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Smooth Second Order Sliding Mode Control of a Class of Underactuated Mechanical Systems

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ABSTRACT This paper investigates a framework for the application of robust and smooth second order sliding mode control to a class of underactuated mechanical systems for the realization of high performance control applications. First, using input and state transformations, the dynamics of the class are transformed into a normal form which consists of a set of reduced order nonlinear subsystems and a set of reduced order linear subsystems. Then we present nonlinear sliding manifold and sliding mode control for the reduced order nonlinear subsystem known as the *Lagrangian zero dynamics* such that stability of the overall system is guaranteed. The control design procedure is illustrated for the Furuta Pendulum, the Overhead Crane, and the Beam-and-Ball system. Numerical simulations verify the effectiveness of the proposed framework. Additionally, we design swingup control law for the Furuta Pendulum to overcome the limitation of the sliding mode control law and achieve global stabilization in the presence of external disturbance.

INDEX TERMS Nonlinear systems, second order sliding mode control, swingup control, underactuated mechanical systems.

I. INTRODUCTION

The last two decades have seen a great deal of interest and active research in the control and analysis of underactuated mechanical systems. Main reasons that have contributed to this interest and research are: i) increased usefulness of these systems in applications of practical importance such as robotics, mechatronics, industry, and aerospace and marine systems; ii) development and inventions of new types of underactuated mechanical systems such as flexible link robots and miniaturized robots for various purposes; iii) a pursuit for the ever demanding benefits such as reduction of cost, weight, complexity, and power consumption; iv) research in nonlinear control theory, many of these systems are used as benchmark nonlinear systems for comparing and evaluating different control design techniques, for examples, the TORA system [1], [2], the Beam-and-Ball system [3], and the Cart-Pole system [4] are the subjects of excellent research works.

Underactuation, i.e., less number of control inputs than the number of degrees to be controlled, causes great difficulties in the realization of high performance control design for these systems. The reason is that the matured and well established nonlinear control techniques, for example, feedback linearization [5] and in many cases direct backstepping, cannot be applied to these systems. Due to underactuation, only partial feedback linearization (PFL) [6], [7] is possible which leave the control design problem still complicated due to control coupling in the actuated and unactuated parts of the dynamics. Normal forms [8] were developed to decouple the actuated and unactuated parts of the dynamics in the control. In most works on underactuated mechanical systems, for example, [4], [9], PFL and normal forms have been used as initial simplifying techniques. Apart from the traditional energy based methods such as [10], methods based on the Lagrangian/Hamiltonian properties of the mechanical system are developed to address the control design problem for underactuated mechanical systems in general, for example IDA-PBC [11] and controlled Lagrangian [12], [13].

Approximate input-output linearization [3] is an other technique applicable to such systems such as the Beam-and-Ball. Higher order compensating sliding mode control (HOCSMC) [14] is based on the concepts of [3]. In [14], a linear sliding manifold is used neglecting the higher order terms which are then compensated in the control law. We will compare our results to this method for the Beam-and-Ball case.

Due to the complex dynamics of underactuated mechanical systems and the uniqueness of the required design approach, in most cases, a system by system control design approach is used, for example, the Furuta Pendulum [15]–[19], the Overhead Crane [20]–[26], and the Beam-and-Ball system [3], [14], [27]. In some excellent works, a class based general control design approach is used that encompasses many systems in a specific class, for example, equivalent-input disturbance [4], energy based [10], IDA-PBC [11], controlled Lagrangian [12], [13], sliding mode [9], [28]–[30], terminal sliding mode [31], dynamic surface control [32], output feedback stabilization [33] and adaptive [34]. The present work is a contribution to the general approaches and applies to a class of systems.

Control design for underactuated systems is also important from actuator/sensor fault point of view. Actuator failure in a fully actuated system renders the system underactuated and failure in an already unactuated system increases the degree of underactuation and the system can be considered as an underactuated system. Similarly, sensor fault causes no or delayed feedback signal and hence leads to ineffective control. Detecting and isolating such faults and designing fault tolerant control laws is an important area of research [35], [36] and sliding mode techniques can effectively address this issue [37]–[40].

Mismatch between a real system and its mathematical model on which control synthesis is based always gives rise to uncertainties. Furthermore, there are always unanticipated and unknown external disturbances in real world applications. Designing control laws that are robust to internal model uncertainties and unknown external disturbances has been the top most priority and ever demanding objective in high performance control applications. Sliding mode control (SMC) techniques are well known for robustness against internal uncertainties and external disturbances. The other important ones are H_∞ which are well known for robustness against external noise and disturbances. In some recent applications [41]–[43], a combination of the two is used for improved performance.

Recent developments in SMC techniques have led to their vast applications in high performance control applications. Capability to control complex nonlinear systems and robustness to uncertainties and external disturbance are the two main reasons making SMC a first choice for the control of nonlinear uncertain systems. Fast response, simple design, and order reduction are extra desirable features. However, the simple two steps SMC design procedure [28] of: i) choosing a *stable linear sliding manifold*, and ii) finding a *control law* to enforce sliding mode in the manifold along system dynamics, is not directly applicable to underactuated mechanical systems. The dynamics of underactuated mechanical systems are complex and a successful application of SMC needs some novel approach.

The application of standard SMC to underactuated mechanical systems was first considered in the most important works of [28] and then in [9]. In [9], two different sliding surfaces were defined depending on whether the relative degree of the system is 1 or 2 and, accordingly, standard SMC laws were selected to enforce sliding mode in the manifold. In essence, the two sliding surfaces defined in [9] are state space equivalent to the sliding surfaces defined in [28]. The two sliding surface approach complicates the design procedure and the standard SMC laws used in [9] suffer from undesired chattering and are not suitable for practical control applications, especially, involving mechanical control systems. In the proposed control design framework, a single sliding surface is used that makes the control design simple. Furthermore, the selected control laws used to enforce sliding mode in the manifold are second order sliding mode that are both smooth and robust. The need of a novel control design framework for the successful application of robust and smooth second order sliding mode control to underactuated mechanical systems using a single sliding manifold provides a strong basis for this paper.

With the motivation mentioned just above, in this article, we consider smooth second order sliding mode control of a class of underactuated mechanical systems. The dynamics of the class are first transformed into a *cascade nontriangular quadratic normal form* [7]. The resulting form consists of a reduced order nonlinear subsystem and a linear subsystem. The proposed framework greatly simplifies control design for systems in the nontriangular quadratic normal form what would otherwise be quite challenging. We design nonlinear sliding manifold and sliding mode control for the reduced order nonlinear subsystem known as the *Lagrangian zero dynamics* such that stability of the overall system is also guaranteed. Additionally, swingup control law is designed in the case of Furuta Pendulum. The designed swingup and sliding mode control law achieve global stabilization of the Furuta Pendulum from the downward stable equilibrium position to the upward unstable equilibrium position in the presence of external disturbance. Illustrative design examples and numerical simulations of the Furuta Pendulum, the Overhead Crane and the Beam-and-Ball system verify the effectiveness of the proposed control design framework.

Organization of the rest of the article is as follows. Problem formulation is presented in Section II and the proposed control design framework in Section III. Section IV presents examples illustrating the design procedure. Simulation results are discussed in Section V and Section VI concludes the paper.

II. PROBLEM FORMULATION

The equations of motion of a mechanical control system with n degrees of freedom are given as:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = F(q)(\tau + d(q, \dot{q}, t)) \tag{1}$$

where $q \in \Re^n$ is the generalized configuration vector, $M(q) \in \Re^{n\times n}$ is the positive definite symmetric inertia matrix, $C(q, \dot{q}) \in \Re^{n\times n}$ contains Coriolis and centrifugal terms, $G(q) \in \Re^{n\times 1}$ contains the gravitational terms, $F(q) \in \Re^{n\times p}$ is the control input matrix, $\tau \in \Re^p$ is the control input vector and $d(q, \dot{q}, t) \in \Re^p$ represents the matched uncertainties. The case, $p = rank(F) = n$, represents fully

TABLE 1. Parameters in Eq. (2) for the design examples.

System	m_{11}	$m_{12}=m_{21}$	m_{22}	c_1	g_1	c_2	g_2
FP	$J_1+m_2(L_1^2+\ell_2^2S_2^2)$	$m_2L_1\ell_2C_2$	$I_2+m_2\ell_2^2$	$2m_2\ell_2^2S_2C_2\dot{q}_1\dot{q}_2-m_2\ell_2L_1S_2\dot{q}_2^2$	0	$-m_2\ell_2^2S_2C_2\dot{q}_1^2$	$-\; m_2\ell_2gS_2$
OC	$M+m$	mLC_2	mL^2	$-mLS_2\dot{q}_2^2$	0	0	$mLgS_2$
BB	$I_1+mq_2^2$	0	$m(1+\frac{I_2}{mr^2})$	$2m\dot{q}_1q_2\dot{q}_2$	mgq_2C_1	$-mq_2\dot{q}_1^2$	mgS_1

$C_i := \cos(q_i), S_i := \sin(q_i), i=1,2, J_1 = I_1 + m_1\ell_1^2, m_0 = m_1l_1 + m_2L_1$
FP := Furuta Pendulum, OC := Overhead Crane, BB := Beam-and-Ball

actuated system and the case, $p = rank(F) < n$, represents underactuated system.

