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A Periodic Observers Synthesis Approach for LDP Systems Based on Iteration

LINGLING LV[1], ZHE ZHANG[1], AND LEI ZHANG[2]
[1]Institute of Electric Power, North China University of Water Resources and Electric Power, Zhengzhou 450011, China
[2]School of Computer and Information Engineering, Institute of Data and Knowledge Engineering, Henan University, Kaifeng 475004, China

Corresponding authors: Lingling Lv (lingling_lv@163.com) and Lei Zhang (zhanglei@henu.edu.cn)

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ABSTRACT The problem of state observer design for the linear discrete-time periodic (LDP) system and its robust consideration are discussed in this paper. Applying the lifting technique and algebraic operations based on the well-known CG-algorithm, an iterative algorithm for periodic observer gain can be generated. By optimizing the free parameter matrix in the proposed algorithm, an algorithm on the minimum norm and robust observer design for the LDP systems is presented. One numerical example is worked out to illustrate the effect of the proposed approaches.

INDEX TERMS Linear discrete-time periodic (LDP) systems, state observer, robustness, iterative algorithm.

I. INTRODUCTION

The controller design requires us to master the state characteristics of the system. However, it is impractical to directly measure all state variables precisely in practical applications. As a result, it requires us to make reliable estimates of the states that cannot be measured directly. The state observer that can accurately reconstruct the state of the system is designed to meet the requirements of this task. State observers, including full order state observers and reduced order state observers, play very important roles in modern control theory and control engineering. They have been widely studied in different fields([1]–[5]). Therefore, state observer design has drawn much attention worldwide and there are many important relevant works existed, one can see [6]–[8] and references therein for instance.

For discrete-time systems, several state observers using quick geometrical forms have been constructed. Based on minimum-volume bounding parallelotopes, an approach to the problem of recursively estimating the state uncertainty set of a discrete-time linear dynamical system is derived in [9]. Algorithms for computing minimal-volume ellipsoidal bounds on the state of a linear, discrete-time dynamical system are presented in [10] and [11]. Paper [12] presents an algorithm to compute a set, which is represented by a zonotope and contains the states consistent with the measured output and the given noise and parameters. These solutions are based on open-loop observers so that the error caused by these methods will be disappointing, thus losing the practical application value.

In this paper, the problem of state observer design is transformed into the solution to the corresponding matrix equation, and a neat iterative algorithm is given based on the well-known CG algorithm. Initially, we consider the state observer design problem for linear discrete-lime periodic systems without disturbances and give the expected algorithm. On this basis, we consider the case where uncertain disturbances existed in the system parameters, and give the algorithm under the consideration of minimum norm and robustness.

The main contribution of this paper is to present the approach to state observer design of linear discrete-time periodic (LDP) systems. Applying lifting technique and algebraic operations, the problem to be considered can be solved by the extend algorithm. In the case of uncertain disturbance existed, the proposed robust optimization algorithm obtains the ideal approximation of the state of the original system. The validity of the proposed algorithms is reflected in the numerical example at the end of the paper.

Here, we give descriptions of some symbols which will be encountered in the rest of this paper. $\mathrm{tr}(A)$ means the trace of matrix A. Norm $\|A\|$ is a Frobenius norm of matrix A. $\Lambda(A)$ means the eigenvalue set of matrix A and Ψ_A denotes the monodromy matrix $A_{K-1}A_{K-2}\cdots A_0$. For a column

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sub-block matrix $X = \left[X_1\ X_2\ \cdots\ X_n\right] \in \mathbb{R}^{m\times n}$, vec($X$) denotes $\left[X_1^{\mathrm{T}}\ X_2^{\mathrm{T}}\ \cdots\ X_n^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{mn}$

II. PRELIMINARIES

Consider the observable linear discrete-time periodic system as the following state-space model:

$$\begin{cases} x_{t+1} = A_t x_t + B_t u_t \\ y_t = C_t x_t \end{cases} \tag{1}$$

where $t \in \mathbb{Z}$ is the set of integers, $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^r$ and $y_t \in \mathbb{R}^m$ are the state vector, the input vector and the output vector respectively, $A_t \in \mathbb{R}^{n\times n}$, $B_t \in \mathbb{R}^{n\times r}$ and $C_t \in \mathbb{R}^{n\times m}$ are coefficient matrices with T-periodic property ($T \geq 1$), which indicate that

$$A_{T+t} = A_t, \quad B_{T+t} = B_t,\ C_{T+t} = C_t.$$

The state observer could give an asymptotic estimation of x_t in cases where the state x_t of system (1) could not be measured under some practice restrictions but the input u_t and output y_t can be utilized. State observer based on state error feedback is widely used, which can be taken the form as

$$\hat{x}_{t+1} = A_t\hat{x}_t + B_t u_t + L_t\left(y_t - \hat{y}_t\right), \tag{2}$$

where $\hat{x}_t \in \mathbb{R}^n$ is the state of the observer, $\hat{y}_t = C_t\hat{x}_t$ is the output of the observer and $L_t \in \mathbb{R}^{n\times m}$ is a real matrix with periodic T.

