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State Estimation for Dynamic Systems With Unknown Process Inputs and Applications

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ABSTRACT The problem of state estimation for discrete-time stochastic time-varying systems in the presence of unknown process inputs or disturbances is addressed in this paper. A Kalman-type filter is proposed, and the optimal oracle filter gain in the sense of minimizing the mean squared error of the state estimate is obtained. To tackle the unknown quantities in the gain matrix, a nonlinear equation is introduced and its solution is taken as the estimate of unknown inputs, and then, a novel nonlinear equation-based unknown input filtering (NEUIF) is proposed. A scalar-based iterative algorithm for related fixed point problem is developed so that the dichotomy method is employed to solve the above nonlinear equation very efficiently. Adopting the same strategy for the dynamic systems with unknown inputs or disturbances, we provide two applications of the proposed state estimation algorithm. One is for a class of nonlinear dynamic systems with linear observations by taking the residual term in linearizing the transition function as an unknown input in the derived linear system. The other is for tracking maneuvering targets in which the bias between the real motion and modeled motions is regarded as an unknown input in the state transition equation. Some numerical simulations demonstrate the effectiveness of the proposed NEUIF method for tackling various uncertainties in complicated dynamic systems.

INDEX TERMS Dynamic systems, state estimation or filtering, unknown inputs and disturbances, nonlinear estimator, nonlinear equation, iterative algorithm.

I. INTRODUCTION

The dynamic systems with various uncertainties are ubiquitous. The state estimation or filtering for dynamic systems with unknown inputs has received a great deal of attentions by researchers in past decades because of their widespread applications in many fields such as maneuvering target tracking, unmanned systems, environmental monitoring, fault detection, and tracking and navigation. Many approaches are developed to deal with such problems in the literatures. One can refer to [1]–[8] and references therein. To the best of our knowledge, the existing methods can be broadly divided into two categories.

1) SOME PRIOR INFORMATION ABOUT UNKNOWN INPUTS IS ASSUMED TO BE KNOWN

If the unknown inputs are time-invariant or can be formulated as an autoregressive model, by augmenting the state to be estimated with the unknown inputs, under certain conditions, some dynamic systems can be converted equivalently to ones that can be solved using the existing state estimation or filtering approaches. This method can result in the estimates of state and unknown inputs simultaneously. In [9], a two-stage Kalman filter is developed by using the state augmentation technique to ensure the optimality under the assumption of constant unknown input. The two-stage strategy is adopted to reduce the computation cost while the augmented state vector has a substantially larger dimension than the original state. Recently, the variational Bayesian filter (VBF) is investigated in many literatures (see, e.g., [10]–[12]). In [11], the posterior joint distribution of state variable and unknown inputs is approximated by variational Bayesian method under some prior distribution information of unknown inputs. In [13], the unknown inputs are assumed to be lied on a linear manifold, the state estimate minimizing the mean squared error (MSE) under some conditions is derived within Bayesian framework. In addition, the asymptotic stability condition of filtering algorithm proposed in [13] is established in [14].

2) NO PRIOR INFORMATION ABOUT UNKNOWN INPUTS IS AVAILABLE

In [3], an unbiased linear minimum variance filter (ULMVF) is proposed and its stability is proved. In the ULMVF approach, a constrained optimization problem must be solved to obtain the gain matrix and the state estimate, but two necessary conditions to ensure the uniqueness of solution of the above optimization problem are often hard to be verified. A parameterizing technique is suggested in [15] to solve such optimization problem. The global optimality of the above filter is verified in [16]. An algorithm proposed in [17], which simultaneously estimates the unknown inputs and the state for linear discrete-time systems, can yield the same state update as the ones in [3] and [15] and the same input estimate as the one in [18]. In [19], the problem of state estimation for linear systems with unknown inputs in both the process and observation equations is discussed, and a recursive filter with global optimality in the sense of unbiased linear minimum variance is developed. In [20], a globally optimal filtering framework is constructed for the dynamic systems in which both the state and observation are affected by unknown inputs, and all filters proposed in [19] and [21]–[24] are proved to be optimal in the sense of unbiased minimum variance. The unbiased information filter is redesigned in [25] on basis of information filter and shares the same estimation accuracy with the ones in [3] and [15] under the same initial conditions.

In this paper, we address the state estimation for dynamic systems in which the state transition equation involves some unknown inputs without any prior information. Unlike many methods in the existing literatures, the novel state estimation algorithm is developed without the unbiasedness assumption so as to reduce the MSE of state estimate. The main work and contributions of this paper are as follows.

Firstly, we propose a Kalman-type recursive estimate without the unbiasedness assumption. In the minimum MSE sense, we derive the optimal oracle gain matrix which involves some unknown terms related to unknown inputs. By introducing a vector-valued equation, we take the fixed point of this nonlinear equation as a nonlinear estimate of the unknown inputs, and then provide an available filtering gain matrix. In theory, we prove that the proposed nonlinear equation-based unknown input filtering (NEUIF) method outperforms than the ULMVF method in the minimum MSE sense in the absence of unknown inputs.

Secondly, to solve the fixed point of the above nonlinear equation, we design a scalar-based iterative method to ensure that the iteration converges to the fixed point. Consequently, the developed iteration can be executed by the dichotomy method, and then the NEUIF can be implemented more efficiently.

Thirdly, we deal with nonlinear filtering and maneuvering target tracking based on the provided NEUIF method. For a nonlinear system with nonlinear process equation and linear observation equation, by taking the residual term in linearizing the nonlinear state transition function as unknown inputs or disturbances of the derived linear systems, we propose a novel nonlinear filtering algorithm. For the multiple model (MM) method for maneuvering target, by treating the bias between the real motion and the modeled motions in the mode set as unknown input in state transition equation, we integrate the NEUIF estimator into the interacting multiple model (IMM) framework and propose an IMM-NEUIF algorithm.

Finally, we evaluate the proposed state estimation approach via some simulation experiments. The results show the comparable performance of the proposed algorithm with the existing filtering algorithms.

This paper is organized as follows. Section II formulates the state estimation problem for stochastic time-varying linear systems with unknown process inputs, and gives the optimal filter gain matrix in the MSE sense. Moreover, an efficient iterative algorithm for estimating unknown inputs is proposed. A theoretical comparison of performance between the proposed filter and the existing ULMVF is also provided. Based on the proposed NEUIF, Section III discusses filtering for nonlinear dynamic systems and tracking for maneuvering targets. In Section IV, the performance of the proposed estimator and filtering is illustrated via some numerical simulations. Section V gives some conclusions. Proofs of some mathematical results are provided in the appendix.

A. NOTATIONS

The notations $\mathbb{R}^m$ and $\mathbb{R}^{m\times n}$ denote the set of all m-dimensional real column vectors and the set of all $m \times n$ real matrices respectively. The notation $\mathbb{N}$ is the set of natural numbers and $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. For a matrix A, A^T, $A^\dagger$, $\rho(A)$ and $\mathcal{R}(A)$ represent its transpose, Moore–Penrose generalized inverse, spectral radius and column space respectively. For two matrices A and B, $A \succeq B$ (or $A \succ B$) means that $A - B$ is symmetric and positive semidefinite (or positive definite). The symbol I stands for the identity matrix with appropriate dimension. The operation $\mathbb{E}[\cdot]$ denotes the mathematical expectation of random variable.

II. PROBLEM FORMULATION AND NOVEL NEUIF APPROACH

In this paper, a class of linear discrete-time stochastic time-varying systems is considered as follows:

$$\begin{aligned} \mathbf{x}_{k+1} &= F_k\mathbf{x}_k + G_k\mathbf{d}_k + \mathbf{w}_k, \\ \mathbf{y}_k &= H_k\mathbf{x}_k + \mathbf{v}_k, \end{aligned} \tag{1}$$

where $k \geq 0$ is time index, $\mathbf{x}_k \in \mathbb{R}^n$ is the state vector, $\mathbf{d}_k \in \mathbb{R}^m$ is an unknown and deterministic input vector, and $\mathbf{y}_k \in \mathbb{R}^p$ is the measurement, $\mathbf{w}_k \in \mathbb{R}^n$ and $\mathbf{v}_k \in \mathbb{R}^p$ are mutually independent white noise processes with mean zeros and known covariance matrices $Q_k = \mathbb{E}[\mathbf{w}_k\mathbf{w}_k^T] \succ 0$ and

$R_k = \mathbb{E}[\mathbf{v}_k \mathbf{v}_k^T] \succ 0$ respectively. The matrices $F_k \in \mathbb{R}^{n\times n}$, $G_k \in \mathbb{R}^{n\times m}$ and $H_{k+1} \in \mathbb{R}^{p\times n}$ are deterministic and known. Without loss of generality, we assume that G_k is full column rank and H_{k+1} is full row rank. The mean $\bar{\mathbf{x}}_0$ and the variance P_0 of the initial state $\mathbf{x}_0$ are assumed to be known.

The state estimator is selected from the set of Kalman-type filters as follows:

$$\hat{\mathbf{x}}_{k+1} = F_k\hat{\mathbf{x}}_k + K_{k+1}(\mathbf{y}_{k+1} - H_{k+1}F_k\hat{\mathbf{x}}_k),$$

where the gain matrix $K_{k+1} \in \mathbb{R}^{n\times p}$ will be determined in the sense of minimizing the MSE of state estimate.