We consider a class of systems described by (1) with $M(q_2)$ and $F(q) = [I_p, 0]^T$. Partitioning the configuration vector $q \in \Re^n$ into actuated $q_1 \in \Re^p$ and unactuated $q_2 \in \Re^{n-p}$ configuration vectors, the nominal dynamics in (1) take the form:

$$m_{11}(q_2)\ddot{q}_1 + m_{12}(q_2)\ddot{q}_2 + c_1(q,\dot{q}) + g_1(q_1,q_2) = \tau \quad (2a)$$
$$m_{21}(q_2)\ddot{q}_1 + m_{22}(q_2)\ddot{q}_2 + c_2(q,\dot{q}) + g_2(q_1,q_2) = 0 \quad (2b)$$

Remark 1: The Furuta Pendulum, the Overhead Crane, the Beam-and-Ball system, the Cart-Pole system, and the Pendubot are well known benchmark underactuated systems described by (2) *with* $n = 2$, $p = 1$.

In general, the dynamics in (2) are a set of n coupled second order nonlinear subsystems. The difficulty of the problem is, that, due to underactuation, the dynamics in (2) are not *exact feedback linearizable* [5] and only partial linearization is possible that leaves the dynamics still coupled in the control input. Moreover, in general, these dynamics are strongly nonlinear (see Table 1 for the dynamics of design examples) and the direct state space form of (2) is not design friendly even in the absence of uncertainties. For the successful and novel realization of sliding mode control, first, the dynamics in (2) are decoupled in the control input and transformed into a canonical form. We use control input and nonlinear state transformation to achieve this simplifying objective first.

Using the following noncollocated partial feedback linearizing control

$$\tau = (m_{12} - m_{11}m_{21}^{-1}m_{22})w + c_1 + g_1 - m_{11}m_{21}^{-1}(c_2 + g_2) \quad (3)$$

where w is a new control input, and, the nonlinear coordinate transformation [7]

$$\begin{aligned} z_1 &= q_1 + \psi(q_2) \\ z_2 &= m_{21}(q_2)\dot{q}_1 + m_{22}(q_2)\dot{q}_2 \\ \xi_1 &= q_2 \\ \xi_2 &= \dot{q}_2 \\ \psi(q_2) &= \int_0^{q_2} m_{21}^{-1}(\theta)m_{22}(\theta)d\theta \end{aligned} \quad (4)$$

transforms the dynamics in (2) into the following *nontriangular quadratic normal form*:

$$\begin{aligned} \dot{z}_1 &= m_{21}^{-1}(\xi_1)z_2 \\ \dot{z}_2 &= -g_2(z_1 - \psi(\xi_1), \xi_1) + \frac{m'_{11}(\xi_1)}{2m_{21}^2(\xi_1)}z_2^2 \\ &\quad + \left(\frac{m_{22}^2(\xi_1)}{2m_{21}^2(\xi_1)}m'_{11}(\xi_1) - \frac{m_{22}(\xi_1)}{m_{21}(\xi_1)}m'_{21}(\xi_1)\right)\xi_2^2 \\ &\quad + \left(\frac{m'_{21}(\xi_1)}{m_{21}(\xi_1)} - \frac{m_{22}(\xi_1)m'_{11}(\xi_1)}{m_{21}^2(\xi_1)}\right)z_2\xi_2 + \frac{1}{2}m'_{22}(\xi_1)\xi_2^2 \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= w \end{aligned} \quad (5)$$

Equation (5) can be written as:

$$\begin{aligned} \ddot{z} &= -m_{21}^{-1}(\xi)g_2(z - \psi(\xi), \xi) \\ &\quad + \frac{1}{2}m'_{11}(\xi)m_{21}^{-1}(\xi)\left(\dot{z} - m_{21}^{-1}(\xi)m_{22}(\xi)\dot{\xi}\right)^2 \\ &\quad - m_{21}^{-2}(\xi)m'_{21}(\xi)m_{22}(\xi)\dot{\xi}^2 + \frac{1}{2}m_{21}^{-1}(\xi)m'_{21}(\xi)\dot{\xi}^2 \\ \ddot{\xi} &= w \end{aligned} \quad (6)$$

where $'$ denotes d/dq_2.

In general, the normal form (6) comprises a block of (p) second order *nonlinear actuated* $z-$*subsystems* and a block of $n-p$ second order *linear unactuated* $\xi-$*subsystems*. The beauty of the normal forms is, that, with ξ as output with global uniform relative degree two, the first block represents the *Lagrangian zero dynamics* for the second block. Treating ξ as control input for the first block, the form reduces the control of the original underactuated nonlinear system (1) to the control of the reduced order $z-$subsystem in (6).

Remark 2: The explicit transformation (4) *applies to two degrees of freedom underactuated mechanical systems. Higher order systems can be reduced to form* (6) *through the procedure outlined in [7].*

Remark 3: It is well known that control design for the nontriangular quadratic normal form (5) *or* (6) *is a challenging problem using traditional design methods. The proposed sliding mode approach solves this challenging problem with much ease and also provides a unified design framework.*

To stabilize the nonlinear underactuated dynamics in Eq. (1), we design sliding manifold and sliding mode control to stabilize their transformed normal form in (6) rewritten as:

$$\ddot{z} = f(z, \dot{z}, \xi, \dot{\xi}) \tag{7a}$$

$$\ddot{\xi} = w + D(z, \dot{z}, \xi, \dot{\xi}, t) \tag{7b}$$

where $z \in \Re^{(p)}$, $\xi \in \Re^{n-p}$, and $D(z, \dot{z}, \xi, \dot{\xi}, t)$ represents the lumped uncertainties after transformation.

The following assumptions are taken into account in the design of sliding manifold and sliding mode control.

Assumption 4: The origin in the system state space is an equilibrium point of the open loop Lagrangian zero dynamics subsystem (7a) *i.e.,* $f(0, 0, 0, 0) = 0$.

The existence of well defined relative degree requires the following assumptions:

Assumption 5: $\frac{\partial f}{\partial \dot{\xi}} \neq 0$.

Assumption 6: $\frac{\partial f}{\partial \xi} \neq 0$.

In addition, we have,

Assumption 7: The transformed uncertainties $D(z, \dot{z}, \xi, \dot{\xi}, t)$ *is bounded as* $\left|D(z, \dot{z}, \xi, \dot{\xi}, t)\right| \leq D_0$.

Remark 8: Most underactuated mechanical systems have natural (open loop) equilibrium points including the origin and it is reasonable to expect that Assumption 4 will hold for these system. Assumptions 5, 6 are related to system dynamics and must be checked for the system in case. The loss of well defined relative degree at the origin for the Beam-and-Ball system is well known. This is discussed in Section IV-C for the Beam-and-Ball system. Assumption 7 is reasonable to hold in any practical scenario.

The next section presents the design of sliding manifold and sliding mode control for the stabilization of system (7).

III. MAIN RESULTS: SLIDING MANIFOLD AND SLIDING MODE CONTROL LAW

Here we assume that stabilization of (7b) does not imply stabilization of the overall system (7). This is true in general and for underactuated mechanical systems in specific. Hence, we investigate stabilization of (7) through stabilization of (7a).

To make the $z-$subsystem (7a) stable, the following condition is needed to be satisfied

$$f(z, \dot{z}, \xi, \dot{\xi}) = -\alpha \dot{z} - \beta z \tag{8}$$

with $\alpha > 0, \beta > 0$ as design constants. To meet condition (8) we design the sliding manifold as:

$$\sigma = f(z, \dot{z}, \xi, \dot{\xi}) + \alpha \dot{z} + \beta z \tag{9}$$

When sliding mode is established, $\sigma = 0$, in (9), condition (8) is met and the dynamics in (7a) become

$$\ddot{z} + \alpha \dot{z} + \beta z = 0 \tag{10}$$

which is a stable linear system for $\alpha > 0, \beta > 0$ and hence z, $\dot{z}$ converge to zero with convergence rate determined by the choice of design constants α, β.

To achieve the desired dynamics (10) for the $z-$subsystem we need a sliding mode control law to enforce sliding mode in the manifold (9). The design of sliding mode control law depends on the relative degree of system (9). The relative degree is determined whether sliding variable σ or the function f explicitly depends on $\dot{\xi}$ or not as investigated below.

A. THE SLIDING VARIABLE σ EXPLICITLY DEPENDS ON $\dot{\xi}$

In this case, the relative degree of system (9) is 1. We take the time derivative of σ in (9) along the dynamics (7) to get:

$$\dot{\sigma} = a(z, \dot{z}, \xi, \dot{\xi}) + u \tag{11}$$

where

$$a(z, \dot{z}, \xi, \dot{\xi}) = \left(\frac{\partial f}{\partial z} + \beta\right)\dot{z} + \left(\frac{\partial f}{\partial \dot{z}} + \alpha\right) f(z, \dot{z}, \xi, \dot{\xi}) + \frac{\partial f}{\partial \xi}\dot{\xi} + \frac{\partial f}{\partial \dot{\xi}} D(z, \dot{z}, \xi, \dot{\xi}, t) \tag{12}$$

$$u = b(z, \dot{z}, \xi, \dot{\xi}) w \tag{13}$$

$$b(z, \dot{z}, \xi, \dot{\xi}) = \frac{\partial f}{\partial \dot{\xi}} \tag{14}$$

One choice for the control law to enforce sliding mode in relative degree 1 system (11) is the following standard SMC law

$$u = -\left(\left(\frac{\partial f}{\partial z} + \beta\right)\dot{z} + \left(\frac{\partial f}{\partial \dot{z}} + \alpha\right) f(z, \dot{z}, \xi, \dot{\xi}) + \frac{\partial f}{\partial \xi}\dot{\xi} + \left|\frac{\partial f}{\partial \dot{\xi}}\right| D_0 \text{sign}(\sigma) + K \text{sign}(\sigma)\right) \tag{15}$$

where K is a strictly positive design constant.

