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A Continuous Reconstruction Observer for Sampled-Data Linear Time Varying Systems

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ABSTRACT In this paper, a novel continuous sampled-data observer for linear time-varying systems is presented. The proposed observer is based on two different impulsive observers. The state estimates of these impulsive observers are fused in a manner such that continuous state estimation is achieved. Another significant contribution is comprehensive stability analysis of the proposed observer. The analysis establishes conditions that guarantee exponential convergence of observers. Contrary to the common understanding, it is revealed that the convergence of associated discrete-time equation for impulsive observers is not a sufficient condition for overall convergence. Contributions are illustrated by simulations.

INDEX TERMS Sampled-data, linear time varying system, impulsive observer.

I. INTRODUCTION

The advancement of digital electronic has attracted interest of researchers in sampled-data systems. The output feedback control of such systems is based on sampled output. Historically, pure discrete-time (DT) observers have been the core of output feedback sampled-data systems. Their simple nature allows ease of implementation. However, their inability to reconstruct continuous system states forces a designer to rely on certain approximations like zero order hold that may compromise system performance [1].

The other type of observers for sampled-data systems which overcome the limitations of DT observers are the impulsive observers. These observers reconstruct the continuous-time (CT) system states within the sampling interval on the basis of system model and sampled output [2]. In literature two different types of sampled data impulsive observers have been discussed these are the current and prediction observers. In current observers, the states are estimated for the current sampling interval using the current output sample. On the other hand, prediction observers estimate the states for the next sampling interval based on the current output. In literature, primarily current impulsive observers have been discussed because of their obvious advantage of incorporating the latest data available for state estimation [3] and [4]. However from an application perspective the estimates cannot be computed within infinitesimally small time, as it is not possible to sample and perform all the computations in such a short duration [5]–[8].

In literature current impulsive observers for a class of linear impulsive system have been dealt in [3], [9], and [10]. In these contributions, the system dynamics were also impulsive with DT measurements of the output. In [11] an impulsive observer is proposed for the stabilization of uncertain non-impulsive system. In [12], an impulsive observer is proposed making use of CT and DT outputs. The same concept of current observer is also extended to the sampled-data nonlinear systems as discussed in [13]–[16]. Output feedback control for a nonlinear CT system using high gain observer with input saturation constraints is discussed in [17] and [18].

The states estimated using impulsive observers contain the so called jumps. These jumps are undesirable in applied control systems. The major contribution of this paper is the development of continuous reconstruction observer for Linear Time Varying (LTV) systems. The state estimates from the current and prediction impulsive observers are fused together with a mathematical relation which is a time dependent fusion function. The advantage of non-impulsive reconstruction observer is eradication of jumps in state estimation.

A rigorous convergence analysis of estimation error for the proposed non-impulsive sampled-data observer is also presented. The analysis is based on derivation of continuous exponential bound for the current and prediction impulsive observers for stable and unstable LTV systems in open loop configuration. The customary analysis for sampled-data systems with impulsive observers presented in literature

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guarantee stability of the closed loop systems. In [3], [10], and [19] stability analysis was carried out with the geometric approach, whereas time varying Lyapunov function is defined for establishing the stability of the closed-loop system in [11]. The advantages of the proposed methodology of convergence analysis in this paper have been elaborated by motivating example.

In this paper, prediction impulsive observer design is presented which to the best of authors' knowledge has not been discussed in literature. The prediction observer like the current observer works in two steps, but the correction in states estimate is introduced on the basis of sampled output at the previous sample. The prediction observer has the advantage of convenient design and implementation, attributed to the availability of complete sampling period for computations.

In addition to the design of prediction impulsive observer, the mathematical framework for current impulsive observers has been reorganized in this paper. The system output at sampling point is used to introduce impulsive correction at t_k^+. This leads to lack of differentiability of the open-loop state estimation dynamic equation from the left hand side. This aspect has been addressed in this paper by the use of D^+ and D^- operator which represents continuous differentiability from the right and left hand side respectively. The non-existence of the derivative at jumps has usually been ignored in the literature.

