z_{yi}) = \det\left(\mathbf{G}(z_{yi})\right) = 0. \quad (26)$$

The degree of this polynomial is $2M_i^2-2$. The K roots closest to the unit circle of the polynomial $D(z_{yi})$, $\hat{z}_{yi}^{(k)}\,|_{k=1,\ldots,K}$, allow to estimate the

$$\hat{v}_i^{(k)} = \text{angle}\left(\hat{z}_{yi}^{(k)}\right)\lambda/2\pi d_i. \quad (27)$$

Then, substitute the roots $\hat{z}_{yi}^{(k)}$ into (24), the following equations can be obtained:

$$\mathbf{a}^H(z_{xi})\mathbf{G}\left(\hat{z}_{yi}^{(k)}\right)\mathbf{a}(z_{xi}) = 0\,|_{k=1,\ldots,K}. \quad (28)$$

In order to solve the above polynomial, the root finding technique can be used again. According to the principle of MUSIC algorithm, the above polynomial become zero if (z_{xi}, z_{yi}) matches with one target. Thus, for each $\hat{z}_{yi}^{(k)}$, the polynomial of degree $2M_i-2$ has p roots close to the unit cycle where p denotes the number of targets having the same $\hat{z}_{yi}^{(k)}$.

The roots $\hat{z}_{xi}^{(k)}$ of the polynomial (28) which is closest to the unit circle determine the

$$\hat{u}_i^{(k)} = \text{angle}\left(\hat{z}_{xi}^{(k)}\right)\lambda/2\pi d_i. \quad (29)$$

It is noted that, the $\hat{u}_i^{(k)}$ and $\hat{v}_i^{(k)}$ are automatically paired.

The transformation of two dimension DOA estimation into two double polynomial root finding procedure (The reason for two is that $i = 1, 2$) can reduce the computational complexity significantly. Comparison of the computational complexity between conventional methods (2D-PSS method and 1D-PSS method) and the proposed method is presented in Subsection A, Section IV.

Through the double polynomial rooting, the ambiguous value is obtained, then the pairing of same target and the elimination of ambiguity can be done. So the double polynomial rooting is the foundation of the proposed method.

D. MULTI TARGET PAIRING AND AMBIGUITY ELIMINATION

Because of the large distance between the array elements, the estimated $\hat{u}_i^{(k)}$ and $\hat{v}_i^{(k)}$ $(i = 1, 2; k = 1, 2, \ldots, K)$ are ambiguous, which needs to be further optimized by referring [24, Th. 1].

Theorem 1: Suppose (θ_k, ϕ_k) is the actual 2D-DOA of the kth target, which presents multiple ambiguous 2D-DOAs for each subarray, i.e., multiple peaks in the 2-D MUSIC spectrum. By intersecting the two MUSIC spectrums of the kth target, there exists and uniquely exists one 2-D DOA $\left(\hat{\theta}_k, \hat{\phi}_k\right)$ that presents a peak in both the spectrums, where $\left(\hat{\theta}_k, \hat{\phi}_k\right)$ is the estimated 2-D DOA of the kth target.

It is noted that, Theorem 1 illustrates that in the situation of single target, the MUSIC spectrum of the two subarrays has only one common peak in the true DOA. While in the situation of multiple targets, except for the true DOA, it may produce common peaks for the spectrum of different targets in different subarrays. Fortunately, if the same targets in different sub arrays is paired, the ambiguity can be successively eliminated.

Next, the relation obtained in Subsection A of Section III is used to pair the same target in different sub arrays.