A. OPTIMAL ORACLE FILTER GAIN

Let $\mathbf{e}_{k+1} = \hat{\mathbf{x}}_{k+1} - \mathbf{x}_{k+1}$ and $P_{k+1} = \mathbb{E}[\mathbf{e}_{k+1}\mathbf{e}_{k+1}^T]$, then

$$\mathbf{e}_{k+1} = (I - K_{k+1}H_{k+1})F_k\mathbf{e}_k + (K_{k+1}H_{k+1} - I)G_k\mathbf{d}_k + (K_{k+1}H_{k+1} - I)\mathbf{w}_k + K_{k+1}\mathbf{v}_{k+1},$$

and

$$P_{k+1} = (I - K_{k+1}H_{k+1}) \cdot \Big(F_kP_kF_k^T + (G_k\mathbf{d}_k)(G_k\mathbf{d}_k)^T + Q_k - E_k\Big) \cdot (I - K_{k+1}H_{k+1})^T + K_{k+1}R_{k+1}K_{k+1}^T,$$

where

$$E_k = (F_k\mathbf{b}_k)(G_k\mathbf{d}_k)^T + (G_k\mathbf{d}_k)(F_k\mathbf{b}_k)^T, \quad \mathbf{b}_k = \mathbb{E}[\hat{\mathbf{x}}_k - \mathbf{x}_k]. \tag{2}$$

Theorem 1: If $\mathbf{b}_k$ and $\mathbf{d}_k$ are known, then in the sense of minimizing the MSE of $\hat{\mathbf{x}}_{k+1}$, i.e., trP_{k+1}, the optimal gain matrix is give as

$$K_{k+1} = \Big(F_kP_kF_k^T + (G_k\mathbf{d}_k)(G_k\mathbf{d}_k)^T + Q_k - E_k\Big)H_{k+1}^T \cdot \Big(H_{k+1}(F_kP_kF_k^T + (G_k\mathbf{d}_k)(G_k\mathbf{d}_k)^T + Q_k - E_k)H_{k+1}^T + R_{k+1}\Big)^{-1}. \tag{3}$$

Proof: See Appendix A. □

Note that the gain K_{k+1} given by (3) cannot be applied directly because it involves the unknown quantities $G_k\mathbf{d}_k$ and $F_k\mathbf{b}_k$ in E_k.

B. ESTIMATING UNKNOWN INPUT AND GAIN MATRIX

Because it is unable to determine the quantity of $F_k\mathbf{b}_k$ in (2), we assume that $E_k = 0$, which is strictly weaker than the unbiasedness assumption on state estimation, then we have an estimate of K_{k+1} as

$$\big(F_k\hat{P}_kF_k^T + (G_k\mathbf{d}_k)(G_k\mathbf{d}_k)^T + Q_k\big)H_{k+1}^T \cdot \big(H_{k+1}(F_k\hat{P}_kF_k^T + (G_k\mathbf{d}_k)(G_k\mathbf{d}_k)^T + Q_k)H_{k+1}^T + R_{k+1}\big)^{-1},$$

where

$$\hat{P}_k = (I - K_kH_k)(F_{k-1}\hat{P}_{k-1}F_{k-1}^T + \hat{\mathbf{z}}_{k-1}\hat{\mathbf{z}}_{k-1}^T + Q_{k-1}) \cdot (I - K_kH_k)^T + K_kR_kK_k^T$$

with the initial value $\hat{P}_0 = P_0$ and an estimate $\hat{\mathbf{z}}_{k-1}$ of $G_{k-1}\mathbf{d}_{k-1}$. Furthermore, to deal with the unknown quantity $G_k\mathbf{d}_k$ in the above, we introduce a matrix-valued function $\Gamma_{k+1} : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ as follows:

$$\Gamma_{k+1}(\mathbf{z}) = (F_k\hat{P}_kF_k^T + \mathbf{z}\mathbf{z}^T + Q_k)H_{k+1}^T \cdot \big(H_{k+1}(F_k\hat{P}_kF_k^T + \mathbf{z}\mathbf{z}^T + Q_k)H_{k+1}^T + R_{k+1}\big)^{-1}, \quad \mathbf{z} \in \mathbb{R}^n. \tag{4}$$

Next, using the function $\Gamma_{k+1}(\cdot)$, we construct an estimate $\hat{\mathbf{z}}_k$ of $G_k\mathbf{d}_k$ such that

$$\hat{\mathbf{z}}_k = G_kG_k^\dagger\Gamma_{k+1}(\hat{\mathbf{z}}_k)(\mathbf{y}_{k+1} - H_{k+1}F_k\hat{\mathbf{x}}_k), \tag{5}$$

and then obtain the filter gain matrix $K_{k+1} = \Gamma_{k+1}(\hat{\mathbf{z}}_k)$.

Remark 1: From

$$\begin{aligned}\hat{\mathbf{x}}_{k+1} &= F_k\hat{\mathbf{x}}_k + K_{k+1}(\mathbf{y}_{k+1} - H_{k+1}F_k\hat{\mathbf{x}}_k)\\ &= F_k(\mathbf{x}_k + \mathbf{e}_k) + K_{k+1}(\mathbf{y}_{k+1} - H_{k+1}F_k\hat{\mathbf{x}}_k)\\ &= \mathbf{x}_{k+1} - (G_k\mathbf{d}_k + \mathbf{w}_k - F_k\mathbf{e}_k) + K_{k+1}(\mathbf{y}_{k+1} - H_{k+1}F_k\hat{\mathbf{x}}_k),\end{aligned}$$

it is very natural to take $K_{k+1}(\mathbf{y}_{k+1} - H_{k+1}F_k\hat{\mathbf{x}}_k)$ as an estimate of $G_k\mathbf{d}_k + \mathbf{w}_k - F_k\mathbf{e}_k$. Moreover, noticing that the vector $G_k\mathbf{d}_k$ to be estimated locates in the column space of G_k, we project $K_{k+1}(\mathbf{y}_{k+1} - H_{k+1}F_k\hat{\mathbf{x}}_k)$ onto $\mathcal{R}(G_k)$ geometrically by the orthogonal projector $G_kG_k^\dagger$.

In summary, we propose the filtering algorithm for the system (1) with unknown inputs as follows:

Algorithm 1 (NEUIF for the System (1) *With Unknown Inputs):* Let $\hat{\mathbf{x}}_0 = \bar{\mathbf{x}}_0$ and $\hat{P}_0 = P_0$. For $k = 0, 1, \ldots$ do

$$\begin{aligned}\hat{\mathbf{x}}_{k+1} &= F_k\hat{\mathbf{x}}_k + K_{k+1}(\mathbf{y}_{k+1} - H_{k+1}F_k\hat{\mathbf{x}}_k),\\ K_{k+1} &= (F_k\hat{P}_kF_k^T + \hat{\mathbf{z}}_k\hat{\mathbf{z}}_k^T + Q_k)H_{k+1}^T \cdot \big(H_{k+1}(F_k\hat{P}_kF_k^T + \hat{\mathbf{z}}_k\hat{\mathbf{z}}_k^T + Q_k)H_{k+1}^T + R_{k+1}\big)^{-1},\\ \hat{P}_{k+1} &= (I - K_{k+1}H_{k+1})(F_k\hat{P}_kF_k^T + \hat{\mathbf{z}}_k\hat{\mathbf{z}}_k^T + Q_k) \cdot (I - K_{k+1}H_{k+1})^T + K_{k+1}R_{k+1}K_{k+1}^T,\end{aligned}$$

where $\hat{\mathbf{z}}_k$ is the solution of the fixed point problem (5).

Notice that, if $\hat{\mathbf{z}}_k = \mathbf{0}$ at each time instant, then the recursive terms $\hat{\mathbf{x}}_k$, $\hat{P}_k$ and K_k in Algorithm 1 coincide perfectly with the standard Kalman filtering.

It is clear that if $\hat{\mathbf{z}}_k \neq \mathbf{0}$, then, as the solution of fixed point problem (5), it nonlinearly depends on the obversion $\mathbf{y}_{k+1}$ in general. Therefore, the gain matrix $K_{k+1} = \Gamma_{k+1}(\hat{\mathbf{z}}_k)$ given by (4) is nonlinear with respect to the observation $\mathbf{y}_{k+1}$ and so does the state estimate $\hat{\mathbf{x}}_{k+1}$.

To solve the fixed point problem (5), the Picard iteration is adopted as Algorithm 2, which can always converge quickly in all our numerical simulations with a variety of situations. And the related theoretical analysis is described in the following section.

Algorithm 2: The pseudocode for solving equation (5).
Input: A tolerance ε
Output: The estimate $\hat{\mathbf{z}}_k$ and the filter gain K_{k+1}

```
1: s ← 1, z^(s) ← 0
2: repeat
3:     s ← s + 1
```

4: $\mathbf{z}^{(s)} \leftarrow G_k G_k^{\dagger} \Gamma_{k+1}(\mathbf{z}^{(s-1)})(\mathbf{y}_{k+1} - H_{k+1} F_k \hat{\mathbf{x}}_k)$
5: until $|\mathbf{z}^{(s)} - \mathbf{z}^{(s-1)}| < \varepsilon$
6: $\hat{\mathbf{z}}_k \leftarrow \mathbf{z}^{(s)}$
7: $K_{k+1} \leftarrow \Gamma_{k+1}(\hat{\mathbf{z}}_k)$

C. ITERATIVE ALGORITHM FOR UNKNOWN INPUTS

Considering that the iteration variable in Algorithm 2 is n-dimensional vector, we further pursue a more efficient algorithm in this subsection. Suppose G_k is nonsingular. For simplicity of notations, the subscripts k, representing the time instant, of all matrices and vectors aforementioned are omitted here.