Obviously, system (2) is equivalent to the following T-periodic closed loop system

$$\hat{x}_{t+1} = \left(A_t - L_t C_t\right)\hat{x}_t + B_t u_t + L_t y_t, \tag{3}$$

whose monodromy matrix is

$$\Psi_A = \widetilde{A}_{T-1}\widetilde{A}_{T-2}\cdots\widetilde{A}_0$$

where $\widetilde{A}_i = A_i - L_i C_i$, $i \in \overline{0, T-1}$. Then the problem of observer design for linear discrete-time periodic system (1) can be represented as

Problem 1: Consider the observable linear discrete-time periodic system (1), seek the periodic matrix $L_t \in \mathbb{R}^{n\times m}$, $t \in \overline{0, T-1}$, such that the observer system (2) gives an asymptotic estimation of state x_t.

The first thing to consider is the existence condition for a full order state observer. A simple proposition is given and its proof is omitted.

Proposition 1: For an observable system, there exists a periodic matrix $L_t \in \mathbb{R}^{n\times m}$, $t \in \overline{0, T-1}$ making Problem 1 solvable if and only if all the eigenvalues of the monodromy matrix Ψ_A of system (3) lie in the open unit disk.

Let $\Gamma = \{s_1, \cdots, s_n, s \in \mathbb{C}\}$ be the predetermined set of poles of system (3), which is symmetric with respect to the real axis. Let $F_t \in \mathbb{R}^{n\times n}$ be the T-periodic matrix satisfying $\Lambda(\Psi_F) = \Gamma$. Clearly, $\Lambda(\Psi_A) = \Gamma$ if and only if there exists a T-periodic invertible matrix X_t such that

$$X_{t+1}^{-1}\widetilde{A}_t X_t = F_t. \tag{4}$$

Because the transpose dose not change eigenvalues of the matrix, equation (4) is equivalent to

$$\widetilde{A}_t^T X_t = X_{t+1}F_t. \tag{5}$$

Obviously, Equation (5) can be rewritten as

$$A_t^{\mathrm{T}}X_t - C_t^{\mathrm{T}}L_t^{\mathrm{T}}X_t = X_{t+1}F_t, \tag{6}$$

which is a variant form of periodic Sylvester matrix equation. Namely, the problem of observer design has been transformed into the problem of solving periodic Sylvester matrix equation.

When the system is disturbed by external environment, the closed loop system matrix will deviate from the nominal matrix $\widetilde{A}_t$, which can be generally expressed as

$$A_t - L_t C_t \mapsto A_t + \Delta_{a,t} - L_t\left(C_t + \Delta_{c,t}\right), \quad t \in \overline{0, T-1},$$

in which $\Delta_{a,t} \in \mathbb{R}^{n\times n}$, $\Delta_{c,t} \in \mathbb{R}^{m\times n}$, $t \in \overline{0, T-1}$ are random small perturbations. Thus, the problem of robust observer design for linear discrete-time periodic system (1) can be portrayed as

Problem 2: Consider the observable linear discrete-time periodic system (1), seek the periodic matrix $L_t \in \mathbb{R}^{n\times m}$, $t \in \overline{0, T-1}$, such that the following conditions are met:

1) Observer system (2) gives an asymptotic estimation of state x_t.
2) Eigenvalues of the closed loop observer are as insensitive as positive to small perturbations.

III. MAIN RESULTS

Starting from the problem of observer design for linear discrete-time periodic system (1), the problem can be transformed into the problem of solving the following periodic Sylvester matrix equation equivalently:

$$A_t^T X_t + X_{t+1}\bar{F}_t = M_t, \quad t \in \overline{0, T-1} \tag{7}$$

in which $\bar{F}_t = -F_t$, $M = C_t^T L_t^T X_t$. Utilizing cyclic lifting technique, equation (7) can be rewritten as following time-invariant equation:

$$A^C X + X\bar{F}^C = M^C, \tag{8}$$

where

$$A^C = \begin{bmatrix} 0 & 0 & 0 & \cdots & A_0^T \\ A_1^T & 0 & 0 & \cdots & 0 \\ 0 & A_2^T & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & A_{T-1}^T & 0 \end{bmatrix},$$

$$\bar{F}^C = \begin{bmatrix} 0 & 0 & 0 & \cdots & \bar{F}_{T-1} \\ \bar{F}_0 & 0 & 0 & \cdots & 0 \\ 0 & \bar{F}_1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & \bar{F}_{T-2} & 0 \end{bmatrix},$$

$$X = \begin{bmatrix} 0 & X_1 & 0 & \cdots & 0 \\ 0 & 0 & X_2 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & & \ddots & X_{T-1} \\ X_0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$M^C = \begin{bmatrix} M_0 & 0 & 0 & \cdots & 0 \\ 0 & M_1 & 0 & \cdots & 0 \\ 0 & 0 & M_2 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_{T-1} \end{bmatrix}.$$

By using the Kronecker product and vectorization operator, the time-invariant equation (8) can be formulated into the following equation:

$$\left[I \otimes A^C \quad \bar{F}^{CT} \otimes I \right] vec(X) = vec(M^C), \tag{9}$$

which is a variant of equation $Ax = b$. Hence, the solving of the equation (8) is equivalent to the solving of the equation (9). In order to solving the equation $Ax = b$, a CG-based method is presented as follows:

Algorithm 1 CG-Based Method for Solving $Ax = b$

1) Choose initial vector $x(0) \in \mathbb{R}^n$, and tolerance ξ;
2) Calculate

$$\begin{aligned} q(0) &= b - Ax(0); \\ r(0) &= A^T q(0); \\ p(0) &= -r(0); \\ k &:= 0; \end{aligned}$$

