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Model Predictive Control With Mixed Performances for Uncertain Positive Systems

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ABSTRACT This paper addresses the model predictive control for positive systems with uncertainty and exogenous disturbance input. By utilizing a linear Lyapunov function, a sequence of model predictive controllers for positive systems is designed based on multi-step control sets guaranteeing robust stability with mixed performances. A sequence of cone sets is chosen as the invariant sets of the model predictive control. With the above idea, a model predictive control algorithm is established to compute the optimal value of the mixed performances. All conditions are in terms of linear programming to cope with large-scale computation with low computation burden. Finally, a numerical example is provided to verify the effectiveness of the proposed design.

INDEX TERMS Model predictive control, positive systems, multi-step control strategy, uncertainty.

I. INTRODUCTION

The past two decades have witnessed an increasing interest in positive systems [1]–[4] since there are lots of real control systems that can be modeled as positive systems. In [5], a communication network that employs drop-tail queueing and additive-increase multiplicative-decrease congestion control algorithms was described as positive system. The problem of establishing an aggregate production plan for a manufacturing plant was considered using a positive system based approach [6]. The routing control problem of dynamic continuous-time networks in each node is described by a positive system [7]. On the other hand, there have been some theory results on positive systems [8]–[12]. In particular, control synthesis of positive systems has attracted a lot of attention. Linear programming was used to design the controller and the observer of positive systems in [13] and [14]. A constrained controller based on linear programming for positive system with delays was constructed in [15]. In [16], stability analysis and control synthesis of positive systems were considered using linear matrix inequalities. A distributed controller [17] was developed for positive systems using a linear Lyapunov function. As we all know, an optimal control design is always desired to improve system performance and there have been various methods dealing with different system performance indexes. L_1- and L_∞-gain performances based robust stability and stabilization [18] were explored in terms to linear programming. Static output-feedback and state-feedback stabilization with optimal L_1/ℓ_1-gain for positive systems were solved in [19] and [20], respectively. Control synthesis with L_1-gain performance was addressed for interconnected positive systems [21]. H_2 suboptimal controllers [22] using linear matrix inequalities were developed for positive systems. An optimal quadratic complementary controller [23] was presented to solve the servomechanism problem of a multivariable positive system.

It is well known that model predictive control (MPC) is a popular control technique [24]–[29]. An MPC controller is designed at each sampling time instant by solving an optimization problem over a prescribed future time instants and the optimization problem is resolved at the next time instant to realize the receding horizon strategy. This property enables MPC to possess a powerful ability to explicitly deal with the constrained problems. In [30], a linear matrix inequalities-based MPC framework was used to design the state-feedback control law that minimizes an upper bound on the robust performance objective subject to input and output constraints. Following the MPC framework [30], the literature [31] and [32] constructed a parameter-dependent Lyapunov function based MPC design and an efficient MPC was also proposed in [33]–[36] to reduce the computation burden [30]. Using mixed H_2/H_∞ design method, the MPC was also extended for the systems with exogenous disturbance input [36]–[38]. In general, the control set used in [30]–[37] is called the single step control set. A new multi-step control set was presented in [39] for constrained polyhedral uncertain systems, where the designed robust controller can achieve a large feasible region and high control performance compared with the single step control set approach.

In [40], the MPC with a single step control set was proposed for interval and polytopic positive systems. The literature [41] considered ℓ_1-gain based MPC for nominal positive systems with exogenous disturbance input. This paper is a continuation of the works in [40] and [41]. The paper investigates the MPC for interval/polytopic positive systems with exogenous input. Compared with existing literature, the contribution of the paper lies in the facts: (i) the designed MPC controller is general without sign and rank constraints; (ii) mixed performance indexes based robust stabilization for interval/polytopic positive systems is solved; and (iii) a sequence of cone invariant sets based multi-step MPC control strategy is constructed for positive systems. The remainder of the paper is organized as follows. Section 2 gives the problem formulation and some preliminaries. In Section 3, the main results are addressed. Section 4 provides a numerical example. Section 5 concludes the paper.

Notations: The symbols $\Re, \Re^n, \Re^{n\times n}$ represent the sets of real numbers, n-dimensional vectors, and $n \times n$ matrices, respectively. Let $\mathbb{N}$ and $\mathbb{N}^+$ denote the nonnegative and positive integers, respectively. The symbol *co* stands for the convex hull. A matrix I is the identity matrix with compatible dimensions. For a vector v, the inequality $v \succeq 0$ ($v \succ 0$) means that all of its components $v_i \geq 0$ ($v_i > 0$). For a matrix A, the inequality $A \succeq 0$ ($A \succ 0$) means that all of its ith row jth column components $a_{ij} \geq 0$ ($a_{ij} > 0$). Then, the inequality $A \succeq B$ ($A \succ B$) means that $a_{ij} \geq b_{ij}$ ($a_{ij} > b_{ij}$), where a_{ij} and b_{ij} are the ith row jth column components of matrices A and B, respectively. Similarly, the inequalities $v \preceq 0$ ($v \prec 0$), $A \preceq 0$ ($A \prec 0$), and $A \preceq B$ ($A \prec B$) can be defined. The vector set $\Re^n_+$ consists of nonnegative nonzero vectors. Set $\mathbf{1}_n = (\underbrace{1, \ldots, 1}_{n})^T$ and $\mathbf{1}_n^{(i)} = (\underbrace{0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{n-i})^T$. The Euclidean norm and 1-norm of a vector $x \in \Re^n$ are defined by $\|x\|_2 = \sqrt[2]{\Sigma_{i=1}^n x_i^2}$ and $\|x\|_1 = \Sigma_{i=1}^n |x_i|$. Given a discrete-time function $w(k) : \mathbb{N} \to \Re^n$, its ℓ_1 norm is defined as $\|w(k)\|_{\ell_1} = \sum_{k=0}^\infty \|w(k)\|_1$. Furthermore, we define the vector space $\ell_1[0, \infty) \triangleq \{w(k)|w(k)$ is measureble in $[0, \infty)$ and $\|w(k)\|_{\ell_1} < \infty\}$.

II. PROBLEM STATEMENT

Consider the linear time-varying system:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) + D(k)w(k), \\ z(k) &= C(k)x(k) + E(k)w(k), \end{aligned} \tag{1}$$

where $x(k) \in \Re^n, z(k) \in \Re^s, u(k) \in \Re^m, w(k) \in \Re^r_+$ represent the system state, the measured output, the control input, and the exogenous disturbance input, respectively. The system matrices $\sum : [A(k)|B(k)|C(k)|D(k)|E(k)]$ are time-varying and hold compatible dimensions. In the paper, the system matrices are assumed to locate into two classed of uncertain sets: interval uncertain set (2), as shown at the top of the next page, and polytope uncertain set (3), as shown at the top of the next page, $\forall k \in \mathbb{N}$, where $\underline{A} \succeq 0, \underline{B} \succeq 0, \underline{C} \succeq 0, \underline{D} \succeq 0, \underline{E} \succeq 0$ and $A_i \succeq 0, B_i \succeq 0, C_i \succeq 0, D_i \succeq 0, E_i \succeq 0$ $\forall i \in \{1, 2, \ldots, l\}$. The exogenous disturbance input $w(k)$ satisfies:

$$w(k)^T \mathbf{1}_r \leq \eta_1, \quad \sum\nolimits_{k=0}^{\infty} w(k)^T \mathbf{1}_r \leq \eta_2, \tag{4}$$

where $\eta_1 > 0, \eta_2 > 0$ are given constants.