The previous analysis in Subsection C of Section III indicates that the roots $\hat{z}_{y1}^{(k)}\big|_{k=1,\ldots,K}$ $(\hat{z}_{y2}^{(j)}\big|_{j=1,\ldots,K})$ and $\hat{z}_{x1}^{(k)}\big|_{k=1,\ldots,K}$ $(\hat{z}_{x2}^{(j)}\big|_{j=1,\ldots,K})$ are intrinsically coupled. Once the roots $\hat{z}_{y1}^{(k)}\big|_{k=1,\ldots,K}$ $(\hat{z}_{y2}^{(j)}\big|_{j=1,\ldots,K})$ and $\hat{z}_{x1}^{(k)}\big|_{k=1,\ldots,K}$ $(\hat{z}_{x2}^{(j)}\big|_{j=1,\ldots,K})$ are solved, the steering vector of subarray 1 (subarray 2) can be obtained as follows

$$\begin{aligned}\boldsymbol{a}_1\left(\hat{u}_k, \hat{v}_k\right) &= \boldsymbol{a}_{y1}\left(\hat{v}_k\right) \otimes \boldsymbol{a}_{x1}\left(\hat{u}_k\right)\\ &= \left[1, \hat{z}_{y1}^{(k)}, \left(\hat{z}_{y1}^{(k)}\right)^2, \ldots, \left(\hat{z}_{y1}^{(k)}\right)^{M_i-1}\right]^T\\ &\quad \otimes \left[1, \hat{z}_{x1}^{(k)}, \left(\hat{z}_{x1}^{(k)}\right)^2, \ldots, \left(\hat{z}_{x1}^{(k)}\right)^{M_i-1}\right]^T. \end{aligned} \tag{30}$$

$$\begin{aligned}\boldsymbol{a}_2\left(\hat{u}_j, \hat{v}_j\right) &= \boldsymbol{a}_{y2}\left(\hat{v}_j\right) \otimes \boldsymbol{a}_{x2}\left(\hat{u}_j\right)\\ &= \left[1, \hat{z}_{y2}^{(j)}, \left(\hat{z}_{y2}^{(j)}\right)^2, \ldots, \left(\hat{z}_{y2}^{(j)}\right)^{M_i-1}\right]^T\\ &\quad \otimes \left[1, \hat{z}_{x2}^{(j)}, \left(\hat{z}_{x2}^{(j)}\right)^2, \ldots, \left(\hat{z}_{x2}^{(j)}\right)^{M_i-1}\right]^T. \end{aligned} \tag{31}$$

Now, pairing the same targets in different sub array is converted to the problem of pairing the sequence of index k and index j.

Then, the relation $\hat{\boldsymbol{a}}_2\left(\hat{u}_k, \hat{v}_k\right) = \boldsymbol{H}_1\boldsymbol{a}_1\left(\hat{u}_k, \hat{v}_k\right)$ is applied to estimate the steering vector of kth target in second subarray. Based on the minimum Euclidean distance criterion, the index k and j can be paired by finding

$$j_k = \underset{j=1,2,\ldots,K}{\arg\min} \left\|\hat{\boldsymbol{a}}_2\left(\hat{u}_k, \hat{v}_k\right) - \boldsymbol{a}_2\left(\hat{u}_j, \hat{v}_j\right)\right\| \quad (k=1, 2, \ldots, K)$$

Now the pairing of same target in the two subarrays is achieved and the ambiguity can be eliminated one by one. The specific operations of ambiguity elimination of each target is same with the operation in [24] and [25].

After the unambiguous values $\left(\tilde{u}_1^{(k)}, \tilde{v}_1^{(k)}\right)$ and $\left(\tilde{u}_2^{(k)}, \tilde{v}_2^{(k)}\right)$ of kth target are obtained by the two subarrays respectively, the following equation can be used to estimate the 2D-DOA of targets.

$$\begin{cases}\tilde{u}^{(k)} = \dfrac{\tilde{u}_1^{(k)} + \tilde{u}_2^{(k)}}{2}\\ \tilde{v}^{(k)} = \dfrac{\tilde{v}_1^{(k)} + \tilde{v}_2^{(k)}}{2}\end{cases} \tag{32}$$

$$\begin{cases}\hat{\theta}^{(k)} = \text{asin}\left(abs\left(\tilde{u}^{(k)} + j\tilde{v}^{(k)}\right)\right)\\ \hat{\phi}^{(k)} = \text{angle}\left(\tilde{u}^{(k)} + j\tilde{v}^{(k)}\right)\end{cases} \tag{33}$$

where $\hat{\theta}^{(k)}$ and $\hat{\phi}^{(k)}$ are the final estimation results of kth target.