To summarizes, the major contribution of the article is to design a continuous (non-impulsive) reconstruction observer and carry out a comprehensive stability analysis for both stable and unstable systems. To accomplish these major objectives a prediction impulsive observer is also proposed, while addressing the ambiguities in the mathematical framework of existing current observer design.

The organization of the paper is as follows. Section II presents sampled-data impulsive reconstruction observers. In Section III, continuous sampled-data reconstruction observer is discussed and Section IV deals with stability analyses. A LTV sampled-data system is illustrated with intuitive second order system example in Section V, to visualize and authenticate the theoretical discussion. Finally, Section VI concludes the paper.

II. BACKGROUND OF SAMPLED-DATA IMPULSIVE RECONSTRUCTION OBSERVERS

Consider the following LTV system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad x(t_o) = x_o,$$
$$y[k] = C[k]x[k], \tag{1}$$

where $x(t) \in R^n$ is CT state, $u(t) \in R^m$ is CT input and $y[k] \in R^p$ is the sampled output. $A(t)$, $B(t)$ and $C[k]$ are time varying bounded system matrices of appropriate dimensions. The sampled output $y[k]$ of the system is only available at sampling points. The discrete indices representing the sampling points are denoted by $k = k_o, k_o + 1, \ldots$. Throughout this paper, parentheses (.) are used for CT whereas square brackets [.] are used for DT quantities and T is a non-pathological sampling time [20]. The initial time t_0 and discrete index k_0 are related by $t_0 = k_0 T$.

The state transition matrix (STM) associated with $A(t)$ in its usual sense is denoted by $\Phi(t_f, t_o)$ where t_o and t_f are the initial and final times respectively [21, p. 386]. DT equivalent of state transition matrix (STM) is defined as $\Phi[k_f, k_o] = \Phi(k_f T, k_o T)$. In case of transition over a sample from k to $k+1$, the DT STM is

$$A[k] = \Phi((k+1)T, kT),$$
$$= \Phi[k+1, k].$$

The invertability of $A[k]$ is guaranteed since STM is always full rank. The $(A[k], C[k])$ pair is assumed to be l step observable [21, p. 485].

The typical reconstruction using sampled output makes use of impulsive correction in the state estimates. The point in time within the sampling interval at which impulsive correction is introduced is dependent upon the observer design methodology. If the impulsive correction immediately follows the observation, the observer is termed as current observer. Alternately, prediction observer applies impulsive correction just before the next observation. The former is more popular in control systems community [3] and [9]. The following subsections briefly discuss the structure of the observers and draw a comparison between the two.

A. CURRENT IMPULSIVE OBSERVERS

The state estimates are impulsively corrected by the current observer immediately after the output is sampled. The state estimates are generated based on system dynamics for the rest of the sampling interval after the impulsive correction. This concept can be found in [3] and [9]. The mathematical formulation of the current impulsive observer is as follows

$$\bar{x}(t^+) = \bar{x}(t) + H_c[k](y[k] - \bar{y}[k]) \quad t = kT, \tag{2}$$
$$\dot{\bar{x}}(t) = A(t)\bar{x}(t) + B(t)u(t) \quad kT < t < (k+1)T, \tag{3}$$
$$D^- \bar{x}(t) = A(t)\bar{x}(t) + B(t)u(t) \quad t = (k+1)T,$$
$$k = k_o, k_o + 1, \ldots, \tag{4}$$

where $\bar{x}(t) \in R^n$ is the current observer state and $H_c[k] \in R^{n\times p}$ is a time varying current observer gain matrix. Let the initial state of the observer be $\bar{x}(t_o) = \bar{x}_o$. We use the notation $\bar{x}(t^+) = \lim_{h\to 0} \bar{x}(t+h) = \bar{x}(kT^+)$, where $h > 0$.