To find a solution of the nonlinear equation

$$\mathbf{z} = \Gamma(\mathbf{z})\mathbf{y}, \tag{6}$$

for any given $\mathbf{y} \in \mathbb{R}^p$, we construct an iterative formula

$$\mathbf{z}_0 = \mathbf{0}_{n\times 1} \text{ and } \mathbf{z}_{s+1} = \Gamma(\mathbf{z}_s)\mathbf{y}, \quad s \in \mathbb{N}. \tag{7}$$

Notice that

$$\begin{aligned}\Gamma(\mathbf{z}) &= \left((FPF^T + Q + \mathbf{z}\mathbf{z}^T)H^T\right) \\ &\quad \cdot \left(H(FPF^T + Q + \mathbf{z}\mathbf{z}^T)H^T + R\right)^{-1} \\ &= (B + \mathbf{z}\mathbf{z}^T H^T)(A^{-1} - t(\mathbf{z})A^{-1}H\mathbf{z}\mathbf{z}^T H^T A^{-1}) \\ &= BA^{-1} + t(\mathbf{z})(I_n - BA^{-1}H)\mathbf{z}\mathbf{z}^T H^T A^{-1},\end{aligned} \tag{8}$$

where

$$\begin{aligned}A &= HCH^T + R, \\ B &= CH^T, \\ C &= FPF^T + Q, \\ t(\mathbf{z}) &= \frac{1}{1 + \mathbf{z}^T H^T A^{-1} H\mathbf{z}}.\end{aligned}$$

Define a function $\phi : \mathbb{R}^{n\times n} \to \mathbb{R}$ as follows:

$$\phi(M) = \frac{\mathbf{y}^T A^{-1} B^T M^T H^T A^{-1}\mathbf{y}}{1 + \mathbf{y}^T A^{-1} B^T M^T H^T A^{-1} HMBA^{-1}\mathbf{y}}. \tag{9}$$

Let

$$M_{s+1} = I_n + \phi(M_s)(I_n - BA^{-1}H)M_s, \quad s \in \mathbb{N}^+$$

with the initial value $M_1 = I_n$. Then, it is easy to verify

$$M_{s+1} = I_n + \sum_{t=1}^{s}\left(\prod_{i=s-t+1}^{s}\phi(M_i)\right)(I_n - BA^{-1}H)^t. \tag{10}$$

Theorem 2: The iteration (7) can be formulated as

$$\mathbf{z}_{s+1} = M_{s+1}BA^{-1}\mathbf{y}, \quad s \in \mathbb{N}.$$

Proof: See Appendix B. □

Theorem 2 gives a characterization of the iterative sequence in (7) through a matrix sequence. Next, we furthermore prove that the limit of the matrix sequence can be represented by a univariate matrix-valued function so as to be determined easily.

Notice that if the iteration (7) converges then $\phi(M_k)$ must converge to a constant in the interval $(-1, 1)$, we furthermore improve the iteration (10) so as to exactly and more efficiently find the fixed point of equation (6). Because $\phi(M_k)$ is nonnegative (see Lemma 2 in Appendix B), the following matrix-valued function

$$\begin{aligned}\Phi : [0, 1) &\to \mathbb{R}^{n\times n} \\ a &\mapsto I_n + \sum_{t=1}^{\infty} a^t (I_n - BA^{-1}H)^t\end{aligned}$$

is well-defined. It is clear that

$$\Phi(a) = \left(I_n - a(I_n - BA^{-1}H)\right)^{-1}. \tag{11}$$

Theorem 3: There exists an $a_0 \in [0, 1)$ such that $\phi(\Phi(a_0)) = a_0$ and

$$\hat{\mathbf{z}} = \Phi(a_0)BA^{-1}\mathbf{y}$$

is the solution of (6).

Proof: See Appendix C. □

It can be seen from the proof of Lemma 4 in Appendix C that a_0 can be obtained by dichotomy, which results in a more efficient computation method as follows.

Algorithm 3: The pseudocode for solving equation (6).
Input: A tolerance ε
Output: The solution $\hat{\mathbf{z}}$ of (6)
1: $l \leftarrow 0, r \leftarrow 1$
2: repeat
3: $m = (l + r)/2$
4: if $m < \phi(\Phi(m))$ then $l = m$
5: else $r = m$
6: end if
7: until $|l - r| < \varepsilon$
8: $\hat{\mathbf{z}} \leftarrow \Phi(m)BA^{-1}\mathbf{y}$

Note that Algorithm 3 has a linear convergence rate, i.e., only $\mathcal{O}(-\log\varepsilon)$ iterative steps can ensure that the iterative algorithm produces a solution around the real fixed point of nonlinear equation (6) within a given deviation ε.

D. PERFORMANCE COMPARISON

In this subsection, we theoretically compare the performance of the NEUIF and ULMVF methods for some dynamic systems that have no unknown inputs and disturbances during some period but no one knows it. Such scenarios are practical and the standard Kalman filtering should not be invoked simply owing to the possible inputs or disturbances.

Theorem 4: Let P_k^N and P_k^U be the mean squared error matrices at the time instant k of the NEUIF and ULMVF methods respectively. For any instant k, if $\mathbf{d}_0 = \cdots = \mathbf{d}_k = \mathbf{0}$, then $P_{k+1}^N \preceq P_{k+1}^U$.

Proof: See Appendix D. □

Theorem 4 shows that if there are no disturbances during the beginning period, then the NEUIF will not be worse than the ULMVF. In fact, the NEUIF is also superior than the ULMVF in many situations with disturbances. It will be presented in Section IV-A by numerical simulations. Consequently, the proposed NEUIF approach performs well no matter whether or not the inputs/disturbances are present.

III. APPLICATIONS BASED ON NEUIF APPROACH

Note that the unknown inputs considered in Section II might be the errors due to the approximation in algorithms or misspecification in model. Based on the proposed NEUIF strategy for state estimation, we suggest new approaches for nonlinear filtering and maneuvering target tracking, which are important and challenging, and have received tremendous attentions.

A. NONLINEAR FILTERING

Consider the following nonlinear filtering problem

$$\begin{aligned} \mathbf{x}_{k+1} &= f_k(\mathbf{x}_k) + \mathbf{w}_k, \\ \mathbf{y}_k &= H_k \mathbf{x}_k + \mathbf{v}_k. \end{aligned} \tag{12}$$

By linearizing the state transition function at $\hat{\mathbf{x}}_k$ and taking all high-order terms as unknown inputs, the nonlinear filter problem (12) is transformed to the following linear system with unknown inputs:

$$\begin{aligned} \mathbf{x}_{k+1} &= F_k \mathbf{x}_k + \mathbf{u}_k + \mathbf{d}_k + \mathbf{w}_k, \\ \mathbf{y}_k &= H_k \mathbf{x}_k + \mathbf{v}_k, \end{aligned} \tag{13}$$

where

$$\begin{aligned} F_k &= \frac{\partial f_k(\mathbf{x}_k)}{\partial \mathbf{x}_k}\Big|_{\hat{\mathbf{x}}_k}, \\ \mathbf{u}_k &= f_k(\hat{\mathbf{x}}_k) - F_k \hat{\mathbf{x}}_k, \\ \mathbf{d}_k &= f_k(\mathbf{x}_k) - f_k(\hat{\mathbf{x}}_k) - F_k(\mathbf{x}_k - \hat{\mathbf{x}}_k). \end{aligned}$$

Applying the state estimation procedure in Section II, we present a novel nonlinear filtering algorithm as follows.

Algorithm 4 (Nonlinear Filtering for the System (12)): Let $\hat{\mathbf{x}}_0 = \bar{\mathbf{x}}_0$ and $\hat{P}_0 = P_0$. For $k = 0, 1, \ldots$ do

1) *Estimate higher order terms.* Solve the fixed point problem

$$\hat{\mathbf{d}}_k = \Gamma_{k+1}(\hat{\mathbf{d}}_k)\tilde{\mathbf{y}}_{k+1},$$

where

$$\tilde{\mathbf{y}}_{k+1} = \mathbf{y}_{k+1} - H_{k+1}\mathbf{x}_{k+1|k}.$$

2) *Prediction.*

$$\begin{aligned} \mathbf{x}_{k+1|k} &= f_k(\hat{\mathbf{x}}_k), \\ P_{k+1|k} &= F_k \hat{P}_k F_k^T + Q_k + \hat{\mathbf{d}}_k \hat{\mathbf{d}}_k^T. \end{aligned}$$

3) *Update.*

$$\begin{aligned} \hat{\mathbf{x}}_{k+1} &= \mathbf{x}_{k+1|k} + K_{k+1}\tilde{\mathbf{y}}_{k+1}, \\ \hat{P}_{k+1} &= (I - \Gamma_{k+1}H_{k+1})P_{k+1|k}(I - \Gamma_{k+1}H_{k+1})^T \\ &\quad + \Gamma_{k+1}R_{k+1}\Gamma_{k+1}^T, \\ K_{k+1} &= P_{k+1|k}H_{k+1}^T S_{k+1}^{-1}, \\ S_{k+1} &= H_{k+1}P_{k+1|k}H_{k+1}^T + R_{k+1}. \end{aligned}$$

Note that, in Algorithm 4, the term $\mathbf{u}_k$ in (13) no longer appears because the original nonlinear function f_k is invoked in the prediction step, i.e., $\mathbf{x}_{k+1|k} = F_k\hat{\mathbf{x}}_k + \mathbf{u}_k = f_k(\hat{\mathbf{x}}_k)$. In addition, the nonlinear equation in the first step of Algorithm 4 can be solved exactly and efficiently using dichotomy given in Algorithm 3.