IV. PERFORMANCE ANALYSIS

A. COMPLEXITY ANALYSIS

The computational complexity of the 2D-MUSIC with UPA is

$$\begin{aligned}&O\left(M_u^2N_u^2L + M_u^3N_u^3 + J_1J_2M_uN_u\left(M_uN_u - K\right)\right)\\ &\quad = O\left(M_u^2N_u^2L + M_u^3N_u^3 + \frac{2M_uN_u\left(M_uN_u - K\right)}{\Delta^2}\right)\end{aligned} \tag{34}$$

where J_1 and J_2 are the times of total search with $u \in (-1, 1)$ and $v \in (0, 1)$. Δ denotes the search step size. So $J_1 = \frac{2}{\Delta}$ and $J_2 = \frac{1}{\Delta}$. $M_uN_u = M^2 + N^2 - 1$.

The computational complexity of the 2D-PSS is

$$\begin{aligned}&O\begin{pmatrix}M^4L + M^6 + N^4L + N^6 + l_1^2M^2\left(M^2 - K\right)\\ +l_2^2N^2\left(N^2 - K\right)\end{pmatrix}\\ &\quad = O\begin{pmatrix}M^4L + M^6 + N^4L + N^6 + \frac{4M^2(M^2-K)}{N^2\Delta^2}\\ +\frac{4N^2(N^2-K)}{M^2\Delta^2}\end{pmatrix}\end{aligned} \tag{35}$$

where l_1 and l_2 are the times of partial search with $v \in (0, 2/N)$ and $v \in (0, 2/M)$, thus $l_1 = \frac{2}{N\Delta}$ and $l_2 = \frac{2}{M\Delta}$

The computational cost of the 1D-PSS is

$$\begin{aligned}&O\begin{pmatrix}M^4L + M^6 + N^4L + N^6 + l_1M^3\left(M^2 - K + 1\right)\\ +l_2N^3\left(N^2 - K + 1\right)\end{pmatrix}\\ &\quad = O\begin{pmatrix}M^4L+M^6+N^4L+N^6+\frac{2M^3(M^2-K+1)}{N\Delta}\\ +\frac{2N^3(N^2-K+1)}{M\Delta}\end{pmatrix}.\end{aligned} \tag{36}$$

The proposed method consists of estimation of covariance matrix and feature decomposition and polynomial root finding procedure. By calculating, the total computational complexity is

$$O\begin{pmatrix}\left(M^2 + N^2\right)^2L + \left(M^2 + N^2\right)^3 + \left(2M^2 - 2\right)^3\\ +, K\left(2M - 2\right)^3 + \left(2N^2 - 2\right)^3 + K\left(2N - 2\right)^3\end{pmatrix}. \tag{37}$$

In Fig. 2, the complexity of these methods is compared through one specific example, where $M = 4$, $L = 100$,

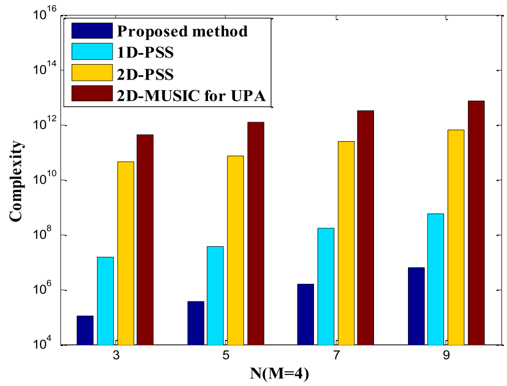

FIGURE 2. Comparison of computational complexity with different *N*.

