IEEE Access Multidisciplinary : Rapid Review : Open Access Journal

SPECIAL SECTION ON DATA-DRIVEN MONITORING, FAULT DIAGNOSIS AND CONTROL OF CYBER-PHYSICAL SYSTEMS

Received January 8, 2018, accepted March 1, 2018, date of publication March 16, 2018, date of current version April 23, 2018.

Digital Object Identifier 10.1109/ACCESS.2018.2815662

Minimum-Variance Unbiased Unknown Input and State Estimation for Multi-Agent Systems by Distributed Cooperative Filters

CHANGQING LIU[1], YOUQING WANG[2], (Senior Member, IEEE), DONGHUA ZHOU[2], (Senior Member, IEEE), AND XIAO SHEN[3]

[1]College of Information Science and Technology, Beijing University of Chemical Technology, Beijing 100029, China
[2]College of Electrical Engineering and Automation, Shandong University of Science and Technology, Qingdao 266590, China
[3]College of Mechanical and Electronic Engineering, Shandong University of Science and Technology, Qingdao 266590, China

Corresponding author: Youqing Wang (wang.youqing@ieee.org)

This work was supported in part by the Shandong Province Science Fund for Distinguished Young Scholars under Grant JQ201812, in part by the National Natural Science Foundation of China under Grant 61751307, and in part by the Research Fund for the Taishan Scholar Project of Shandong Province of China.

ABSTRACT This paper addressed the problem of the simultaneous estimation of unknown inputs and states in a multi-agent system with time-invariant and time-varying topology. A group of distributed cooperative recursive filters, in the sense of minimum-variance unbiased, was developed, where the estimations of unknown input and state were combined. A necessary and sufficient existing condition is presented and proven for the proposed distributed cooperative filters. Theoretical and numerical analyses demonstrate that the existing condition of the proposed filters is significantly relaxed, in comparison to that of conventional decentralized filters.

INDEX TERMS Distributed cooperative filter, estimation, multi-agent system.

I. INTRODUCTION

Unknown inputs can affect system performance significantly. In many situations, the direct measurement of unknown inputs is very difficult or even impossible. Hence, the estimation of the unknown inputs in a system is a significant problem.

In this study, a group of distributed cooperative filters is proposed for an uncertain multi-agent system in order to estimate its unknown inputs and states. Its advantages over conventional decentralized methods are analyzed and presented. The main motivation for this study arose from the development of consensus theory [1]–[15]. To provide the background for this study, consensus theory, unbiased estimation of minimum-variance, and distributed cooperative filters are respectively introduced in the following three subsections.

A. CONSENSUS THEORY AND COOPERATIVE STRATEGY

For a multi-agent system, a cooperative strategy involves achieving a common objective through cooperation among individual agents in the system. This issue has attracted considerable attention in the field of computation and optimization [1] since the 1990s. The consensus theory is a fundamental cooperative strategy for multi-agent systems. It requires that the state of every agent in the system reach a common value through communicating with each other. Consensus theory has received much attention over the last decade. Jadbabaie *et al.* [2] analyzed the consensus of the Vicsek model [3] theoretically. Since then, many studies on consensus theory with regard to multi-agent systems have emerged. Thus far, there have been many interesting results such as, for example, a survey paper [4] and a book [5]. Moreover, the study on consensus theory has not been confined at the stage of theoretical research, but has rather advanced to actual applications on wireless sensor networks [6], flocking problems [7], and so on. Chen *et al.* [16] proposed distributed cooperative adaptive laws in order to estimate the unknown parameters of multi-agent systems. Inspired by these studies, we explore a new application of consensus theory in order to estimate unknown inputs and states in a multi-agent system. Specifically, we propose new distributed cooperative filters in order to estimate the unknown inputs and states of a heterogeneous multi-agent system. In comparison to conventional filters, the proposed distributed cooperative filters have a much relaxed existing condition. The details will be provided in Section 3.

B. MINIMUM-VARIANCE UNBIASED ESTIMATION

In [17]–[19] the necessary and sufficient conditions for the existence of an optimal state estimator in continuous-time systems was established. Moreover, considerable attention [20], [21] was given to the design methods for the reconstruction of unknown input. The earliest approaches toward reconstructing the unknown input for discrete systems were based on augmenting the state vector along with the unknown input vector by using a model of the unknown input. To reduce the amount of calculation for the state filter, a two-stage Kalman filter is proposed [22]. It should be noted that the estimations of state and the unknown input are decoupled in [22]. Although [22] has many successful applications, the result is limited because it ignores the dynamical evolution of unknown input.

In [23] an optimal recursive state filter was proposed without using prior information for the unknown input. Then, [23] was extended in [24], which proposed the stability and convergence conditions and found a new design method for the filter. In [25], it was found that the two-stage Kalman filter is closely related to the Kitanidis filter [23], in the sense of being able to derive the Kitanidis filter by making the two-stage filter independent of the underlying input model. Furthermore, paper [25] obtained an estimate of the unknown input, whereas, paper [25] did not prove the optimality of the unknown input estimation.

Paper [26] proposed a recursive filter which can simultaneously obtain the minimum-variance unbiased (MVU) estimations of the unknown input and the state. Inspired by [16] and [26], the current study proposes a new distributed cooperative filter in order to obtain the MVU estimations of the unknown input and the state of a heterogeneous multi-agent system, and to find a more relaxed existing condition compared to that of the conventional filter proposed in [26].

C. DISTRIBUTED COOPERATIVE FILTER

The main advantage of the distributed multi-agent system is that it has adaptive and learning abilities. The information shared between agents can be utilized in order to collaboratively solve inference and optimization problems [27]. In comparison to traditional centralized solutions, distributed solutions do not require a powerful fusion center in order to process the data from every agent. As a result, distributed solutions can effectively reduce both computation and communications. On the other hand, in a centralized solution, if the fusion center breaks down, this will lead to a failure of the entire network. By comparison, distributed solutions can avoid this problem and are more robust to agent and link failure [28]. As a result, many studies have proposed with a distributed cooperative filter [29]–[33]. Paper [29] proposed a distributed Kalman filter scheme in order to estimate actuator faults for deep space formation flying satellites in the form of an overlapping block-diagonal state space representation. Based on the linear matrix inequality method, paper [30] considered a robust distributed state estimation and fault detection, as well as isolation problems based on an unknown input observer for a network of heterogeneous multi-agent LPV systems. Additionally, based on the LMI method, [31] focuses on the design of fault detection and isolation filters for multi-agent systems, where limited communication exists among the agents and extend the formulation to a class of linear parameter-varying systems. Using the FIR model, the problem of distributed bias-compensated recursive least-squares estimation over multi-agent networks was investigated in [32]. In [33], a robust unknown input observer-based fault estimation was proposed. It used the relative output information in order to utilize the communication topology for multi-agent systems with undirected graphs. Inspired by previous work, we propose a new distributed cooperative filter in order to estimate the unknown input for heterogeneous multi-agent systems.

To the authors' best knowledge, this study makes the following contributions. First, we propose a new distributed cooperative filter for a heterogeneous multi-agent system in order to obtain the MVU estimation of unknown input and the states in the system, which has not been previously studied. The previous work on the distributed method mostly depends on the LMI method. Whereas, our work get the distributed MVU filters of the multi-agent systems. Secondly, a necessary and sufficient condition for the existence of the proposed filter has been presented and proven. Furthermore, in comparison to the traditional decentralized filter [26], the existing condition of our filter was significantly relaxed, and this conclusion was proven by theoretical analysis.

The rest of this paper is organized as follows. Section 2 includes a preliminary discussion, which will be used in the following sections. In Section 3, the problem is formulated and the structure of the distributed cooperative filter is presented. In Section 4 the optimal reconstruction of the unknown input is investigated. Subsequently, the state estimation problem is solved in Section 5. The main results of this paper are provided in Section 6. In Section 7, some simulation results are provided in order to verify the theory. Section 8 offers the conclusion of this study.

II. PRELIMINARIES

A. ALGEBRAIC GRAPH THEORY

In this study, the network topology among N agents was used in order to describe their interconnections and was modeled as a weighted graph $G = (V, \varepsilon, A)$ with a set of nodes V, set of edges ε, and adjacency matrix A with non-negative adjacent elements. The i-th agent is denoted by node v_i. The edge in graph G is denoted by an unordered pair $e_{ij} = (i, j)$. $e_{ij} \in \varepsilon$ if and only if there is information exchange between agent i and agent j, and $e_{ij} \in \varepsilon \Leftrightarrow e_{ji} \in \varepsilon$. The adjacency element a_{ij} represents agent-agent communication. Note that $e_{ij} \in \varepsilon \Leftrightarrow a_{ij} = 1$. Otherwise, $a_{ij} = 0$. It is assumed that $a_{ij} = a_{ji}$, which means that A is symmetric [34]. If there a path exists between any two nodes $v_i, v_j \in V$, then G is connected. Every agent j

that has a connection with an agent i is considered a neighbor of agent i. N_i denotes the set of all neighbors for agent i.

B. MINIMUM-VARIANCE UNBIASED ESTIMATOR

Consider the following linear discrete-time system

$$x_{k+1} = B_k x_k + G_k d_k + \omega_k \tag{1}$$
$$y_k = C_k x_k + v_k \tag{2}$$

where $x_k \in R^q$ is the state vector, $d_k \in R^m$ is an unknown input vector, and $y_k \in R^p$ is the measurement. Process noise $\omega_k \in R^q$ and measurement noise $v_k \in R^p$ are assumed to be mutually uncorrelated, zero-mean, white random signals.

Below is a recursive filter:

$$\hat{x}_{k|k-1} = B_{k-1}\hat{x}_{k-1|k-1} \tag{3}$$
$$\hat{d}_{k-1} = M_k(y_k - C_k\hat{x}_{k|k-1}) \tag{4}$$
$$\hat{x}^*_{k|k} = \hat{x}_{k|k-1} + G_{k-1}\hat{d}_{k-1} \tag{5}$$
$$\hat{x}_{k|k} = \hat{x}^*_{k|k} + K_k(y_k - C_k\hat{x}^*_{k|k}) \tag{6}$$

where $M_k \in R^{m*p}$ and $K_k \in R^{q*p}$ are the design gain matrices. Then, the following existing conditions of the filter are provided.

Lemma 1[26]: Given that $\hat{x}_{k-1|k-1}$ is unbiased, there exist M_k and K_k such that the recursive filter (3)-(6) can achieve the MVU estimations of d_{k-1} and x_k in systems (1)-(2), if and only if:

$$rank\ (C_k G_{k-1}) = m \tag{7}$$

III. PROBLEM FORMULATION

Consider the following linear discrete-time heterogeneous multi-agent system:

$$x^i_{k+1} = B^i_k x^i_k + G^i_k d_k + \omega^i_k \tag{8}$$
$$y^i_k = C^i_k x^i_k + v^i_k \tag{9}$$

where $x^i_k \in R^q$ is the state vector of agent i, $d_k \in R^m$ denotes the unknown input, and $y^i_k \in R^p$ is the measurement of system i. Process noise $\omega^i_k \in R^q$ and measurement noise $v^i_k \in R^p$ are assumed to be mutually uncorrelated, zero-mean, white random signals with known covariance matrices $Q^i_k = \mathbf{E}\left[\omega^i_k\ \omega^{iT}_k\right]$ and $R^i_k = \mathbf{E}\left[v^i_k\ v^{iT}_k\right]$, respectively, where $\mathbf{E}$ denotes the mathematical expectation.

Remark 1: The main property of system (8) and (9) is that although various systems have different time-varying system structures, the unknown input vector is the same for all agents. There exist many real-world systems that can be represented by (8) and (9). For example, a group of agents working together in the same environment may have the same unknown input, which is related to the temperature or gravity; thus, these agents can be described in the form of (8) and (9). A practical example is that of a group of aircrafts flying in formation in the same sky. They may have different kinetics, but they all have the same unknown input, such as wind power. If one wants to estimate the power of the wind in real time, distributed cooperative filters can be used instead of the traditional decentralized one. These practical examples provided the main motivation for addressing the systems (8) and (9).