3) For $\|r(k)\| > \xi$, calculate

$$\begin{aligned} \alpha(k) &= \frac{tr\left[r^T(k)p(k)\right]}{\|Ap(k)\|^2}; \\ x(k+1) &= x(k) + \alpha(k)p(k); \\ q(k+1) &= b - Ax(k+1); \\ r(k+1) &= A^T q(k+1); \\ p(k+1) &= -r(k+1) + \frac{\|r(k+1)\|^2}{\|r(k)\|^2} p(k); \\ k &= k+1; \end{aligned}$$

What can be seen is that Algorithm 1 is a neat method to solve the equation $Ax = b$. However, the large-size and the non-sparsity of coefficient matrices of (9) restrict the application of the algorithm. In order to solve the periodic Sylvester matrix equation (7) and further solve Problem 1 by the CG-based method, we extend Algorithm 1 to a periodic iterative algorithm which can be used in the case of solving Problem 1.

Firstly, for the following index:

$$J = \frac{1}{2} \sum_{t=0}^{T-1} \left\| C_t^{\mathrm{T}} G_t - A_t^{\mathrm{T}} X_t(k) - X_{t+1}(k) \bar{F}_t \right\|^2, \tag{10}$$

seek the minimizer periodic matrix X_t^* such that

$$\left. \frac{\partial J}{\partial X_t} \right|_{X_t = X_t^*} = 0.$$

Remark 1: Since the function J is a quadratic function, there is only one extreme point, and the extreme point is the global minimum point. This means that the index function will not fall into the local optimum for every choice of initial value.

Remark 2: What should be pointed out is that the variable $C_t^{\mathrm{T}} G_t$ in equation (10) is equivalent to the M_t in equation (7). Since the G_t is a given free parameter matrix, $C_t^{\mathrm{T}} G_t$ can be seen as a known matrix.

The algorithm to seek the matrix L_t can be presented as follows:

Algorithm 2 Periodic CG-Based Algorithm

1) Let $F_t \in \mathbb{R}^{n \times n}, t \in \overline{0, T-1}$ be a real periodic matrix, which satisfies $\Lambda(\Psi_F) = \Gamma$ and $\Lambda(\Psi_F) \cap \Lambda(\Psi_A) = 0$. Further, let $G_t = L_t^T X_t \in \mathbb{R}^{r \times n}, t \in \overline{0, T-1}$ be a real parametric matrix such that periodic matrix pair (F_t, G_t) is completely observable;
2) Set tolerance ε; Choose arbitrary initial periodic matrix $X_t(0) \in \mathbb{R}^{n \times n}, t \in \overline{0, T-1}$; calculate

$$\begin{aligned} Q_t(0) &= C_t^T G_t - A_t^T X_t(0) - X_{t+1}(0) \bar{F}_t; \\ R_t(0) &= A_t Q_t(0) + Q_{t-1}(0) \bar{F}_{t-1}^{\mathrm{T}}; \\ P_t(0) &= -R_t(0); \\ k &:= 0; \end{aligned}$$

3) While $\|R_t(k)\| \geq \varepsilon, t \in \overline{0, T-1}$, calculate

$$\begin{aligned} \alpha(k) &= \frac{\sum_{t=0}^{T-1} \mathrm{tr}\left[R_t^{\mathrm{T}}(k) P_t(k)\right]}{\sum_{t=0}^{T-1} \left\| A_t^T P_t(k) + P_{t+1}(k) \bar{F}_t \right\|^2}; \\ X_t(k+1) &= X_t(k) + \alpha(k) P_t(k) \in \mathbb{R}^{n \times n}; \\ Q_t(k+1) &= C_t^T G_t - A_t^T X_t(k+1) - X_{t+1}(k+1) \bar{F}_t; \\ R_t(k+1) &= A_t Q_t(k+1) + Q_{t-1}(k+1) \bar{F}_{t-1}^{\mathrm{T}}; \\ P_t(k+1) &= -R_t(k+1) + \frac{\sum_{t=0}^{T-1} \|R_t(k+1)\|^2}{\sum_{t=0}^{T-1} \|R_t(k)\|^2} P_t(k); \\ k &= k+1; \end{aligned}$$

4) Let $X_t = X_t(k)$. The real periodic matrix L_t can be obtained as

$$L_t = \left(G_t X_t^{-1} \right)^T, t \in \overline{0, T-1}.$$

Remark 3: The main part of the algorithm does not contain nested loops, so the computational complexity of the algorithm is $O(n)$.

Next, the convergence and correctness of the algorithm are proved.

Lemma 1: For sequences $\{R_t(k)\}$, $\{P_t(k)\}$, $t \in \overline{0, T-1}$, the following relations hold for $k \geq 0$:

$$\sum_{t=0}^{T-1} \mathrm{tr}\left[R_t^{\mathrm{T}}(k+1)P_t(k)\right] = 0 \tag{11}$$

$$\sum_{t=0}^{T-1} \mathrm{tr}\left[R_t^{\mathrm{T}}(k)P_t(k)\right] + \sum_{t=0}^{T-1} \|R_t(k)\|^2 = 0 \tag{12}$$

$$\sum_{j>0} \frac{\left(\sum_{t=0}^{T-1} \|R_t(k)\|^2\right)^2}{\sum_{t=0}^{T-1} \|P_t(k)\|^2} < \infty \tag{13}$$

Proof: According to the expression of $R_t(k+1)$, the following deduction is established.