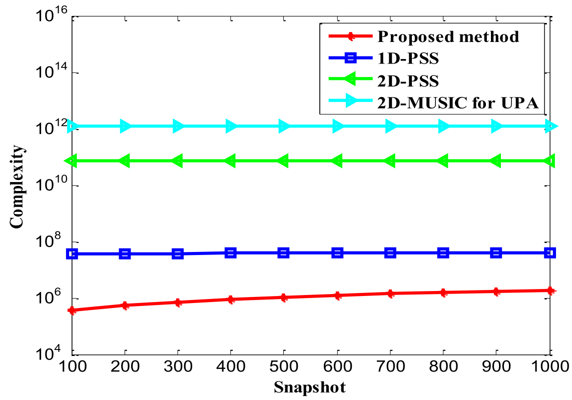

FIGURE 3. Comparison of computational complexity versus different snapshots.

$K = 2$, $\Delta = 0.00005$, and for pair comparison, $M_uN_u = M^2+N^2-1$. As shown in the figure, the proposed algorithm is the fastest compared with the other algorithms.

In Fig. 3, the complexity of these algorithms versus different snapshots is compared. The parameters are set as $M = 5$, $N = 4$, $M_u = 8$,$N_u = 5$, $K = 2$, $\Delta = 0.00005$. In comparison, the proposed algorithm has the lowest computational complexity among these algorithms which proves the efficiency of the proposed method.

Moreover, the computation time of these methods are given in Table 1. They are computed by the MATLAB R2014a with Inter Core i7-3770 @3.4 GHz and 16GB RAM, and the parameters are set as $M = 4, N = 3, M_u = 6, N_u = 4, K = 2$, $L = 100$ $\Delta = 0.00005$. It is obvious that the computation time of the proposed algorithm is the shortest compared with the other algorithms.

B. MAXIMUM ESTIMATED NUMBER OF TARGETS

The maximum number of targets that can be estimated by the proposed algorithm is connected with (25), i.e., $\text{Rank}\left(\boldsymbol{Q}_i\right) \geq M_i$. Because of $\text{Rank}\left(\boldsymbol{Q}_i\right) \leq \min\left[\left(M^2+N^2-1-K\right), M_i^2\right]$. So the maximum number of sources can be expected to be estimated by the proposed method is $\min\left[\left(M^2+N^2-1-M\right), \left(M^2+N^2-1-N\right)\right]$. While the number of sources that can be resolved by the 2D-PSS and 1D-PSS approach is at most $\left\{\min\left(M^2, N^2\right)-1\right\}$. If M and N are very large, the proposed method can expect to estimate more targets.

TABLE 1. Computation time of different methods.

	Complex multiplications	Computation time, s
Proposed method	1.0978×10^5	1.1788
1D-PSS	1.4990×10^7	2.3370
2D-PSS	4.6122×10^{10}	2.1786×10^4
2D-MUSIC with UPA	4.2240×10^{11}	8.58519×10^4