The result of this study was obtained under the assumption that $\left(C^i_k\ \ B^i_k\right)$ is observable and x^i_0 is independent of ω^i_k and v^i_k for all k and i. Moreover, we assume that $p \geq m$ and $q \geq m$.

This study aimed to estimate the MVU of system state x^i_k and unknown input d_{k-1} by using $Y^i_k = \left\{y^i_0, y^i_1, \cdots, y^i_k\right\}$ under the condition that d_{k-1} is unavailable. Therefore, the unknown input d_{k-1} had no restricted conditions.

We considered a distributed cooperative filter in the following form

$$\hat{x}^i_{k|k-1} = B^i_{k-1}\hat{x}^i_{k-1|k-1} \tag{10}$$
$$\hat{d}^i_{k-1} = M^{ii}_k(y^i_k - C^i_k\hat{x}^i_{k|k-1}) + \sum_{j\in N_i} a_{i,j}M^{ij}_k(y^j_k - C^j_k\hat{x}^j_{k|k-1}) \tag{11}$$
$$\hat{x}^i_{k|k}{}^* = \hat{x}^i_{k|k-1} + G^i_{k-1}\hat{d}^i_{k-1} \tag{12}$$
$$\hat{x}^i_{k|k} = \hat{x}^i_{k|k}{}^* + K^{ii}_k(y^i_k - C^i_k\hat{x}^i_{k|k}{}^*) + \sum_{j\in N_i} a_{i,j}K^{ij}_k(y^j_k - C^j_k\hat{x}^j_{k|k}{}^*) \tag{13}$$

where $M^{ii}_k, M^{ij}_k \in R^{m*p}$ and $K^{ii}_k, K^{ij}_k \in R^{n*p}$ are design gain matrices, and $a_{i,j}$ is the element of the adjacency matrix of the multi-agent system graph G. Under the assumption that $\hat{x}^i_{k-1|k-1}$ is an unbiased estimate of $x^i_{k-1}, \hat{x}^i_{k|k-1}$ is biased due to the existence of unknown system input. Therefore, we need to estimate the unknown input in the sense of the MVU in (11), and then use it to obtain the unbiased state estimation $\hat{x}^i_{k|k}{}^*$ in (12). Finally, (13) minimizes the variance of $\hat{x}^i_{k|k}{}^*$ with regard to the l_1 matrix norm.

Matrices M^{ii}_k and M^{ij}_k, which are used in order to obtain the unbiased and MVU estimates of the unknown input are presented in Section 4. Gain matrices K^{ii}_k and K^{ij}_k that obtain the unbiased and MVU estimation of the state are computed in Section 5.

IV. UNKNOWN INPUT ESTIMATION

In this section, the estimation of unknown input is investigated. In Subsection A, matrices M^{ii}_k and M^{ij}_k are determined such that (11) is an unbiased estimator of d_{k-1}. In Subsection B, we extend this computation to the multi-agent system with time-varying topology. In Subsection C, we consider the condition of the multi-agent system having time-invariant topology; however, we select M^{ii}_k and M^{ij}_k such that (11) is an MVU estimator of d_{k-1}. In Subsection D, we extend this computation to the multi-agent system with time-varying topology.

A. UNBIASED UNKNOWN INPUT ESTIMATION UNDER TIME-INVARIANT TOPOLOGY

First, we define the compact formulation for the multi-agent system as one consisting of n agents:

$$X_{k+1} = \overline{B}_k X_k + \overline{G}_k d_k + W_k \tag{14}$$
$$Y_k = \overline{C}_k X_k + V_k \tag{15}$$

where

$$X_k=\left[x_k^{1T},x_k^{2T},\cdots,x_k^{nT}\right]^T,\quad Y_k=\left[y_k^{1T},y_k^{2T},\cdots,y_k^{nT}\right]^T,$$
$$W_k=\left[\omega_k^{1T},\omega_k^{2T},\cdots,\omega_k^{nT}\right]^T,\quad V_k=\left[v_k^{1T},v_k^{2T},\cdots,v_k^{nT}\right]^T,$$
$$\overline{B}_k=diag(B_k^1,B_k^2,\cdots,B_k^n),$$
$$\overline{C}_k=diag(C_k^1,C_k^2,\cdots,C_k^n),\quad \overline{G}_k=\left[G_k^{1T},G_k^{2T},\cdots,G_k^{nT}\right]^T.$$

Define $\overline{Q}_k = diag(Q_k^1,Q_k^2,\cdots,Q_k^n), \overline{R}_k = diag(R_k^1,R_k^2,\cdots,R_k^n)$.

Then, by writing (10) to (13) in the form of augmented multi-agent system, we can obtain a distributed cooperative filter in the following form.

$$\hat{X}_{k|k-1}=\overline{B}_{k-1}\hat{X}_{k-1|k-1} \tag{16}$$
$$\hat{D}_{k-1}=(I_n+A)\otimes 1_{(m*p)}.*\overline{M}_k(Y_k-\overline{C}_k\hat{X}_{k|k-1}) \tag{17}$$
$$\hat{X}^*_{k|k}=\hat{X}_{k|k-1}+\overline{G}'_{k-1}\hat{D}_{k-1} \tag{18}$$
$$\hat{X}_{k|k}=\hat{X}^*_{k|k}+(I_n+A)\otimes 1_{(q*p)}.*\overline{K}_k(Y_k-\overline{C}_k\hat{X}^*_{k|k}) \tag{19}$$

where

$$\hat{X}_{k|k-1}=\left[\hat{x}_{k|k-1}^{1}{}^T,\hat{x}_{k|k-1}^{2}{}^T,\cdots,\hat{x}_{k|k-1}^{n}{}^T\right]^T,$$
$$\hat{X}_{k-1|k-1}=\left[\hat{x}_{k-1|k-1}^{1}{}^T,\hat{x}_{k-1|k-1}^{2}{}^T,\cdots,\hat{x}_{k-1|k-1}^{n}{}^T\right]^T,$$
$$\hat{D}_{k-1}=\left[\hat{d}_{k-1}^{1}{}^T,\hat{d}_{k-1}^{2}{}^T,\cdots,\hat{d}_{k-1}^{n}{}^T\right]^T,$$
$$\overline{G}'_k=diag(G_k^1,G_k^2,\cdots,G_k^n),$$
$$\overline{M}_k=\begin{bmatrix} M_k^{11} & M_k^{12} & \cdots & M_k^{1n}\\ M_k^{21} & M_k^{22} & \cdots & M_k^{2n}\\ \vdots & \vdots & \ddots & \vdots\\ M_k^{n1} & M_k^{n2} & \cdots & M_k^{nn}\end{bmatrix}^{mn*pn},$$
$$\hat{X}^*_{k|k}=\left[\hat{x}_{k|k}^{1}{}^{*T},\hat{x}_{k|k}^{2}{}^{*T},\cdots,\hat{x}_{k|k}^{n}{}^{*T}\right]^T,$$
$$\hat{X}_{k|k}=\left[\hat{x}_{k|k}^{1}{}^T,\hat{x}_{k|k}^{2}{}^T,\cdots,\hat{x}_{k|k}^{1}{}^T\right]^T,$$
$$\overline{K}_k=\begin{bmatrix} K_k^{11} & K_k^{12} & \cdots & K_k^{1n}\\ K_k^{21} & K_k^{22} & \cdots & K_k^{2n}\\ \vdots & \vdots & \ddots & \vdots\\ K_k^{n1} & K_k^{n2} & \cdots & K_k^{nn}\end{bmatrix}^{qn*pn}.$$

A is the adjacency matrix of system graph G, and $1_{(q*q)}$ denotes the q-order square matrix, whose elements are all one. The operation. * means that the corresponding elements of the two matrices with the same dimension multiply together directly. The operation. * means that the corresponding elements of the two matrices with the same dimension multiply together directly. The operation $\otimes$ means the Kronecter product.

Remark 2: Note that matrices $\overline{G}_k$ and $\overline{G}'_k$ have different demission. The reason is that, in our filter, every agent can only use its own estimator of $\hat{d}_{k-1}^i$ in order to estimate $\hat{x}_{k|k}^{i}{}^*$. However, every agent in the multi-agent system has the same unknown input d_k in (14). Although there is only one unknown input, different agents estimate it differently. Hence, system matrix $\overline{G}_k$ can be written as this form.

Subsequently, we consider the estimation of the unknown input. $\tilde{Y}_k$ as follows:

$$\tilde{Y}_k=Y_k-\overline{C}_k\hat{X}_{k|k-1} \tag{20}$$

From (14) and (15), one obtains:

$$\tilde{Y}_k=\overline{C}_k\overline{G}_{k-1}d_{k-1}+E_k \tag{21}$$

where E_k is given by:

$$E_k=\overline{C}_k(\overline{B}_{k-1}\tilde{X}_{k-1}+W_{k-1})+V_k \tag{22}$$

with $\tilde{X}_k=X_k-\hat{X}_{k|k}$.

Now we assume that $\hat{X}_{k-1|k-1}$ is unbiased, which means that $\mathbf{E}\left(E^k\right)=0$. Then it follows from (22) and consequently (21) that:

$$\mathbf{E}\left[\tilde{Y}_k\right]=\overline{C}_k\overline{G}_{k-1}d_{k-1} \tag{23}$$

From Equation (23), one can achieve an unbiased estimation of the unknown input d_{k-1}.

By substituting (21) into (17), one can obtain the following formula.

$$\hat{D}_{k-1}=(I_n+A)\otimes 1_{(m*p)}.*\overline{M}_k(\overline{C}_k\overline{G}_{k-1}d_{k-1}+E_k) \tag{24}$$

Then one obtains:

$$\mathbf{E}[\hat{D}_{k-1}]=(I_n+A)\otimes 1_{(m*p)}.*\overline{M}_k\overline{C}_k\overline{G}_{k-1}\mathbf{E}[d_{k-1}] \tag{25}$$

Since $\hat{D}_{k-1}\in R^{(m*n)*1}, d_{k-1}\in R^{m*1}, (I_n+A)\otimes 1_m.*\overline{M}_k\overline{C}_k\overline{G}_{k-1}\in R^{(m*n)*m}$, therefore, if one wants to obtain the unbiased estimation of d_{k-1}, one obtains:

$$(I_n+A)\otimes 1_{(m*p)}.*\overline{M}_k\overline{C}_k\overline{G}_{k-1}=1^n\otimes I_m \tag{26}$$

where 1^n denotes the n-dimensional column vector, of which all elements are one.