The second term on the right hand side of (2) is the impulsive correction. Equation (3) predicts the states for time $kT < t < (k+1)T$ based on the estimate at $t = kT$. Due to impulsive correction, the derivative of $\bar{x}(t)$ at $t = (k+1)T$ does not exist. Accordingly we have used left sided derivative $D^-\bar{x}(t)$ in (4), defined as

$$D^-\bar{x}(t) = \lim_{h\to 0^-} \frac{\bar{x}(t) - \bar{x}(t-h)}{h},$$

where h is a positive real number. Response of such an observer is illustrated in Fig. 1. It may be noted here that the convention used in this paper is slightly different

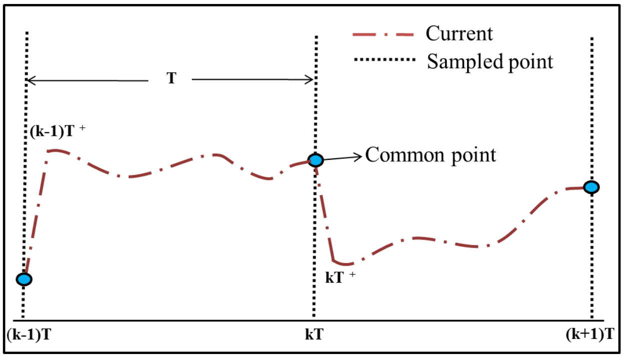

FIGURE 1. Impulsive function of current observer.

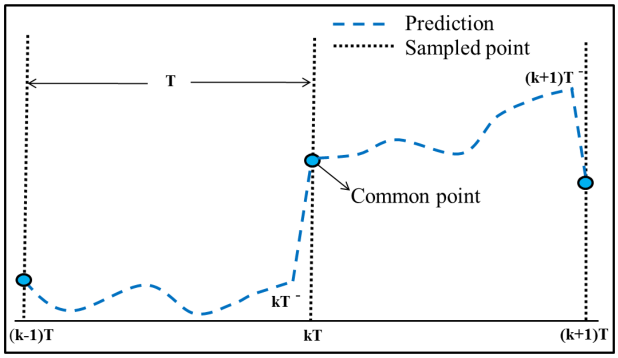

FIGURE 2. Impulsive function of prediction observer.

from [3] and [9] in the sense that the samples are assumed to be available at exact sampling instants kT instead of $(kT)^-$.

Error state vector of current impulsive observer is written as

$$\bar{e}(t) = x(t) - \bar{x}(t) \quad \forall t \geq t_0 \tag{5}$$

derivative of (5) is

$$D^-\bar{e}(t) = A(t)\bar{e}(t) \quad t = (k+1)T, \tag{6}$$
$$\dot{\bar{e}}(t) = A(t)\bar{e}(t) \quad kT < t < (k+1)T, \tag{7}$$

DT error dynamics using current observer is written as

$$\bar{e}(t^+) = x(t^+) - [\bar{x}(t) + H_c[k](y[k] - \bar{y}[k])] \quad t = kT. \tag{8}$$

Continuity of system state imply $x(t) = x(t^+)$ leading to

$$\bar{e}(t^+) = \bar{e}(t) - H_c[k]C[k](x[k] - \bar{x}[k]) \quad t = kT, \tag{9}$$

after simplification (9) is expressed as

$$\bar{e}(t^+) = (I - H_c[k]C[k])\bar{e}[k] \quad t = kT, \tag{10}$$

where $H_c[k]$ is chosen such that (10) is uniformly exponentially stable (UES). Methods for designing $H_c[k]$ can be found in [3], [11], [21, p. 548], and [22, p. 426]. Additional constraints on the rate of convergence of (10) will be discussed in Section IV.