$$\begin{aligned}
&R_t(k+1)\\
&= A_t Q_t(k+1) + Q_{t-1}(k+1)\bar{F}_{t-1}^T\\
&= A_t\left(C_t G_t - A_t^T X_t(k+1) - X_{t+1}(k+1)\bar{F}_t\right)\\
&\quad + \left(C_{t-1}G_{t-1} - A_{t-1}^T X_{t-1}(k) - X_t(k)\bar{F}_{t-1}\right)\bar{F}_{t-1}^T\\
&= A_t\left(C_t G_t - A_t^T X_t(k) - X_t \bar{F}_t\right)\\
&\quad + \left(C_{t-1}G_{t-1} - A_{t-1}^T X_{t-1} - X_t(k)\bar{F}_{t-1}\right)\bar{F}_{t-1}^T\\
&\quad - \alpha(k)A_t\left(A_t^T P_t(k) + P_{t+1}(k)\bar{F}_t\right)\\
&\quad - \alpha(k)\left(A_{t-1}^T P_{t-1}(k) + P_t(k)\bar{F}_{t-1}\right)\bar{F}_{t-1}^T\\
&= R_t(k) - \alpha(k)A_t\left(A_t^T P_t(k) + P_{t+1}(k)\bar{F}_t\right)\\
&\quad - \alpha(k)\left(A_{t-1}^T P_{t-1}(k) + P_t(k)\bar{F}_{t-1}\right)\bar{F}_{t-1}^T
\end{aligned}$$

Then, based on the deduction mentioned above and the expression of $\alpha(k)$ as well as the definition of Frobenius norm, the following deduction holds.

$$\begin{aligned}
&\sum_{t=0}^{T-1} \mathrm{tr}\left[R_t^{\mathrm{T}}(k+1)P_t(k)\right]\\
&= \sum_{t=0}^{T-1} \mathrm{tr}\left[R_t^{\mathrm{T}}(k)P_t(k)\right]\\
&\quad - \alpha(k)\sum_{t=0}^{T-1}\left[\left(A_t^{\mathrm{T}}P_t(k) + P_{t+1}(k)\bar{F}_t\right)^{\mathrm{T}} A_t^{\mathrm{T}}P_t(k)\right]\\
&\quad - \alpha(k)\sum_{t=0}^{T-1}\left[\left(A_{t-1}^{\mathrm{T}}P_{t-1}(k) + P_t(k)\bar{F}_{t-1}\right)^{\mathrm{T}} P_t(k)\bar{F}_{t-1}\right]\\
&= \sum_{t=0}^{T-1} \mathrm{tr}\left[R_t^{\mathrm{T}}(k)P_t(k)\right]\\
&\quad - \alpha(k)\sum_{t=0}^{T-1}\left\|A_t^{\mathrm{T}}P_t(k) + P_{t+1}(k)\bar{F}_t\right\|^2\\
&= 0
\end{aligned}$$

Thus, Equation (11) holds. Obviously, Equation (12) holds for $k = 0$. Then, according to the expression of $P_t(k+1)$ and Equation (11), the following deduction holds:

$$\begin{aligned}
&\sum_{t=0}^{T-1} \mathrm{tr}\left[R_t^{\mathrm{T}}(k+1)P_t(k+1)\right]\\
&= -\sum_{t=0}^{T-1} \mathrm{tr}\left[R_t^{\mathrm{T}}(k+1)R_t(k+1)\right]\\
&\quad + \frac{\sum_{t=0}^{T-1}\|R_t(k+1)\|^2}{\sum_{t=0}^{T-1}\|R_t(k)\|^2}\sum_{t=0}^{T-1} \mathrm{tr}\left[R_t^{\mathrm{T}}(k+1)P_t(k)\right]\\
&= -\sum_{t=0}^{T-1}\|R_t(k+1)\|^2
\end{aligned}$$

That's to say, Equation (12) holds. Applying Kronecker product, we can conduct as Equation (14), shown at the bottom of the next page, where,

$$\pi = \left\|\begin{bmatrix} E\otimes A_0^{\mathrm{T}} & \bar{F}_0^T\otimes E & & & \\ & E\otimes A_1^{\mathrm{T}} & \bar{F}_1^T\otimes E & & \\ & & E\otimes A_2^{\mathrm{T}} & & \\ & & & \ddots & \bar{F}_{T-2}^T\otimes E \\ \bar{F}_{T-1}^T\otimes E & & & & E\otimes A_{T-1}^{\mathrm{T}} \end{bmatrix}\right\|^2.$$

Review the Index (10) and again using the expression of $\alpha(k)$, the following deduction holds for $k \geq 0$.

$$\begin{aligned}
&J(k+1)\\
&= \frac{1}{2}\sum_{t=0}^{T-1}\left\|Q_t(k) - \alpha(k)\left[A_t^{\mathrm{T}}P_t(k) + P_{t+1}(k)\bar{F}_t\right]\right\|^2\\
&= \frac{1}{2}\sum_{t=0}^{T-1}\|Q_t(k)\|^2 + \frac{1}{2}\alpha^2(k)\sum_{t=0}^{T-1}\left\|A_t^{\mathrm{T}}P_t(k) + P_{t+1}(k)\bar{F}_t\right\|^2\\
&\quad - \alpha(k)\sum_{t=0}^{T-1}\mathrm{tr}\left[Q_t^T(k)\left(A_t^{\mathrm{T}}P_t(k) + P_{t+1}(k)\bar{F}_t\right)\right]\\
&= J(k) + \frac{1}{2}\alpha(k)\sum_{t=0}^{T-1}\mathrm{tr}\left[P_t^T(k)R_t(k)\right]\\
&\quad - \alpha(k)\sum_{t=0}^{T-1}\mathrm{tr}\left[P_t^T(k)A_tQ_t(k) + P_t^T(k)Q_{t-1}(k)\bar{F}_{t-1}^T\right]\\
&= J(k) - \frac{1}{2}\alpha(k)\sum_{t=0}^{T-1}\mathrm{tr}\left[P_t^T(k)R_t(k)\right].
\end{aligned}$$