Then, from Equation (26), one can obtain the following equation:

$$\overline{M}_k^i\overline{C}_k^i\overline{G}_{k-1}^i=I_m \tag{27}$$

holds for all agents, where i denotes the i-th agent and the definitions of the matrix that we need to use bellow are as follows.

$$X_k^i=\left[x_k^{1T},x_k^{2T},\cdots,x_k^{iT}\right]^T,$$
$$Y_k^i=\left[y_k^{1T},y_k^{2T},\cdots,y_k^{jT}\right]^T,$$
$$W_k^i=\left[\omega_k^{1T},\omega_k^{2T},\cdots,\omega_k^{jT}\right]^T,$$
$$V_k^i=\left[v_k^{1T},v_k^{2T},\cdots,v_k^{jT}\right]^T,$$
$$\overline{B}_k^i=diag(B_k^1,B_k^2,\cdots,B_k^j),$$
$$\overline{C}_k^i=diag(C_k^1,C_k^2,\cdots,C_k^n),$$
$$\overline{G}_k^i=\left[G_k^{1T},G_k^{2T},\cdots,G_k^{jT}\right]^T,$$
$$\overline{M}_k^i=\left[M_k^{i1}\ M_k^{i2}\ \ldots\ M_k^{ij}\right]^{m*pj},$$

$$\begin{aligned}
\overline{K}_k^i &= \left[K_k^{i1} \quad K_k^{i2} \quad \ldots \quad K_k^{ij} \right]^{m*pj}, \\
\hat{X}_{k|k-1}^i &= \left[\hat{x}_{k|k-1}^{1}{}^T, \hat{x}_{k|k-1}^{2}{}^T, \cdots, \hat{x}_{k|k-1}^{jT} \right]^T, \\
\hat{X}_{k-1|k-1}^i &= \left[\hat{x}_{k-1|k-1}^{1}{}^T, \hat{x}_{k-1|k-1}^{2}{}^T, \cdots, \hat{x}_{k-1|k-1}^{jT} \right]^T, \\
\hat{X}_{k|k}^i{}^* &= \left[\hat{x}_{k|k}^{1}{}^{*T}, \hat{x}_{k|k}^{2}{}^{*T}, \cdots, \hat{x}_{k|k}^{j}{}^{*T} \right]^T, \\
\hat{X}_{k|k}^i &= \left[\hat{x}_{k|k}^{1}{}^T, \hat{x}_{k|k}^{2T}, \cdots, \hat{x}_{k|k}^{jT} \right]^T, \\
\overline{Q}_k^i &= diag(Q_k^1, Q_k^2, \cdots, Q_k^j), \\
\overline{R}_k^i &= diag(R_k^1, R_k^2, \cdots, R_k^j), j \in N^i.
\end{aligned}$$

Theorem 1: Given that $\hat{X}_{k-1|k-1}$ is unbiased, there exists a gain matrix $\overline{M}_k$ such that (16)-(17) is an unbiased estimator of d_{k-1}, if and only if

$$rank\left(\overline{C}_k^i \overline{G}_{k-1}^i\right) = m \tag{28}$$

holds for all agents i.

Proof: Equation (27) indicates that $\hat{D}_{k-1}$ is unbiased if and only if $\overline{M}_k^i$ satisfies $\overline{M}_k^i \overline{C}_k^i \overline{G}_{k-1}^i = I_m$.

Sufficiency: First, it is obvious that matrix $\overline{G}_{k-1}^{iT} \overline{C}_k^{iT} \overline{C}_k^i \overline{G}_{k-1}^i$ is reversible. Then, one can see that when $\overline{M}_k^i = (\overline{G}_{k-1}^{iT} \overline{C}_k^{iT} \overline{C}_k^i \overline{G}_{k-1}^i)^{-1} \overline{G}_{k-1}^{iT} \overline{C}_k^{iT}$ and $rank\left(\overline{C}_k^i \overline{G}_{k-1}^i\right) = m$, $\overline{M}_k^i \overline{C}_k^i \overline{G}_{k-1}^i = I_m$, then $\mathbf{E}\left[\hat{d}_k^i\right] = d_k$.

This concludes the proof.

Necessity: Because $\mathbf{E}[E_k] = 0$, when $\mathbf{E}\left[\hat{d}_k^i\right] = d_k$, one can see that $\overline{M}_k^i \overline{C}_k^i \overline{G}_{k-1}^i = I_m$.

If $\overline{M}_k^i \overline{C}_k^i \overline{G}_{k-1}^i = I_m$, because $\overline{M}_k^i \in R^{m*(p*j)}$, $\overline{C}_k^i \in R^{(p*j)*(q*j)} \overline{G}_{k-1}^i \in R^{(q*j)*m}$, then $p * j > m$; therefore, $rank\left(\overline{C}_k^i \overline{G}_{k-1}^i\right) = m$.

This concludes the proof.

For convenience, (28) is termed as a cooperative existing condition. One can find that the cooperative existing condition is much more relaxed than the conventional filter $rank\left(C_k^i G_{k-1}^i\right) = m$ that holds for all i, which means that it is not required that every agent satisfies condition (7). Only the augmented multi-agent system needs to satisfy (28). This is a significantly relaxed condition, which means that every agent in the system does not need to satisfy the rank condition, but rather only the augmented multi-agent system needs to satisfy the rank condition.

Remark 3: From the perspective of physical significance, one can easily understand the importance of this study. $rank\left(C_k^i G_{k-1}^i\right) = m$ means that every dimension of unknown input d_{k-1} can be shown in $\tilde{y}_k^i$. Under this assumption, one can easily estimate the unknown input d_{k-1}. However, for systems (8) to (9), this condition does not need to be satisfied since the agent can use the information from its neighbors in order to estimate the unknown input d_{k-1}. $rank\left(\overline{C}_k^i \overline{G}_{k-1}^i\right) = m$ means that as long as all the dimensions of unknown input d_{k-1} can be shown in $\tilde{Y}_k$ once, we can obtain the unbiased estimation of d_{k-1}. This is the fundamental reason for why we can obtain this relaxed condition.

Although we have obtained the unbiased estimation of unknown input d_{k-1}, E_k does not have a unit variance. Therefore, (21) does not satisfy the assumptions of the Gauss-Markov theorem, and thus, we still do not obtain the MVU estimator of d_{k-1}. However, the variance of E_k can be calculated from the covariance matrices of state estimation.

In Subsection C, we propose the MVU estimator of d_{k-1} by using the matrix $\left(\mathbf{E}\left[E_k \; E_k^T\right]\right)^{-1}$ through weighted least-squares (WLS) estimation.

B. UNBIASED UNKNOWN INPUT ESTIMATION UNDER TIME-VARYING TOPOLOGY

In this subsection, we extend the former result to the multi-agent system with time-varying topology.

For the multi-agent system with time-varying topology, system equations (14) to (15) are the same, with regard to the distributed cooperative filter (10) to (13). The only difference is that when we substitute (10) to (13) into (14) and (15), we obtain a different augmented form of the distributed cooperatives (29) to (32).

$$\hat{X}_{k|k-1} = \overline{B}_{k-1} \hat{X}_{k-1|k-1} \tag{29}$$

$$\hat{D}_{k-1} = (I_n + A_k) \otimes 1_{(m*p)} .* \overline{M}_k (Y_k - \overline{C}_k \hat{X}_{k|k-1}) \tag{30}$$

$$\hat{X}_{k|k}^* = \hat{X}_{k|k-1} + \overline{G}_{k-1}' \hat{D}_{k-1} \tag{31}$$

$$\hat{X}_{k|k} = \hat{X}_{k|k}^* + (I_n + A_k) \otimes 1_{(q*p)} .* \overline{K}_k (Y_k - \overline{C}_k \hat{X}_{k|k}^*) \tag{32}$$

where A_k is the adjacency matrix of system graph G at time k.

Then, one can obtain the new form of $\hat{D}_{k-1}$.

$$\hat{D}_{k-1} = (I_n + A_k) \otimes 1_{(m*p)} .* \overline{M}_k (\overline{C}_k \overline{G}_{k-1} d_{k-1} + E_k)$$

Then one obtains:

$$\mathbf{E}[\hat{D}_{k-1}] = (I_n + A_k) \otimes 1_{(m*p)} .* \overline{M}_k \overline{C}_k \overline{G}_{k-1} \mathbf{E}[d_{k-1}]$$

and if one wants to obtain the unbiased estimation of d_{k-1}, one can obtain:

$$(I_n + A_k) \otimes 1_{(m*p)} .* \overline{M}_k \overline{C}_k \overline{G}_{k-1} = 1^n \otimes I_m \tag{33}$$

Then from equation (33), one can obtain the same result (28) as a time-invariant one. The only difference is that the neighbors of agent i are time-varying.

Remark 4: Note that, unlike many studies on time-varying multi-agent systems, the presented result does not need the union of system graph G over the connection period. The reason is that, although the proposed filters are distributed, for agent i at time k, the necessary and sufficient condition for the unbiased estimation of the filter is that $\hat{X}_{k-1|k-1}^i$ is unbiased, and that the system matrix satisfies the condition $rank\left(\overline{C}_k^i \overline{G}_{k-1}^i\right) = m$. For agent i, at time $k+1$, the necessary and sufficient condition for the unbiased estimation of the filter is that $\hat{X}_{k|k}^i$ is unbiased and that the system matrix satisfies the condition $rank\left(\overline{C}_{k+1}^i \overline{G}_k^i\right) = m$. Therefore, one can see that there is no neighbor connection for agent i between time $k - 1$ and time k. In other words, as long as agent i satisfies the necessary and sufficient condition for the unbiased estimation of the filter, the neighbors of agent i at time k and time $k+1$ can be completely different. However, at every time point, agent i only uses the information of itself and its neighbors. It has been pointed out in [31] that the distributed filter means that the computations for the estimation of the filters are shared among the agents. According to this definition, the proposed new filter is distributed. This is the reason for the proposed distributed cooperative filter not needing the union of system graph G over the connection period.

C. MINIMUM-VARIANCE UNBIASED UNKNOWN INPUT ESTIMATION UNDER TIME-INVARIANT TOPOLOGY

Similarly with the definition of $\overline{M}_k^i$ in (27), we now define $\tilde{Y}_k^i$, E_k^i and $\tilde{R}_k^i$.

$$\begin{aligned}\tilde{Y}_k^i &= Y_k^i - \overline{C}_k^i \hat{X}_{k|k-1}^i \\ E_k^i &= \overline{C}_k^i(\overline{B}_{k-1}^i \tilde{X}_{k-1}^i + W_{k-1}^i) + V_k^i\end{aligned} \tag{34}$$

where $\tilde{X}_k^i = X_k^i - \hat{X}_{k|k}^i$.

By denoting the variance of E_k^i as $\tilde{R}_k^i$, a straightforward calculation yields:

$$\tilde{R}_k^i = \mathbf{E}\left[E_k^i \ E_k^{iT}\right] = \overline{C}_k^i(\overline{B}_{k-1}^i P_{k-1|k-1}^i \overline{B}_{k-1}^{iT} + \overline{Q}_k^i)\overline{C}_k^{iT} + \overline{R}_k^i \tag{35}$$

where $P_{k|k}^i = \mathbf{E}\left[\tilde{X}_k^i \ \tilde{X}_k^{iT}\right]$.

For convenience, we also define $\tilde{Y}_k$, E_k, and $\tilde{R}_k$.

$$\begin{aligned}\tilde{Y}_k &= Y_k - \overline{C}_k \hat{X}_{k|k-1} \\ E_k &= \overline{C}_k(\overline{B}_{k-1}\tilde{X}_{k-1} + W_{k-1}) + V_k\end{aligned}$$

where $\tilde{X}_k = X_k - \hat{X}_{k|k|}$.

By denoting the variance of E_k as $\tilde{R}_k$, a straightforward calculation yields:

$$\tilde{R}_k = \mathbf{E}\left[E_k \ E_k^T\right] = \overline{C}_k(\overline{B}_{k-1} P_{k-1|k-1} \overline{B}_{k-1T} + \overline{Q}_k)\overline{C}_k^T + \overline{R}_k \tag{36}$$

where $P_{k|k} = \mathbf{E}\left[\tilde{X}_k \ \tilde{X}_k^T\right]$.

Furthermore, by defining

$$P_{k|k-1}^i = \overline{B}_{k-1}^i P_{k-1|k-1}^i \overline{B}_{k-1}^{iT} + \overline{Q}_{k-1}^i,$$

It can be rewritten as

$$\tilde{R}_k^i = \overline{C}_k^i P_{k|k-1}^i \overline{C}_k^{iT} + \overline{R}_k^i$$

The MVU estimation of unknown input is then obtained as follows.

Theorem 2: Given that $\hat{X}_{k-1|k-1}$ is unbiased and $\tilde{R}_k^i$ is positive definite. We define $\overline{M}_k^i$ as follows:

$$\overline{M}_k^i = (F_k^{iT}(\tilde{R}_k^i)^{-1}F_k^i)^{-1}F_k^{iT}(\tilde{R}_k^i)^{-1} \tag{37}$$

where $F_k^i = \overline{C}_k^i \overline{G}_{k-1}^i$. Then, given the innovation $\tilde{Y}_k^i$, (17) is the MVU estimator of d_{k-1}. The variance of the unknown input estimate is given by $(F_k^{iT}(\tilde{R}_k^i)^{-1}F_k^i)^{-1}$.