It is clear that the extended Kalman filter (EKF) is obtained by linearizing state transition function and ignoring higher order terms, whereas Algorithm 4 takes the existence of higher order terms into account. Obviously, the two algorithms coincide if $\hat{\mathbf{d}}_k = \mathbf{0}$ at each time instant.

B. MANEUVERING TARGET TRACKING

The IMM filter method is widely adopted in maneuvering target tracking [26]. The standard IMM method has some weaknesses. On the one hand, the real motion modes of a target cannot be obtained completely and exactly in many applications. Therefore, the transition equation cannot always characterize exactly the actual motion rule of the target. On the other hand, the real motion modes of a target might be also various, therefore, it is impossible to construct a large set of modes because of huge computational burden. Conversely, if the set of modes is too small to portray real motion of target properly, the IMM filter may performed poorly. Therefore, a balanced approach is to choose a relative small mode set, and introduce a minor disturbance on each mode which tolerates a proper deviation between the real motion of target and the selected modes.

Consider the following dynamic system with r motion modes

$$\begin{aligned} \mathbf{x}_{k+1}^j &= F_k^j \mathbf{x}_k^j + \mathbf{d}_k^j + \mathbf{w}_k^j, \quad j = 1, 2, \ldots, r, \\ \mathbf{y}_k &= H_k \mathbf{x}_k + \mathbf{v}_k, \end{aligned}$$

where the process noise $\mathbf{w}_k^j \in \mathbb{R}^n$ and observation noise $\mathbf{v}_k$ are mutually independent white noises with mean zero and known covariance matrices $Q_k^j = \mathbb{E}[\mathbf{w}_k^j(\mathbf{w}_k^j)^T] \succeq 0, j = 1, 2, \ldots, r$ and $R_k = \mathbb{E}[\mathbf{v}_k\mathbf{v}_k^T] \succ 0$ respectively.

We treat the bias between the real motion and modeled motions in the predetermined mode set as an unknown input in process equation, and integrate the NEUIF method into the IMM framework by replacing Kalman filtering in the latter by the NEUIF.

Denote $P = (p_{ij})$ as the transition probability matrix and $\mu_k = (\mu_k^i)$ the mode probability, $i, j = 1, \ldots, r$. Let $\hat{\mathbf{x}}_k^j$, K_k^j and P_k^j be the state estimate, gain matrix and state estimate covariance of the j-th mode-matched filter respectively, and $\hat{\mathbf{x}}_k$ and $\hat{P}_k$ be the combined state estimate and its covariance. We present an IMM-NEUIF algorithm as follows.

Algorithm 5 (IMM-NEUIF): For $k = 0, 1, \ldots$ do

1) *Interaction:* For all $j = 1, \ldots, r$, compute
   a) The mixing probability from mode i to mode j

$$\mu_k^{i|j} = p_{ij}\mu_k^i/\bar{c}_j, \quad i = 1, \ldots, r,$$

where

$$\bar{c}_j = \sum_{i=1}^{r} p_{ij}\mu_k^i;$$

b) The mixing state estimate of mode j

$$\hat{\mathbf{x}}_k^{0j} = \sum_{i=1}^r \hat{\mathbf{x}}_k^i \mu_k^{i|j};$$

c) The mixing covariance of state estimate of mode j

$$P_k^{0j} = \sum_{i=1}^r \mu_k^{i|j}\big(P_k^i + (\hat{\mathbf{x}}_k^i - \hat{\mathbf{x}}_k^{0j})(\hat{\mathbf{x}}_k^i - \hat{\mathbf{x}}_k^{0j})^T\big).$$

2) *NEUIF filtering:* For all $j = 1, \ldots, r$ do

a) Compute $\hat{\mathbf{d}}_k$ by solving the fixed point problem

$$\hat{\mathbf{d}}_k^j = \Gamma_{k+1}(\hat{\mathbf{d}}_k^j)(\mathbf{y}_{k+1} - H_{k+1}F_k^j\hat{\mathbf{x}}_k^{0j});$$

b) Filtering

$$\begin{aligned}
\hat{\mathbf{x}}_{k+1}^j &= F_k^j\hat{\mathbf{x}}_k^j + K_{k+1}^j(\mathbf{y}_{k+1} - H_{k+1}F_k^j\hat{\mathbf{x}}_k^{0j}),\\
K_{k+1}^j &= \big(F_k^j P_k^{0j}(F_k^j)^T + \hat{\mathbf{d}}_k^j(\hat{\mathbf{d}}_k^j)^T + Q_k^j\big)\\
&\quad \cdot H_{k+1}^T\big(H_{k+1}(F_k^j P_k^{0j}(F_k^j)^T + \hat{\mathbf{d}}_k^j(\hat{\mathbf{d}}_k^j)^T\\
&\quad + Q_k^j)H_{k+1}^T + R_{k+1}\big)^{-1},\\
P_{k+1}^j &= (I - K_{k+1}^j H_{k+1})\big(F_k^j P_k^{0j}(F_k^j)^T,\\
&\quad + \hat{\mathbf{d}}_k^j(\hat{\mathbf{d}}_k^j)^T + Q_k^j\big)(I - K_{k+1}^j H_{k+1})^T\\
&\quad + K_{k+1}^j R_{k+1}(K_{k+1}^j)^T.
\end{aligned}$$

3) *Mode probability update:* For all $j = 1, \ldots, r$ do

$$\mu_{k+1}^j = \Lambda_{k+1}^j \bar{c}_j / c,$$

where

$$\begin{aligned}
c &= \sum_{j=1}^r \Lambda_{k+1}^j \bar{c}_j,\\
\Lambda_{k+1}^j &= \frac{1}{(2\pi)^{n/2}\det(S_{k+1}^j)}\exp\Big(-\frac{1}{2}\mathbf{v}_j^T(S_{k+1}^j)^{-1}\mathbf{v}_j\Big),\\
\mathbf{v}_{k+1}^j &= \mathbf{y}_{k+1} - H_{k+1}F_k^j\hat{\mathbf{x}}_k^{0j},\\
S_{k+1}^j &= H_{k+1}\big(F_k^j P_k^{0j}(F_k^j)^T + Q_k^j\big)H_{k+1}^T + R_{k+1}.
\end{aligned}$$

4) *Combination:*

$$\begin{aligned}
\hat{\mathbf{x}}_{k+1} &= \sum_{j=1}^r \hat{\mathbf{x}}_{k+1}^j \mu_{k+1}^j,\\
\hat{P}_{k+1} &= \sum_{j=1}^r \mu_{k+1}^j\big(P_{k+1}^j + (\hat{\mathbf{x}}_{k+1}^j - \hat{\mathbf{x}}_{k+1})\\
&\qquad \times (\hat{\mathbf{x}}_{k+1}^j - \hat{\mathbf{x}}_{k+1})^T\big).
\end{aligned}$$

IV. EXAMPLES

In this section, simulation experiments are provided to demonstrate the efficiency of the proposed filtering strategy for some complicated dynamic systems.

A. FILTERING FOR LINEAR SYSTEMS WITH UNKNOWN PROCESS INPUTS

Consider the motion and observation models of an object in three-dimensional space as [27]:

$$\begin{aligned}
\mathbf{x}_{k+1} &= F_k\mathbf{x}_k + G_k\mathbf{d}_k + \mathbf{w}_k,\\
\mathbf{y}_k &= H_k\mathbf{x}_k + \mathbf{v}_k,
\end{aligned}$$

where $\mathbf{x}_k = [x_k, \dot{x}_k, \ddot{x}_k, y_k, \dot{y}_k, \ddot{y}_k, z_k, \dot{z}_k, \ddot{z}_k]^T$ with $[x_k, \dot{x}_k, \ddot{x}_k]^T$, $[y_k, \dot{y}_k, \ddot{y}_k]^T$ and $[z_k, \dot{z}_k, \ddot{z}_k]^T$ being the vectors of the positions, velocities and accelerations at the time instant k along with x, y and z axes respectively, and

$$\begin{aligned}
F_k &= \mathrm{diag}(T, T, T),\\
T &= \begin{bmatrix} 1 & \dfrac{\sin\omega t}{\omega} & \dfrac{1-\cos\omega t}{\omega^2}\\ 0 & \cos\omega t & \dfrac{\sin\omega t}{\omega}\\ 0 & -\omega\sin\omega t & \cos\omega t \end{bmatrix},\\
H_k &= \begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0 \end{bmatrix},
\end{aligned}$$

which means only the position of object can be observed at each time instant. The covariance of noise $\mathbf{w}_k$ is $Q_k = \beta\,\mathrm{diag}(\tilde{Q}, \tilde{Q}, \tilde{Q})$ in which the (i, j)-th entry of $\tilde{Q}$ is given by

$$\begin{cases} U_{i,j}, & i \neq j,\\ \alpha(|U_{i,1}| + |U_{i,2}| + |U_{i,3}|), & i = j, \end{cases}$$

and $U_{i,j}$ is the (i, j)-th entry of the following matrix U as shown at the bottom of this page. Note that $\tilde{Q}$ is a positive definite matrix with appropriate parameter α. The covariance of noise $\mathbf{v}_k$ is