Then, one has

$$J(k+1) - J(k) = -\frac{1}{2}\frac{\left(\sum_{t=0}^{T-1}\mathrm{tr}\left[P_t^T(k)R_t(k)\right]\right)^2}{\sum_{t=0}^{T-1}\left\|A_t^{\mathrm{T}}P_t(k) + P_{t+1}(k)\bar{F}_t\right\|^2} \leq 0, \tag{15}$$

which means that $\{J(k)\}$ is a descent sequence, so that

$$J(k+1) \leq J(0)$$

holds for all $k \geq 0$. Then

$$\sum_{k=0}^{\infty}[J(k) - J(k+1)] = J(0) - \lim_{k\to\infty} J(k) < \infty. \tag{16}$$

In view of Equation (12), (14) and (16), the following deduction holds:

$$\begin{aligned}
&\sum_{k\geq 0}\frac{\left(\sum_{t=0}^{T-1}\|R_t(k)\|^2\right)^2}{\sum_{t=0}^{T-1}\|P_t(k)\|^2}\\
&=\sum_{k\geq 0}\frac{\left(\sum_{t=0}^{T-1}\operatorname{tr}\left[R_t^T(k)P_t(k)\right]\right)^2}{\sum_{t=0}^{T-1}\|P_t(k)\|^2}\\
&\leq \pi\sum_{k\geq 0}\frac{\left(\sum_{t=0}^{T-1}\operatorname{tr}\left[R_t^T(k)P_t(k)\right]\right)^2}{\sum_{t=0}^{T-1}\left\|A_t^{\mathrm{T}}P_t(k)+P_{t+1}(k)\bar{F}_t\right\|^2}\\
&=2\pi(J(0)-\lim_{k\to\infty}J(k))\\
&<\infty.
\end{aligned}$$

To summarize, the Lemma 1 has been proved. ■

Based on the above lemma, the following conclusion could be drawn as:

Theorem 1: Consider the completely observable periodic discrete-time linear system (1), the T-periodic matrix L_t, $t\in\overline{0,T-1}$, derived from Algorithm 2 is a solution of Problem 1.

Proof: To explain that matrices $L_t, t\in\overline{0,T-1}$, which derived from Algorithm 2 are solutions to Problem 1, we first prove the convergence of matrix sequence $\{R_t(k)\}$, $t\in\overline{0,T-1}$ generated from Algorithm 2.

By Lemma 1 and the expressions of $P_t(k+1)$ in Algorithm 2, we have

$$\begin{aligned}
&\sum_{t=0}^{T-1}\|P_t(k+1)\|^2\\
&=\sum_{t=0}^{T-1}\left\|-R_t(k+1)+\frac{\sum_{t=0}^{T-1}\|R_t(k+1)\|^2}{\sum_{t=0}^{T-1}\|R_t(k)\|^2}P_t(k)\right\|^2\\
&=\left(\frac{\sum_{t=0}^{T-1}\|R_t(k+1)\|^2}{\sum_{t=0}^{T-1}\|R_t(k)\|^2}\right)^2\sum_{t=0}^{T-1}\|P_t(k)\|^2\\
&\quad+\sum_{t=0}^{T-1}\|R_t(k+1)\|^2. \qquad (17)
\end{aligned}$$

Equation (17) can be written as

$$t(k+1)=t(k)+\frac{1}{\sum_{t=0}^{T-1}\|R_t(k+1)\|^2} \qquad (18)$$

equivalently, where

$$t(k)=\frac{\sum_{t=0}^{T-1}\|P_t(k)\|^2}{\left(\sum_{t=0}^{T-1}\|R_t(k)\|^2\right)^2}.$$