System (1) with uncertain sets Ω_1 and Ω_2 are called interval and polytope systems, respectively. In [39], it has been verified that the two classes of uncertain systems are suitable for describing time-varying system (1), especially time-varying positive system (1). The objective of the paper is to design a sequence of control laws $u(k+i|k) = K_i x(k+i|k)$, $i = 0, 1, \ldots, N$ such that the system (1) is robustly stable with ℓ_1-gain performance:

$$\|z(k)\|_{\ell_1} \leq \gamma_1 \|w(k)\|_{\ell_1} \tag{5}$$

with $x(0) = 0$ and 1-norm based bounded output:

$$\|z(k)\|_1 \leq \gamma_2, \tag{6}$$

where $x(k+i|k)$, $u(k+i|k)$ are the state and control law predicted at time instant k, $N \in \mathbb{N}^+$ is the predicted step, and $\gamma_1 > 0$, $\gamma_2 > 0$. The value of γ_1 represents the disturbance attenuation level from the disturbance $w(k)$ to the output $z(k)$. Minimizing the value of γ_1 will increase the ability to attenuate the disturbance. From (6), the 1-morm $\|z(k)\|_1$ shows the dynamic performance from the initial state to the output. Then, minimizing the value of γ_2 will obtain better system performance. In view of the aforementioned facts, the formulas (5) and (6) are called mixed performance indexes of MPC for system (1). In [36]–[38], mixed H_∞/H_2 performance indexes were used for the MPC design of general systems. Considering the positivity property of positive systems, we introduce the mixed performance indexes (5) and (6), which are natural extensions of mixed H_∞/H_2 performance.

The system state $x(k)$ and the controller gain $K_i = K_i^- + K_i^+$ proposed in later sections are subject to the constraints:

$$x(k)^T \mathbf{1}_n \leq \delta, \tag{7a}$$

$$K_i^{+T} \mathbf{1}_m - K_i^{-T} \mathbf{1}_m \preceq \theta, \tag{7b}$$

where $K_i^- \preceq 0$, $K_i^+ \succeq 0$, and $\delta > 0$, $\theta \succ 0$, $\theta \in \Re^n$ are given. We will explain why the condition (7) is used in Section III(C).

In the following, we introduce some preliminaries of positive systems.

$$\Omega_1 \triangleq \{\sum \left| \underline{A} \preceq A(k) \preceq \overline{A}, \underline{B} \preceq B(k) \preceq \overline{B}, \underline{C} \preceq C(k) \preceq \overline{C}, \underline{D} \preceq D(k) \preceq \underline{D}, \underline{E} \preceq E(k) \preceq \overline{E} \right\} \quad (2)$$

$$\Omega_2 \triangleq \{\sum \left| \sum \in co\{[A_1|B_1|C_1|D_1|E_1], [A_1|B_1|C_1|D_1|E_1], \ldots, [A_l|B_l|C_l|D_l|E_l], \}, \forall k \in \mathbb{N}, l \in \mathbb{N}^+ \right\} \quad (3)$$

Definition 1 [1], [2]: A discrete-time system

$$x(k+1) = Ax(k) + Bu(k) + Dw(k), \quad z(k) = Cx(k) + Ew(k), \quad (8)$$

is positive if the state $x(k) \succeq 0$ and the output $z(k) \succeq 0$ hold $\forall u(k) \succeq 0, \forall w(k) \succeq 0$ under any nonnegative initial condition $x(k_0) \succeq 0$.

Lemma 1 [1], [2]: System (8) is positive if and only if $A \succeq 0, B \succeq 0, C \succeq 0, D \succeq 0, E \succeq 0$.

For interval system (1), it can be derived that $A(k) \succeq 0$, $B(k) \succeq 0$, $C(k) \succeq 0$, $D(k) \succeq 0$, $E(k) \succeq 0$, $\forall k \in \mathbb{N}$ using the interval uncertain set (2). For polytope system (1), we have $[A(k)|B(k)|C(k)|D(k)|E(k)] = \sum_{i=1}^{l} \lambda_i [A_i|B_i|C_i|D_i|E_i] \succeq 0$, $\sum_{i=1}^{l} \lambda_i = 1, \lambda_i \geq 0$ by the polytope uncertain set (3). Then, we can obtain that $A(k) \succeq 0$, $B(k) \succeq 0$, $C(k) \succeq 0$, $D(k) \succeq 0$, $E(k) \succeq 0, \forall k \in\in \mathbb{N}$. Finally, interval and polytope system (1) are positive by Lemma 1.

Lemma 2 [1], [2]: Let $A \succeq 0$, then the following statements are equivalent:

(a) The matrix A is Schur;

(b) There is a vector $\nu \succ 0$ in $\Re^n$ such that $(A - I)\nu \prec 0$.

(c) There is a vector $\nu \succ 0$ in $\Re^n$ such that $(A - I)^T \nu \prec 0$.

III. MAIN RESULTS

This section will present the MPC design for interval/polytope systems described by (1) via three steps: firstly, the MPC controller is designed; secondly, a sequence of cone invariant sets based on the multi-step control approach is constructed; finally, the robust stability of the considered systems is achieved.

A. MPC CONTROLLER DESIGN

Theorem 1: (a) If there exist constants $\hbar > 1$, $\varsigma_i > 0$, $\gamma_2 > 0$ and vectors $\nu^{(i)} \succ 0$, $\nu^{(i)} \in \Re^n$, $\xi^{(\iota)} \in \Re^n$, $\overline{\xi} \prec 0$, $\overline{\xi} \in \Re^n$, $\zeta^{(\iota)} \succeq 0$, $\zeta^{(\iota)} \in \Re^n$, $\overline{\zeta} \in \Re^n$ such that

$$\hbar \underline{A} \mathbf{1}_m^T \underline{B}^T \nu^{(i+1)} + \hbar \overline{B} \sum_{\iota=1}^{m} \mathbf{1}_m^{(\iota)} \xi^{(\iota)T} + \underline{B} \sum_{\iota=1}^{m} \mathbf{1}_m^{(\iota)} \zeta^{(\iota)T} \succeq 0, \quad (9a)$$

$$\overline{A}^T \nu^{(i+1)} + \overline{\xi} + \overline{\zeta} - \nu^{(i)} + \overline{C}^T \mathbf{1}_s \prec 0, \quad (9b)$$

$$\overline{D}^T \nu^{(i+1)} - \varsigma_i \mathbf{1}_r + \overline{E}^T \mathbf{1}_s \prec 0, \quad (9c)$$

$$\mathbf{1}_m^T \overline{B}^T \nu^{(i+1)} \leq \hbar \mathbf{1}_m^T \underline{B}^T \nu^{(i+1)}, \quad (9d)$$

$$\xi^{(\iota)} \preceq \overline{\xi}, \quad \zeta^{(\iota)} \preceq \overline{\zeta}, \quad \iota = 1, 2, \ldots, m, \quad (9e)$$

$$\varsigma_i \geq \varsigma_{i+1}, \quad (9f)$$

and

$$x(0)^T \nu^{(0)} + \varsigma_0 \eta_2 \leq \gamma_2 \quad (10)$$

hold for $i = 0, 1, \ldots, N-1$ and $\nu^{(i)} = \nu^{(N)}$, $\varsigma_i = \varsigma_N$ for $i \geq N$, then under the MPC control law

$$u(k+i|k) = K_i x(k+i|k) = (K_i^- + K_i^+) x(k+i|k) \quad (11)$$

with

$$K_i^- = \frac{\sum_{\iota=1}^{m} \mathbf{1}_m^{(\iota)} \xi^{(\iota)T}}{\mathbf{1}_m^T \underline{B}^T \nu^{(i+1)}}, \quad K_i^+ = \frac{\sum_{\iota=1}^{m} \mathbf{1}_m^{(\iota)} \zeta^{(\iota)T}}{\hbar \mathbf{1}_m^T \underline{B}^T \nu^{(i+1)}}, \quad (12)$$

interval system (1) is positive and satisfies the mixed performance indexes (5) and (6), where $x(0)$ is the initial condition.