Proof: One can always find an invertible matrix satisfying $\tilde{S}_k^i \tilde{S}_k^{iT} = \tilde{R}_k^i$ under the assumption that $\tilde{R}_k^i$ is positive definite. Cholesky factorization is one example of how we can achieve this. Then, one can transform (34) to:

$$(\tilde{S}_k^i)^{-1}\tilde{Y}_k^i = (\tilde{S}_k^i)^{-1}\overline{C}_k^i \overline{G}_{k-1}^i d_{k-1} + (\tilde{S}_k^i)^{-1}E_k^i \tag{38}$$

Under the assumption that $(\tilde{S}_k^i)^{-1}\overline{C}_k^i \overline{G}_{k-1}^i$ has full column rank, the least-squares (LS) solution of (38) is:

$$\hat{d}_{k-1}^i = (F_k^{iT}(\tilde{R}_k^i)^{-1}F_k^i)^{-1}F_k^{iT}(\tilde{R}_k^i)^{-1}\tilde{Y}_k^i \tag{39}$$

This completes the proof.

Once one have the optimal gain matrix $\overline{M}_k^i$ for agent i, one can obtain the extended optimal gain matrix $\overline{M}_k$ for the multi-agent system.

It should be noted that solving (38) by using LS estimation is equivalent to solving (34) by using WLS estimation with the weighting matrix $(\tilde{R}_k^i)^{-1}$. Furthermore, because the weighting matrix is chosen such that $(\tilde{S}_k^i)^{-1}E_k^i$ has unit variance, Equation (39) satisfies the assumptions of the Gauss-Markov theorem. Therefore, (39) is the MVU estimate of d_{k-1}. The variance of the WLS solution (39) is given by $(F_k^{iT}(\tilde{R}_k^i)^{-1}F_k^i)^{-1}$.

D. MINIMUM-VARIANCE UNBIASED UNKNOWN INPUT ESTIMATION UNDER TIME-VARYING TOPOLOGY

Based on subsection B, we can obtain the MVU estimation of d_k. Since we can obtain the unbiased estimation of d_k, according to Theorem 2, we can obtain the MVU estimation of d_k as long as $\hat{X}_{k-1|k-1}$ is unbiased and $\tilde{R}_k^i$ is positive definite.

The proof is shown in Subsection B. The only difference is that the graph of the multi-agent system is time-varying. Similarly, we also do not need the union of system graph G over the connection period. The reason is that as long as we can guarantee the unbiasedness of the $\hat{X}_{k-1|k-1}$ and the positivity of $\tilde{R}_k^i$, we can always obtain the MVU estimation of d_k. In other words, if at time $k-1$ agent i has a connection with agent j, and from time k to infinity agent i does not have a connection with agent j, agent i can also obtain an MVU estimation of d_k. The reason is that at time $k-1$ agent i has obtained the unbiased estimation of $\hat{x}_{k-1|k-1}^i$; therefore, the subsequent estimation is also unbiased.

According to Theorem 2, one can also obtain the MVU estimation of d_k under a multi-agent system with time-varying topology.

V. STATE ESTIMATION

Consider a state estimator of system (14) and (15) that takes the recursive form (16) to (19) (in the cases of time-varying topology (29) to (32)). In Subsection A, one calculate the gain matrix $\overline{K}_k$ in order to obtain the unbiased estimator of X_k in (19). In Subsection B, we extend this result to a multi-agent system with time-varying topology. In Subsection C, we obtain the MVU estimation of X_k. In Subsection D, we extend this result to a multi-agent system with time-varying topology.

A. UNBIASED STATE ESTIMATION UNDER TIME-INVARIANT TOPOLOGY

By defining $\tilde{X}_k^* = X_k - \hat{X}_{k|k}^*$, it follows from (14) to (16) and (18) that:

$$\tilde{X}_k^* = \overline{B}_{k-1}\tilde{X}_{k-1} + \overline{G}_{k-1}\tilde{D}_{k-1} + W_{k-1} \tag{40}$$

where $\tilde{D}_k = D_k - \hat{D}_k$, $D_{k-1} = [d_{k-1T}, d_{k-1T}, \cdots, d_{k-1T}]^T$. The following theorem is a direct consequence of (39).

Theorem 3: Given that $\hat{X}_{k-1|k-1}$ and $\hat{D}_{k-1}$ are unbiased estimations, (18) to (19) are unbiased estimators of X_k for any value of $\overline{K}_k$.

Proof: Substituting (17) and (18) in (19), yields:

$$\begin{aligned}\hat{X}_{k|k} &= \hat{X}_{k|k-1} + (I_n + A) \otimes 1_{(q*p)}. * \overline{K}_k\tilde{Y}_k \\ &\quad + [I_{qn} - (I_n + A) \otimes 1_{(q*p)}. * \overline{K}_k\overline{C}_k]\overline{G}'_{k-1}\hat{D}_{k-1} \quad (41)\\ &= \hat{X}_{k|k-1} + (I_n + A) \otimes 1_{(q*p)}. * \overline{K}_k\tilde{Y}_k \\ &\quad + [I_{qn} - (I_n + A) \otimes 1_{(q*p)}. * \overline{K}_k\overline{C}_k] \\ &\quad * \overline{G}'_{k-1}(I_n + A) \otimes 1_{(m*p)}. * \overline{M}_k\tilde{Y}_k \quad (42)\end{aligned}$$

By defining

$$\begin{aligned}L_k &= (I_n+A) \otimes 1_{(q*p)}. * \overline{K}_k \\ &\quad + [I_{qn}-(I_n+A) \otimes 1_{(q*p)}. * \overline{K}_k\overline{C}_k]\overline{G}'_{k-1}(I_n + A) \\ &\quad \otimes 1_{(m*p)}. * \overline{M}_k\end{aligned}$$

Eq. (41) is rewritten as follows:

$$\hat{X}_{k|k} = \hat{X}_{k|k-1} + L_k(Y_k - \overline{C}_k\hat{X}_{k|k-1}) \tag{43}$$

which is the kind of update considered in [23].

This completes the proof.

B. UNBIASED STATE ESTIMATION UNDER TIME-VARYING TOPOLOGY

In this subsection, we extend the former result to a case where the multi-agent system has time-varying topology. As was done in subsection A, we defined $\tilde{X}_k^*$ and now we simply introduce the theorem.

Theorem 4: Given that $\hat{X}_{k-1|k-1}$ and $\hat{D}_{k-1}$ are unbiased, (31) to (32) are unbiased estimators of X_k for any value of $\overline{K}_k$.

Proof: Substituting (30) and (31) in (32) yields:

$$\begin{aligned}\hat{X}_{k|k} &= \hat{X}_{k|k-1} + (I_n+A_k) \otimes 1_{(q*p)}. * \overline{K}_k\tilde{Y}_k \\ &\quad + [I_{qn}-(I_n+A_k) \otimes 1_{(q*p)}. * \overline{K}_k\overline{C}_k]\overline{G}'_{k-1}\hat{D}_{k-1} \quad (44)\\ &= \hat{X}_{k|k-1} + (I_n+A_k) \otimes 1_{(q*p)}. * \overline{K}_k\tilde{Y}_k \\ &\quad + [I_{qn}-(I_n+A_k) \otimes 1_{(q*p)}. * \overline{K}_k\overline{C}_k] \\ &\quad * \overline{G}'_{k-1}(I_n+A_k) \otimes 1_{(m*p)}. * \overline{M}_k\tilde{Y}_k \quad (45)\end{aligned}$$

By defining

$$\begin{aligned}L_k &= (I_n+A_k) \otimes 1_{(q*p)}. * \overline{K}_k \\ &\quad + [I_{qn}-(I_n+A_k) \otimes 1_{(q*p)}. * \overline{K}_k\overline{C}_k]\overline{G}'_{k-1}(I_n + A_k) \\ &\quad \otimes 1_{(m*p)}. * \overline{M}_k\end{aligned}$$

Eq. (45) is rewritten as follows:

$$\hat{X}_{k|k} = \hat{X}_{k|k-1} + L_k(Y_k - \overline{C}_k\hat{X}_{k|k-1})$$

which is the kind of update considered in [23].

This completes the proof.

C. MINIMUM-VARIANCE UNBIASED STATE ESTIMATION UNDER TIME-INVARIANT TOPOLOGY

In this subsection, we compute the optimal gain matrix $\overline{K}_k$ based on the previously obtained matrix $\overline{M}_k$. Specifically, any matrix $\overline{M}_k$ satisfying (26) and used in (17) can be used in order to obtain the optimal gain matrix $\overline{K}_k$, and furthermore in order to obtain the MVU estimate $\hat{X}_{k|k}$ of X_k. First, we calculate matrix $\tilde{D}_{k-1}$. From (17) and (21) to (22), we obtain:

$$\begin{aligned}\tilde{D}_{k-1} &= D_{k-1} - ((I_n + A) \otimes 1_{(m*p)}. * \overline{M}_k\overline{C}_k\overline{G}_{k-1})d_{k-1} \\ &\quad - (I_n + A) \otimes 1_{(m*p)}. * \overline{M}_k E_k \\ &= (1^n \otimes I_m - (I_n + A) \otimes 1_{(m*p)}. * \overline{M}_k\overline{C}_k\overline{G}_{k-1})d_{k-1} \\ &\quad - (I_n + A) \otimes 1_{(m*p)}. * \overline{M}_k E_k \\ &= -(I_n + A) \otimes 1_{(m*p)}. * \overline{M}_k E_k \quad (46)\end{aligned}$$

which also proves that the unknown input estimator is unbiased. Substituting (46) in (40) yields:

$$\tilde{X}_k^* = \overline{B}_{k-1}^*\tilde{X}_{k-1} + W_{k-1}^* \tag{47}$$

where

$$\begin{aligned}\overline{B}_{k-1}^* &= (I_{qn} - \overline{G}'_{k-1}(I_n + A) \otimes 1_{(m*p)}\overline{M}_k\overline{C}_k)\overline{B}_{k-1} \quad (48)\\ W_{k-1}^* &= (I_{qn} - \overline{G}'_{k-1}(I_n + A) \otimes 1_{(m*p)}. * \overline{M}_k\overline{C}_k)W_{k-1} \\ &\quad - \overline{G}'_{k-1}(I_n + A) \otimes 1_{(m*p)}. * \overline{M}_k V_k \quad (49)\end{aligned}$$

Then, one can obtain the error covariance matrix $P_{k|k}^* = \mathbf{E}\left[\tilde{X}_k^* \ \tilde{X}_k{}^{*T}\right]$ from (47) to (49),

$$\begin{aligned}P_{k|k}^* &= \overline{B}_{k-1}^* P_{k-1|k-1}\overline{B}_{k-1}{}^{*T} + \overline{Q}_{k-1}^* \\ &= (I_{qn} - \overline{G}'_{k-1}(I_n + A) \otimes 1_{(m*p)}. * \overline{M}_k\overline{C}_k) * P_{k|k-1} \\ &\quad * (I_{qn} - \overline{G}'_{k-1}(I_n + A) \otimes 1_{(m*p)}. * \overline{M}_k\overline{C}_k)^T \\ &\quad + \overline{G}'_{k-1}(I_n + A) \otimes 1_{(m*p)}. * \overline{M}_k R_k \\ &\quad * [(I_n + A) \otimes 1_{(m*p)}. * \overline{M}_k]^T\overline{G}'^T_{k-1} \quad (50)\end{aligned}$$

where $\overline{Q}_k^* = \mathbf{E}\left[W_k^* \ W_k{}^{*T}\right]$.