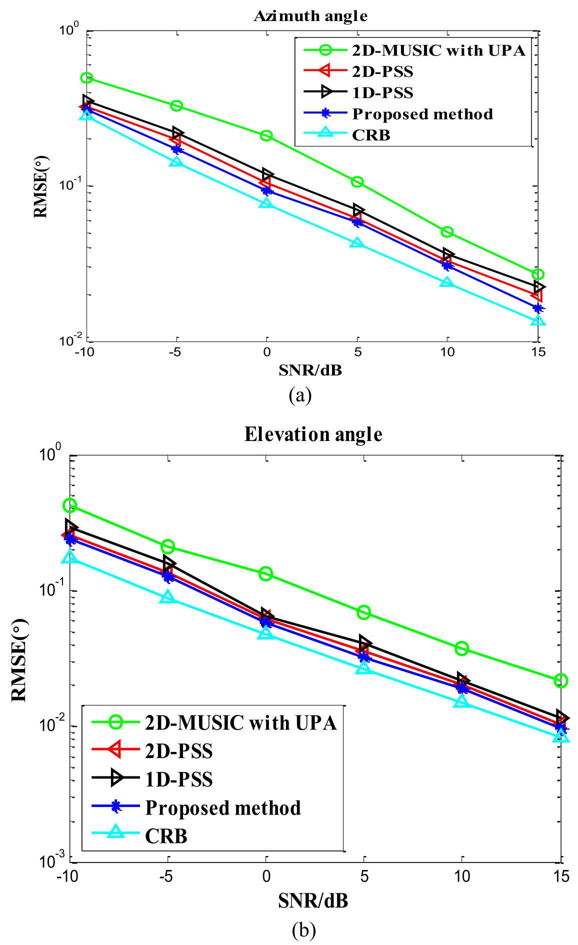

FIGURE 4. Comparison of estimation accuracy of different algorithms: (a) Azimuth angle and (b) Elevation angle

C. ADVANTAGES

The advantages of the proposed algorithm are shown as follows:

(i) The proposed method needs two double polynomial root finding procedure and does not require any search operation which has low complexity;

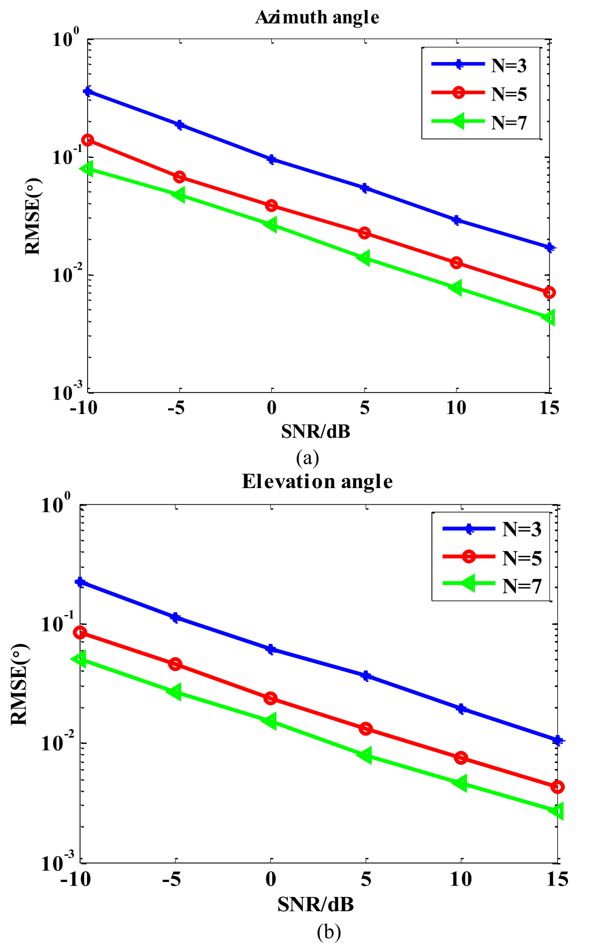

FIGURE 5. Estimation accuracy comparison with different N and $M = 4$: (a) Azimuth angle and (b) Elevation angle.