(b) If there exist constants $\hbar > 1$, $\varsigma_i > 0$ and vectors $\nu^{(i)} \succ 0$, $\nu^{(i)} \in \Re^n$, $\xi^{(\iota)} \in \Re^n$, $\overline{\xi} \prec 0$, $\overline{\xi} \in \Re^n$, $\zeta^{(\iota)} \succeq 0$, $\zeta^{(\iota)} \in \Re^n$, $\overline{\zeta} \in \Re^n$ such that

$$\hbar A_j \mathbf{1}_m^T \widehat{B}^T \nu^{(i+1)} + \hbar B_j \sum_{\iota=1}^{m} \mathbf{1}_m^{(\iota)} \xi^{(\iota)T} + B_j \sum_{\iota=1}^{m} \mathbf{1}_m^{(\iota)} \zeta^{(\iota)T} \succeq 0, \quad (13a)$$

$$A_j^T \nu^{(i+1)} + \overline{\xi} + \overline{\zeta} - \nu^{(i)} + C_j^T \mathbf{1}_s \prec 0, \quad (13b)$$

$$D_j^T \nu^{(i+1)} - \varsigma_i \mathbf{1}_r + E_j^T \mathbf{1}_s \prec 0, \quad (13c)$$

$$\mathbf{1}_m^T B_j^T \nu^{(i+1)} \leq \hbar \mathbf{1}_m^T \widehat{B}^T \nu^{(i+1)}, \quad (13d)$$

$$\xi^{(\iota)} \preceq \overline{\xi}, \quad \zeta^{(\iota)} \preceq \overline{\zeta}, \quad \iota = 1, 2, \ldots, m, \quad (13e)$$

$$\varsigma_i \geq \varsigma_{i+1}, \quad (13f)$$

and (10) hold for $j = 1, 2, \ldots, l$, $i = 0, 1, \ldots, N-1$ and $\nu^{(i)} = \nu^{(N)}$, $\varsigma_i = \varsigma_N$ for $i \geq N$, then under the MPC control law

$$u(k+i|k) = K_i x(k+i|k) = (K_i^- + K_i^+) x(k+i|k) \quad (14)$$

with

$$K_i^- = \frac{\sum_{\iota=1}^{m} \mathbf{1}_m^{(\iota)} \xi^{(\iota)T}}{\mathbf{1}_m^T \widehat{B}^T \nu^{(i+1)}}, \quad K_i^+ = \frac{\sum_{\iota=1}^{m} \mathbf{1}_m^{(\iota)} \zeta^{(\iota)T}}{\hbar \mathbf{1}_m^T \widehat{B}^T \nu^{(i+1)}}, \quad (15)$$

polytopic system (1) is positive and satisfies the mixed performance indexes (5) and (6), where

$$\widehat{B} = [\widehat{b}_{\iota\jmath}], \quad B_j = [b_{\iota\jmath}^{(j)}], \ \widehat{b}_{\iota\jmath} = \min_{j=1,2,\ldots,l} \{b_{\iota\jmath}^{(j)}\}, \quad (16)$$

for $\iota = 1, 2, \ldots, n$, $\jmath = 1, 2, \ldots, m$.

Proof: (a) First, we prove that interval system (1) is positive. Since $\hbar \mathbf{1}_n^T \lambda > 0$, we have

$$\underline{A} + \overline{B} \frac{\sum_{\iota=1}^{m} \mathbf{1}_m^{(\iota)} \xi^{(\iota)T}}{\mathbf{1}_m^T \underline{B}^T \nu^{(i+1)}} + \underline{B} \frac{\sum_{\iota=1}^{m} \mathbf{1}_m^{(\iota)} \zeta^{(\iota)T}}{\hbar \mathbf{1}_m^T \underline{B}^T \nu^{(i+1)}} \succeq 0. \quad (17)$$

From (12), it is shown that $\underline{A} + \overline{B} K_i^- + \underline{B} K_i^+ \succeq 0$ with $K_i^- \prec 0$ and $K_i^+ \succ 0$. Then, $A(k) + B(k) K_i \succeq \underline{A} + \overline{B} K_i^- + \underline{B} K_i^+ \succeq 0$. By Lemma 1, interval system (1) is positive, which means that $x(k) \succeq 0, \forall k \in \mathbb{N}$.

Next, we prove that mixed performances (5) and (6) are satisfied. Choose a linear Lyapunov function:

$$V(k+i) = x(k+i|k)^T v^{(i)} + \varsigma_i \sum_{n=i}^{\infty} w(k+n)^T \mathbf{1}_r. \quad (18)$$

Then

$$\begin{aligned} V(k+i+1) - V(k+i) = {} & x(k+i|k)^T \big(A(k+i)^T v^{(i+1)} \\ & + K_i^T B(k+i)^T v^{(i+1)} - v^{(i)}\big) \\ & + w(k+i)^T D(k+i)^T v^{(i+1)} \\ & + \varsigma_{i+1} \sum_{n=i+1}^{\infty} w(n)^T \mathbf{1}_r \\ & - \varsigma_i \sum_{n=i}^{\infty} w(n)^T \mathbf{1}_r. \end{aligned} \quad (19)$$

Owing to (2) and $K_i^- \prec 0$, $K_i^+ \succ 0$, we obtain

$$\begin{aligned} K_i^T B(k+i)^T v^{(i+1)} & \preceq K_i^{-T} \underline{B}^T v^{(i+1)} + K_i^{+T} \overline{B}^T v^{(i+1)} \\ & = \frac{\sum_{\iota=1}^m \xi^{(\iota)} \mathbf{1}_m^{(\iota)T} \underline{B}^T v^{(i+1)}}{\mathbf{1}_m^T \underline{B}^T v^{(i+1)}} \\ & \quad + \frac{\sum_{\iota=1}^m \zeta^{(\iota)} \mathbf{1}_m^{(\iota)T} \overline{B}^T v^{(i+1)}}{\hbar \mathbf{1}_m^T \underline{B}^T v^{(i+1)}}. \end{aligned} \quad (20)$$

By (9d) and (9e), (20) is transformed into

$$\begin{aligned} K_i^T B(k+i)^T v^{(i+1)} & \preceq \frac{\overline{\xi} \sum_{\iota=1}^m \mathbf{1}_m^{(\iota)T} \underline{B}^T v^{(i+1)}}{\mathbf{1}_m^T \underline{B}^T v^{(i+1)}} \\ & \quad + \frac{\overline{\zeta} \sum_{\iota=1}^m \mathbf{1}_m^{(\iota)T} \overline{B}^T v^{(i+1)}}{\hbar \mathbf{1}_m^T \underline{B}^T v^{(i+1)}} \\ & = \frac{\overline{\xi} \mathbf{1}_m^T \underline{B}^T v^{(i+1)}}{\mathbf{1}_m^T \underline{B}^T v^{(i+1)}} + \frac{\overline{\zeta} \mathbf{1}_m^T \overline{B}^T v^{(i+1)}}{\hbar \mathbf{1}_m^T \underline{B}^T v^{(i+1)}} \\ & = \overline{\xi} + \frac{\overline{\zeta} \mathbf{1}_m^T \overline{B}^T v^{(i+1)}}{\hbar \mathbf{1}_m^T \underline{B}^T v^{(i+1)}} \\ & \preceq \overline{\xi} + \overline{\zeta}. \end{aligned} \quad (21)$$

Combining (2), (9b), and (21) yields

$$\begin{aligned} & A(k+i)^T v^{(i+1)} + K_i^T B(k+i)^T v^{(i+1)} - v^{(i)} \\ & \quad \preceq \overline{A}^T v^{(i+1)} + \overline{\xi} + \overline{\zeta} - v^{(i)} \preceq -\overline{C}^T \mathbf{1}_s. \end{aligned} \quad (22)$$

By (9f) and (22), (19) becomes

$$\begin{aligned} & V(k+i+1) - V(k+i) \\ & \quad \leq -x(k+i|k)^T \overline{C}^T \mathbf{1}_s + w(k+i)^T \\ & \quad \times D(k+i)^T v^{(i+1)} \\ & \quad + \varsigma_i \sum_{n=i+1}^{\infty} w(k+n)^T \mathbf{1}_r \\ & \quad - \varsigma_i \sum_{n=i}^{\infty} w(k+n)^T \mathbf{1}_r \\ & \quad \leq -x(k+i|k)^T \overline{C}^T \mathbf{1}_s \\ & \quad + w(k+i)^T D(k+i)^T v^{(i+1)} \\ & \quad - \varsigma_i w(k+i)^T \mathbf{1}_r \\ & \quad \leq -x(k+i|k)^T \overline{C}^T \mathbf{1}_s \\ & \quad + w(k+i)^T (\overline{D}^T v^{(i+1)} - \varsigma_i \mathbf{1}_r), \end{aligned} \quad (23)$$