The above standard SMC law, which consists of an equivalent control term and a discontinuous term, can be achieved by taking the Lyapunov function candidate $V = \frac{1}{2}\sigma^2$ for (11) and taking its time derivative along the dynamics (7). However, the standard SMC law (15) suffers from chattering which is undesired, especially, for mechanical control systems. Moreover, the above standard SMC law can not be applied if the relative degree of the system is greater than 1.

To avoid chattering. we need a control law that is smooth. Furthermore, to deal with internal uncertainties and external disturbances, the control law must be robust. We choose the following well known smooth second order sliding mode (SSOSM) control law [44] to enforce sliding mode in relative degree 1 system (11):

$$\begin{aligned} u &= -s_1 - K_1 |\sigma|^{m/(m+1)} \text{sign}(\sigma) + u_0 \\ \dot{u}_0 &= -K_2 |\sigma|^{(m-1)/(m+1)} \text{sign}(\sigma) \end{aligned} \tag{16}$$

where $m \geq 1$ and $K_1 > 0, K_2 > 0$ are design constants.

The term $s_1 = \hat{a}(z, \dot{z}, \xi, \dot{\xi})$ in (16) is used to cancel the the uncertain bounded term $a(z, \dot{z}, \xi, \dot{\xi})$ in (11) and is estimated via the following observer [44] ($m = 2$):

$$\begin{aligned}\dot{s}_0 &= v_0 + u\\ v_0 &= -\lambda_0|\Lambda|^{1/3}|s_0-\sigma|^{2/3}\text{sign}(s_0-\sigma)+s_1\\ \dot{s}_1 &= v_1\\ v_1 &= -\lambda_1|\Lambda|^{1/2}|s_1-v_0|^{1/2}\text{sign}(s_1-v_0)+s_2\\ \dot{s}_2 &= -\lambda_2|\Lambda|\text{sign}(s_2-v_1)\end{aligned} \tag{17}$$

where $\lambda_0, \lambda_1, \lambda_2$ are design parameters and $\Lambda > 0$ is Lipshitz constant of $\dot{a}(z, \dot{z}, \xi, \dot{\xi})$.

Theorem 9: The closed loop system (11), (16), (17) *is finite time stable and hence* σ, $\dot{\sigma}$ *converge to* 0 *in finite time.*

Proof: The proof can be found in [44]. □

Once sliding mode is established, $\sigma = 0$, condition (8) is met and $z, \dot{z}$ converge to zero in accordance with (10). With ($z = 0, \dot{z} = 0$) the Lagrangian zero dynamics are given by (7a):

$$f(0, 0, \xi, \dot{\xi}) = 0 \tag{18}$$

If the *first order zero dynamics* in (18) are stable (see Assumption 4) then stabilization of the $z-$subsystem (7a) will render the overall system (7) stable.

Remark 10: Eqs. (11), (13) *and* (14) *and hence the control laws* (15) *and* (16) *needs the validity of Assumption 5.*

B. THE SLIDING VARIABLE σ DOES NOT EXPLICITLY DEPEND ON $\dot{\xi}$

In this case the relative degree of system (9) is 2. We take twice the time derivative of σ in (9) along the dynamics (7) to achieve:

$$\ddot{\sigma} = a(z, \dot{z}, \xi, \dot{\xi}) + u \tag{19}$$

where

$$\begin{aligned}a(z, \dot{z}, \xi, \dot{\xi}) = {}& \frac{\partial^2 f}{\partial z^2}\dot{z}^2 + \frac{\partial^2 f}{\partial \dot{z}^2}f^2 + \frac{\partial^2 f}{\partial \xi^2}\dot{\xi}^2 + 2\frac{\partial^2 f}{\partial z \partial \dot{z}}f\dot{z}\\ &+ 2\frac{\partial^2 f}{\partial z \partial \xi}\dot{z}\dot{\xi} + 2\frac{\partial^2 f}{\partial \dot{z} \partial \xi}f\dot{\xi} + \left(\frac{\partial f}{\partial z}+\beta\right)f\\ &+ \left(\frac{\partial f}{\partial \dot{z}}+\alpha\right)\left(\frac{\partial f}{\partial z}\dot{z} + \frac{\partial f}{\partial \dot{z}}f + \frac{\partial f}{\partial \xi}\dot{\xi}\right)\\ &+ \frac{\partial f}{\partial \xi}D(z, \dot{z}, \xi, \dot{\xi}, t)\end{aligned} \tag{20}$$

$$u = b(z, \dot{z}, \xi, \dot{\xi})w \tag{21}$$

$$b(z, \dot{z}, \xi, \dot{\xi}) = \frac{\partial f}{\partial \xi} \tag{22}$$

We choose the following smooth second order sliding mode (SSOSM) control law [45] to enforce sliding mode in relative degree 2 system (19):

$$u = -s_2 - K_1|\sigma|^{(\rho-2)/\rho}\text{sign}(\sigma) - K_2|\dot{\sigma}|^{(\rho-2)/(\rho-1)}\text{sign}(\dot{\sigma}) \tag{23}$$

where $\rho \geq 2$ and $K_1 > 0, K_2 > 0$ are design constants.

The term $s_2 = \hat{a}(z, \dot{z}, \xi, \dot{\xi})$ in the control law (23) is used to cancel the the uncertain bounded term $a(z, \dot{z}, \xi, \dot{\xi})$ in (19) and is estimated via the observer [45] ($m = 2$):

$$\begin{aligned}\dot{s}_0 &= s_1\\ \dot{s}_1 &= v_1 + u\\ v_1 &= -\lambda_2|\Lambda|^{1/3}|s_1-\dot{\sigma}|^{2/3}\text{sign}(s_1-\dot{\sigma})+s_2\\ \dot{s}_2 &= -\lambda_1|\Lambda|\text{sign}(s_2-v_1)\end{aligned} \tag{24}$$

where λ_2 and λ_1 are design parameters and $\Lambda > 0$ is Lipshitz constant of $\ddot{a}(z, \dot{z}, \xi, \dot{\xi})$. Further the observer also estimate $\dot{\sigma}$ as $s_1 = \hat{\dot{\sigma}}$.

Theorem 11: The closed loop system (19), (23), (24) *is finite time stable and hence* σ, $\dot{\sigma}$ *converge to* 0 *in finite time.*

Proof: The proof can be found in [45]. □

Once sliding mode is established, $\sigma = 0$, condition (8) is met and $z, \dot{z}$ converge to zero in accordance with (10). With ($z = 0, \dot{z} = 0$) the Lagrangian zero dynamics are given by (7a):

$$f(0, 0, \xi) = 0 \tag{25}$$

which is an *algebraic equation*. By Assumption 4 $f(0, 0, 0) = 0$ and the solution to this equation is $\xi = 0$ and hence ξ tends to zero as well and, consequently, the overall system (7) becomes stable.

Remark 12: Eqs. (19), (21) *and* (22) *and hence the control law* (23) *needs the validity of Assumption 6.*

Remark 13: The convergence rate of the proposed algorithm can be controlled at two levels: i) by the choice of controller gains K_1, K_2 *in the control laws* (16) *and* (23) *for the desired convergence rate of the sliding variable* σ*; and, ii) by the choice of design constants* α, β *in the sliding variable* (9) *for the desired convergence rate of the zero dynamics* (10).

Remark 14: To use the SSOSM control law (16) *and observer* (17)*, the order of an n degrees of freedom underactuated mechanical system increases from 2n to 2n + 4. In the case of SSOSM control law* (23) *and observer* (24)*, the order increases from 2n to 2n + 3. This overhead in computational cost can be overlooked keeping in view the overall high performance the control laws offer and their implementation in today's high performance platforms.*

The next section presents illustrative design examples.

IV. DESIGN EXAMPLES

We illustrate the design procedure for the Furuta Pendulum, the Overhead Crane, and the Beam-and-Ball system as examples of the class. Fig. 1 shows the schematics and parameters in the equation of motions are shown in Table 1. Each of these systems is unique in its dynamics and has been the subject of excellent research works mostly standalone.

A. THE FURUTA PENDULUM

The Furuta Pendulum is shown in Fig. 1a. It consists of an inverted pendulum and a rotating arm. The control ojcetive

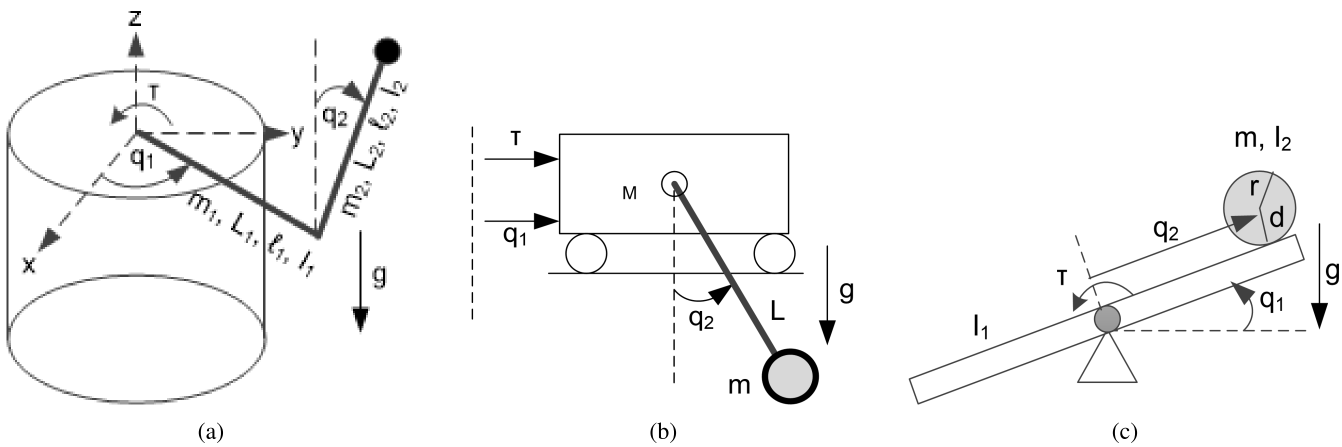

FIGURE 1. Schematics of design examples. (a) The Furuta Pendulum. (b) The Overhead Crane. (c) The Beam-and-Ball system.