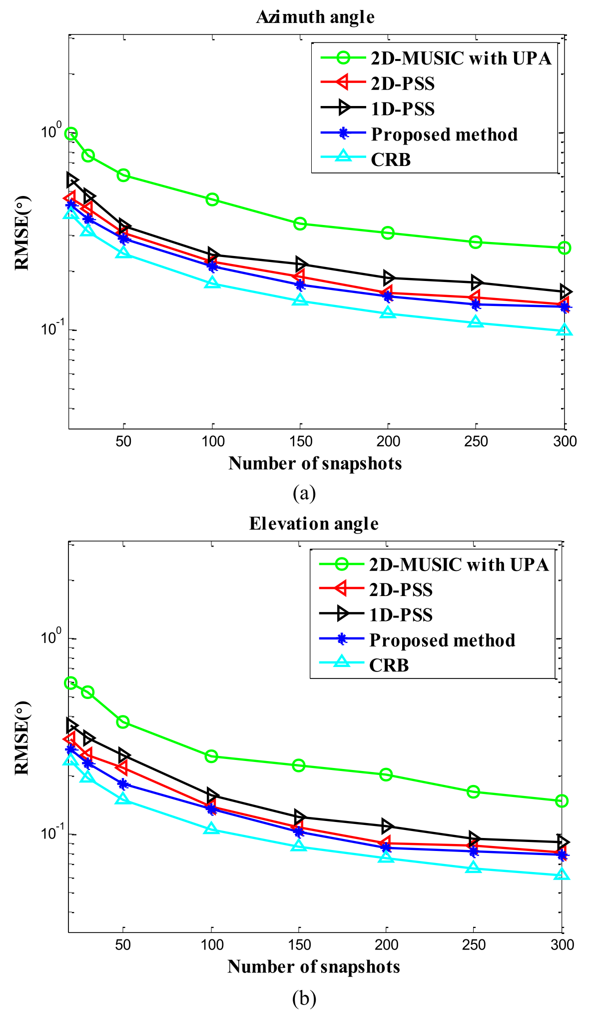

FIGURE 6. Comparison of estimation accuracy with different snapshots: (a) Azimuth angle and (b) Elevation angle.

(ii) The proposed algorithm utilizes the signal subspace and noise subspace of all data to estimate the DOA of targets, which obtains high estimation accuracy;

(iii) The proposed method avoids the false targets by utilizing the relation between the two direction matrices;

(iv) The proposed method obtains automatically paired DOA estimation.

It is noted that, small distance between adjacent elements results in mutual coupling which restricts the accuracy of DOA estimation. Therefore, large values of M and N should be adopted to reduce the effect of mutual coupling.

D. CRAMÉR-RAO BOUND (CRB)

The derivation of the CRB can be found in [22].

V. SIMULATION RESULTS

In this section, several simulation experiments are conducted to test the performance of the proposed method.

The CPA consists of two uniform planar subarrays with 4×4 and 3×3 sensors, respectively. The inter-element spacing are $d_1 = 3\lambda/2$ and $d_2 = 4\lambda/2 = 2\lambda$, and the inter-element spacing of UPA is set to half wavelength. For fair comparison, the number of sensors in UPA is 6×4. The search step $\Delta = 0.00005$. The noise is white Gaussian noise. Assume there are 2 targets. The 2D-DOA of them are $(\theta, \phi) = (20°, 60°)$ and $(\theta, \phi) = (50°, 35°)$ respectively.

The root mean square error (RMSE) is defined as follows

$$RMSE = \sqrt{\frac{1}{CK}\sum_{c=1}^{C}\sum_{k=1}^{K}\left(\alpha_k - \hat{\alpha}_{k,c}\right)^2}$$

where C denotes the simulation times and $\hat{\alpha}_{k,c}$ represents the estimation result of the kth target in the cth simulation. The simulation times is set as $C = 200$. α_k represents the true value of θ or ϕ.

A. COMPARISON OF ESTIMATION ACCURACY OF DIFFERENT ALGORITHMS

Firstly, the estimation accuracy is compared between conventional methods (2D-MUSIC with UPA, 2D-PSS, 1D-PSS) and the proposed method. $L = 500$ and SNR varies from −10dB to 15dB. As shown in Fig. 4, the RMSE curve of the