Substituting (9c) into (23) gives

$$\begin{aligned} & V(k+i+1) - V(k+i) \\ & \quad \leq -x(k+i|k)^T \overline{C}^T \mathbf{1}_s - w(k+i)^T \overline{E}^T \mathbf{1}_s \\ & \quad \leq -x(k+i|k)^T C(k+i)^T \mathbf{1}_s \\ & \qquad - w(k+i)^T E(k+i)^T \mathbf{1}_s \\ & \quad = -\|z(k+i)\|_1. \end{aligned} \quad (24)$$

Summing both sides of (24) from $i = 0$ to ∞ yields

$$V(\infty) - V(k) \leq -\sum_{i=0}^{\infty} \|z(k+i)\|_1. \quad (25)$$

Let $k = 0$, we have

$$\sum_{i=0}^{\infty} \|z(i)\|_1 \leq V(0) - V(\infty) \leq V(0) \quad (26)$$

owing to the fact $V(\infty) \geq 0$. Thus, we get $\sum_{i=0}^{\infty} \|z(i)\|_1 \leq \varsigma_0 \sum_{i=0}^{\infty} \|w(i)\|_1$ when $x(0) = 0$. This reveals that the performance (5) is satisfied as long as one chooses $\gamma_1 = \varsigma_0$.

Noting the condition (4), then

$$\begin{aligned} \sum_{i=0}^{\infty} \|z(i)\|_1 & \leq V(0) = x(0)^T v^{(0)} + \varsigma_0 \sum_{n=0}^{\infty} w(n)^T \mathbf{1}_r \\ & \leq x(0)^T v^{(0)} + \varsigma_0 \eta_2. \end{aligned} \quad (27)$$

Thus, the performance (6) is satisfied as long as (10) holds.

(b) We first prove that polytopic system (1) is positive. Under the control law (14), the resulting closed-loop polytopic system (1) with the control law (14) is

$$\begin{aligned} x(k+i+1) = {} & \big(A(k+i) + B(k+i)K_i\big)x(k+i|k) \\ & + D(k+i)w(k+i) \\ = {} & \sum_{j=1}^{l} \lambda_j (A_j + B_j K_i) x(k+i|k) \\ & + \sum_{j=1}^{l} \lambda_j D_j w(k+i), \end{aligned} \quad (28)$$

where $\sum_{j=1}^{l} \lambda_j = 1, \lambda_j \geq 0$. By (13a) and (16), we obtain $A_j + B_j \frac{\sum_{\iota=1}^m \mathbf{1}_m^{(\iota)} \xi^{(\iota)T}}{\mathbf{1}_m^T \widehat{B}^T v^{(i+1)}} + B_j \frac{\sum_{\iota=1}^m \mathbf{1}_m^{(\iota)} \zeta^{(\iota)T}}{\hbar \mathbf{1}_m^T \widehat{B}^T v^{(i+1)}} \succeq 0$. Using (14) and (15) leads to $A_j + B_j K_i \succeq 0$. Then, $A(k+i) + B(k+i)K_i \succeq 0$. By (3), it follows that $D(k+i) \succeq 0$. Therefore, the closed-loop system is positive by Lemma 1.

By (16) and (13d), we have $\frac{\mathbf{1}_m^T B_j^T v^{(i+1)}}{\mathbf{1}_m^T \widehat{B}^T v^{(i+1)}} \geq 1$ and $\frac{\mathbf{1}_m^T B_j^T v^{(i+1)}}{\hbar \mathbf{1}_m^T \widehat{B}^T v^{(i+1)}} \leq 1$. From (13e), (14), and (15), we deduce

$$\begin{aligned} K_i^T B_j^T v^{(i+1)} & \preceq \frac{\overline{\xi} \mathbf{1}_m^T B_j^T v^{(i+1)}}{\mathbf{1}_m^T \widehat{B}^T v^{(i+1)}} + \frac{\overline{\zeta} \mathbf{1}_m^T B_j^T v^{(i+1)}}{\hbar \mathbf{1}_m^T \widehat{B}^T v^{(i+1)}} \\ & \preceq \overline{\xi} + \overline{\zeta}. \end{aligned} \quad (29)$$

Choose a linear Lyapunov function candidate as (18), then

$$\begin{aligned} & V(k+i+1) - V(k+i) \\ & \quad = x(k+i|k)^T \sum_{j=1}^{l} \lambda_j \big(A_j^T v^{(i+1)} + K_i^T B_j^T v^{(i+1)} - v^{(i)}\big) \\ & \quad + w(k+i)^T \sum_{j=1}^{l} \lambda_j D_j^T v^{(i+1)} \\ & \quad + \varsigma_{i+1} \sum_{n=i+1}^{\infty} w(n)^T \mathbf{1}_r \\ & \quad - \varsigma_i \sum_{n=i}^{\infty} w(n)^T \mathbf{1}_r \end{aligned}$$

$$\le x(k+i|k)^T \sum_{j=1}^{l} \lambda_j\big(A_j^T v^{(i+1)} + K_i^T B_j^T v^{(i+1)} - v^{(i)}\big) + w(k+i)^T \sum_{j=1}^{l} \lambda_j\big(D_j^T v^{(i+1)} - \varsigma_i \mathbf{1}_r\big). \quad (30)$$

from (3) and (13f), where $\sum_{j=1}^{l} \lambda_j = 1, \lambda_j \ge 0$. Substituting (29) into (30) yields

$$\begin{aligned} &V(k+i+1) - V(k+i) \\ &\quad \le x(k+i|k)^T \sum_{j=1}^{l} \lambda_j\big(A_j^T v^{(i+1)} + \overline{\xi} + \overline{\zeta} - v^{(i)}\big) \\ &\qquad + w(k+i)^T \sum_{j=1}^{l} \lambda_j\big(D_j^T v^{(i+1)} - \varsigma_i \mathbf{1}_r\big). \end{aligned} \quad (31)$$

By (13b) and (13c), (30) is transformed into

$$\begin{aligned} V(k+i+1) - V(k+i) &\le -x(k+i|k)^T \sum_{j=1}^{l} \lambda_j C_j^T \mathbf{1}_s \\ &\quad - w(k+i)^T \sum_{j=1}^{l} \lambda_j E_j^T \mathbf{1}_s \\ &= -x(k+i|k)^T C(k+i)^T \mathbf{1}_s \\ &\quad - w(k+i)^T E(k+i)^T \mathbf{1}_s \\ &= -\|z(k+i)\|_1. \end{aligned} \quad (32)$$

The rest of the proof can be given using a similar method to (24)–(27) and is omitted. □

Remark 1: Given $\hbar$, the conditions (9) and (13) are linear programming problems, which are solvable by using the Linprog toolbox in Matlab. It is necessary to point out that the given value $\hbar$ does not increase the conservativeness of the MPC design in Theorem 1. From (9d), it is easy to choose the value $\hbar \ge \max_{\substack{i=1,2,\ldots,n,\\ j=1,2,\ldots,m}} \{\frac{\overline{b}_{ij}}{\underline{b}_{ij}}\}$, where $\overline{b}_{ij}$ and $\underline{b}_{ij}$ are the ith row jth column element of the matrices $\overline{B}$ and $\underline{B}$, respectively. Similarly, the value $\hbar$ in (13d) can be chosen.

Remark 2: In [36]–[38], the MPC based on mixed H_2/H_∞ performances was investigated for general systems (non-positive). In [40], the MPC for nominal positive systems, i.e., system (1) without interval and polytopic uncertainties, was proposed. Theorem 1 in the paper develops the mixed H_2/H_∞ MPC control approach to positive systems with mixed performances (5) and (6). Theorem 1 further solves the MPC of interval/polytopic system (1).