Subsequently, we calculate the error covariance matrix $P_{k|k}$. It follows from (19) that:

$$\begin{aligned}\tilde{X}_k &= (I_{qn} - (I_n + A) \otimes 1_{(q*p)}. * \overline{K}_k\overline{C}_k)\tilde{X}_k^* \\ &\quad - (I_n + A) \otimes 1_{(q*p)}. * \overline{K}_k V_k \quad (51)\end{aligned}$$

Substituting (47) in (51) yields:

$$\begin{aligned}\tilde{X}_k &= (I_{qn} - (I_n + A) \otimes 1_{(q*p)}. * \overline{K}_k\overline{C}_k) \\ &\quad \times(\overline{B}_{k-1}^*\tilde{X}_{k-1} + W_{k-1}^*) - (I_n+A) \otimes 1_{(q*p)}. * \overline{K}_k V_k \quad (52)\end{aligned}$$

where

$$\mathbf{E}\left[W_{k-1}^* \ V_k^T\right] = -\overline{G}'_{k-1}(I_n + A) \otimes 1_{(m*p)}. * \overline{M}_k R_k.$$

It should be noted that (52) is closely related to the Kalman filter. This result denotes the dynamic evolution of the state estimation error for a Kalman filter with a gain matrix $\overline{K}_k$ for system $(\overline{B}_k^*, \overline{C}_k)$, where process noise W_{k-1}^* is correlated with measurement noise V_k. Therefore, the computation of matrix $\overline{K}_k$ can be transformed into a standard Kalman filter problem.

From (51) and (50), we can obtain the error covariance matrix $P_{k|k}$

$$\begin{aligned} P_{k|k} &= (I_n + A) \otimes 1_{(q*p)}. * \overline{K}_k \tilde{R}_k^* [(I_n + A) \otimes 1_{(q*p)}. * \overline{K}_k]^T \\ &\quad - V_k^* [(I_n + A) \otimes 1_{(q*p)}. * \overline{K}_k]^T \\ &\quad - [(I_n + A) \otimes 1_{(q*p)}. * \overline{K}_k] V_k^{*T} + P_{k|k}{}^* \end{aligned} \tag{53}$$

where

$$\begin{aligned} \tilde{R}_k^* &= \overline{C}_k P_{k|k}^* \overline{C}_k^T + \overline{R}_k + \overline{C}_k S_k^* + S_k^{*T} \overline{C}_k^T \\ V_k^* &= P_{k|k}^* \overline{C}_k + S_k^* \\ S_k^* &= \mathbf{E}\left[\tilde{X}_k^* \;\; V_k^T\right] = -\overline{G}_{k-1}'(I_n + A) \otimes 1_{(m*p)}. * \overline{M}_k \overline{R}_k \end{aligned} \tag{54}$$

Note that $\tilde{R}_k^*$ is equal to the variance of the zero-mean signal $\tilde{Y}_k^*$, $\tilde{R}_k^* = \mathbf{E}\left[\tilde{Y}_k^* \;\; \tilde{Y}_k{}^{*T}\right]$

where

$$\begin{aligned} \tilde{Y}_k^* &= Y_k - \overline{C}_k \hat{X}_{k|k}^* \\ &= (I_{pn} - \overline{C}_k \overline{G}_{k-1}'(I_n + A) \otimes 1_{(m*p)}. * \overline{M}_k) E_k \end{aligned} \tag{55}$$

By using (55) and (36), (54) can be rewritten as follows:

$$\begin{aligned} \tilde{R}_k^* = (I_{pn} - \overline{C}_k \overline{G}_{k-1}'(I_n + A) \otimes 1_{(m*p)}. * \overline{M}_k) * \\ \tilde{R}_k (I_{pn} - \overline{C}_k \overline{G}_{k-1}'(I_n + A) \otimes 1_{(m*p)}. * \overline{M}_k)^T. \end{aligned}$$

We define:

$$(I_n + A) \otimes 1_{(q*p)}. * \overline{K}_k = \overline{K}_k' \tag{56}$$

and also define the optimal gain matrix as $\overline{K}_k''$.

From Kalman filtering theory, we know that the uniqueness of the optimal gain matrix $\overline{K}_k''$ requires that $\tilde{R}_k^*$ is invertible. However, we find that $rank(I_{pn} - \overline{C}_k \overline{G}_{k-1}'(I_n + A) \otimes 1_{(m*p)}. * \overline{M}_k) \leq pn$; therefore, $rank(\tilde{R}_k^*) \leq pn$. For example, when there is only one agent in the system and at this time the multi-agent system changes to one single system, [26] proves that:

$$\begin{aligned} &rank(I_{pn} - \overline{C}_k \overline{G}_{k-1}'(I_n + A) \otimes 1_{(m*p)}. * \overline{M}_k) \\ &\qquad = (p - m) * n = p - m \end{aligned}$$

Therefore, the optimal gain matrix $\overline{K}_k''$ is not unique. Let r be the rank of $\tilde{R}_k^*$; then, we propose a gain matrix $\overline{K}_k''$ in the following form:

$$\overline{K}_k'' = \tilde{\overline{K}}_k'' \alpha_k \tag{57}$$

where $\tilde{\overline{K}}_k'' \in R^{(pn)*r}$ and $\alpha_k \in R^{r*(pn)}$ is a matrix that makes matrix $\alpha_k \tilde{R}_k^* \alpha_k^T$ have a full rank. The optimal gain matrix $\tilde{\overline{K}}_k''$ is presented below.

Theorem 5: If $\overline{M}_k$ satisfies $(I_n + A) \otimes 1_{(m*p)}. * \overline{M}_k \overline{C}_k \overline{G}_{k-1} = 1^n \otimes I_m$, then the following gain matrix $\overline{K}_k''$ can minimize the variance of $\hat{X}_{k|k}$:

$$\overline{K}_k'' = (P_{k|k}^* \overline{C}_k^T + S_k^*) \alpha_k^T (\alpha_k \tilde{R}_k^* \alpha_k^T)^{-1} \alpha_k \tag{58}$$

where $r = rank(\tilde{R}_k^*)$ and $\alpha_k \in R^{r*(p*n)}$ is an arbitrary matrix that makes matrix $\alpha_k \tilde{R}_k^* \alpha_k^T$ have a full rank.

Proof: Substituting (57) in (53) and minimizing the trace of $P_{k|k}$ over $\tilde{\overline{K}}_k''$ yields (58). By substituting (58) in (53), one obtains the error covariance matrix,

$$P_{k|k} = P_{k|k}^* - \overline{K}_k''(P_{k|k}^* \overline{C}_k^T + S_k^*)^T \tag{59}$$

which is the same result as that in [24].

This completes the proof.

It should be noted that expression (58) depends only on matrix $\overline{M}_k$. According to equation (26) and Theorem 1, the matrix $\overline{M}_k$ satisfies the condition $(I_n + A) \otimes 1_{(m*p)}. * \overline{M}_k \overline{C}_k \overline{G}_{k-1} = 1^n \otimes I_m$ means that the estimation of the unknown input $\hat{D}_{k-1}$ is unbiased. We obtain gain matrix $\overline{K}_k''$ in the form of (58) by minimizing the variance of $\hat{X}_{k|k}$ based on matrix $\overline{M}_k$ used in (17).

Since we obtained the optimal gain matrix $\overline{K}_k'$ in the form of (56), we know that once the matrix $\overline{M}_k$ is determined, we can obtain $\overline{K}_k''$ by (58). However, the form of the gain matrix is defined as (56). From graph theory we know that if and only if all nodes in graph G are connected to all other nodes in graph G, all the elements of matrix $\overline{K}_k'$ can be nonzero. Otherwise, there will always be some elements in matrix $\overline{K}_k'$ that must be zero. This can be easily understood in the physical sense. Some elements can be zero in matrix $\overline{K}_k'$ and this means that agent i could only receive information from its neighbors instead of all the other agents in the system. However, the result obtained from (58) needs all the elements of matrix $\overline{K}_k''$ in order to be nonzero; therefore, we cannot realize the condition obtained from (58), and thus, we can only use the sub-optimal gain matrix $\overline{K}_k'$ in order to obtain the MVU estimate $\hat{X}_{k|k}$ of X_k. Then, we can use the l_1 matrix norm in order to obtain the sub-optimal gain matrix $\overline{K}_k'$.

First, we define matrix $T = \overline{K}_k'' - \overline{K}_k'$, then we obtain the l_1 matrix norm of T in the following form:

$$\|T\|_1 = \sum_{i=1}^{m} \sum_{j=1}^{n} |t_{ij}|$$

where t_{ij} is the elements of matrix T.

Since we know that the only differences between $\overline{K}_k''$ and $\overline{K}_k'$ is that some elements of $\overline{K}_k'$ must be zero and that the corresponding elements in $\overline{K}_k''$ must be nonzero, therefore, we can obtain the following form of $\overline{K}_k'$ in order to minimize the l_1 matrix norm of T, as follows:

$$\overline{K}_k' = (I_n + A) \otimes 1_{(m*p)}. * \overline{K}_k'' \tag{60}$$

Theorem 6: If $\overline{M}_k$ satisfies $(I_n + A) \otimes 1_{(m*p)}. * \overline{M}_k \overline{C}_k \overline{G}_{k-1} = 1^n \otimes I_m$, then the following gain matrix $\overline{K}_k'$ can minimize the variance of $\hat{X}_{k|k}$ with regard to the l_1 matrix norm, as follows:

$$\overline{K}_k' = (I_n + A) \otimes 1_{m*p}. * (P_{k|k}^* \overline{C}_k^T + S_k^*) * \alpha_k^T (\alpha_k \tilde{R}_k^* \alpha_k^T)^{-1} \alpha_k \quad (61)$$

where $r = rank(\tilde{R}_k^*)$ and $\alpha_k \in R^{r*(p*n)}$ is an arbitrary matrix, which makes the matrix $\alpha_k \tilde{R}_k^* \alpha_k^T$ have a full rank.

This proof is similar to the proof of Theorem 5.

D. MINIMUM-VARIANCE UNBIASED STATE ESTIMATION UNDER TIME-VARYING TOPOLOGY

In subsection D, we compute the optimal gain matrix $\overline{K}_k$ based on matrix $\overline{M}_k$ that we obtained previously. Specifically, any matrix $\overline{M}_k$ satisfying (33) and used in (30) can be used to obtain the optimal gain matrix $\overline{K}_k$, and furthermore to obtain the MVU estimate $\hat{X}_{k|k}$ of X_k. First, we search for an expression of $\tilde{D}_{k-1}$ follows from (30) and (21) to (22) that:

$$\begin{aligned}\tilde{D}_{k-1} &= (I_{m*n} - (I_n + A_k) \otimes 1_{(m*p)}. * \overline{M}_k \overline{C}_k \overline{G}_{k-1}')D_{k-1} \\ &\quad - (I_n + A_k) \otimes 1_{(m*p)}. * \overline{M}_k E_k \\ &= (1^n \otimes I_m - (I_n + A_k) 1_{(m*p)}. * \overline{M}_k \overline{C}_k) d_{k-1} \\ &\quad - (I_n + A_k) \otimes 1_{(m*p)}. * \overline{M}_k E_k \\ &= -(I_n + A_k) \otimes 1_{(m*p)}. * \overline{M}_k E_k \end{aligned} \quad (62)$$

which also proves that the unknown input estimator is unbiased. Substituting (62) in (40) yields:

$$\tilde{X}_k^{*'} = \overline{B}_{k-1}' \tilde{X}_{k-1} + W_{k-1}' \quad (63)$$

where

$$\overline{B}_{k-1}' = (I_{qn} - \overline{G}_{k-1}'(I_n + A_k) \otimes 1_{(m*p)} \overline{M}_k \overline{C}_k) \overline{B}_{k-1} \quad (64)$$

$$\begin{aligned}W_{k-1}' &= (I_{qn} - \overline{G}_{k-1}'(I_n + A_k) \otimes 1_{(m*p)}.\overline{M}_k \overline{C}_k) W_{k-1} \\ &\quad - \overline{G}_{k-1}'(I_n + A_k) \otimes 1_{(m*p)}.\overline{M}_k V_k \end{aligned} \quad (65)$$

Then, we can obtain the error covariance matrix $P_{k|k}^*$ from (63) to (65), as follows:

$$\begin{aligned}P_{k|k}^{*'} &= \overline{B}_{k-1}' P_{k-1|k-1} \overline{B}_{k-1}'^T + \overline{Q}_{k-1}' \\ &= (I_{qn} - \overline{G}_{k-1}'(I_n + A_k) \otimes 1_{(m*p)}.\overline{M}_k \overline{C}_k) \\ &\quad * P_{k|k-1} * (I_{qn} - \overline{G}_{k-1}'(I_n + A_k) \otimes 1_{(m*p)}.\overline{M}_k \overline{C}_k)^T \\ &\quad + \overline{G}_{k-1}'(I_n + A_k) \otimes 1_{(m*p)}.\overline{M}_k \overline{R}_k [(I_n + A_k) \\ &\quad \otimes 1_{(m*p)}.\overline{M}_k]^T \overline{G}_{k-1}'^T \end{aligned}$$

where $\overline{Q}_k' = \mathbf{E}\left[W_k' \ W_k'^T\right]$.