$$R_k = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz}\\ \sigma_{xy} & \sigma_y^2 & \sigma_{yz}\\ \sigma_{xz} & \sigma_{yz} & \sigma_z^2 \end{bmatrix},$$

where

$$\begin{aligned}
\sigma_x^2 &= \sigma_r^2\sin^2\gamma\cos^2\eta + \sigma_\gamma^2 r^2\cos^2\gamma\cos^2\eta\\
&\quad + \sigma_\eta^2 r^2\sin^2\gamma\sin^2\eta,\\
\sigma_y^2 &= \sigma_r^2\sin^2\gamma\sin^2\eta + \sigma_\gamma^2 r^2\cos^2\gamma\sin^2\eta\\
&\quad + \sigma_\eta^2 r^2\sin^2\gamma\cos^2\eta,
\end{aligned}$$

$$U = \begin{bmatrix} \dfrac{6\omega t - 8\sin\omega t + \sin 2\omega t}{4\omega^5} & \dfrac{2\sin^4(\omega t/2)}{\omega^4} & \dfrac{-2\omega t + 4\sin\omega t - \sin 2\omega t}{4\omega^3}\\ \dfrac{2\sin^4(\omega t/2)}{\omega^4} & \dfrac{2\omega t - \sin 2\omega t}{4\omega^3} & \dfrac{\sin^2\omega t}{2\omega^2}\\ \dfrac{-2\omega t + 4\sin\omega t - \sin 2\omega t}{4\omega^3} & \dfrac{\sin^2\omega t}{2\omega^2} & \dfrac{2\omega t + \sin 2\omega t}{4\omega} \end{bmatrix}.$$

$$\begin{aligned}
\sigma_z^2 &= \sigma_r^2 \cos^2 \gamma + \sigma_\gamma^2 r^2 \sin^2 \gamma, \\
\sigma_{xy} &= \sigma_r^2 \sin^2 \gamma \sin \eta \cos \eta + \sigma_\gamma^2 r^2 \cos^2 \gamma \sin \eta \cos \eta \\
&\quad - \sigma_\eta^2 r^2 \sin^2 \gamma \sin \eta \cos \eta, \\
\sigma_{xz} &= (\sigma_r^2 - \sigma_\gamma^2 r^2) \sin \gamma \cos \gamma \cos \eta, \\
\sigma_{yz} &= (\sigma_r^2 - \sigma_\gamma^2 r^2) \sin \gamma \cos \gamma \sin \eta,
\end{aligned}$$

$(r, \gamma, \eta) \in [0, \infty) \times [0, \pi] \times [0, 2\pi)$ gives a triple of the range, elevation angle, and azimuth angle of object, and σ_r, σ_γ and σ_η are the measurement noise standard deviations of r, γ and η respectively.

Let d_k^i denote the i-th component of unknown input $\mathbf{d}_k$ at the k time instant, $i = 1, \ldots, m$. Suppose there are N change points for each component of unknown input vector. Given the tuning parameters N and σ^2, for each $i = 1, \ldots, m$, the sequence $\{d_k^i\}$ is specified as follows:

1) Set $d_0^i = 0$;
2) Generate N time points $t_1, t_2, \ldots, t_N$;
3) For $k = 1, 2, \ldots, d_k^i$ is generated from the distribution $\mathcal{N}(0, \sigma^2)$ if $k = t_j$ $(j = 1, \ldots, N)$; otherwise, $d_k^i = d_{k-1}^i$.

In the next simulations, we set $\omega = 0.5$, $t = 1$, $\alpha = 1.5$, $\sigma_r = 15$, $\sigma_\gamma = 0.002$, $\sigma_\eta = 0.002$, $N = 4$, and take the initial values

$$\bar{\mathbf{x}}_0 = [0, 1, 1, 0, 1, 1, 0, 1, 1]^T$$

and

$$P_0 = \text{diag}(1, 0.2, 0.2, 1, 0.2, 0.2, 1, 0.2, 0.2).$$

As mentioned in the introduction section, the existing methods can be roughly divided into two categories. We select the ULMVF and VBF methods to compare with the NEUIF in the root mean squared error (RMSE) sense through 100 Monte Carlo trials. The reason for choosing the ULMVF is that the filters given in [3], [15], [17], and [25] are equivalent on the same initial conditions.

1) CASE 1

Suppose

$$G_k^T = \begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.5 & 0 \end{bmatrix}.$$

Note $\text{rank}(H_{k+1}G_k) = \text{rank}(G_k) = 3$, which implies that there exists an unbiased state estimator [3] and the ULMVF has a global optimality of unbiased linear minimum-variance estimate [16]. Figure 1 gives the components and magnitude of the disturbance vector $G_k\mathbf{d}_k$, and Figure 2 provides a comparison of the RMSEs of the ULMVF, NEUIF and VBF methods. Both are with $\beta = 2$ and $\sigma = 6$. As depicted in Figure 2, the NEUIF has a smaller RMSE than the ULMVF at every time instant. This phenomenon can be explained by the well-known bias–variance tradeoff, i.e., a lower bias may lead to a higher variance [28]. In addition, the NEUIF has a superior stability to the VBF whose performance is probably affected by the norm of disturbance.

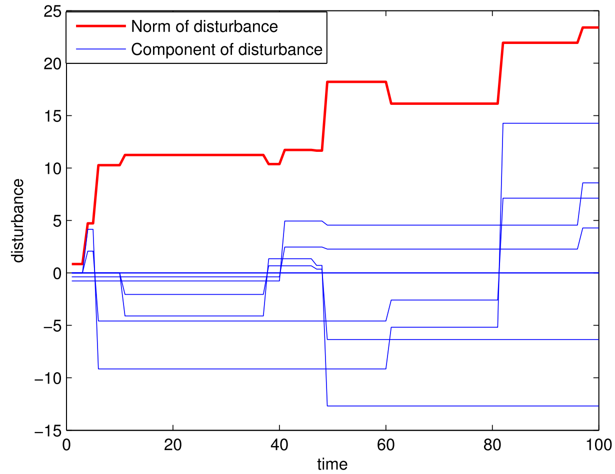

FIGURE 1. The components and magnitude of disturbance vector in Case 1.

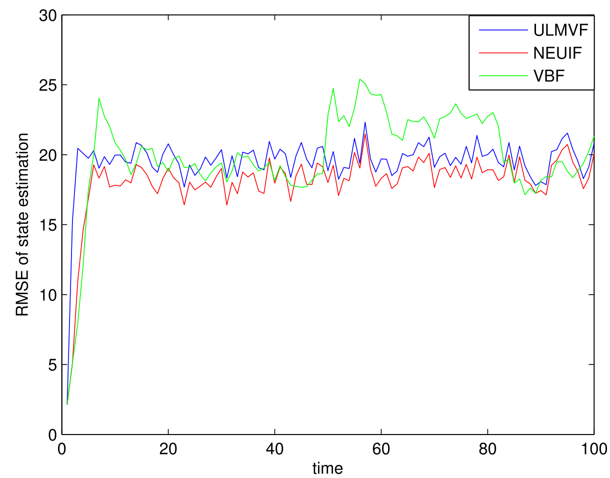

FIGURE 2. Comparison of RMSEs of ULMVF, NEUIF and VBF in Case 1.

TABLE 1. RMSEs of NEUIF with different parameter settings in Case 1.

	σ				
β	1	2	3	4	5
2	16.73	16.93	16.94	17.01	17.58
5	19.74	19.74	19.98	20.07	20.17
10	23.45	23.47	23.53	23.65	23.75

Table 1 reports the RMSEs, which are averaged by the simulation time 100, of the NEUIF with different β and σ settings, which relate to the process noise level and magnitude of disturbance vector respectively. It can be seen that the RMSE of NEUIF grows gradually with the increasing of β or σ, but the amount of change is small.

2) CASE 2

Let

$$G_k = I_9.$$

Note that there is no longer a linear unbiased estimator. The ULMVF and relevant methods perform poorly in this situation because some invertible matrices may degenerate.

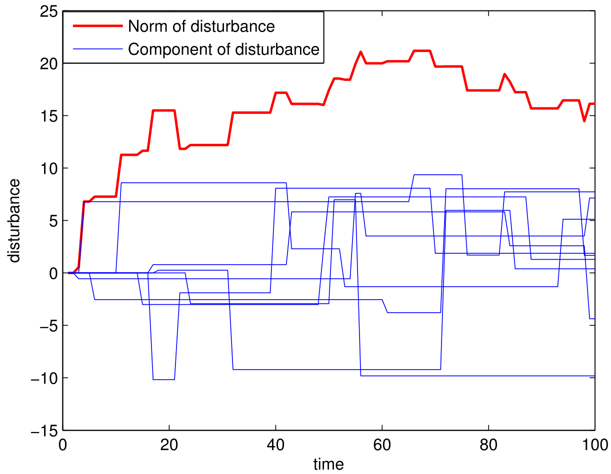

FIGURE 3. The components and magnitude of disturbance vector in Case 2.

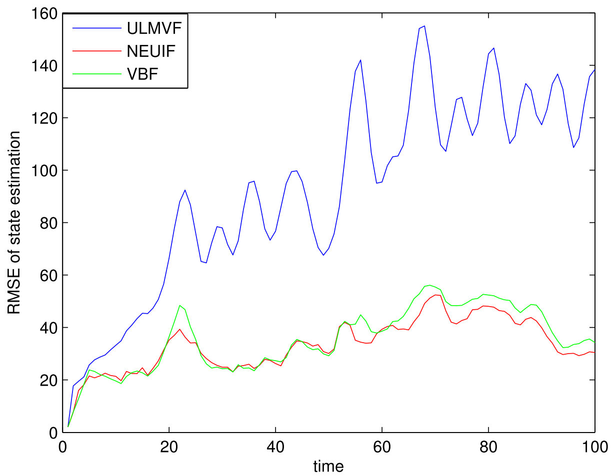

FIGURE 4. Comparison of RMSEs of ULMVF, NEUIF and VBF in Case 2.