Remark 3: In [40], the MPC controller gain rank is 1. In practice, there always exist some control systems that cannot be stabilized using the controller with the gain matrix rank 1. Additionally, the MPC is essentially an optimization problem. It is not suitable to use a restricted control law to achieve the optimal system performances. Theorem 1 removes the rank constraint in [40].

Remark 4: In Theorem 1, the open-loop systems are required to be positive. It is necessary to point out that Theorem 1 can be extended to general systems (non-positive). It is a complex but straight process. We do not repeat it here.

B. INVARIANT SET

In Theorem 1, the multi-step control approach is employed for the MPC controller design of positive systems. The multi-step control approach releases the conservativeness of the single step control based MPC. In the multi-step control design, the state stays in a sequence of control sets. Under the MPC control law, the state is steered from a control set to another control set and finally stays in a classic invariant set. For the MPC of general systems, the ellipsoidal set is usually chosen as the invariant set. Owing to the speciality of positive systems, the ellipsoidal set is not very suitable for the invariant set of positive systems. In [39] and [40], a cone set (or a sequence of cone sets) is verified to be more suitable for the MPC of positive systems. Here, we further introduce a sequence of cone sets as the control invariant set of the MPC.

Theorem 2: (a) If there exist constants $0 < \sigma < 1, \tau > 0$, $\rho_i > 0$ and vectors $v^{(i)} \succ 0, v^{(i)} \in \Re^n, \xi^{(i)} \in \Re^n, \overline{\xi} \prec 0$, $\overline{\xi} \in \Re^n, \zeta^{(i)} \succeq 0, \zeta^{(i)} \in \Re^n, \overline{\zeta} \in \Re^n$ such that

$$\overline{A}^T v^{(i+1)} + \overline{\xi} + \overline{\zeta} - \sigma v^{(i)} \prec 0, \quad (33a)$$
$$\overline{D}^T v^{(i+1)} - \tau \mathbf{1}_r \prec 0, \quad (33b)$$
$$\sigma\rho_i + \eta_1\tau \le \rho_{i+1}, \quad (33c)$$
$$\rho_{i+1} \le \rho_i, \quad (33d)$$
$$x(k)^T v^{(0)} \le \rho_0, \quad (33e)$$

hold for $i = 0, 1, \ldots, N-1$, and $v^{(i)} = v^{(N)}$ for $i > N$, then, under the MPC control law (11) with (12), the sets $\Gamma_i = \{x | x^T v^{(i)} \le \rho_i\}$ are the multi-step control sets of the MPC for interval system (1).

(b) If there exist constants $0 < \sigma < 1, \tau > 0, \rho_i > 0$ and vectors $v^{(i)} \succ 0, v^{(i)} \in \Re^n, \xi^{(i)} \in \Re^n, \overline{\xi} \prec 0, \overline{\xi} \in \Re^n$, $\zeta^{(i)} \succeq 0, \zeta^{(i)} \in \Re^n, \overline{\zeta} \in \Re^n$ such that

$$A_j^T v^{(i+1)} + \overline{\xi} + \overline{\zeta} - \sigma v^{(i)} \prec 0, \quad (34a)$$
$$D_j^T v^{(i+1)} - \tau \mathbf{1}_r \prec 0, \quad (34b)$$
$$\sigma\rho_i + \eta_1\tau \le \rho_{i+1}, \quad (34c)$$
$$\rho_{i+1} \le \rho_i, \quad (34d)$$
$$x(k)^T v^{(0)} \le \rho_0, \quad (34e)$$

hold for $i = 0, 1, \ldots, N-1$, and $v^{(i)} = v^{(N)}$ for $i > N$, then, under the MPC control law (14) with (15), the sets $\Gamma_i = \{x | x^T v^{(i)} \le \rho_i\}$ are the multi-step control sets of the MPC for polytopic system (1).

Proof: (a) The proof is given via the induction method. When $i = 0$, $x(k) \in \Gamma_0$ by (33e). Assume $x(k+i) \in \Gamma_i$. Combining (2) and the proofs in (20) and (21), we have

$$\begin{aligned} x(k+i+1)^T v^{(i+1)} &= x(k+i)^T (A(k+i)^T v^{(i+1)} \\ &\quad + K_i^{-T} B(k+i)^T v^{(i+1)} \\ &\quad + K_i^{+T} B(k+i)^T v^{(i+1)}) \\ &\quad + w(k+i)^T D(k+i)^T v^{(i+1)} \\ &\le x(k+i)^T (\overline{A}^T v^{(i+1)} + \overline{\xi} + \overline{\zeta}) \\ &\quad + w(k+i)^T \overline{D} v^{(i+1)}. \end{aligned} \quad (35)$$

By (33a) and (33b), (35) is transformed into

$$x(k+i+1)^T v^{(i+1)} \le \sigma x(k+i)^T v^{(i)} + \tau w(k+i)^T \mathbf{1}_r. \quad (36)$$

By (4), we obtain

$$x(k+i+1)^T v^{(i+1)} \le \sigma\rho_i + \tau\eta_1. \quad (37)$$

Substituting (33c) into (37) yields $x(k+i+1)^T v^{(i+1)} \leq \rho_{i+1}$, which means that $x(k+i+1) \in \Gamma_{i+1}$.

(b) By using a similar method in (a), we have

$$\begin{aligned} x(k+i+1)^T v^{(i+1)} &= x(k+i)^T (A(k+i)^T v^{(i+1)} \\ &\quad + K_i^{-T} B(k+i)^T v^{(i+1)} \\ &\quad + K_i^{+T} B(k+i)^T v^{(i+1)}) \\ &\quad + w(k+i)^T D(k+i)^T v^{(i+1)} \\ &\leq x(k+i)^T \sum\nolimits_{j=1}^{l} \lambda_j (A_j^T v^{(i+1)} \\ &\quad + K_i^{-T} B_j^T v^{(i+1)} + K_i^{+T} B_j^T v^{(i+1)}) \\ &\quad + w(k+i)^T \sum\nolimits_{j=1}^{l} \lambda_j D_j^T v^{(i+1)} \\ &\leq x(k+i)^T \sum\nolimits_{j=1}^{l} \lambda_j (A_j^T v^{(i+1)} + \overline{\xi} + \overline{\zeta}) \\ &\quad + \tau w(k+i)^T \sum\nolimits_{j=1}^{l} \lambda_j \mathbf{1}_r \\ &\leq \sigma x(k+i)^T \sum\nolimits_{j=1}^{l} \lambda_j v^{(i)} \\ &\quad + \tau w(k+i)^T \sum\nolimits_{j=1}^{l} \lambda_j \mathbf{1}_r \\ &\leq \sigma x(k+i)^T v^{(i)} + \tau \eta_1 \\ &\leq \sigma \rho_i + \tau \eta_1 \\ &\leq \rho_{i+1} \end{aligned} \tag{38}$$

from (34a)-(34d) and the proof in (29). This completes the proof. □

In [40], the multi-step control sets $\Upsilon_i = \{x | x^T v^{(i)} \leq \rho\}$ are used. Theorem 2 in the paper constructs improved multi-step control sets Γ_i by introducing an additional condition (33d) (or, (34d)). Thus, a series of free variables ρ_i is added to the recursive conditions. These variables release the conservativeness in Υ_i. By (33d) or (34d), we also have $\Gamma_i \subseteq \Upsilon_i$ for each fixed i.

Remark 5: Take the conditions (9) and (33) into account. Assume that the conditions (9b) and (9c) hold. Then the conditions (33a) and (33b) must hold if one chooses arbitrarily $0 < \sigma < 1$ and $\tau = \varsigma_N$. This implies that the conditions (33a) and (33b) can be removed. Thus, the recursive condition (33) is easier to be computed. The objective of adding the conditions (33a) and (33b) is to increase the feasibility of the recursive condition (33) by introducing some more free variables. A similar issue exists in the conditions (13) and (34).