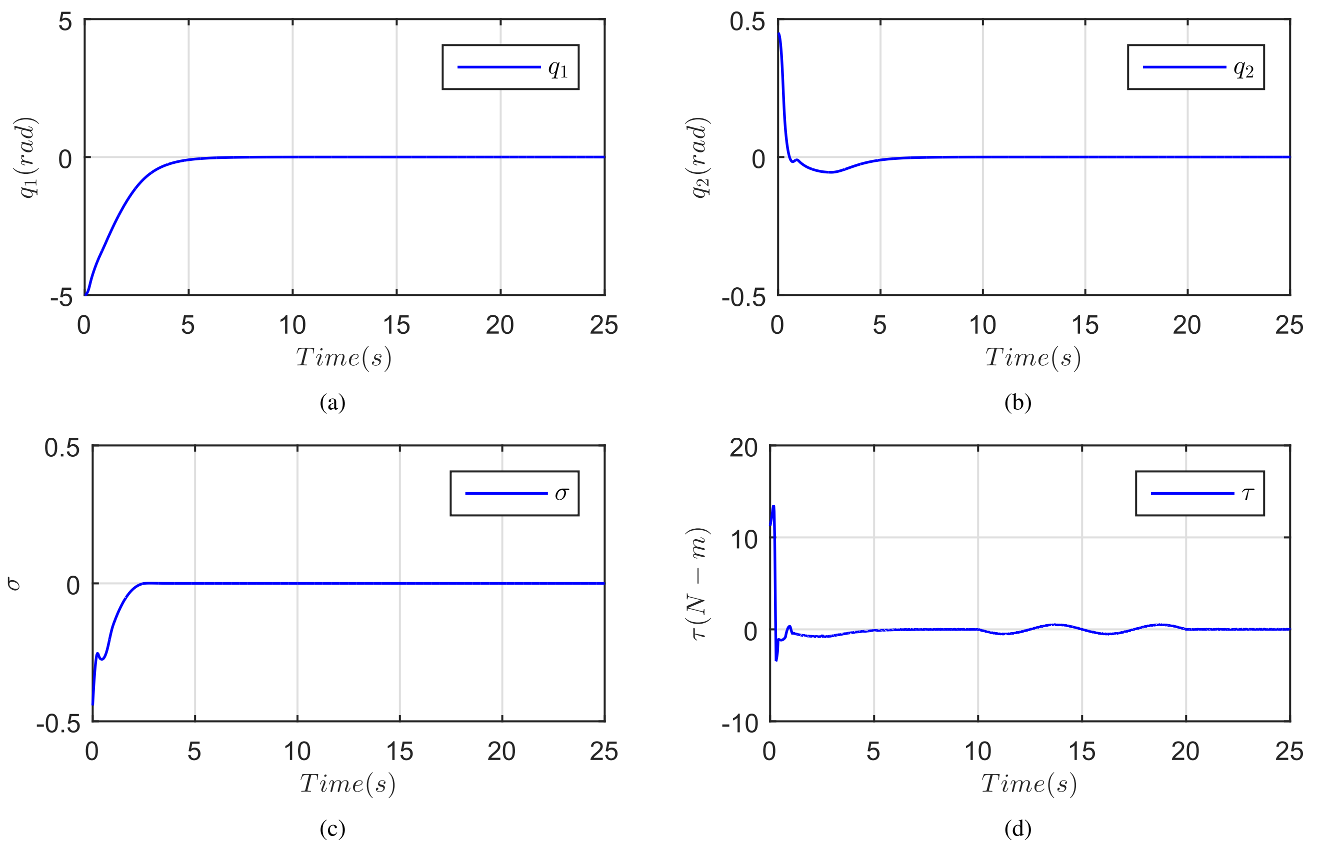

FIGURE 2. The Furuta Pendulum - response with SSOSM control u (23), $q(0) = [-5, 0, 0.45, 0]^T$, applied disturbance $d(t) = 0.5\sin(0.4\pi t)$ from $t = 10$ (s) to $t = 20$ (s). (a) Arm angle q_1 (rad). (b) Pendulum angle q_2 (rad). (c) Sliding surface σ. (d) Control effort τ (N-m).

is to stabilize the inverted pendulum at the upward unstable equilibrium position $q_2 = 0$ by applying control torque τ to the rotating arm. Some excellent research works can be referred to as [7] and [15]–[19].

Physical parameters of the Furuta Pendulum are chosen as in [7] as:

$m_1 = 1.0$ (kg), $L_1 = 1.0$ (m), $\ell_1 = 0.5$ (m), $m_2 = 1.0$ (kg), $L_2 = 1.5$ (m), $\ell_2 = 0.75$ (m) and $g = 9.8$ (m.s^{-2}).

The dynamics of the Furuta Pendulum are:

$$\left(J_1 + m_2\left(L_1^2 + \ell_2^2\sin^2(q_2)\right)\right)\ddot{q}_1 + m_2L_1\ell_2\cos(q_2)\ddot{q}_2 + 2m_2\ell_2^2\sin(q_2)\cos(q_2)\dot{q}_1\dot{q}_2 - m_2\ell_2L_1\sin(q_2)\dot{q}_2^2 = \tau \tag{26a}$$

$$m_2L_1\ell_2\cos(q_2)\ddot{q}_1 + \left(I_2 + m_2\ell_2^2\right)\ddot{q}_2 - m_2\ell_2^2\sin(q_2)\cos(q_2)\dot{q}_1^2 - m_2\ell_2 g\sin(q_2) = 0 \tag{26b}$$

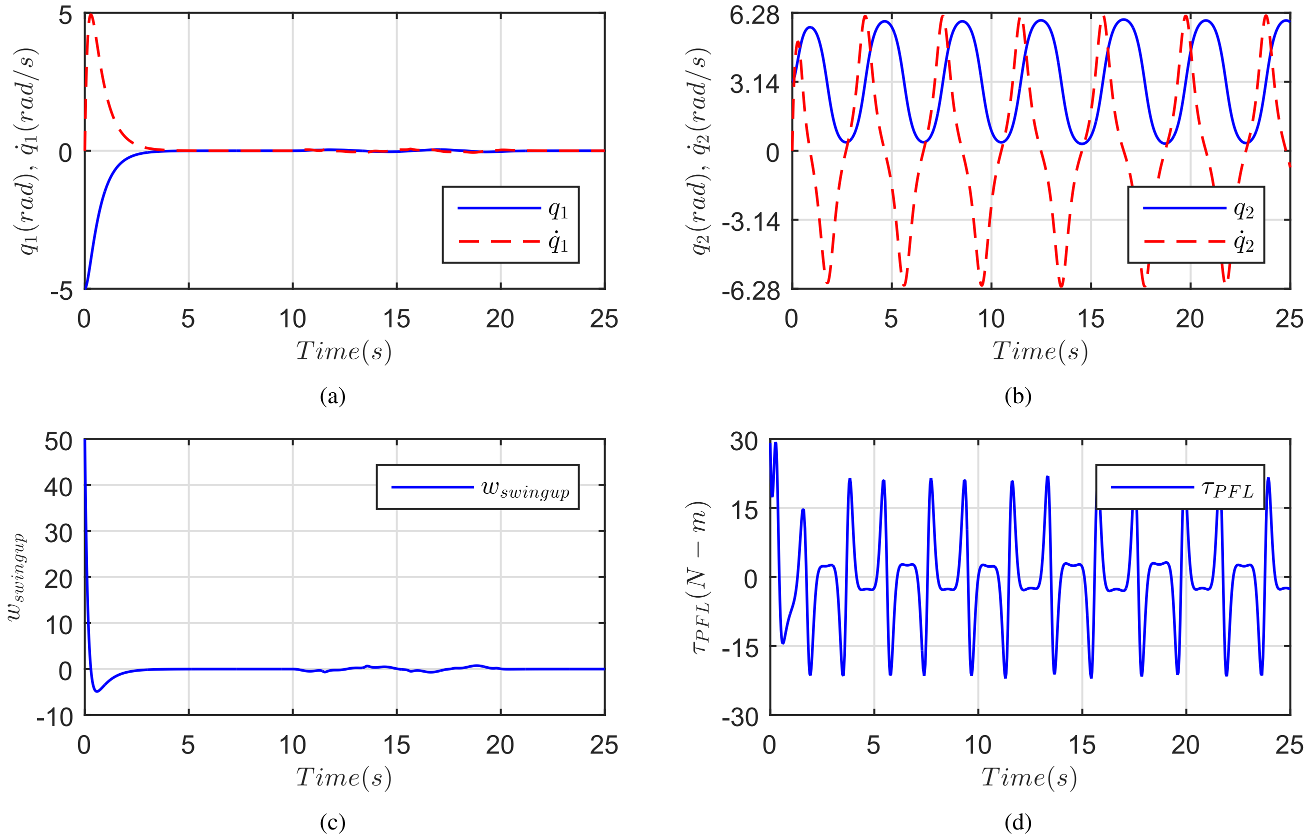

FIGURE 3. **The Furuta Pendulum - response with swingup control $w_{swingup}$ (36), $q(0) = [-5, 0, \pi, 0]^T$, applied disturbance $d(t) = 0.5\sin(0.4\pi t)$ from $t = 10$ (s) to $t = 20$ (s). (a) Arm angle q_1 (rad) and velocity $\dot{q}_1$ (rad/s). (b) Pendulum angle q_2 (rad) and velocity $\dot{q}_2$ (rad/s). (c) Swingup control $w_{swingup}$. (d) Control effort τ_{PFL} (N-m).**

where q_1 is the arm angle in radians and q_2 is the pendulum angle in radians as shown in Fig. 1a.