Next, we calculate the error covariance matrix $P_{k|k}$. From Equation (32) we obtain:

$$\begin{aligned}\tilde{X}_k' &= (I_{qn} - (I_n + A_k) \otimes 1_{(q*p)}. * \overline{K}_k \overline{C}_k) \tilde{X}_k^{*'} \\ &\quad - (I_n + A_k) \otimes 1_{(q*p)}. * \overline{K}_k V_k \end{aligned} \quad (66)$$

Substituting (63) in (66) yields:

$$\begin{aligned}\tilde{X}_k' &= (I_{qn} - (I_n + A_k) \otimes 1_{(q*p)}. * \overline{K}_k \overline{C}_k) \\ &\quad * (\overline{B}_{k-1}' \tilde{X}_{k-1} + W_{k-1}') - (I_n + A_k) \otimes 1_{(q*p)}. * \overline{K}_k V_k \end{aligned} \quad (67)$$

where $E\left[W_{k-1}' \ V_k^T\right] = -\overline{G}_{k-1}'(I_n + A_k) \otimes 1_{(m*p)}. * \overline{M}_k \overline{R}_k$.

It should be noted that (67) is closely related to the Kalman filter, which was discussed in Subsection C. Therefore, the computation of matrix $\overline{K}_k$ can be transformed into a standard Kalman filtering problem.

From (67) and (66), we can obtain the error covariance matrix $P_{k|k}$, as follows:

$$\begin{aligned}P_{k|k}' &= (I_n + A_k) \otimes I_q \overline{K}_k \tilde{R}_k' [(I_n + A_k) \otimes I_q \overline{K}_k]^T \\ &\quad - V_k' [(I_n + A_k) \otimes I_q \overline{K}_k]^T - [(I_n + A_k) \otimes I_q \overline{K}_k] V_k'^T \\ &\quad + P_{k|k}^{*'} \end{aligned} \quad (68)$$

where

$$\begin{aligned}\tilde{R}_k' &= \overline{C}_k P_{k|k}^* \overline{C}_k^T + \overline{R}_k + \overline{C}_k S_k' + S_k'^T \overline{C}_k^T \\ V_k' &= P_{k|k}^* \overline{C}_k + S_k' \\ S_k' &= \mathbf{E}\left[\tilde{X}_k^* \ V_k^T\right] = -\overline{G}_{k-1}'(I_n + A_k) \otimes I_m \overline{M}_k \overline{R}_k \end{aligned} \quad (69)$$

Note that $\tilde{R}_k'$ equals to the variance of the zero-mean signal $\tilde{Y}_k^{*'}, \tilde{R}_k^{*'} = \mathbf{E}\left[\tilde{Y}_k^{*'} \ \tilde{Y}_k^{*'T}\right]$, where

$$\begin{aligned}\tilde{Y}_k^{*'} &= Y_k - \overline{C}_k \hat{X}_{k|k}^* \\ &= (I_{pn} - \overline{C}_k \overline{G}_{k-1}'(I_n + A_k) \otimes I_m \overline{M}_k) E_k \end{aligned} \quad (70)$$

By using (70) and (36), (69) can be rewritten as follows:

$$\begin{aligned}\tilde{R}_k' &= (I_{pn} - \overline{C}_k \overline{G}_{k-1}'(I_n + A_k) \otimes I_m \overline{M}_k) \\ &\quad * \tilde{R}_k (I_{pn} - \overline{C}_k \overline{G}_{k-1}'(I_n + A_k) \otimes I_m \overline{M}_k)^T. \end{aligned}$$

We define

$$(I_n + A_k) \otimes 1_{(q*q)}. * \overline{K}_k = \overline{\overline{K}}_k' \quad (71)$$

and define the optimal gain matrix as $\overline{\overline{K}}_k''$.

From Kalman filtering theory, we know that the uniqueness of the optimal gain matrix $\overline{\overline{K}}_k''$ requires that $\tilde{R}_k'$ is invertible. However, in subsection C we showed that $\tilde{R}_k'$ is singular.

Therefore, the optimal gain matrix $\overline{\overline{K}}_k''$ is not unique. Let r' be the rank of $\tilde{R}_k'$. Then, we propose a gain matrix $\overline{\overline{K}}_k''$ in the form of:

$$\overline{\overline{K}}_k'' = \tilde{\overline{\overline{K}}}_k'' \alpha_k' \quad (72)$$

where $\tilde{\overline{\overline{K}}}_k'' \in R^{(pn)*r}$ and $\alpha_k' \in R^{r*(pn)}$ are arbitrary matrices that make matrix $\alpha_k' \tilde{R}_k^{*'} \alpha_k'^T$ have a full rank. The optimal gain matrix $\tilde{\overline{\overline{K}}}_k''$ is then provided by the following theorem.

Theorem 7: If $\overline{M}_k$ satisfies $(I_n + A_k) \otimes 1_{(m*m)}. * \overline{M}_k \overline{C}_k \overline{G}_{k-1} = 1^n \otimes I_m$, then the following gain matrix $\overline{\overline{K}}_k''$ can minimize the variance of $\hat{X}_{k|k}$, as follows:

$$\overline{\overline{K}}_k'' = (P_{k|k}^{*'} \overline{C}_k^T + S_k') \alpha_k'^T (\alpha_k' \tilde{R}_k' \alpha_k'^T)^{-1} \alpha_k' \quad (73)$$

where $r' = rank(\tilde{R}_k')$ and $\alpha_k' \in R^{r*(p*n)}$ is an arbitrary matrix that makes matrix $\alpha_k' \tilde{R}_k' \alpha_k'^T$ have a full rank.

Proof: Substituting (72) in (68) and minimizing the trace of $P_{k|k}$ over $\overline{\overline{K}}''_k$ yields (73). By substituting (73) in (68), we obtain the error covariance matrix, as follows:

$$P_{k|k} = P^{*'}_{k|k} - \overline{\overline{K}}''_k (P^{*'}_{k|k}\overline{C}^T_k + S'_k)^T \tag{74}$$

which is the same result as the result in [8].

This completes the proof.

It should be noted that expression (73) depends only on the choice of $\overline{M}_k$. According to equation (33) and Theorem 1, the matrix $\overline{M}_k$ satisfies $(I_n + A) \otimes 1_{(m*m)}. * \overline{M}_k\overline{C}_k\overline{G}_{k-1} = 1^n \otimes I_m$ means that the estimation of the unknown input $\hat{D}_{k-1}$ is unbiased. We can obtain the gain matrix $\overline{K}''_k$ in the form expressed in (73), which minimizes the variance of $\hat{X}_{k|k}$ based on the matrix $\overline{M}_k$ used in (30).

As we have shown in subsection C, we cannot use $\overline{\overline{K}}''_k$; therefore, we provide the following theorem in order to show the gain matrix $\overline{\overline{K}}'_k$ that we are able to use.

Theorem 8: If $\overline{M}_k$ satisfies $(I_n + A_k) \otimes 1_{(m*p)}. * \overline{M}_k\overline{C}_k\overline{G}_{k-1} = 1^n \otimes I_m$, then, the following gain matrix $\overline{\overline{K}}'_k$ can minimize the variance of $\hat{X}_{k|k}$ with regard to the l_1 matrix norm,

$$\overline{\overline{K}}'_k = (I_n + A_k) \otimes 1_{(m*p)}. * (P^{*'}_{k|k}\overline{C}^T_k + S'_k) * \alpha'^T_k(\alpha'_k\tilde{R}'_k\alpha'^T_k)^{-1}\alpha'_k \tag{75}$$

where $r' = rank(\tilde{R}'_k)$ and $\alpha'_k \in R^{r*(p*n)}$ are arbitrary matrices that make matrix $\alpha'_k\tilde{R}'_k\alpha'^T_k$ have full rank.

The proof is similar to the proof of Theorem 5.

VI. MAIN RESULT

In Sections 4 and 5, we presented the results of the unknown MVU input and state estimations, respectively, under a time-invariant and time-varying multi-agent system. Now, we can just come to a conclusion with regard the former results and summarize them to one theorem in order to make the conclusion clear.

Subsection A provides the results for the multi-agent system under time-invariant topology. Subsection B provides the results of the multi-agent system under time-varying topology.

A. MINIMUM-VARIANCE UNBIASED UNKNOWN INPUT AND STATE ESTIMATIONS UNDER TIME-INVARIANT TOPOLOGY

Theorem 9: If and only if multi-agent system (14) to (15) satisfies condition (28), the distributed cooperative filters (16) to (19) can achieve the MVU estimation of the unknown input and state, where the gain matrices $\overline{M}_k$ and $\overline{K}_k$ are given as Equations (37) and (61), respectively.

Proof: Theorem 1 means that if and only if the system matrix satisfies condition (28), we can obtain the unbiased estimation of the unknown input for multi-agent system d_k. Theorem 2 indicates that when $\overline{M}_k$ has a specific form, we can obtain the MVU estimation of d_k. Theorem 3 shows that if and only if $\hat{X}_{k-1|k-1}$ and $\hat{D}_{k-1}$ are unbiased. Then (18) to (19) are the unbiased estimators of X_k for any value of $\overline{K}_k$. Theorem 5 and Theorem 6 show that when $\overline{K}_k$ has a specific form we can obtain the MVU estimation of X_k.

This completes the proof.

B. MINIMUM-VARIANCE UNBIASED UNKNOWN INPUT AND STATE ESTIMATION UNDER TIME-VARYING TOPOLOGY

Theorem 10: If and only if the multi-agent systems (14) to (15) satisfy condition (28), the distributed cooperative filters (29) to (32) can achieve the MVU estimation of the unknown input and state, where the gain matrices $\overline{M}_k$ and $\overline{K}_k$ are given as Equations (37) and(61), respectively.

Proof: Theorem 1 means that if and only if the system matrix satisfies condition (28), then we can obtain an unbiased estimation for the unknown input of the multi-agent system d_k. Theorem 2 indicates that when $\overline{M}_k$ has a specific form, we can obtain the MVU estimation of d_k. Theorem 4 shows that if and only if $\hat{X}_{k-1|k-1}$ and $\hat{D}_{k-1}$ are unbiased, then, (31) to (32) are unbiased estimators of X_k for any value of $\overline{K}_k$. Theorems 7 and 8 show that when $\overline{K}_k$ has a specific form, we can obtain the MVU estimation of X_k.

This completes the proof.

VII. SIMULATION

In this section, a numerical example is provided in order to demonstrate that our filter is considerably better than the conventional decentralized filter. Subsection A provides a numerical example in order to verify the proposed method. Subsection B provides a practical example.

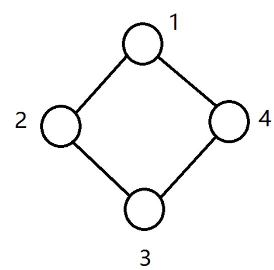


FIGURE 1. Topology of Graph *G*.