C. CONSTRAINT HANDLING

In this subsection, we will consider how to handle the constraint (7). First, it is necessary to explain why the constraint (7) is used. In the classical MPC literature [30]–[39], the constraint condition is based on quadratic forms $x^T Rx \leq \varrho_1$ and $u^T Qu \leq \varrho_2$, where R and Q are positive definite matrices and $\varrho_1 > 0, \varrho_2 > 0$. On one hand, the quadratic forms describe the constraints on $\|x\|_2$ and $\|u\|_2$. On the other hand, the quadratic forms are consistent with the quadratic performance index used for the MPC of general systems. Based on the special feature of positive systems, we employ the linear performance indexes (5) and (6). Naturally, we use (7) to describe the constraints on states and control inputs. The condition (7a) can be rewritten as $\|x\|_1 \leq \delta$, which is consistent with the classic quadratic constraint condition by the equivalence of norms. For the constraint on $u(k)$, we have $\|u(k)\|_1 = \|K_i^- x(k) + K_i^+ x(k)\|_1 \leq (\|K_i^-\|_1 + \|K_i^+\|_1)\|x(k)\|_1$. Therefore, the bound of $\|u(k)\|_1\|$ can be obtained as long as the bounds of $\|K_i^-\|_1 + \|K_i^+\|_1$ and $\|x(k)\|_1$ are given. The condition (7a) gives the bound of $\|x(k)\|_1$. This is why the controller gain constraint condition (7b) is utilized to describe the constraint on the control input.

Theorem 3: (a) If there exist a constant $\varepsilon > 0$ and a vector $\underline{\xi} \prec 0, \underline{\xi} \in \Re^n$ such that

$$v^{(i)} \succeq \varepsilon \mathbf{1}_n, \tag{39a}$$

$$\rho_i \leq \varepsilon\delta, \tag{39b}$$

$$m\hbar\overline{\zeta} - m\underline{\xi} \preceq \hbar\theta \mathbf{1}_m^T \underline{B}^T v^{(i+1)}, \tag{39c}$$

$$\underline{\xi} \preceq \xi^{(\imath)}, \quad \imath = 1, 2, \ldots, m, \tag{39d}$$

hold for $i = 0, 1, \ldots, N-1$, and $v^{(i)} = v^{(N)}$ for $i > N$, then, under the MPC control law (11) with (12), the constraint (7) is satisfied for interval system (1), where m is the dimension of $u(k)$.

(b) If there exist a constant $\varepsilon > 0$ and a vector $\underline{\xi} \prec 0$, $\underline{\xi} \in \Re^n$ such that

$$v^{(i)} \succeq \varepsilon \mathbf{1}_n, \tag{40a}$$

$$\rho_i \leq \varepsilon\delta, \tag{40b}$$

$$m\hbar\overline{\zeta} - m\underline{\xi} \preceq \hbar\theta \mathbf{1}_m^T \widehat{B}^T v^{(i+1)}, \tag{40c}$$

$$\underline{\xi} \preceq \xi^{(\imath)}, \quad \imath = 1, 2, \ldots, m, \tag{40d}$$

hold for $i = 0, 1, \ldots, N-1$, and $v^{(i)} = v^{(N)}$ for $i > N$, then, under the MPC control law (14) with (15), the constraint (7) is satisfied for polytopic system (1), where m is the dimension of $u(k)$.

Proof: (a) By (39a), (39b), and Theorem 2, we have $\varepsilon x(k+i)^T \mathbf{1}_n \leq x(k+i)^T v^{(i)} \leq \rho_i \leq \varepsilon\delta$, which ensures the validity of the condition (7a). Combining (12), (39c), and (39d) gives

$$\begin{aligned} K_i^{+T}\mathbf{1}_m - K_i^{-T}\mathbf{1}_m &= \frac{\sum_{\imath=1}^{m} \zeta^{(\imath)} \mathbf{1}_m^{(\imath)T}}{\mathbf{1}_m^T \underline{B}^T v^{(i+1)}} \mathbf{1}_m - \frac{\sum_{\imath=1}^{m} \xi^{(\imath)} \mathbf{1}_m^{(\imath)T}}{\hbar \mathbf{1}_m^T \underline{B}^T v^{(i+1)}} \mathbf{1}_m \\ &\preceq \overline{\zeta} \frac{\sum_{\imath=1}^{m} \mathbf{1}_m^{(\imath)T}}{\mathbf{1}_m^T \underline{B}^T v^{(i+1)}} \mathbf{1}_m - \underline{\xi} \frac{\sum_{\imath=1}^{m} \mathbf{1}_m^{(\imath)T}}{\hbar \mathbf{1}_m^T \underline{B}^T v^{(i+1)}} \mathbf{1}_m \\ &= \frac{m\overline{\zeta}}{\mathbf{1}_m^T \underline{B}^T v^{(i+1)}} - \frac{m\underline{\xi}}{\hbar \mathbf{1}_m^T \underline{B}^T v^{(i+1)}} \\ &\preceq \theta, \end{aligned} \tag{41}$$

which implies the condition (7b).

The proof of (b) can be given by using a similar method to (a) and is omitted. □

Remark 6: Given $\hbar$ as chosen in Remark 1, the conditions (39) and (40) are linear programming.

Theorems 1, 2, and 3 have considered the MPC controller design, invariant set, and the constraint handling, respectively. In the following, we will establish the robust stability criterion on interval/polytopic system (1).

Theorem 4: (a) If there exist constants $\hbar > 1, \varsigma_i > 0, \gamma_2 > 0, 0 < \sigma < 1, \tau > 0, \rho_i > 0, \varepsilon > 0$ and vectors $\nu^{(i)} \succ 0, \nu^{(i)} \in \Re^n, \xi^{(i)} \in \Re^n, \overline{\xi} \in \Re^n, \underline{\xi} \prec 0, \underline{\xi} \in \Re^n, \zeta^{(i)} \succeq 0, \zeta^{(i)} \in \Re^n, \overline{\zeta} \in \Re^n$ such that (9), (10), (33), and (39) hold for $i = 0, 1, \ldots, N-1$, and $\nu^{(i)} = \nu^{(N)}$ for $i > N$, then, under the MPC control law (11) with (12), the resulting closed-loop interval system (1) is positive and robustly stable with mixed performances (5) and (6).

(b) If there exist constants $\hbar > 1, \varsigma_i > 0, 0 < \sigma < 1, \tau > 0, \rho_i > 0, \varepsilon > 0$ and vectors $\nu^{(i)} \succ 0, \nu^{(i)} \in \Re^n, \xi^{(i)} \in \Re^n, \overline{\xi} \in \Re^n, \underline{\xi} \prec 0, \underline{\xi} \in \Re^n, \zeta^{(i)} \succeq 0, \zeta^{(i)} \in \Re^n, \overline{\zeta} \in \Re^n$ such that (13), (10), (34), and (40) hold for $i = 0, 1, \ldots, N-1$, and $\nu^{(i)} = \nu^{(N)}$ for $i > N$, then, under the MPC control law (14) with (15), the resulting closed-loop polytopic system (1) is positive and robustly stable with mixed performances (5) and (6).

Theorem 4 is a direct result of Theorems 1, 2, and 3. Theorem 2 ensures that the MPC in Theorem 1 is a multi-step control design, Theorem 3 guarantees the robust property of the MPC design, and Theorem 1 reveals that the consider system is stable with desired mixed performances. Based on these issues, we do not repeat the proof of Theorem 4.

To obtain the optimal values of γ_1 and γ_2, the following optimization algorithm can be implemented:

Algorithm 1: $\min\limits_{\substack{\varsigma_i,\gamma_2,\tau,\rho_i \\ \nu^{(i)},\xi^{(i)},\overline{\xi},\underline{\xi},\zeta^{(i)},\overline{\zeta}}} \gamma_1 + \gamma_2$ (or, $\varsigma_0 + \gamma_2$) that is subject to (9), (10), (33), and (39) (or, (13), (10), (34), and (40)), where $\hbar > 1$ and $0 < \sigma < 1$ are given constants.