Assume that

$$\lim_{k\to\infty}\sum_{t=0}^{T-1}\|R_t(k)\|^2\neq 0, \qquad (19)$$

which implies that there exists a constant $\delta>0$ such that

$$\sum_{t=0}^{T-1}\|R_t(k)\|^2\geq\delta$$

$$\begin{aligned}
\sum_{t=0}^{T-1}\left\|A_t^{\mathrm{T}}P_t(k)+P_{t+1}(k)\bar{F}_t\right\|^2&=\sum_{t=0}^{T-1}\left\|\left(E\otimes A_t^{\mathrm{T}}\right)\operatorname{vec}(P_t(k))+\left(\bar{F}_t^{\mathrm{T}}\otimes E\right)\operatorname{vec}(P_{t+1}(k))\right\|^2\\
&=\left\|\begin{array}{l}\left(E\otimes A_0^{\mathrm{T}}\right)\operatorname{vec}(P_0(k))+\left(\bar{F}_0^T\otimes E\right)\operatorname{vec}(P_1(k))\\ \left(E\otimes A_1^{\mathrm{T}}\right)\operatorname{vec}(P_1(k))+\left(\bar{F}_1^T\otimes E\right)\operatorname{vec}(P_2(k))\\ \qquad+\left(\bar{F}_{T-1}^T\otimes E\right)\operatorname{vec}(P_0(k))\end{array}\right\|^2\\
&=\left\|\begin{bmatrix}E\otimes A_0^{\mathrm{T}} & \bar{F}_0^T\otimes E & & & \\ & E\otimes A_1^{\mathrm{T}} & \bar{F}_1^T\otimes E & & \\ & & E\otimes A_2^{\mathrm{T}} & & \\ & & & \ddots & \bar{F}_{T-2}^T\otimes E\\ \bar{F}_{T-1}^T\otimes E & & & & E\otimes A_{T-1}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\operatorname{vec}(P_0(k))\\ \operatorname{vec}(P_1(k))\\ \operatorname{vec}(P_2(k))\\ \vdots\\ \operatorname{vec}(P_{T-1}(k))\end{bmatrix}\right\|^2\\
&\leq\left\|\begin{bmatrix}E\otimes A_0^{\mathrm{T}} & \bar{F}_0^T\otimes E & & & \\ & E\otimes A_1^{\mathrm{T}} & \bar{F}_1^T\otimes E & & \\ & & E\otimes A_2^{\mathrm{T}} & & \\ & & & \ddots & \bar{F}_{T-2}^T\otimes E\\ \bar{F}_{T-1}^T\otimes E & & & & E\otimes A_{T-1}^{\mathrm{T}}\end{bmatrix}\right\|^2\left\|\begin{bmatrix}\operatorname{vec}(P_0(k))\\ \operatorname{vec}(P_1(k))\\ \operatorname{vec}(P_2(k))\\ \vdots\\ \operatorname{vec}(P_{T-1}(k))\end{bmatrix}\right\|^2\\
&=\pi\sum_{j=0}^{T-1}\left\|P_j(k)\right\|^2 \qquad (14)
\end{aligned}$$

for all $k \geq 0$. It follows from (18) and (19) that

$$t(k+1) \leq t(k) + \frac{1}{\delta} \leq \cdots \leq t(0) + \frac{k+1}{\delta},$$

which means

$$\frac{1}{t(k+1)} \geq \frac{\delta}{\delta t(0) + k + 1}.$$

So we have

$$\sum_{k=1}^{\infty} \frac{1}{t(k)} \geq \sum_{k=1}^{\infty} \frac{\delta}{\delta t(0) + k + 1} = \infty.$$

However, according to Equation (13) that

$$\sum_{j=1}^{\infty} \frac{1}{t(k)} < \infty.$$

This gives a contradiction. Thus, there holds

$$\lim_{k\to\infty} \sum_{t=0}^{T-1} \|R_t(k)\|^2 = 0,$$

which indicates that the T-periodic matrix X_t generated by Algorithm 2 is one solution to the equation (7). Thereby, the T-periodic matrix L_t derived from Algorithm 2 is one solution to Problem 1. ∎

A. MINIMUM NORM AND ROBUST CONSIDERATION

The design of state observer also need to consider the problem of robustness. In previous work, we discussed the robustness of small gain and immunity to system uncertainty [13]. Therefore, we propose an index function, which takes into account both minimum norm processing and anti-interference processing.

Lemma 2 [13]: Let $\Psi = A(T-1)A(T-2)\cdots A(0) \in \mathbb{R}^{n\times n}$ be diagonalizable and $Q \in \mathbb{C}^{n\times n}$ be a nonsingular matrix such that $\Psi = Q^{-1}\Lambda Q \in \mathbb{R}^{n\times n}$, where $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is the Jordan canonical form of matrix Ψ. For a real scalar $\varepsilon > 0$, $\Delta_i(\varepsilon) \in \mathbb{R}^{n\times n}$, $i \in \overline{0, T-1}$, are matrix functions of ε satisfying

$$\lim_{\varepsilon\to 0^+} \frac{\Delta_i(\varepsilon)}{\varepsilon} = \Delta_i,$$

where $\Delta_i \in \mathbb{R}^{n\times n}$, $i \in \overline{0, T-1}$ are constant matrices. Then for any eigenvalue λ of matrix

$$\Psi(\varepsilon) = (A(T-1) + \Delta_{T-1}(\varepsilon)) \cdots (A(0) + \Delta_0(\varepsilon)),$$

the following relation holds:

$$\min_i\{|\lambda_i - \lambda|\} \leq \varepsilon n \kappa_{\mathrm{F}}(Q) \left(\sum_{i=0}^{T-1} \|A(i)\|_{\mathrm{F}}^{T-1} \right) \max_i\{\|\Delta_i\|_{\mathrm{F}}\} + \mathrm{O}(\varepsilon^2). \quad (20)$$

Then, one could take the index of anti-interference property as

$$J_1(G_t) = \kappa_{\mathrm{F}}(X(0)) \sum_{t=0}^{T-1} \|A_t - L_t C_t\|_{\mathrm{F}}^{T-1}. \quad (21)$$

When it comes to minimum norm processing, the paper takes the index as

$$J_1(G_t) = \sum_{t=0}^{T-1} \|L_t\|_{\mathrm{F}}^2. \quad (22)$$

Under the consideration of minimum norm and robustness, takes a tradeoff between J_1 and J_2:

$$J(G_t) = \alpha J_1(G_t) + (1-\alpha) J_2(G_t), \quad (23)$$

where $0 \leq \alpha \leq 1$ is a weighting factor. The algorithm for robust and minimum norm observer design can be presented as follows.