Using transformations (3) and (4), the normal form (6) for the Furuta Pendulum is:

$$\ddot{z} = \left(k_1 + k_2\frac{(\dot{z} - k_3\dot{\xi})^2}{\cos(\xi)} + k_4\frac{\dot{\xi}^2}{\cos(\xi)}\right)\tan(\xi) \quad (27a)$$

$$\ddot{\xi} = w \quad (27b)$$

where $k_1 = \dfrac{g}{L_1}$, $k_2 = \dfrac{1}{m_2^2 L_1^3 \ell_2}$, $k_3 = \left(I_2 + m_2\ell_2^2\right)$ and $k_4 = \dfrac{(I_2 + m_2\ell_2^2)}{m_2 L_1 \ell_2}$.

Remark 15: Eq. (27a) *shows that for the Furuta Pendulum: Assumption 4 holds; Assumption 5 does not hold and hence control laws* (15) *and* (16) *can not be applied; Assumption 6 holds and hence control law* (23) *can be applied.*

The dynamics in (27a) show that, after z converges to zero, the $\xi-$dynamics are governed by the algebraic equation $\tan(\xi) = 0$. We note that in (27a), the term in the parenthesis in front of $\tan(\xi)$ is strictly positive for $-\dfrac{\pi}{2} < \xi < \dfrac{\pi}{2}$, and hence, to achieve the stable system in (10), we choose the sliding manifold as:

$$\sigma = \tan(\xi) + \alpha\dot{z} + \beta z \quad (28)$$

The relative degree of σ is 2 and the design procedure in Section III-B is applicable. The derivative of σ is

$$\dot{\sigma} = \alpha\left(k_1 + k_2\left(\dot{z} - k_3\dot{\xi}\right)^2\sec(\xi) + k_4\dot{\xi}^2\sec(\xi)\right)\tan(\xi) + \dot{\xi}\sec^2(\xi) + \beta\dot{z} \quad (29)$$

In terms of coordinates $(q_1, \dot{q}_1, q_2, \dot{q}_2)$ of system (26) we have

$$\sigma = \alpha\dot{q}_1 + \alpha k_4\dot{q}_2\sec(q_2) + \beta q_1 + \beta k_4 \ln\left(\sec(q_2) + \tan(q_2)\right) + \tan(q_2) \quad (30)$$

$$\dot{\sigma} = \alpha k_2\left(\alpha\dot{q}_1 + \alpha k_4\dot{q}_2\sec(q_2) - k_3\dot{q}_2\right)^2\sec(q_2)\tan(q_2) + \alpha k_1\tan(q_2) + \alpha k_4\dot{q}_2^2\sec(q_2)\tan(q_2) + \dot{q}_2\sec^2(q_2) + \alpha\beta\dot{q}_1 + \alpha\beta k_4\dot{q}_2\sec(q_2) \quad (31)$$

In Eq. (21), $b(z, \dot{z}, \xi, \dot{\xi})$ in terms of $(q_1, \dot{q}_1, q_2, \dot{q}_2)$ is:

$$b(z, \dot{z}, \xi, \dot{\xi}) = \sec^2(q_2) + 2\alpha\dot{q}_2(k_4 + k_2k_3^2)\sec(q_2)\tan(q_2) - 2\alpha k_2k_3(\dot{q}_1 + k_4\dot{q}_2\sec(q_2))\sec(q_2)\tan(q_2) \quad (32)$$

The final control τ for the system (26) is given by (3) with w given by (21) and u given by (23) and (24). Fig. 2 shows simulation results for the Furuta Pendulum (26).

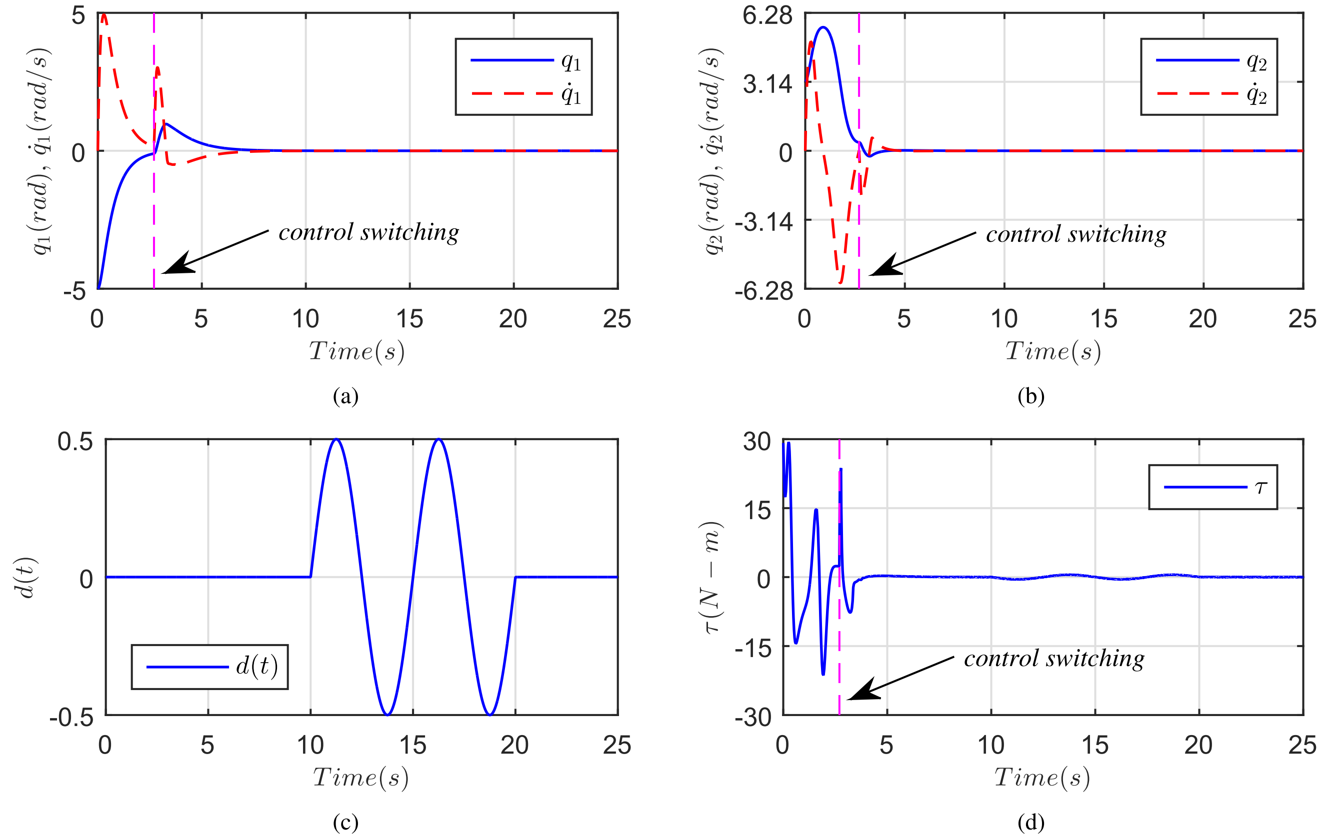

FIGURE 4. The Furuta Pendulum - swing up with $w_{swingup}$ (36) and balancing with u (23), $q(0) = [-5, 0, \pi, 0]^T$, applied disturbance $d(t) = 0.5\sin(0.4\pi t)$ from $t = 10$ (s) to $t = 20$ (s). (a) Arm angle q_1 (rad) and velocity $\dot{q}_1$ (rad/s). (b) Pendulum angle q_2 (rad) and velocity $\dot{q}_2$ (rad/s). (c) Disturbance $d(t) = 0.5\sin(0.4\pi t)$. (d) Control effort τ (N-m).

Swingup Control Law: The controller u in (23) cannot stabilize the Furuta Pendulum globally first due to the assumption $-\frac{\pi}{2} < \xi = q_2 < \frac{\pi}{2}$ in its synthesis and second due to a singularity in the PFL control in (3) at $q_2 = \frac{\pi}{2}$. For global stabilization we design swingup control law.