A. NUMERICAL EXAMPLE

In this subsection, we consider a multi-agent system with four agents and time-invariant topology. Graph G is shown in Fig. 1. The system matrix is presented below. We find that none of the agents satisfy the conventional existing condition(7); however, the augmented multi-agent system satisfies the relaxed existing condition (28). Fig. 2 shows the state estimation error using the proposed distributed cooperative filters. Fig. 3 shows the estimation error of the unknown inputs using the proposed distributed cooperative filters. Fig. 4 shows the state estimation error using the conventional decentralized filters. Fig. 5 shows the estimation error of the unknown inputs using the conventional decentralized filters.

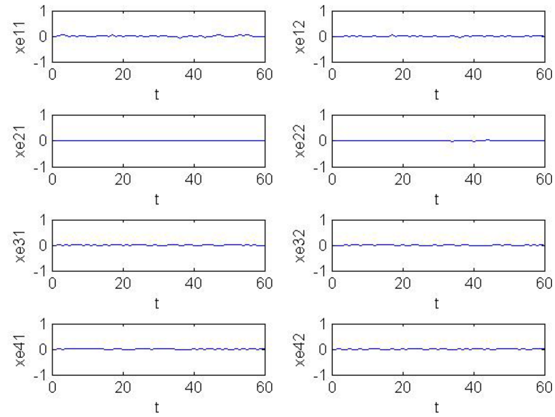


FIGURE 2. State estimation error *X* using distributed cooperative filters.

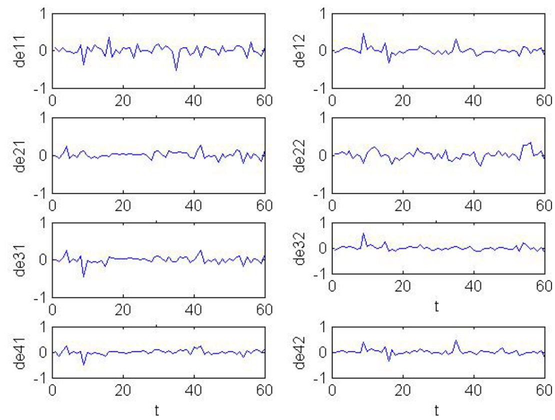


FIGURE 3. Estimation error of unknown input *d* using distributed cooperative filters.

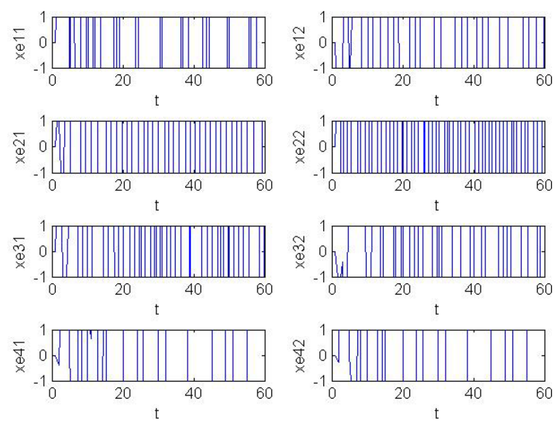


FIGURE 4. State estimation error *X* using conventional decentralized filters.

In the simulation,

$$B_k^1 = \begin{bmatrix} 1 & 0 \\ 0 & \sin(k)+1 \end{bmatrix}, \quad B_k^2 = \begin{bmatrix} 1 & 0 \\ 0 & \cos(k)+1 \end{bmatrix},$$

$$B_k^3 = \begin{bmatrix} 1 & 0 \\ 0 & -\sin(k)-1 \end{bmatrix}, \quad B_k^4 = \begin{bmatrix} 1 & 0 \\ 0 & -\cos(k)-1 \end{bmatrix}.$$

$$C_k^1 = \begin{bmatrix} 2 & \sin(k) \\ \sin(k) & 2 \end{bmatrix}, \quad C_k^2 = \begin{bmatrix} 3 & \cos(k) \\ \cos(k) & 3 \end{bmatrix},$$

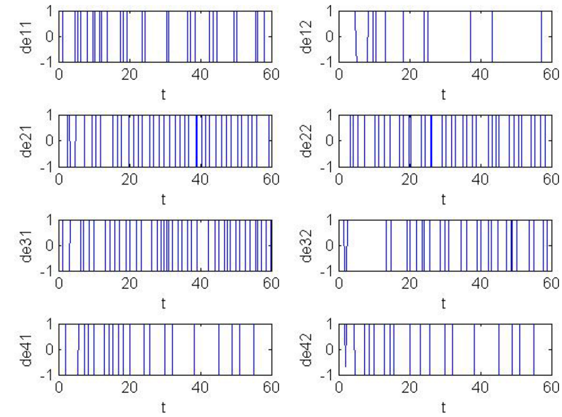


FIGURE 5. Estimation error of unknown input *d* using conventional decentralized filters.

$$C_k^3 = \begin{bmatrix} 4 & \sin(k) \\ \sin(k) & 4 \end{bmatrix}, \quad C_k^4 = \begin{bmatrix} 5 & \cos(k) \\ \cos(k) & 5 \end{bmatrix}.$$

$$G_k^1 = \begin{bmatrix} 1 & 1+\sin(k) \\ 0 & 0 \end{bmatrix}, \quad G_k^2 = \begin{bmatrix} 0 & 0 \\ 1+\sin(k) & 1 \end{bmatrix},$$

$$G_k^3 = \begin{bmatrix} 1+\cos(k) & 0 \\ 1 & 0 \end{bmatrix}, \quad G_k^4 = \begin{bmatrix} 0 & 1 \\ 0 & 1+\cos(k) \end{bmatrix}.$$

$d_1 = k$ and $d_2 = \sin(k)$.

Model noise and measurement noise are:

$$w^{11}, w^{12}, w^{21}, w^{22}, w^{31}, w^{32}, w^{41}, w^{42} \sim N(0, 0.1),$$
$$v^{11}, v^{12}, v^{21}, v^{22}, v^{31}, v^{32}, v^{41}, v^{42} \sim N(0, 0.01).$$

In Figures 2 to 5, $xe_{ij} = \hat{x}_{ij} - x_{ij}$, where $\hat{x}_{ij}$ and x_{ij} denote the estimation and the true value for the j-th dimension state of the i-th agent, respectively. The definition of de_{ij} is similar to that of xe_{ij}.

From the system matrices, we can see that $rank(C_k^i G_{k-1}^i) = 1 \neq 2$; however, $rank(\overline{C}_k \overline{G}_{k-1}) = 2$. In other words, even though an agent does not satisfy the existing condition(7), the augmented multi-agent system will satisfy the cooperative existing condition(28). From Figures 2 and 3, one can see that the distributed cooperative filter can estimate the unknown input and the state correctly. Whereas, form Figures 4 and 5, one can see that the conventional decentralized filter cannot estimate the unknown input and the state correctly, which proves that our result is a significantly looser existing condition, in comparison to the conventional decentralized condition.

B. PRACTICAL EXAMPLE

In this subsection, we consider a linearized dynamic model of a vertical takeoff and landing aircraft in the vertical plane [35]. In [35], the states are defined as follows:

$$x = \begin{bmatrix} \text{horizontal velocity[in knots]} \\ \text{vertical velocity[in knots]} \\ \text{pitch rate[in degrees per second]} \\ \text{pitch angle[in degrees]} \end{bmatrix}.$$

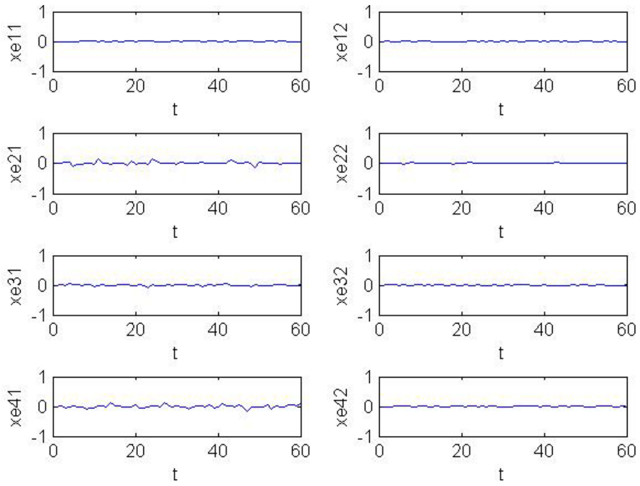


FIGURE 6. State estimation error ***X***.

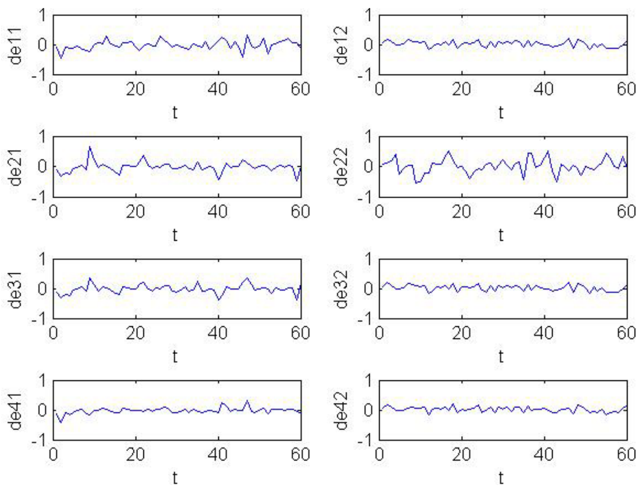


FIGURE 7. Estimation error of unknown input ***d***.

For convenience, we chose the first two state dimensions and simplified this problem as planar rather than spatial. The states of the system are defined as follows:

$$x = \begin{bmatrix} \text{horizontal velocity[in knots]} \\ \text{vertical velocity[in knots]} \end{bmatrix}$$

The unknown input d was chosen as the wind power that influences the velocity of the aircraft.

The state matrix was as follows:

$$B = \begin{bmatrix} -0.0366 & 0.0271 \\ 0.0482 & -1.0100 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, we set the system matrix of the four agents as follows:

$$B_k^1 = B_k^2 = B_k^3 = B_k^4 = \begin{bmatrix} -0.0366 & 0.0271 \\ 0.0482 & -1.0100 \end{bmatrix}.$$

$$C_k^1 = C_k^2 = C_k^3 = C_k^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is clear that $C_k^i B_k^i$ was observable.

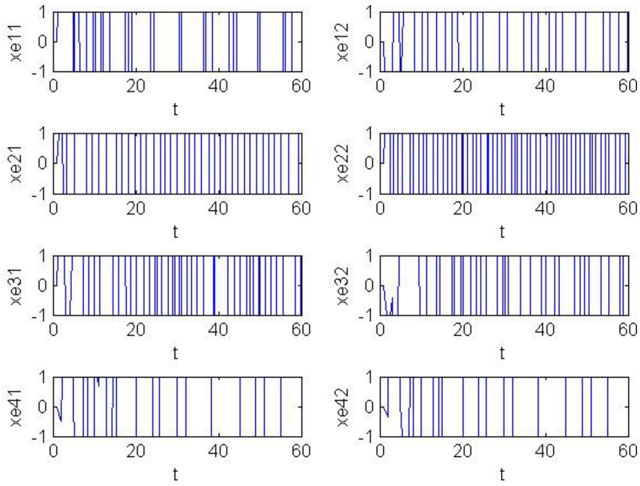


FIGURE 8. State estimation error ***X*** using conventional decentralized filters.

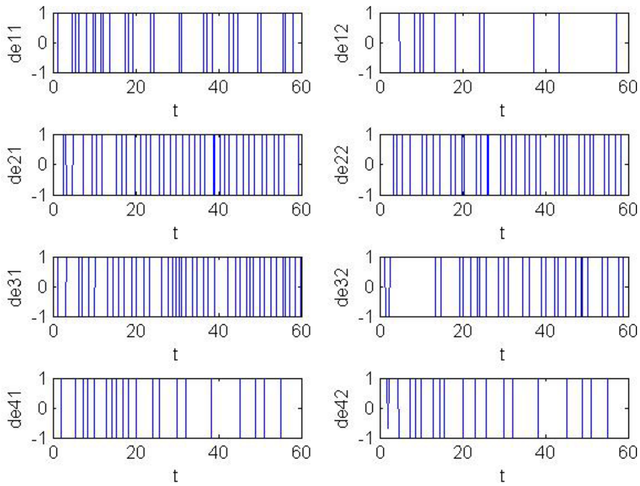


FIGURE 9. Estimation error of unknown input ***d*** using conventional decentralized filters.