TABLE 2. RMSEs of NEUIF with different parameter settings in Case 2.

	σ				
β	1	2	3	4	5
2	17.10	20.44	24.18	28.72	33.35
5	19.76	22.37	24.12	30.01	34.52
10	22.50	24.65	27.96	31.06	34.57

The Moore–Penrose generalized inverse instead of the inverse is thus invoked for the singular matrices in our simulations. In contrast, some filtering algorithms such as the VBF, by adding some assumptions on unknown inputs, are relatively capable to overcome this problem. Figure 3 gives the components and magnitude of the disturbance vector. Figure 4 provides a comparison of the RMSEs of the ULMVF, NEUIF and VBF methods. Both are with $\beta = 5$ and $\sigma = 5$. This simulation shows that the NEUIF outperforms the VBF at most time instants.

Table 2 reports the RMSEs of NEUIF with different β and σ settings. The variation trend of RMSE is the same as that in Case 1 with the varyings of β and σ. And the change range of RMSE is also acceptable.

In summary, from the simulation results in the above two scenarios, we can conclude that the NEUIF shares the stability of the ULMVF and the applicability of the VBF.

B. NONLINEAR FILTERING

In this subsection, we provide a comparison of the novel nonlinear filtering based on the proposed NEUIF strategy with the EKF [24], [29] and unscented Kalman filter (UKF) [30].

As [11], consider a one-dimensional dynamic system with the nonlinear motion model

$$x_{k+1} = \frac{x_k}{2} + \frac{25x_k}{1+x_k^2} + 8\cos(1.2k) + w_k, \tag{14}$$

and the linear observation model

$$y_{k+1} = H_{k+1}x_{k+1} + v_{k+1}, \tag{15}$$

where $H_k = 1$, $Q_k = 2.5^2$, $R_k = 1$ and the initial values are taken as $x_0 = 10$, $P_0 = 2.5^2$.

Figure 5 illustrates the comparison of MSEs of the EKF, UKF and NEUIF approaches. It can be seen that the NEUIF performs quite well.

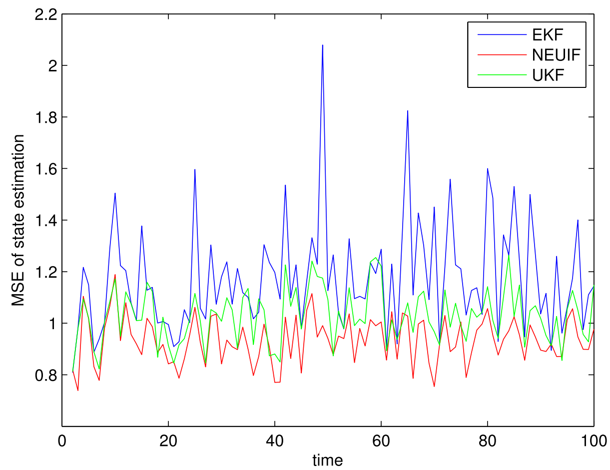

FIGURE 5. Comparison of MSEs of EKF, NEUIF and UKF.

C. TRACKING OF MANEUVERING TARGET

In this subsection, we consider the tracking problem of a maneuvering target in two-dimensional space considered in [31] and [32].

As depicted in Figure 6, the target starts from (29320m, 34820m) at the time instant $t = 0$s with velocity of 330m/s, and implements a variable accelerated motion along direction $\mathbf{v} = (-1/\sqrt{2}, -1/\sqrt{2})$ for 10s with acceleration of $\frac{330\times0.8\pi}{10}\cos(\frac{\pi t}{10})$m/s^2. Then it makes a counterclockwise turning with acceleration of 3g for 12s, before 4s constant velocity moves. Subsequently, a clockwise turning with acceleration of 3g for 12s is taken. Finally, it moves with acceleration of $\frac{330\times0.5\pi}{12}\cos(\frac{\pi t}{12})$m/s^2 for 12s. The radar is located at the original point with range and bearing measurement noise standard deviations of $\sigma_r = 15$m and $\sigma_\varepsilon = 0.002$rad respectively. Suppose the measurements are available at a sampling interval of $T = 1$s.

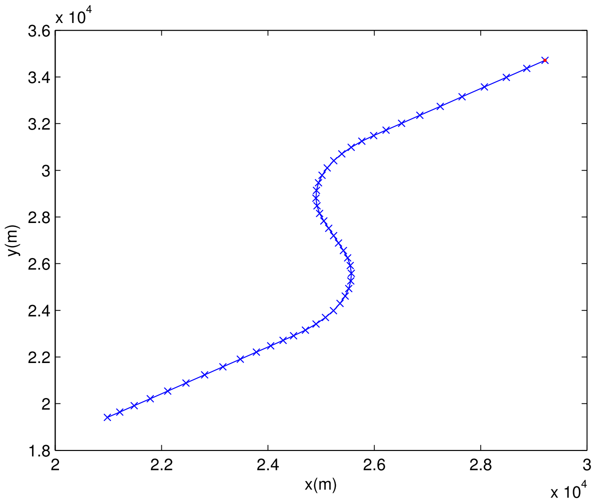

FIGURE 6. **Trajectory of target and snapshots at every 1 second.**

To show the performance of the proposed filter, as [32], we assume that the target maneuvers with two motion modes. Here, the unknown inputs in the transition equation are employed. It is noteworthy that the unknown inputs may be the actual disturbances or the errors owing to the motion mode misspecification.

1) MOTION MODE I

The first motion mode is

$$\mathbf{x}_{k+1}^{(1)} = F_k^{(1)}\mathbf{x}_k^{(1)} + \mathbf{d}_k^{(1)} + \mathbf{w}_k^{(1)},$$

where

$$\mathbf{x}_k^{(1)} = [x_k, \dot{x}_k, y_k, \dot{y}_k]^T,$$
$$F_k^{(1)} = \begin{bmatrix} F^{(1)} & 0 \\ 0 & F^{(1)} \end{bmatrix} \text{with } F^{(1)} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix},$$

and $\mathbf{w}_k^{(1)}$ is white noise process with mean zero and covariance

$$Q_k^{(1)} = \sigma_1^2 \begin{bmatrix} Q^{(1)} & 0 \\ 0 & Q^{(1)} \end{bmatrix}$$

with

$$Q^{(1)} = \begin{bmatrix} \frac{1}{3}T^3 & \frac{1}{2}T^2 \\ \frac{1}{2}T^2 & T \end{bmatrix} \text{and } \sigma_1^2 = 0.25\text{m}^2/\text{s}^3.$$

2) MOTION MODE II

The second motion mode is

$$\mathbf{x}_{k+1}^{(2)} = F_k^{(2)}\mathbf{x}_k^{(2)} + \mathbf{d}_k^{(2)} + \mathbf{w}_k^{(2)},$$

where

$$\mathbf{x}_k^{(2)} = [x_k, \dot{x}_k, \ddot{x}_k, y_k, \dot{y}_k, \ddot{y}_k]^T,$$
$$F_k^{(2)} = \begin{bmatrix} F^{(2)} & 0 \\ 0 & F^{(2)} \end{bmatrix} \text{with } F^{(2)} = \begin{bmatrix} 1 & T & \frac{1}{2}T^2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix},$$

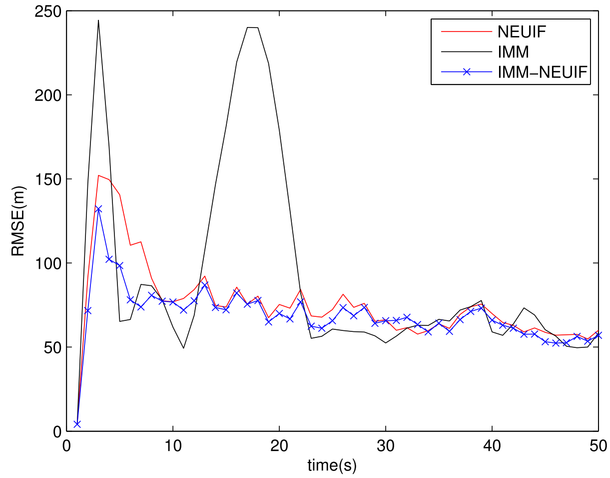

FIGURE 7. **Comparison of RMSEs of target position estimates.**

and $\mathbf{w}_k^{(2)}$ is white noise process with mean zero and covariance

$$Q_k^{(2)} = \sigma_2^2 \begin{bmatrix} Q^{(2)} & 0 \\ 0 & Q^{(2)} \end{bmatrix}$$

with

$$Q^{(2)} = \begin{bmatrix} \frac{1}{20}T^5 & \frac{1}{8}T^4 & \frac{1}{6}T^3 \\ \frac{1}{8}T^4 & \frac{1}{3}T^3 & \frac{1}{2}T^2 \\ \frac{1}{6}T^3 & \frac{1}{2}T^2 & T \end{bmatrix} \text{and } \sigma_2^2 = 9\text{m}^2/\text{s}^5.$$

The transition probability matrix and the initial mode probability for IMM filter and IMM-NEUIF are set respectively as

$$P = \begin{bmatrix} 0.95 & 0.05 \\ 0.1 & 0.9 \end{bmatrix} \text{and } \mu_1 = (0.8, 0.2).$$

For the aforementioned motion modes of maneuvering target, we consider the following two measurement models as [33]:

$$\mathbf{y}_k^{(i)} = H_k^{(i)}\mathbf{x}_k^{(i)} + \mathbf{v}_k, \quad i = 1, 2,$$

where

$$H_k^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$
$$H_k^{(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$\mathbf{v}_k$ is white noise process with mean zero and covariance

$$R_k = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix},$$

where

$$\sigma_x^2 = \sigma_r^2 \cos^2\gamma + r^2\sigma_\gamma^2 \sin^2\gamma,$$
$$\sigma_y^2 = \sigma_r^2 \sin^2\gamma + r^2\sigma_\gamma^2 \cos^2\gamma,$$
$$\sigma_{xy} = (\sigma_r^2 - r^2\sigma_\gamma^2)\sin\gamma\cos\gamma,$$

and r and γ denote the range and elevation angle respectively.