We partially linearize the dynamics of Furuta Pendulum with respect to q_1. Solving Eq. (2b) for $\ddot{q}_2$ as

$$\ddot{q}_2 = -m_{22}^{-1}\left(c_2 + g_2 + m_{21}\ddot{q}_1\right) \tag{33}$$

putting the result in (2a) and using the following collocated Partial Feedback Linearizng control

$$\tau_{PFL} = (m_{11} - m_{12}m_{22}^{-1}m_{21})w_{swingup} - m_{12}m_{22}^{-1}(c_2 + g_2) + c_1 + g_1 \tag{34}$$

where $w_{swingup}$ is a new control to be designed, the dynamics of the Furuta Pendulum become:

$$\ddot{q}_1 = w_{swingup} \tag{35a}$$

$$m_{22}\ddot{q}_2 + c_2 + g_2 = -m_{21}w_{swingup} \tag{35b}$$

To stabilize (35a), we choose the following state feedback control law

$$w_{swingup} = -K_d\dot{q}_1 - K_p q_1 \tag{36}$$

with $K_d > 0$, $K_p > 0$ as design constants. Once q_1 is stabilized, $w_{swingup}$ becomes zero and the q_2-dynamics become:

$$\left(I_2 + m_2\ell_2^2\right)\ddot{q}_2 - m_2\ell_2 g\sin(q_2) = 0 \tag{37}$$

Fig. 3 shows closed loop response of the Furuta Pendulum (26) with $w_{swingup}$ (36) with $K_d = 8$, $K_p = 10$ for the initial condition $q(0) = [-5, 0, \pi, 0]^T$. The pendulum behavior in (37) is shown in Fig. 3b. Fig. 4 shows a successful swing up from $q_2 = \pi$ to $q_2 = 0$ using $w_{swingup}$ (36) and then balancing with u (23) in the presence of external disturbance.

B. THE OVERHEAD CRANE

The Overhead Crane is shown in Fig. 1b. It consists of a trolley of mass M and a payload of mass m suspended with a massless (assumed) rope of length L. The control task is the fast and precise transportation of the payload with minimum swing. Some works can be referred to as [20]–[26].

The physical parameters of the Crane are chosen as in [20]: $M = 30.0$ (kg), $L = 2.0$ (m), $m = 20.0$ (kg), and $g = 9.8$ (m.s^{-2}). The dynamics of the Overhead Crane are:

$$(M + m)\ddot{q}_1 + mL\cos(q_2)\ddot{q}_2 - mL\sin(q_2)\dot{q}_2^2 = \tau \tag{38a}$$

$$mL\cos(q_2)\ddot{q}_1 + mL^2\ddot{q}_2 + mLg\sin(q_2) = 0 \tag{38b}$$

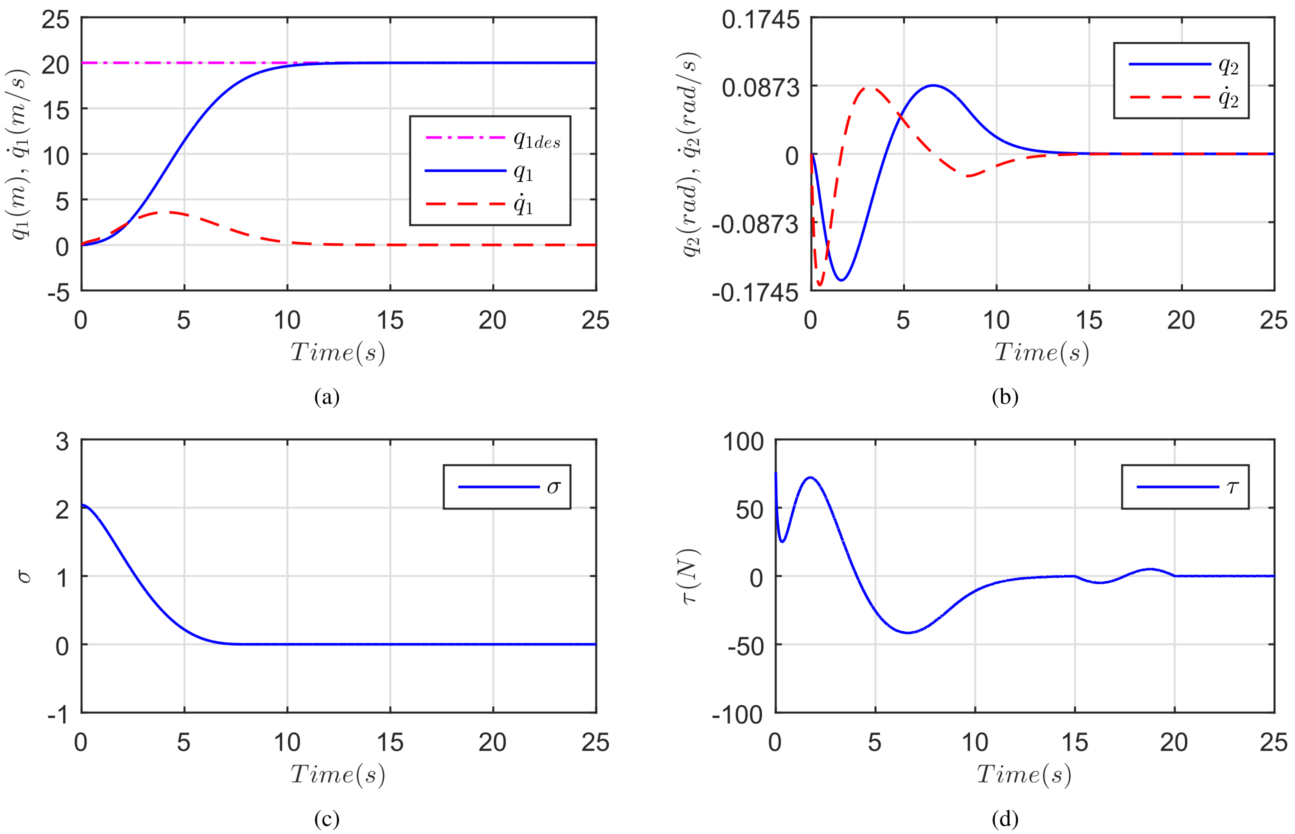

FIGURE 5. The Overhead Crane - response with SSOSM control u (23), applied disturbance $d(t) = 5\sin(0.4\pi t)$ from $t = 15$ (s) to $t = 20$ (s). (a) Crane Trolly position q_1 (m) and velocity $\dot{q}_1$ (m/s). (b) Payload swing angle q_2 (rad) and velocity $\dot{q}_2$ (rad/s). (c) Sliding surface σ. (d) Control effort τ (N).

where q_1 is the trolley position (meters) and q_2 is the payload angle (radians) as shown in Fig. 1b.

Using transformations (3) and (4), the normal form (6) for the Overhead Crane becomes:

$$\ddot{z} = \left(-g + L\frac{\dot{\xi}^2}{\cos(\xi)}\right)\tan(\xi) \tag{39a}$$

$$\ddot{\xi} = w \tag{39b}$$

Remark 16: Eq. (39a) *shows that for the Overhead Crane: Assumption 4 holds; Assumption 5 does not hold and hence control laws* (15) *and* (16) *can not be applied; Assumption 6 holds and hence control law* (23) *can be applied.*

The dynamics in (39a) show that, after z converges to zero, the $\xi-$dynamics are governed by the algebraic equation $\tan(\xi) = 0$. We note that in (39a), the term $\left(-g + L\frac{\dot{\xi}^2}{\cos(\xi)}\right)$ is not strictly positive or negative. To solve this problem we present the following assumptions and explanation.

First, for practical crane systems, the payload swing angle $q_2 = \xi$ is usually less than $\frac{\pi}{18}$ radians and the payload swing velocity $|\dot{q}_2| = |\dot{\xi}| < 1$ radians/second and hence $\frac{\dot{\xi}^2}{\cos(\xi)} \ll 1$. Second, we assume $L < g$. Therefore $\left(-g + L\frac{\dot{\xi}^2}{\cos(\xi)}\right)$ can be assumed to be strictly negative.

With the above assumptions and explanation in hand, to achieve the stable system in (10), we choose the sliding manifold as

$$\sigma = \tan(\xi) - \alpha\dot{z} - \beta z \tag{40}$$

The relative degree of σ is 2 and the design procedure in Section III-B is applicable. The derivative of σ is

$$\dot{\sigma} = \dot{\xi}\sec^2(\xi) - \alpha\left(-g + L\dot{\xi}^2\sec(\xi)\right)\tan(\xi) - \beta\dot{z} \tag{41}$$

In terms of coordinates $(q_1, \dot{q}_1, q_2, \dot{q}_2)$ of system (38) we have

$$\begin{aligned}\sigma &= -\alpha\dot{q}_1 - \alpha L\dot{q}_2\sec(q_2) - \beta q_1 \\ &\quad - \beta L\ln\left(\sec(q_2) + \tan(q_2)\right) + \tan(q_2) \end{aligned} \tag{42}$$

$$\begin{aligned}\dot{\sigma} &= \sec^2(q_2)\dot{q}_2 - \alpha\left(-g + L\dot{q}_2^2\sec(q_2)\right)\tan(q_2) \\ &\quad - \beta\dot{q}_1 - \beta L\dot{q}_2\sec(q_2) \end{aligned} \tag{43}$$

In Eq. (21), $b(z, \dot{z}, \xi, \dot{\xi})$ in terms of $(q_1, \dot{q}_1, q_2, \dot{q}_2)$ is:

$$b(z, \dot{z}, \xi, \dot{\xi}) = \sec^2(q_2) - 2\alpha L\dot{q}_2\sec(q_2)\tan(q_2) \tag{44}$$

The final control τ for system (38) is given by (3) with w given by (21) and u given by (23) and (24). Fig. 5 shows simulation results for the Overhead Crane (38).