Algorithm 3 Robust and Minimum Norm Observer Design

1) Let $F_t \in \mathbb{R}^{n\times n}, t \in \overline{0, T-1}$ be a real periodic matrix, which satisfies $\Lambda(\Psi_F) = \Gamma$ and $\Lambda(\Psi_F) \cap \Lambda(\Psi_A) = 0$. Further, let $G_t = L_t^{\mathrm{T}} X_t \in \mathbb{R}^{r\times n}, t \in \overline{0, T-1}$ be a real parametric matrix such that periodic matrix pair (F_t, G_t) is completely observable;
2) Set tolerance ε; Choose arbitrary initial periodic matrix $X_t(0) \in \mathbb{R}^{n\times n}$, $t \in \overline{0, T-1}$; calculate
$$\begin{aligned} Q_t(0) &= C_t^{\mathrm{T}} G_t - A_t^{\mathrm{T}} X_t(0) - X_{t+1}(0)\bar{F}_t; \\ R_t(0) &= A_t Q_t(0) + Q_{t-1}(0)\bar{F}_{t-1}^{\mathrm{T}}; \\ P_t(0) &= -R_t(0); \\ k &:= 0; \end{aligned}$$
3) While $\|R_t(k)\| \geq \varepsilon$, $t \in \overline{0, T-1}$, calculate
$$\begin{aligned} \alpha(k) &= \frac{\sum_{t=0}^{T-1} \mathrm{tr}\left[P_t^{\mathrm{T}}(k) R_t(k)\right]}{\sum_{t=0}^{T-1} \left\|A_t^{\mathrm{T}} P_t(k) + P_{t+1}(k)\bar{F}_t\right\|^2}; \\ X_t(k+1) &= X_t(k) + \alpha(k) P_t(k) \in \mathbb{R}^{n\times n}; \\ Q_t(k+1) &= C_t^{\mathrm{T}} G_t - A_t^{\mathrm{T}} X_t(k+1) - X_{t+1}(k+1)\bar{F}_t; \\ R_t(k+1) &= A_t Q_t(k+1) + Q_{t-1}(k+1)\bar{F}_{t-1}^{\mathrm{T}}; \\ P_t(k+1) &= -R_t(k+1) + \frac{\sum_{t=0}^{T-1} \|R_t(k+1)\|^2}{\sum_{t=0}^{T-1} \|R_t(k)\|^2} P_t(k); \\ k &= k+1; \end{aligned}$$
4) Based on gradient-based search methods and the index (23), choosing the appropriate weighting factor α, solve the optimization problem
$$\text{Minimize } J(G_t),$$
and denote the optimal decision matrix by $G_{opt,t}$.
5) Substitute $G_{opt,t}$ into step 2-3 gives optimization solution $X_{opt,t}(k)$
6) Let $X_{opt,t} = X_{opt,t}(k)$. The robust and minimum norm periodic matrix $L_{opt,t}$ can be obtained as
$$L_{opt,t} = \left(G_{opt,t} X_{opt,t}^{-1}\right)^{\mathrm{T}}, t \in \overline{0, T-1}.$$

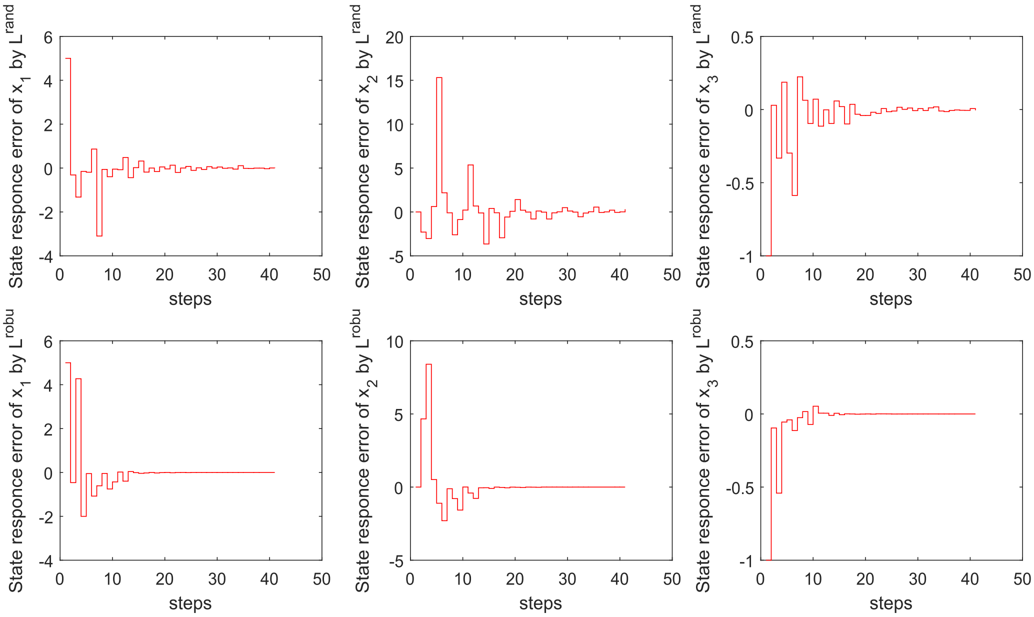

FIGURE 1. State response error polyline respectively corresponding to L_t^{rand} and L_t^{robu}.