The RMSEs of NEUIF, IMM filter and IMM-NEUIF through 100 Monte Carlo trials are depicted in Figure 7. Here, the IMM filter is invoked by taking no misspecified motion mode and completely ignoring the unknown inputs, and the NEUIF is evaluated only for the first motion mode for simplicity.

It is well known that the IMM filter is effective in maneuvering target tracking if the motion modes are suitably specified. However, while the actual motion of target cannot be matched precisely or approximated properly by combining the modes in the predetermined mode set, the IMM filter may perform somewhat poorly. This phenomenon can be seen clearly by the RMSE of IMM filter during the time interval [12s, 22s] in which the selected mode differs greatly from the real motion mode. Meanwhile, Figure 7 shows that the proposed NEUIF and IMM-NEUIF can improve the performance to some extent, and the IMM-NEUIF outperforms the NEUIF in most situations.

V. CONCLUSIONS

This paper addresses the state estimation for discrete-time stochastic time-varying linear systems in the presence of unknown process inputs or disturbances. Without the unbiasedness assumption on the state estimate as in the existing literatures, the proposed estimator has general applicability and good performance.

The key technique in this paper is that we introduce a nonlinear equation to estimate the unknown inputs and develop an efficient iterative algorithm for solving the fixed point of this nonlinear equation. A theoretical analysis on the performance of the proposed filter is presented. Using the proposed state estimation algorithm, we propose the novel approaches for filtering of nonlinear systems and tracking of maneuvering targets.

Numerical simulations, including the state estimation for linear system with unknown process inputs, nonlinear filtering and maneuvering target tracking, demonstrate the effectiveness of the proposed filtering method for uncertainties in complicated dynamic systems.

The proposed state estimation approach can be further applied to more complicated dynamic systems such as environmental monitoring and fault-tolerant control.

APPENDIX

A. PROOF OF THEOREM 1

It is clear that minimizing $\mathrm{tr}P_k$ is equivalent to minimizing the MSE of the state estimate $\hat{\mathbf{x}}_{k+1}$.

By simple deduction, we have

$$\begin{aligned}\frac{\partial \mathrm{tr}P_{k+1}}{\partial K_{k+1}} &= -2\big(F_kP_kF_k^T + (G_k\mathbf{d}_k)(G_k\mathbf{d}_k)^T \\ &\quad + Q_k - E_k\big)H_{k+1}^T \\ &\quad + 2K_{k+1}H_{k+1}\big(F_kP_kF_k^T + (G_k\mathbf{d}_k)(G_k\mathbf{d}_k)^T \\ &\quad + Q_k - E_k\big)H_{k+1}^T \\ &\quad + 2K_{k+1}R_{k+1}. \end{aligned} \tag{16}$$

The fact that

$$\begin{aligned}F_kP_kF_k^T &+ (G_k\mathbf{d}_k)(G_k\mathbf{d}_k)^T - E_k \\ &= \mathbb{E}[(F_k\mathbf{e}_k - G_k\mathbf{d}_k)(F_k\mathbf{e}_k - G_k\mathbf{d}_k)^T]\end{aligned}$$

is positive semi-definite and R_{k+1} is positive definite implies

$$H_{k+1}\big(F_kP_kF_k^T + (G_k\mathbf{d}_k)(G_k\mathbf{d}_k)^T + Q_k - E_k\big)H_{k+1}^T + R_{k+1}$$

is non-singular. Letting the right hand side of (16) be equal to zero and solving such equation, we obtain (3).

B. PROOF OF THEOREM 2

The proof of this theorem needs the following two lemmas.

Lemma 1: If H is nonsingular, then $\rho(I_n - BA^{-1}H) < 1$; and if H is singular but full row rank, then $\rho(I_n - BA^{-1}H) = 1$.

Proof: Notice that $BA^{-1}H = CH^T(HCH^T + R)^{-1}H$ has the same characteristic roots as the positive semi-definite matrix $D = C^{\frac{1}{2}} H^T(HCH^T + R)^{-1}HC^{\frac{1}{2}}$. Furthermore,

$$\begin{aligned} D &= D^2 + C^{\frac{1}{2}} H^T(HCH^T + R)^{-1}R(HCH^T + R)^{-1}HC^{\frac{1}{2}} \\ &\succeq D^2. \end{aligned}$$

It is easy to verify, from the Rayleigh–Ritz theorem [34], that $\rho(D) \geq \rho(D^2)$, and then $\rho(BA^{-1}H) = \rho(D) \in (0, 1]$. This lemma thus follows. □

Lemma 2: For any $s \in \mathbb{N}$, HM_sB is positive definite. Thus, $\phi(M_s) \geq 0$.

Proof: It is clear that $HM_1\,B = HCH^T$ is positive definite.

For any $t = 1, \ldots, s$, let $J_t = (I_n - BA^{-1}H)^tC$, then from

$$(I_n - CH^TA^{-1}H)^{t-1}C^{\frac{1}{2}} = C^{\frac{1}{2}}(I_n - C^{\frac{1}{2}}\,H^TA^{-1}HC^{\frac{1}{2}})^{t-1},$$

we have

$$\begin{aligned} J_t &= (I_n - CH^TA^{-1}H)^tC \\ &= (I_n - CH^TA^{-1}H)^{t-1}C^{\frac{1}{2}}(I_n - C^{\frac{1}{2}}\,H^TA^{-1}HC^{\frac{1}{2}})C^{\frac{1}{2}} \\ &= C^{\frac{1}{2}}(I_n - C^{\frac{1}{2}}\,H^TA^{-1}HC^{\frac{1}{2}})^tC^{\frac{1}{2}} \\ &= C^{\frac{1}{2}}(I_n - C^{\frac{1}{2}}\,H^T(HCH^T + R)^{-1}HC^{\frac{1}{2}})^tC^{\frac{1}{2}} \\ &= C^{\frac{1}{2}}(I_n + C^{\frac{1}{2}}\,H^TR^{-1}HC^{\frac{1}{2}})^{-t}C^{\frac{1}{2}}. \end{aligned}$$

Therefore, from (10), we conclude that

$$HM_{s+1}B = HCH^T + \sum_{t=1}^{s}\left(\prod_{i=s-t+1}^{s}\phi(M_i)\right)HJ_tH^T$$

is positive definite, and then $\phi(M_{s+1}) \geq 0$ from the definition of the function $\phi(\cdot)$ given by (9). □

Now, we provide the proof of Theorem 2. Obviously,

$$\mathbf{z}_1 = \Gamma(\mathbf{z}_0)\mathbf{y} = \Gamma(\mathbf{0})\mathbf{y} = M_1\,BA^{-1}\mathbf{y}.$$

Suppose that for any given $s \in \mathbb{N}^+$, $\mathbf{z}_s = M_sBA^{-1}\mathbf{y}$, then

$$\begin{aligned} \mathbf{z}_{s+1} &= \Gamma(\mathbf{z}_s)\mathbf{y} \\ &= BA^{-1}\mathbf{y} + t(\mathbf{z}_s)(I_n - BA^{-1}H)\mathbf{z}_s\mathbf{z}_s^TH^TA^{-1}\mathbf{y} \\ &= BA^{-1}\mathbf{y} + t(\mathbf{z}_s)(I_n - BA^{-1}H) \\ &\quad \cdot M_sBA^{-1}\mathbf{y}\mathbf{y}^TA^{-1}B^TM_s^TH^TA^{-1}\mathbf{y} \end{aligned}$$

$$\begin{aligned}&= \big(I_n + t(\mathbf{z}_s)\mathbf{y}^T A^{-1} B^T M_s^T H^T A^{-1}\mathbf{y} \\ &\quad \cdot (I_n - BA^{-1}H)M_s\big)BA^{-1}\mathbf{y} \\ &= M_{s+1}BA^{-1}\mathbf{y}.\end{aligned}$$

This theorem thus follows using mathematical induction.

C. PROOF OF THEOREM 3

Firstly, we provide two lemmas.