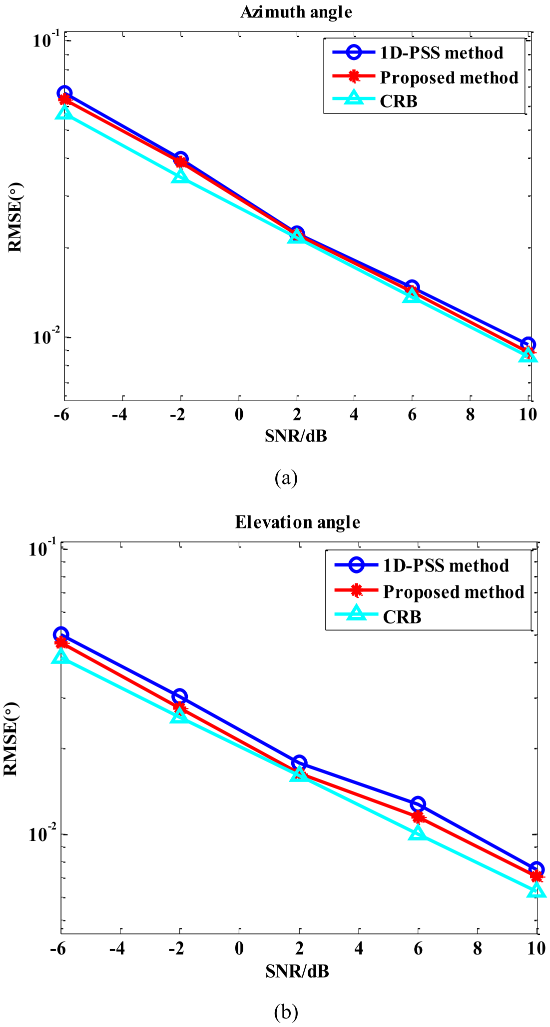

FIGURE 7. **The comparison of estimation performance under a special case: (a) Azimuth angle and (b) Elevation angle.**

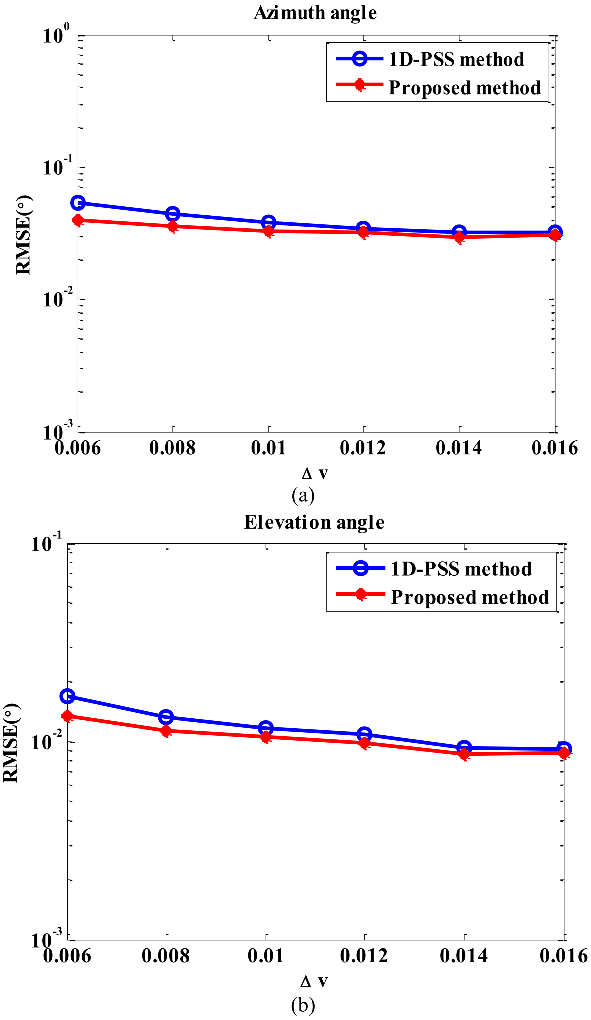

FIGURE 8. **The estimation performance with two adjacent sources: (a) Azimuth angle and (b) Elevation angle.**

proposed algorithm is the lowest among these methods which proves the effectiveness of the proposed method.

B. COMPARISON OF ESTIMATION ACCURACY WITH DIFFERENT N

Secondly, the estimation performance of the proposed algorithm with different N is presented in Fig. 5. $M = 4$. The other parameters are same with the first simulation. It can be seen that the estimation accuracy of the proposed method increases with the increasing of N which is caused by the diversity gain.