C. THE BEAM-AND-BALL SYSTEM

Fig. 1c shows the well known system of Beam-and-Ball [3]. The control objective is to bring the Ball from an initial

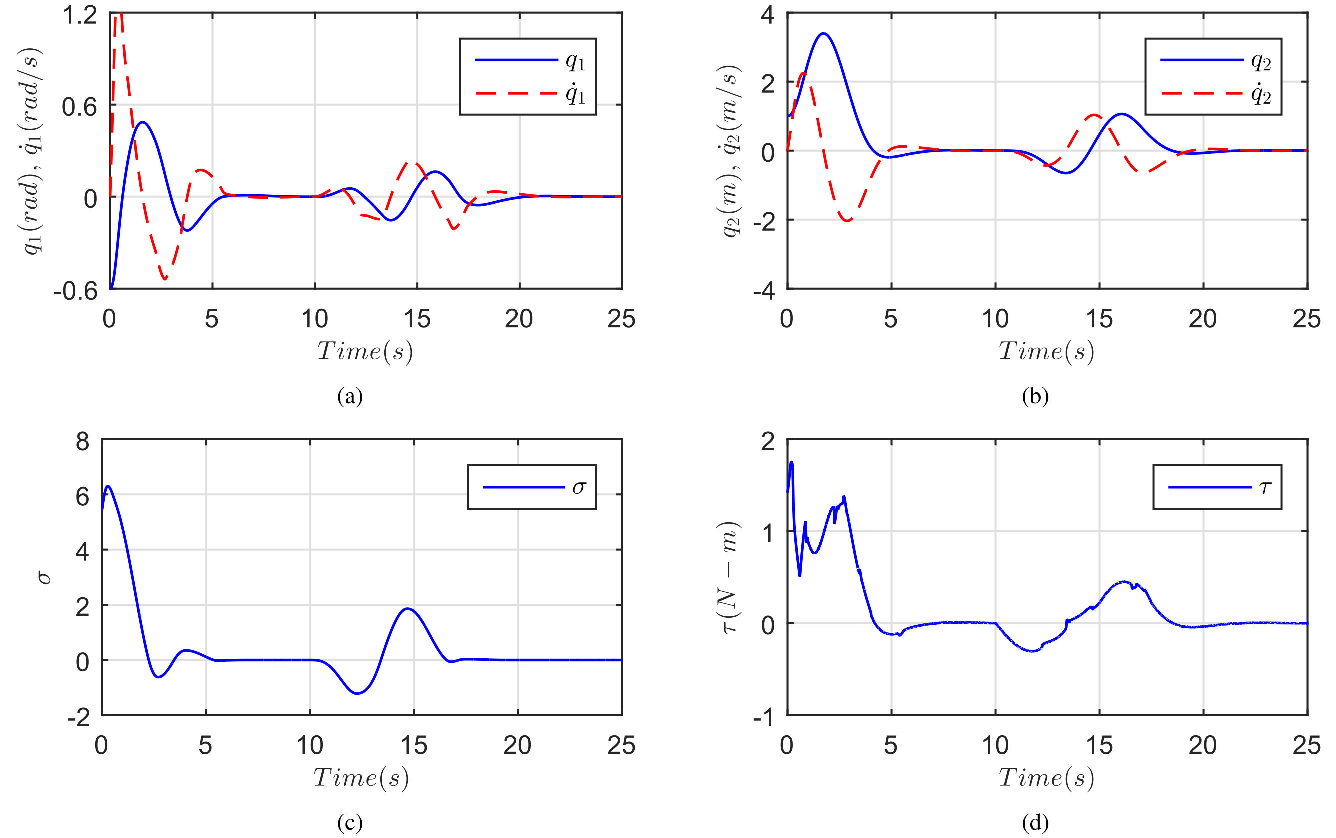

FIGURE 6. The Beam-and-Ball - response with SSOSM control u (23), $q(0) = [-0.6, 0, 1.0, 0]^T$, applied disturbance $d(t) = 0.25\sin(0.4\pi t)$ from $t = 10$ (s) to $t = 15$ (s). (a) Beam angle q_1 (rad) and velocity $\dot{q}_1$ (rad/s). (b) Ball position q_2 (m) and velocity $\dot{q}_2$ (m/s). (c) Sliding surface σ. (d) Control effort u (N-m).

position to the origin, the center of beam. The Beam-and-Ball has been studied in numerous excellent research works for example [3], [7], [11], [14], [27], and [46]–[50].

For the Beam-and-Ball system, we chose the physical parameters as in [3] and [14]: $m = 0.05$ (kg), $I_1 = 0.02$ (kg.m^2), $I_2 = 2\times10^{-6}$ (kg.m^2), $r = 0.01$ (m), $g = 9.8$ (m.s^{-2}). The dynamics of the Beam-and-Ball system are:

$$\left(I_1 + mq_2^2\right)\ddot{q}_1 + 2m\dot{q}_1 q_2 \dot{q}_2 + mgq_2\cos(q_1) = \tau \tag{45a}$$

$$\left(m + \frac{I_2}{r^2}\right)\ddot{q}_2 - mq_2\dot{q}_1^2 + mg\sin(q_1) = 0 \tag{45b}$$

where q_1 represents the Beam angle (radians) and q_2 represents the Ball position (meters) as shown in Fig. 1c.

Using the control input transformation

$$\tau = \left(I_1 + mq_2^2\right)w + 2m\dot{q}_1 q_2 \dot{q}_2 + mgq_2\cos(q_1) \tag{46}$$

where w is a new control input and writing $[z_1, z_2, \xi_1, \xi_2]^T = [q_2, \dot{q}_2, q_1, \dot{q}_1]^T$, the *normal form* for the Beam-and-Ball system (45) can be written as:

$$\begin{aligned}\dot{z}_1 &= z_2\\ \dot{z}_2 &= \frac{1}{1+\frac{I_2}{mr^2}}\left(z_1\xi_2^2 - g\sin(\xi_1)\right)\\ \dot{\xi}_1 &= \xi_2\\ \dot{\xi}_2 &= w\end{aligned} \tag{47}$$

The normal form (47) can be rewritten in the form of (7) as

$$\ddot{z} = k_0\dot{\xi}^2 z - k_0 g\sin(\xi) \tag{48a}$$

$$\ddot{\xi} = w \tag{48b}$$

where $k_0 = \dfrac{1}{1+\dfrac{I_2}{mr^2}} = 0.7143.$

Remark 17: Eq. (48a) *shows that for the beam-and-ball: Assumption 4 holds; Assumption 5 does not hold and hence control laws* (15) *and* (16) *can not be applied; Assumption 6 holds and hence control law* (23) *can be applied.*

The dynamics in (48a) show that, after z converges to zero, the $\xi-$dynamics are governed by the algebraic equation $-k_0 g\sin(\xi) = 0$. We choose the sliding manifold as

$$\sigma = -k_0 g\sin(\xi) + \alpha\dot{z} + \beta z \tag{49}$$

The relative degree of σ is 2 and the design procedure in Section III-B is applicable. The derivative of σ is

$$\dot{\sigma} = -k_0 g\dot{\xi}\cos(\xi) + \alpha k_0\dot{\xi}^2 z - \alpha k_0 g\sin(\xi) + \beta\dot{z} \tag{50}$$