IV. A NUMERICAL EXAMPLE

Example 1: Consider the observable discrete periodic system (1) with following parameters:

$$A_t = \begin{cases} \begin{bmatrix} -4.5 & -1 & 2 \\ 2.5 & 0.5 & 1 \\ 0.2 & 0.4 & 0.1 \end{bmatrix}, & t = 3k \\ \begin{bmatrix} 0 & 1 & 0.5 \\ 1 & 2 & 1.2 \\ 1.2 & 0 & 1 \end{bmatrix}, & t = 3k+1 \\ \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 0.1 & 0.3 \end{bmatrix}, & t = 3k+2 \end{cases},$$

$$B_t = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\mathrm{T}}, \quad t = 0, 1, \cdots,$$

$$C_t = \begin{cases} \begin{bmatrix} 2 & 0.5 & 1 \end{bmatrix}, & t = 3k \\ \begin{bmatrix} -1 & 0.3 & 1 \end{bmatrix}, & t = 3k+1 \\ \begin{bmatrix} 0 & 3 & 1 \end{bmatrix}, & t = 3k+2 \end{cases}$$

where $k = 0, 1, \cdots$, and let the pole set of the observer be $\Gamma = \{-0.1 \pm 0.1i, -0.1\}$. Let $G(t) = [1.5 \;\; 1 \; -1.5]$ randomly and substitute the parameters into Algorithm 2, the following periodic observer gain can be obtained:

$$L_t^{\mathrm{rand}} = \begin{cases} \begin{bmatrix} -1.9047 & 1.2999 & 0.3712 \end{bmatrix}^{\mathrm{T}}, & t = 3k \\ \begin{bmatrix} 3.4261 & 6.7995 & -0.0732 \end{bmatrix}^{\mathrm{T}}, & t = 3k+1 \\ \begin{bmatrix} 0.7874 & -7.2746 & 0.2120 \end{bmatrix}^{\mathrm{T}}, & t = 3k+2. \end{cases}$$

Giving $\alpha = 0.5$ and applying Algorithm 3 give the following robust periodic observer gain:

$$L_t^{\mathrm{robu}} = \begin{cases} \begin{bmatrix} -2.2170 & 1.1513 & 0.3406 \end{bmatrix}^{\mathrm{T}}, & t = 3k \\ \begin{bmatrix} 0.1512 & 0.1517 & -0.1218 \end{bmatrix}^{\mathrm{T}}, & t = 3k+1 \\ \begin{bmatrix} 0.6359 & -0.2399 & 0.0857 \end{bmatrix}^{\mathrm{T}}, & t = 3k+2. \end{cases}$$

As far as minimum norm is concerned, by computing $\|L\| = \sqrt{\sum_{t=0}^{3} \|L_t\|_F^2}$ for L_t^{rand} and L_t^{robu} respectively, we can obtain that $\left\|L^{rand}\right\| = 10.8174$ and $\left\|L^{robu}\right\| = 2.6242$. What can be concluded is that the minimum norm processing of the observer gain is successful.

Let the periodic close-loop system matrix be perturbed by random disturbance $\Delta_t^A \in \mathbb{R}^{3\times3}$ and $\Delta_t^C \in \mathbb{R}^{3\times1}$, which satisfy $\left\|\Delta_t^A\right\|_F = 1, \left\|\Delta_t^C\right\|_F = 1, t = 0, 1, \cdots$. Then the disturbed close-loop system matrix can be represented as:

$$A_t + \mu\Delta_t^A - L_t\left(C_t + \mu\Delta_t^C\right), t = 0, 1, \cdots,$$

where $\mu > 0$ is a factor representing the disturbance level. Let discrete reference input $v(t) = 0.1\sin(\frac{\pi}{2} + t)$ and error $e_t = \hat{x}_t - x_t$. Then the results of the state response error polyline respectively corresponding to L_t^{rand} and L_t^{robu} in step 40 can be obtained in Figure 1. By comparison, the robust observer design algorithm proposed in this paper is effective.

V. CONCLUSION

The paper presents an ideal approach to design state observers for linear discrete-time periodic systems. The algorithm can also be applied in the cases where uncertain disturbances existed. It is reasonable and successful to transform the observer design problem into the solution to the corresponding matrix equation. Simulation results have shown the correctness and efficiency of the proposed algorithm. Observer-based robust stabilization of linear discrete-time periodic systems will be the focus of the next step.

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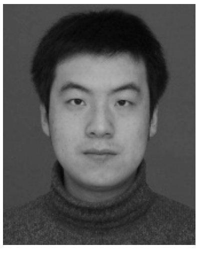
LINGLING LV received the Ph.D. degree from the Harbin Institute of Technology, Harbin, China, in 2010. She is currently an Associate Professor with North China University of Water Resources and Electric Power, Zhengzhou, China. Her research interests cover periodic systems, optimal control, and robust control.

ZHE ZHANG received the bachelor's degree from Central South University, Changsha, China, in 2015. He is currently pursuing the degree at the North China University of Water Resources and Electric Power, Zhengzhou, China. His research interests cover periodic systems and robust control.



LEI ZHANG received the Ph.D. degree from the Harbin Institute of Technology, Harbin, China, in 2015. He is currently an Associate Professor with Henan University, Kaifeng, China. His research interests cover information safety, big data, and robust control.