Lemma 3: For any $a \in [0, 1)$, we have

$$\lim_{a\to 1^-} H\Phi(a)B = A. \tag{17}$$

Proof: From (11), we can derive

$$\begin{aligned}H\Phi(a)B &= HC^{\frac12}\big(I_n - a(I_n - C^{\frac12} H^T A^{-1}HC^{\frac12})\big)^{-1}C^{\frac12} H^T \\ &= HC^{\frac12}\big(I_n - a(I_n + C^{\frac12} H^T R^{-1}HC^{\frac12})^{-1}\big)^{-1} \\ &\quad \cdot C^{\frac12} H^T \\ &= HC^{\frac12}\big(I_n + a\big(I_n + C^{\frac12} H^T R^{-1}HC^{\frac12} - aI_n\big)^{-1}\big) \\ &\quad \cdot C^{\frac12} H^T \\ &= HCH^T + aHC^{\frac12}\big((1-a)I_n \\ &\quad + C^{\frac12} H^T R^{-1}HC^{\frac12}\big)^{-1}C^{\frac12} H^T \\ &= HCH^T + \frac{a}{1-a}\big(HCH^T \\ &\quad - HCH^T((1-a)R + HCH^T)^{-1}HCH^T\big) \\ &= HCH^T + \frac{a}{1-a}\Big((HCH^T)^{-1} + \frac{1}{1-a}R^{-1}\Big)^{-1} \\ &= HCH^T + a\big((1-a)(HCH^T)^{-1} + R^{-1}\big)^{-1}.\end{aligned}$$

The continuousness of the above with respect to a implies $\lim_{a\to 1^-} H\Phi(a)B = HCH^T + R = A$. □

Lemma 4: There exists an $a_0 \in [0, 1)$ such that $\phi(\Phi(a_0)) = a_0$.

Proof: If $\mathbf{y} = \mathbf{0}_{p\times 1}$, then the result is obvious. Assume that $\mathbf{y} \neq \mathbf{0}_{p\times 1}$, from the definition of ϕ give by (9), the function

$$\phi(\Phi(a)) = \frac{\mathbf{y}^T A^{-1}B^T\Phi(a)^T H^T A^{-1}\mathbf{y}}{1 + \mathbf{y}^T A^{-1}B^T\Phi(a)^T H^T A^{-1}H\Phi(a)BA^{-1}\mathbf{y}}$$

is continuous with respect to a.

It is clear that

$$\phi(\Phi(0)) = \frac{\mathbf{y}^T A^{-1}HCH^T A^{-1}\mathbf{y}}{1 + \mathbf{y}^T A^{-1}HCH^T A^{-1}HCH^T A^{-1}\mathbf{y}} > 0.$$

From equation (17) and the positive definiteness of A, we have

$$\lim_{a\to 1^-}\phi(\Phi(a)) = \frac{\mathbf{y}^T A^{-1}\mathbf{y}}{1 + \mathbf{y}^T A^{-1}\mathbf{y}} < 1.$$

Therefore, the equation $(\phi \circ \Phi)(a) = \phi(\Phi(a)) = a$ has a fixed point $a_0 \in (0, 1)$. □

Secondly, we give the proof of this theorem.

Taking a_0 in Lemma 4 such that

$$\phi(\Phi(a_0)) = a_0 \text{ and } \Phi(a_0)^{-1}\hat{\mathbf{z}} = BA^{-1}\mathbf{y}.$$

From the definition of function $\phi(\cdot)$ given by (9), we have

$$\phi(\Phi(a_0)) = \frac{\hat{\mathbf{z}}^T H^T A^{-1}\mathbf{y}}{1 + \hat{\mathbf{z}}^T H^T A^{-1}H\hat{\mathbf{z}}} = t(\hat{\mathbf{z}})\hat{\mathbf{z}}^T H^T A^{-1}\mathbf{y},$$

and then from (8) and (11),

$$\begin{aligned}\Gamma(\hat{\mathbf{z}})\mathbf{y} &= \Phi(a_0)^{-1}\hat{\mathbf{z}} + t(\hat{\mathbf{z}})(I_n - BA^{-1}H)\hat{\mathbf{z}}\cdot\hat{\mathbf{z}}^T H^T A^{-1}\mathbf{y} \\ &= \Phi(a_0)^{-1}\hat{\mathbf{z}} + \phi(\Phi(a_0))(I_n - BA^{-1}H)\hat{\mathbf{z}} \\ &= \Phi(a_0)^{-1}\hat{\mathbf{z}} + a_0(I_n - BA^{-1}H)\hat{\mathbf{z}} \\ &= \hat{\mathbf{z}}.\end{aligned}$$

D. PROOF OF THEOREM 4

Lemma 5: Suppose the matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{p\times p}$ are positive semi-definite and positive definite respectively, and $H \in \mathbb{R}^{p\times n}$. Let

$$g(X) = (I - XH)A(I - XH)^T + XBX^T, \quad X \in \mathbb{R}^{n\times p},$$

then the following two optimization problems

$$\min_{X\in\mathbb{R}^{n\times p}} \operatorname{tr}(g(X)) \quad \text{and} \quad \min_{X\in\mathbb{R}^{n\times p}} g(X)$$

are equivalent and share the same optimal solution.

Proof: Note $HAH^T + B$ is positive definite. From

$$\begin{aligned}g(X) &= X(HAH^T + B)X^T - XHA - AH^T X^T + A \\ &= \Big(X(HAH^T + B) - AH^T\Big)(HAH^T + B)^{-1} \\ &\quad \cdot \Big(X(HAH^T + B) - AH^T\Big)^T \\ &\quad + A - AH^T(HAH^T + B)^{-1}HA,\end{aligned}$$

we immediately know that $X = AH^T(HAH^T + B)^{-1}$ minimizes $g(X)$. The existence of optimal solution of optimization problem with matrix-valued objective function implies the equivalence of two problems $\min \operatorname{tr}(g(X))$ and $\min g(X)$. □

Next, we give the proof of this theorem.

Obviously, $\hat{P}_0^N = P_0^N = P_0^U = P_0$. For the time instant k, assume that

$$P_k^N \preceq \hat{P}_k^N \preceq P_k^U,$$

where $\hat{P}_k^N$ is an estimate of P_k^N in Algorithm 1. It is to see

$$\begin{aligned}P_{k+1}^N &= (I - K_{k+1}H_{k+1})\Big(F_k P_k^N F_k^T + Q_k\Big) \\ &\quad \cdot (I - K_{k+1}H_{k+1})^T + K_{k+1}R_{k+1}K_{k+1}^T \\ &\preceq (I - K_{k+1}H_{k+1})(F_k\hat{P}_k^N F_k^T + \hat{\mathbf{z}}_k\hat{\mathbf{z}}_k^T + Q_k) \\ &\quad \cdot (I - K_{k+1}H_{k+1})^T + K_{k+1}R_{k+1}K_{k+1}^T \\ &= \hat{P}_{k+1}^N.\end{aligned}$$

Furthermore, denote $\mathcal{K}_{k+1} = \{K_{k+1} \in \mathbb{R}^{n\times p} : K_{k+1}H_{k+1}G_k = G_k\}$. The fact that the estimate $\hat{\mathbf{z}}_k$ given by (5) is a vector in the column space of G_k implies $(I - K_{k+1}H_{k+1})^T\hat{\mathbf{z}}_k = \mathbf{0}$ for any $K_{k+1} \in \mathcal{K}_{k+1}$. Therefore,

$$\begin{aligned}&\operatorname{tr}P_{k+1}^U \\ &= \min_{K_{k+1}\in\mathcal{K}_{k+1}} \operatorname{tr}\big((I - K_{k+1}H_{k+1})(F_k P_k^U F_k^T + Q_k) \\ &\quad \cdot (I - K_{k+1}H_{k+1})^T + K_{k+1}R_{k+1}K_{k+1}^T\big)\end{aligned}$$

$$
\begin{aligned}
&= \min_{K_{k+1}\in\mathcal{K}_{k+1}} \operatorname{tr}\big((I - K_{k+1}H_{k+1})(F_k P_k^U F_k^T + \hat{\mathbf{z}}_k\hat{\mathbf{z}}_k^T + Q_k) \\
&\quad \cdot (I - K_{k+1}H_{k+1})^T + K_{k+1}R_{k+1}K_{k+1}^T\big) \\
&\geq \min_{K_{k+1}\in\mathbb{R}^{n\times p}} \operatorname{tr}\big((I - K_{k+1}H_{k+1})(F_k P_k^U F_k^T + \hat{\mathbf{z}}_k\hat{\mathbf{z}}_k^T + Q_k) \\
&\quad \cdot (I - K_{k+1}H_{k+1})^T + K_{k+1}R_{k+1}K_{k+1}^T\big) \\
&\geq \min_{K_{k+1}\in\mathbb{R}^{n\times p}} \operatorname{tr}\big((I - K_{k+1}H_{k+1})(F_k P_k^N F_k^T + \hat{\mathbf{z}}_k\hat{\mathbf{z}}_k^T + Q_k) \\
&\quad \cdot (I - K_{k+1}H_{k+1})^T + K_{k+1}R_{k+1}K_{k+1}^T\big) \\
&= \operatorname{tr}\hat{P}_{k+1}^N.
\end{aligned}
$$

Applying Lemma 5 twice, we have $\hat{P}_{k+1}^N \preceq P_{k+1}^U$. In result,

$$P_{k+1}^N \preceq \hat{P}_{k+1}^N \preceq P_{k+1}^U.$$

By mathematical induction, we complete the proof.

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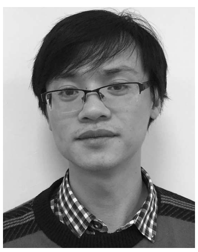

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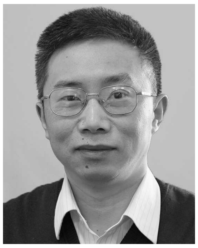

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