In classical MPC design, the linear matrix inequalities technique is usually used. However, the computation burden of the corresponding MPC algorithm will be increased with the dimension of the system. Meanwhile, the linear matrix inequalities technique will gets into trouble when dealing with large-scale computation. Efficient MPC algorithms have been one of the difficulties in MPC design. Algorithm 1 is a linear programming problem. As we all know, linear programming possesses powerful ability for solving the large-scale computation and a linear condition is also simpler than other ones. This feature overcomes the drawbacks of MPC algorithms based on linear matrix inequalities. These are also the reasons why we employ the linear programming technique, the linear performance indexes, and the linear Lyapunov function.

Remark 7: It is necessary to point out that Algorithm 1 has some constraints. In Remark 1, a scope of value $\hbar$ is suggested. The value of σ is restricted into the interval (0, 1). However, we do not know how to choose a couple of proper values $\hbar, \sigma$ such that $\gamma_1 + \gamma_2$ is optimal. We provide a reference algorithm as follows:

Step 1: Given a value $\hbar_0 = \max\limits_{\substack{i=1,2,\ldots,n, \\ j=1,2,\ldots,m}} \{\frac{\overline{b}_{ij}}{\underline{b}_{ij}}\}$ and $\sigma = 0.5$, implement Algorithm 1. If Algorithm 1 is infeasible, then choose the value of $\hbar$ larger than $\hbar_0$ (one could choose the value along the power function $\hbar_k = \hbar_0^k$, where k is the search time) until Algorithm 1 is feasible. Denote the feasible value $\hbar_k$.

Step 2: Implement Algorithm 1. Denote the value of $\gamma_1+\gamma_2$ as α_0.

Step 3: Change the value of σ as 0.75 and 0.25, respectively (here, the value of σ is given by using a dichotomy search). Skip the values of σ that makes Algorithm 1 infeasible. Implement Algorithm 1 for the values of σ that makes Algorithm 1 feasible (in general, we could prescribe the computation times).

Step 4: Assign the value of $\gamma_1 + \gamma_2$ smaller than the last one to α_0 until completing all prescribe computation times. Denote the value of $\sigma = \sigma_0$.

Step 5: Fix the value σ_0 and an interval $(\hbar_{k-1}, \hbar_{k+1})$. Change the value of $\hbar$ by a dichotomy search in the fixed interval. Then, go to Step 4.

Remark 8: The reference algorithm in Remark 7 can reduce but cannot remove the conservativeness. In the future work, how to overcome the constraints from these two parameters $\hbar$ and σ is an interesting topic.

IV. EXAMPLE

In a social or biologic environment, the dynamics of the survival rates and fertility rates plays a key role in investigating the environment change, the resource utilization, etc. An age-structured population model is usually used to describe the population dynamics and the Leslie modelling method [42] is effective for characterizing the age-structured population dynamics. In [1], a Leslie population model is established via positive systems. The literature [20] further improved the model by adding the exogenous disturbance input. Indeed, system (1) is also an improved model of the classic Lesilie model. In system (1), we impose the interval/polytopic uncertainty on the system. On one hand, almost all control systems contain uncertainty in practice. On the other hand, the interval/polytopic constraint can describe the uncertainty of the systems well.

Consider interval system (1) with

$$\underline{A} = \begin{pmatrix} 0.52 & 0.31 & 0.94 \\ 0.45 & 0.62 & 1.00 \\ 0.75 & 0.51 & 0.86 \end{pmatrix}, \quad \overline{A} = \begin{pmatrix} 0.61 & 0.43 & 1.05 \\ 0.55 & 0.70 & 1.27 \\ 0.81 & 0.63 & 0.96 \end{pmatrix},$$

$$\underline{B} = \begin{pmatrix} 1.22 & 1.43 & 0.36 \\ 0.57 & 1.46 & 0.27 \\ 0.08 & 0.70 & 1.10 \end{pmatrix}, \quad \overline{B} = \begin{pmatrix} 1.29 & 1.52 & 0.45 \\ 0.60 & 1.53 & 0.34 \\ 0.11 & 0.79 & 1.30 \end{pmatrix},$$

$$\underline{C} = \begin{pmatrix} 0.01 & 0.01 & 0.01 \\ 0.02 & 0.03 & 0.02 \\ 0.02 & 0.03 & 0.02 \end{pmatrix}, \quad \overline{C} = \begin{pmatrix} 0.02 & 0.01 & 0.02 \\ 0.03 & 0.03 & 0.03 \\ 0.02 & 0.05 & 0.04 \end{pmatrix},$$

$$\underline{D} = \begin{pmatrix} 0.02 & 0.01 & 0.04 \\ 0.02 & 0.06 & 0.02 \\ 0.03 & 0.03 & 0.05 \end{pmatrix}, \quad \overline{D} = \begin{pmatrix} 0.04 & 0.03 & 0.04 \\ 0.03 & 0.07 & 0.04 \\ 0.04 & 0.04 & 0.06 \end{pmatrix},$$

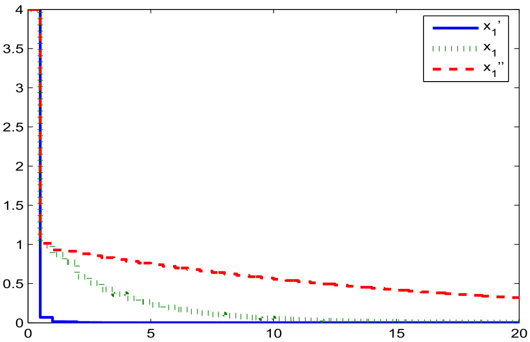

FIGURE 1. The simulation results of the states x_1', x_1, and x_1''.

$$\underline{E} = \begin{pmatrix} 0.01 & 0.07 & 0.01 \\ 0.02 & 0.05 & 0.02 \\ 0.01 & 0.03 & 0.01 \end{pmatrix}, \quad \overline{E} = \begin{pmatrix} 0.03 & 0.08 & 0.03 \\ 0.04 & 0.06 & 0.03 \\ 0.02 & 0.04 & 0.02 \end{pmatrix},$$

$$w(k) = \begin{pmatrix} e^{-2k} \\ e^{-2k} \\ e^{-2k} \end{pmatrix}, \quad \delta = 20,\ \theta = \begin{pmatrix} 200 \\ 200 \\ 200 \end{pmatrix}.$$

It follows that $\eta_1 = 3$ and $\eta_2 = \frac{e^2}{e^2-1}$. Choosing the predictive step $N = 2$ and implementing Algorithm 1, we obtain

$$v^{(0)} = \begin{pmatrix} 0.0197 \\ 0.0247 \\ 0.0264 \end{pmatrix}, \quad v^{(1)} = \begin{pmatrix} 0.0120 \\ 0.0141 \\ 0.0153 \end{pmatrix},$$

$$v^{(2)} = \begin{pmatrix} 0.0321 \\ 0.0001 \\ 0.0001 \end{pmatrix}, \quad \underline{\xi} = \begin{pmatrix} -6.0135 \\ -6.0239 \\ -6.0322 \end{pmatrix},$$

$$\xi^{(1)} = \begin{pmatrix} -0.0152 \\ -0.0094 \\ -0.0280 \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} -0.0151 \\ -0.0093 \\ -0.0281 \end{pmatrix},$$

$$\xi^{(3)} = \begin{pmatrix} -0.0151 \\ -0.0093 \\ -0.0280 \end{pmatrix}, \quad \overline{\xi} = \begin{pmatrix} -0.0150 \\ -0.0092 \\ -0.0279 \end{pmatrix},$$

$$\zeta^{(1)} = \begin{pmatrix} 0.0001 \\ 0.0002 \\ 0.0001 \end{pmatrix}, \quad \zeta^{(2)} = \begin{pmatrix} 0.0002 \\ 0.0001 \\ 0.0001 \end{pmatrix},$$

$$\zeta^{(3)} = \begin{pmatrix} 0.0002 \\ 0.0002 \\ 0.0001 \end{pmatrix}, \quad \overline{\zeta} = \begin{pmatrix} 0.0003 \\ 0.0003 \\ 0.0002 \end{pmatrix},$$

$$\varsigma_0 = \gamma_1 = 0.0200, \quad \varsigma_1 = 0.0190, \quad \varsigma_2 = 0.0095,$$
$$\rho_0 = 0.2162, \quad \rho_1 = 0.2145, \quad \rho_2 = 0.2128, \quad \varepsilon = 0.0114,$$
$$\hbar = 1.375, \quad \sigma = 0.9375, \quad \gamma_2 = 0.2377, \quad \tau = 0.0034.$$