The other parameters such as G_k^i and d were the same as the numerical example. Fig. 6 shows the state estimation error. Fig. 7 shows the estimation error of unknown input.

From Figures 6 and 7, one can see that the distributed cooperative filter can estimate the unknown input and the state correctly. Form Figures 8 and 9, one can see that the conventional decentralized filter cannot estimate the unknown input and the state correctly. From the results, one can see that compared to the conventional decentralized filters, the distributed cooperative filter can estimate the states and unknown input correctly, which proves that our filter can also work in practice.

VIII. CONCLUSION

A distributed cooperative filter was developed, with regard to the MVU, which simultaneously estimates the unknown inputs and states of a linear discrete-time heterogeneous multi-agent system. The estimate of the unknown inputs was obtained by innovating on LS estimation. The problem of state estimation was transformed into a standard Kalman filtering problem for a system with a correlated process and measurement noise. Most significantly, the proposed filter had a looser existing condition, in comparison to the conventional filter. The presented numerical example demonstrates the effectiveness of the proposed filter. In the future, multi-agent system could be studied in two-dimensional system framework [36], [37].

REFERENCES

[1] M. A. Potter and K. A. De Jong, "A cooperative coevolutionary approach to function optimization," in *Proc. Int. Conf. Parallel Problem Solving Nature*, 1994, pp. 249–257.

[2] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 988–1001, Jun. 2003.

[3] T. Vicsek and A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, "Novel type of phase transition in a system of self-driven particles," *Phys. Rev. Lett.*, vol. 75, no. 6, p. 1226, 1995.

[4] W. Ren, R. W. Beard, and E. M. Atkins, "Information consensus in multivehicle cooperative control," *IEEE Control Syst. Mag.*, vol. 27, no. 2, pp. 71–82, Apr. 2007.

[5] W. Ren and Y. Cao, *Distributed Coordination of Multi-Agent Networks: Emergent Problems, Models, and Issues*. Berlin, Germany: Springer-Verlag, 2011.

[6] S. Kar and J. M. F. Moura, "Distributed consensus algorithms in sensor networks: Quantized data and random link failures," *IEEE Trans. Signal Process.*, vol. 58, no. 3, pp. 1383–1400, Mar. 2010.

[7] F. Cucker and S. Smale, "Emergent behavior in flocks," *IEEE Trans. Autom. Control*, vol. 52, no. 5, pp. 852–862, May 2007.

[8] P. De Lellis, M. Di Bernardo, and F. Garofalo, "Novel decentralized adaptive strategies for the synchronization of complex networks," *Automatica*, vol. 45, no. 5, pp. 1312–1318, 2009.

[9] P. De Lellis, M. Di Bernardo, F. Sorrentino, and A. Tierno, "Adaptive synchronization of complex networks," *Int. J. Comput. Math.*, vol. 85, no. 8, pp. 1189–1218, 2008.

[10] J. Zhou, T. Chen, and L. Xiang, "Adaptive synchronization of coupled chaotic delayed systems based on parameter identification and its applications," *Int. J. Bifurcation Chaos*, vol. 16, no. 10, pp. 2923–2933, 2006.

[11] J. Zhou, T. Chen, and L. Xiang, "Robust synchronization of delayed neural networks based on adaptive control and parameters identification," *Chaos Solitons Fractals*, vol. 27, no. 4, pp. 905–913, Feb. 2006.

[12] L. Cheng, Y. Wang, Z.-G. Hou, M. Tan, and Z. Cao, "Sampled-data based average consensus of second-order integral multi-agent systems: Switching topologies and communication noises," *Automatica*, vol. 49, no. 5, pp. 1458–1464, 2013.

[13] W. Hu, L. Liu, and G. Feng, "Consensus of linear multi-agent systems by distributed event-triggered strategy," *IEEE Trans. Cybern.*, vol. 46, no. 1, pp. 148–157, Jan. 2016.

[14] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Trans. Autom. Control*, vol. 50, no. 5, pp. 655–661, May 2005.

[15] Z. Li, W. Ren, X. Liu, and L. Xie, "Distributed consensus of linear multi-agent systems with adaptive dynamic protocols," *Automatica*, vol. 49, no. 7, pp. 1986–1995, 2013.

[16] W. Chen, C. Wen, S. Hua, and C. Sun, "Distributed cooperative adaptive identification and control for a group of continuous-time systems with a cooperative PE condition via consensus," *IEEE Trans. Autom. Control*, vol. 59, no. 1, pp. 91–106, Jan. 2014.

[17] M. Darouach, M. Zasadzinski, and S. J. Xu, "Full-order observers for linear systems with unknown inputs," *IEEE Trans. Autom. Control*, vol. 39, no. 3, pp. 606–609, Mar. 1994.

[18] M. Hou and P. C. Müller, "Design of observers for linear systems with unknown inputs," *IEEE Trans. Autom. Control*, vol. 37, no. 6, pp. 871–875, Jun. 1992.

[19] P. Kudva, N. Viswanadham, and A. Ramakrishna, "Observers for linear systems with unknown inputs," *IEEE Trans. Autom. Control*, vol. AC-25, no. 1, pp. 113–115, Feb. 1980.

[20] M. Hou and R. J. Patton, *Input Observability and Input Reconstruction*. New York, NY, USA: Pergamon, 1998.

[21] Y. Xiong and M. Saif, "Unknown disturbance inputs estimation based on a state functional observer design," *Automatica*, vol. 39, pp. 1389–1398, Aug. 2003.

[22] B. Friedland, "Treatment of bias in recursive filtering," *IEEE Trans. Autom. Control*, vol. AC-14, no. 4, pp. 359–367, Aug. 1969.

[23] P. K. Kitanidis, "Unbiased minimum-variance linear state estimation," *Automatica*, vol. 23, no. 6, pp. 775–778, 1987.

[24] M. Darouach and M. Zasadzinski, "Unbiased minimum variance estimation for systems with unknown exogenous inputs," *Automatica*, vol. 33, no. 4, pp. 717–719, 1997.

[25] C.-S. Hsieh, "Robust two-stage Kalman filters for systems with unknown inputs," *IEEE Trans. Autom. Control*, vol. 45, no. 12, pp. 2374–2378, Dec. 2000.

[26] S. Gillijns and B. De Moor, "Unbiased minimum-variance input and state estimation for linear discrete-time systems," *Automatica*, vol. 43, no. 1, pp. 111–116, 2007.

[27] A. H. Sayed, "Adapation, learning, and optimization over networks," *Found. Trends Mach. Learn.*, vol. 7, no. 4, pp. 311–801, 2014.

[28] F. S. Cattivelli and A. H. Sayed, "Analysis of spatial and incremental LMS processing for distributed estimation," *IEEE Trans. Signal Process.*, vol. 59, no. 4, pp. 1465–1480, Apr. 2011.

[29] S. M. Azizi and K. Khorasani, "A distributed Kalman filter for actuator fault estimation of deep space formation flying satellites," in *Proc. IEEE Int. Syst. Conf.*, Mar. 2009, pp. 354–359.

[30] M. Chadli, M. Davoodi, and N. Meskin, "Distributed state estimation, fault detection and isolation filter design for heterogeneous multi-agent linear parameter-varying systems," *IET Control Theory Appl.*, vol. 11, no. 2, pp. 254–262, 2017.

[31] M. Chadli, M. Davoodi, and N. Meskin, "Distributed fault detection and isolation filter design for heterogeneous multi-agent LPV systems," in *Proc. Amer. Control Conf.* May 2017, pp. 1610–1615.

[32] J. Lou *et al.*, "Distributed incremental bias-compensated RLS estimation over multi-agent networks," *Sci. China, Inf. Sci.*, vol. 60, no. 3, p. 032204, 2017.

[33] K. Zhang, G. Liu, and B. Jiang, "Robust unknown input observer-based fault estimation of leader–follower linear multi-agent systems," *Circuits, Syst., Signal Process.*, vol. 36, no. 2, pp. 1–18, 2016.

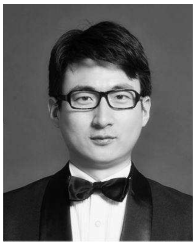
[34] R. Diestel, *Graph Theory*. New York, NY, USA: Springer-Verlag, 2000.

[35] T. G. Park and K. S. Lee, "Process fault isolation for linear systems with unknown inputs," *IEE Proc.-Control Theory Appl.*, vol. 151, no. 6, pp. 720–726, Nov. 2004.

[36] Y. Wang, D. Zhao, Y. Li, and S. X. Ding, "Unbiased minimum variance fault and state estimation for linear discrete time-varying two-dimensional systems," *IEEE Trans. Autom. Control*, vol. 62, no. 10, pp. 5463–5469, Oct. 2017.

[37] Y. Wang, H. Zhang, S. Wei, D. Zhou, and B. Huang, "Control performance assessment for ILC-controlled batch processes in a 2-D system framework," *IEEE Trans. Syst., Man, Cybern., Systems*, to be published, doi: 10.1109/TSMC.2017.2672563.

CHANGQING LIU received the B.S. degree from the Beijing University of Chemical Technology, Beijing, China, in 2014, where he is currently pursuing the Ph.D. degree with the Beijing University of Chemical Technology. His research interests include multi-agent system, fault-tolerant control, and fault diagnosis and isolation.



YOUQING WANG (M'09–SM'12) received the B.S. degree from Shandong University, Jinan, Shandong, China, in 2003, and the Ph.D. degree in control science and engineering from Tsinghua University, Beijing, China, in 2008. He was a Research Assistant with the Department of Chemical Engineering, The Hong Kong University of Science and Technology, from 2006 to 2007. From 2008 to 2010, he was a Senior Investigator with the Department of Chemical Engineering, University of California at Santa Barbara, Santa Barbara, USA. In 2015, he was a Visiting Professor with the Department of Chemical and Materials Engineering, University of Alberta, Canada. He is currently a Full Professor with the Shandong University of Science and Technology. His research interests include fault-tolerant control, state monitoring, modeling and control of biomedical processes (e.g., artificial pancreas system), and iterative learning control. He holds membership of two IFAC Technical Committees (TC6.1 and TC8.2). He was a recipient of several research awards (including the *Journal of Process Control* Survey Paper Prize and ADCHEM2015 Young Author Prize). He is an Associate Editor of *Multidimensional Systems and Signal Processing* and the *Canadian Journal of Chemical Engineering*.

DONGHUA ZHOU (SM'99) received the B.Eng., M.Sc., and Ph.D. degrees in electrical engineering from Shanghai Jiao Tong University, China, in 1985, 1988, and 1990, respectively. He was an Alexander von Humboldt Research Fellow with the University of Duisburg, Germany, from 1995 to 1996, and a Visiting Scholar with Yale university, USA, from 2001 to 2002. He joined Tsinghua University in 1997, where he was a Professor and the Head of the Department of Automation from 2008 to 2015. He is currently the Vice President with the Shandong University of Science and Technology. He has authored and coauthored over 140 peer-reviewed international journal papers and six monographs in the areas of process identification, fault diagnosis, fault-tolerant control, reliability prediction, and predictive maintenance. He is a member of the IFAC Technical Committee on Fault Diagnosis and Safety of Technical Processes. He was also the NOC Chair of the 6th IFAC Symposium on SAFEPROCESS 2006. He is an Associate Editor of the *Journal of Process Control*, the Vice Chairman of the Chinese Association of Automation.

XIAO SHEN received the B.Eng., M.Sc., and Ph.D. degrees in mechanical and electronic engineering from the Shandong University of Science and Technology, China, in 2004, 2007, and 2014, respectively. She is currently a Lecturer with the College of Mechanical and Electronic Engineering, Shandong University of Science and Technology. Her research interests include multi-agent system, fault-tolerant control, and fault diagnosis and isolation.