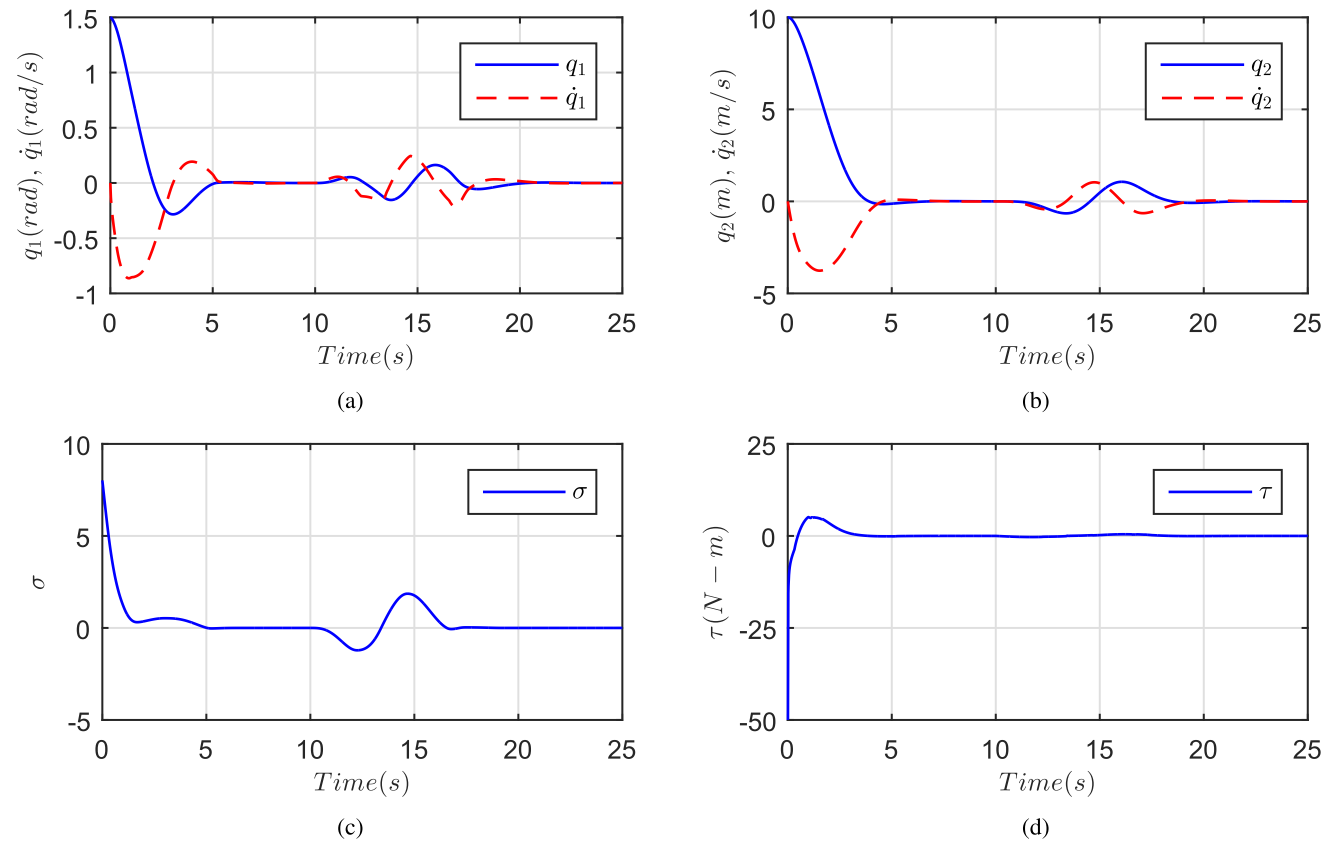

FIGURE 7. The Beam-and-Ball - response with SSOSM control u (23), $q(0) = [1.5, 0, 10.0, 0]^T$, applied disturbance $d(t) = 0.25\sin(0.4\pi t)$ from $t = 10$ (s) to $t = 15$ (s). (a) Beam angle q_1 (rad) and velocity $\dot{q}_1$ (rad/sec). (b) Ball position q_2 (m) and velocity $\dot{q}_2$ (m/sec). (c) Sliding surface σ. (d) Control effort u (N-m).

In terms of coordinates $(q_1, \dot{q}_1, q_2, \dot{q}_2)$ of system (45) we have

$$\sigma = -k_0 g\sin(q_1) + \alpha\dot{q}_2 + \beta q_2 \quad (51)$$

$$\dot{\sigma} = -k_0 g\dot{q}_1\cos(q_1) + \alpha k_0\dot{q}_1^2 q_2 - \alpha k_0 g\sin(q_1) + \beta\dot{q}_2 \quad (52)$$

The first term $k_0\dot{\xi}^2 z$ in (48a) is taken into account in controller synthesis but excluded in the design of sliding manifold in (49) for the following reasons:

i. including this term in the sliding manifold results in undefined relative degree at the origin and Assumption 5 becomes invalid.
ii. being third order, is small near the origin.
iii. To achieve the stable system in (10), the coefficient β of z in the sliding manifold (49) can be chosen sufficiently large to dominate the strictly positive state dependent coefficient $k_0\dot{\xi}^2$ of z in (48a). The Beam velocity $\dot{q}_1 = \dot{\xi}$ is usually less than 2 radians/second, and hence, $k_0\dot{\xi}^2 < 2$ and choosing $\beta \geq 2$ is sufficient.

Reasons (ii) and (iii) are crude assumptions but simulation results justify their validity.

In Eq. (21), $b(z, \dot{z}, \xi, \dot{\xi})$ in terms of $(q_1, \dot{q}_1, q_2, \dot{q}_2)$ is:

$$b(z, \dot{z}, \xi, \dot{\xi}) = 2k_0\alpha q_2\dot{q}_2 - k_0 g\cos(q_1) \quad (53)$$

The final control τ for system (45) is given by (46) with w given by (21) and u given by (23) and (24). Figs. 6 and 7 show simulation results for the Beam-and-Ball system (45).

V. SIMULATION RESULTS AND DISCUSSION

A. THE FURUTA PENDULUM

Fig. 2 shows closed loop response of the Furuta Pendulum with SSOSM control u (23) and observer (24). The controller parameters are chosen as $\rho = 3$, $K_1 = 8$, $K_2 = 10$ and the observer parameters as $\lambda_1 = 1$, $\lambda_2 = 3$. The sliding parameters are chosen as $\alpha = 0.3061$, $\beta = 0.2041$. The controller stabilizes the system from the initial condition $q(0) = [-5, 0, 0.45, 0]^T$ to the upward unstable equilibrium position $q = [0, 0, 0, 0]^T$ in less than 6 seconds. The control effort is smooth. Fig. 3 shows closed loop response with swingup control (36) with parameters $K_d = 8.0$, $K_p = 10.0$. The controller stabilizes the q_1-dynamics as desired but the q_2-dynamics behave as in (37). Fig. 4 shows successful swing up from $q_2 = \pi$ to $q_2 = 0$ with $w_{swingup}$ (36) and then balancing with u (23) in the presence of external disturbance $d(t) = 0.5\sin(0.4\pi t)$. Figs. 2, 4 show system response, under sliding mode control law (23), is robust to applied disturbance and even to a large magnitude disturbance. Fig. 3 shows system response, under feedback control law (36), is affected by applied disturbance.

B. THE OVERHEAD CRANE

Fig. 5 shows closed loop response of the Overhead Crane with SSOSM control u (23) and observer (24). The controller parameters are chosen as $\rho = 3$, $K_1 = 1$, $K_2 = 1.6$ and the observer parameters as $\lambda_1 = 1$, $\lambda_2 = 3$. The sliding parameters are chosen as $\alpha = 0.2041$, $\beta = 0.1020$. The Crane transports the payload to the desired position $q_{1des} = 20$ (m) in 10 seconds while the payload swing angle q_2 remains within the desired range of $|q_2| < \frac{\pi}{18} = 0.1745$ radians, i.e., within 10 degrees. System response is robust to a large applied disturbance $d(t) = 5\sin(0.4\pi t)$.

C. THE BEAM-AND-BALL SYSTEM

Figs. 6 and 7 show closed loop response of the Beam-and-Ball with SSOSM control (23) and observer (24). The controller parameters are chosen as $\rho = 3$, $K_1 = 5$, $K_2 = 6$ and the observer parameters as $\lambda_1 = 4$, $\lambda_2 = 5$. The sliding variable parameters are chosen as $\alpha = 1.5$, $\beta = 1.5$. The controller stabilizes the system in less than 6 seconds. Moreover the control effort is smooth and within acceptable range. The effects of applied disturbance on stability are made visible in simulation. Small variations in Beam angle due to disturbance cause large variation in Ball position, which is unactuated, and then the control takes action by changing the Beam angle to bring the Ball back to desired position.

D. COMPARISON OF RESULTS

It is interesting to compare the results with some existing results reported in the literature. The comparison shows the improved performance of the proposed control design framework.

The physical parameters of the Overhead Crane are the same as in [20]. Fig. 5 shows the results are in agreement with and improved to [20]. We set the desired position of the Crane to a larger value of 20(m) in contrast to 14(m) of [20]. Still the peak control effort is less, i.e., 76(N) as compared [20] where it is greater than 200(N) and the payload swing angle is contained within 10 degrees in contrast to [20] where it is greater than 10 degrees. However, the settling time in our case is 10 seconds, because of greater desired position, compared to 6 seconds of [20]. Moreover, system response is robust to sinusoidal disturbance with large magnitude (5) and large time period (5 seconds) compared to magnitude 0.54 and time period 0.5 seconds in [20]. Disturbance with small magnitude and high frequency are simply absorbed by mechanical systems especially with large physical parameters as is the case. Furthermore, the design in this article is simple and complete.

The physical parameters of the Beam-and-Ball system are the same as in [3] and [14]. The initial condition in Figure 6 is the same as in [14]. Fig. 6 shows the results are in agreement with and improved to [14]. Comparing to [14], the overshoot in the Ball position and the settling time are remarkably improved. The settling time is 7 seconds in our case and it is 12 seconds in [14]. The overshoot in the Ball position is 3.4 and 6.67 in the cited reference. Fig. 7 shows response for initial condition similar to but much larger than that in [14]. Again we see that the settling time and the undershoot are remarkably improved in our case. Shown in [14], the SMC law based on approximate input-output linearization [3] becomes unstable for these initial conditions.

VI. CONCLUSION

We presented a framework based on smooth second order sliding mode for the control of a class of underactuated mechanical systems. A nonlinear sliding manifold was designed to stabilize the reduced order Lagrangian zero dynamics of the underactuated mechanical system such that stability of the overall system is also guaranteed. The effectiveness of the proposed framework was shown through illustrative design examples of the Furuta Pendulum, the Overhead Crane, and the Beam-and-Ball system. The control action is smooth as desired and demanded for mechanical control systems. The results are in general agreement with and improved to previous works while the design procedure is comprehensive and simple. The proposed control design framework is general and can be applied to nonlinear systems other than underactuated mechanical systems. Additionally, to overcome the limitations of sliding mode control law, swingup control law was designed in the case of Furuta Pendulum. Successful swingup and stabilization in the presence of external disturbance was demonstrated using the designed control laws.

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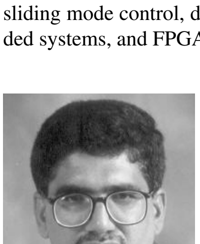

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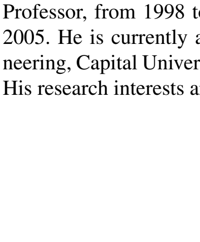

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