C. COMPARISON OF ESTIMATION ACCURACY WITH DIFFERENT L

Thirdly, the estimation accuracy of these algorithms versus different L is compared. SNR=0dB. L varies from 20 to 300. The other parameters are same with the first simulation. Fig. 6 gives the simulation results. It is shown that the estimation accuracy increases with the increasing of snapshots which is caused by the more accuracy estimation of covariance matrix. It is noted that the estimation accuracy of the proposed method is the best among these methods which proves the effectiveness of the proposed method.

D. COMPARISON OF ESTIMATION ACCURACY UNDER A SPECIAL CASE

Fourthly, the DOA estimation performance under a special case is researched, where $M = 4$, $N = 5$, $L = 500$, $(\theta_1, \phi_1) = (34.2°, 65.5°)$, $(\theta_2, \phi_2) = (38.7°, 77.7°)$ and convert the 2D-DOA into the transformed domain $(u_1, v_1) = (0.2331, 0.5116)$, $(u_2, v_2) = (0.1331, 0.6116)$. For $v_1 - 2/N = v_2 - 2/M$ and $u_1 + 2/N = u_2 + 2/M$, in this situation, the 1D-PSS method and 2D-PSS method will result in false targets when eliminating the ambiguous angle. And they need additional operation to eliminate them. However the proposed method does not need this operation. And the simulation result is shown in Fig. 7. It can be seen that the proposed method can obtain correct estimation under this special situation and it achieves a lower RMSE than the 1D-PSS method.

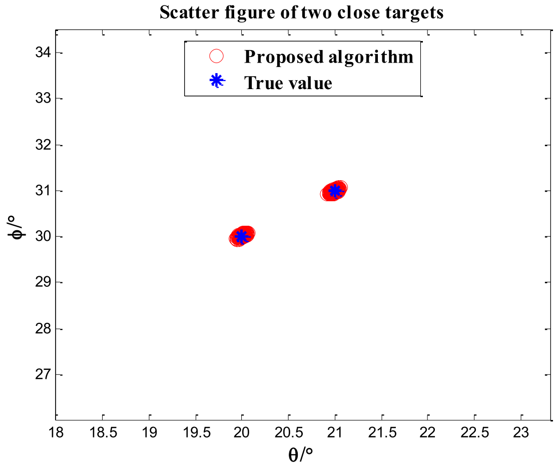

FIGURE 9. Scatter figure of two close sources.

E. COMPARISON OF DOA ESTIMATION ACCURACY WITH ADJACENT TARGETS

The DOA estimation performance of two adjacent targets is shown in Fig. 8. The parameters are set as $\Delta v = [0.016, 0.018, 0.02, 0.022, 0.024, 0.026]$, $M = 7$, $N = 5$, $K = 2$, $L = 100$ and SNR $= 8$dB. The scatter figure of two adjacent targets is plotted in Fig. 9, where $(\theta_1, \phi_1) = (20^\circ, 30^\circ)$ and $(\theta_2, \phi_2) = (21^\circ, 31^\circ)$. The two figures show that the proposed algorithm can work well and has a better DOA estimation accuracy than the 1D-PSS method.

VI. CONCLUSION

In this paper, the 2D-DOA estimation problem for coprime planar arrays is considered and a high accuracy and computational efficient method based on polynomial root finding technique is proposed. It make use of the signal space and noise space of all data to improve the estimation accuracy. In addition, the computational complexity is reduced by converting the 2D spectral peak searching problem into two double polynomial root finding problem. Furthermore, the matching error problem in the situation of multiple targets is avoided. Simulation results prove that the proposed method achieves not only higher estimation accuracy but also lower computational complexity than conventional methods.

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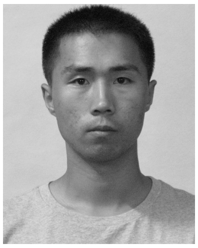

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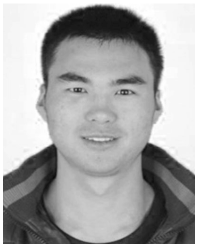

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