Then, implement

the first obtained control law at the first predicted step:

$$u(0) = \begin{pmatrix} -0.1563 & -0.0950 & -0.2876 \\ -0.1545 & -0.0948 & -0.2886 \\ -0.1545 & -0.0940 & -0.2876 \end{pmatrix} x(0).$$

The lower and upper bound of the closed-loop system matrix

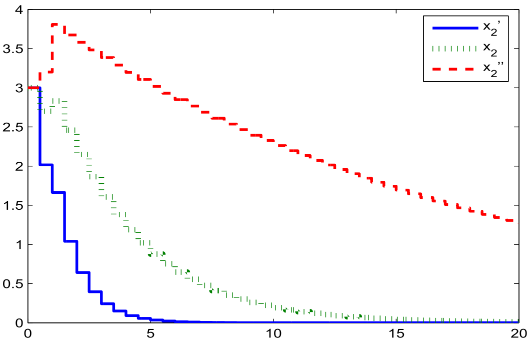

FIGURE 2. The simulation results of the states x_2', x_2, and x_2''.

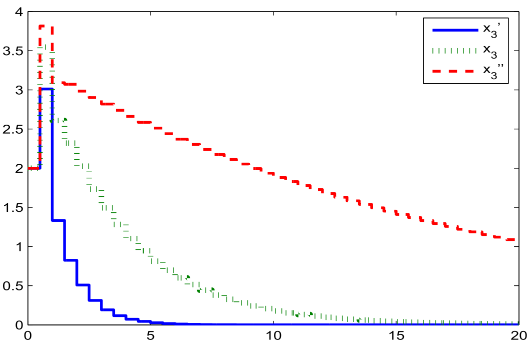

FIGURE 3. The simulation results of the states x_3', x_3, and x_3''.

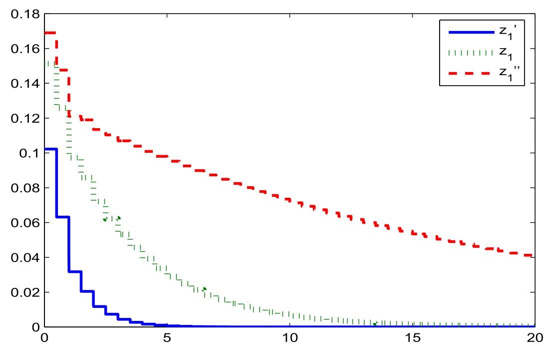

FIGURE 4. The simulation results of the outputs z_1', z_1, and z_1''.

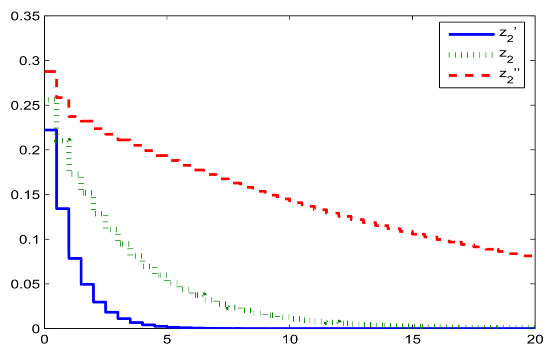

FIGURE 5. The simulation results of the outputs z_2', z_2, and z_2''.

are

$$\underline{A} + \overline{B}K_0^{(1)-} + \underline{B}K_0^{(1)+} = \begin{pmatrix} 0.0137 & 0.0008 & 0.0007 \\ 0.0671 & 0.3859 & 0.2880 \\ 0.4094 & 0.3021 & 0.2263 \end{pmatrix},$$

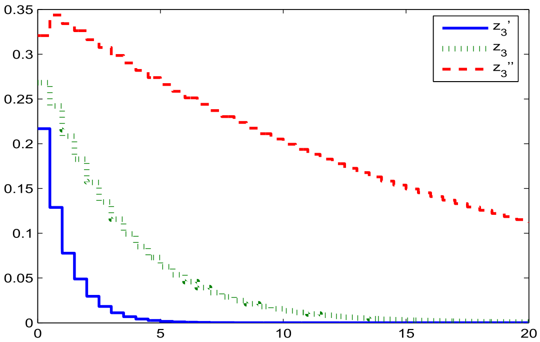

FIGURE 6. The simulation results of the outputs z_3', z_3, and z_3''.

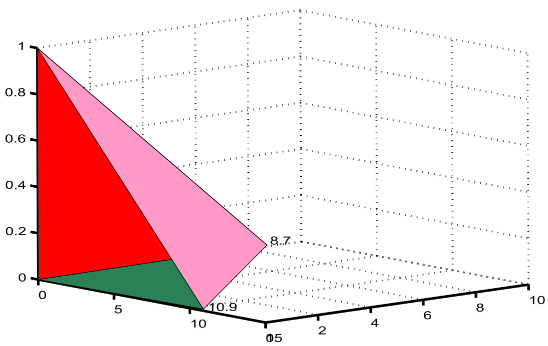

FIGURE 7. The feasible region of the initial states.

$$\overline{A}+\underline{B}K_0^{(1)-}+\overline{B}K_0^{(1)+}=\begin{pmatrix}0.1037 & 0.1208 & 0.1107\\ 0.1671 & 0.4659 & 0.5580\\ 0.4694 & 0.4221 & 0.3263\end{pmatrix}.$$

Implement the first obtained control law at the second predicted step:

$$u(1)=\begin{pmatrix}-0.1563 & -0.0950 & -0.2876\\ -0.1545 & -0.0948 & -0.2886\\ -0.1545 & -0.0940 & -0.2876\end{pmatrix}x(1).$$

The lower and upper bound of the closed-loop system matrix are

$$\underline{A}+\overline{B}K_0^{(1)-}+\underline{B}K_0^{(1)+}=\begin{pmatrix}0.0137 & 0.0008 & 0.0007\\ 0.0671 & 0.3859 & 0.2880\\ 0.4094 & 0.3021 & 0.2263\end{pmatrix},$$

$$\overline{A}+\underline{B}K_0^{(1)-}+\overline{B}K_0^{(1)+}=\begin{pmatrix}0.1037 & 0.1208 & 0.1107\\ 0.1671 & 0.4659 & 0.5580\\ 0.4694 & 0.4221 & 0.3263\end{pmatrix}.$$

The controller gain matrix and the closed-loop system matrices in the second sampling time instant are the same as those in the first sampling time instant. This case is possible in the optimization problem. Denote $x(k)'=(x_1(k)'\ x_2(k)'\ x_3(k)')^T$ and $z(k)'=(z_1(k)'\ z_2(k)'\ z_3(k)')^T$ as the lower bounded state and output of interval system (1) with corresponding matrices, $x(k)''=(x_1(k)''\ x_2(k)''\ x_3(k)'')^T$ and $z(k)''=(z_1(k)''\ z_2(k)''\ z_3(k)'')^T$ as the upper bounded state and output of interval system (1) with corresponding matrices, and $x(k)=(x_1(k)\,x_2(k)\,x_3(k))^T$ and $z(k)=(z_1(k)\,z_2(k)\,z_3(k))^T$ as the state and output of interval system (1), respectively break Figs. 1-3 show the simulations of the states and Figs. 4-6 show the simulations of the outputs. Fig. 7 shows the feasible region of the initial conditions.

V. CONCLUSION

This paper has solved the MPC design for interval/polytopic positive systems with exogenous disturbance input. A new MPC framework for positive systems with desired mixed performances is established. Using a linear technique, the MPC controller, the cone based multi-step control set, and the constraint handling are proposed, respectively. Under the MPC design, the systems are robust stability with mixed linear performance indexes. An implementable MPC algorithm is formulated into linear programming.

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