B. PREDICTION IMPULSIVE OBSERVERS

An alternate approach to observer design is to delay the impulsive correction till just before the next sample point. States are constructed using open loop extrapolation based on system model before the correction. Such an observer is termed as prediction observer. Impulsive function of prediction observer is illustrated in Fig. 2 and is mathematically represented as

$$D^+\hat{x}(t) = A(t)\hat{x}(t) + B(t)u(t) \quad t = kT, \tag{11}$$
$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) \quad kT < t < (k+1)T, \tag{12}$$
$$\hat{x}[k+1] = \hat{x}(t^-) + H_p[k](y[k] - \hat{y}[k]) \quad t = (k+1)T, \quad k = k_o, k_o+1, \ldots, \tag{13}$$

where $\hat{x} \in R^n$ is the prediction observer state, $H_p[\mathrm{k}] \in R^{n\times p}$ is a time varying prediction observer gain matrix and $\hat{x}(t_o) = \hat{x}_o$ is the initial state of the observer. We also use the notation $\hat{x}(t^-) = \lim_{h\to 0}\hat{x}(t-h)$, where $h > 0$. $D^+\hat{x}(t)$ is the right sided derivative defined as

$$D^+\hat{x}(t) = \lim_{h\to 0^+}\frac{\hat{x}(t+h) - \hat{x}(t)}{h}.$$

Equation (12) predicts the states for time $kT < t < (k+1)T$ based on the estimate at $t = kT$.

Error state vector of prediction impulsive observer is written as

$$\hat{e}(t) = x(t) - \hat{x}(t), \tag{14}$$

derivative of (14) is

$$D^+\hat{e}(t) = A(t)\hat{e}(t) \quad t = kT, \tag{15}$$
$$\dot{\hat{e}}(t) = A(t)\hat{e}(t) \quad kT < t < (k+1)T, \tag{16}$$

where (15) signifies the discontinuous nature of $\hat{e}(t)$. DT error dynamics using prediction observer is written as

$$\hat{e}[k+1] = x(t) - [\hat{x}(t^-) + H_p[k](y[k] - \hat{y}[k])] \quad t = (k+1)T. \tag{17}$$

Continuity of system states imply $x(t) = x(t^-)$ leading to

$$\hat{e}[k+1] = \hat{e}(t^-) - H_p[k](y[k] - \hat{y}[k]) \quad t = (k+1)T. \tag{18}$$

Substituting $\hat{e}(t^-) = A[k]\hat{e}[k]$ in (18) gives

$$\hat{e}[k+1] = (A[k] - H_p[k]C[k])\hat{e}[k], \tag{19}$$

where $H_p[k]$ is chosen such that (19) is UES.

C. RELATION BETWEEN CURRENT AND PREDICTION IMPULSIVE OBSERVERS

The following analysis reveals an interesting relationship between the current and prediction impulsive observers. The solution of (3) and (4) results in

$$\bar{x}(t) = \Phi(t, kT^+)\bar{x}(kT^+) + \int_{kT^+}^{t}\Phi(t,\sigma)\beta(\sigma)u(\sigma)d(\sigma) \quad kT < t \leq (k+1)T. \tag{20}$$

Evaluating (20) at $t = (k+1)T$

$$\bar{x}[k+1] = A[k]\bar{x}[k] + \int_{kT}^{(k+1)T} \Phi((k+1)T, \sigma)B(\sigma) \times u(\sigma)d(\sigma) + A[k]H_c[k]C[k](x[k] - \hat{x}[k]). \tag{21}$$

Similarly for prediction observer

$$\hat{x}[k+1] = A[k]\hat{x}[k] + \int_{kT}^{(k+1)T} \Phi((k+1)T, \sigma)B(\sigma) \times u(\sigma)d(\sigma) + H_p[k]C[k](x[k] - \hat{x}[k]). \tag{22}$$

The expressions (21) and (22) with

$$H_c[k] = A^{-1}[k]H_p[k], \tag{23}$$

and $\bar{x}(k_oT) = \hat{x}(k_oT)$, guarantee the following

$$\bar{x}(t) = \hat{x}(t), \quad t = kT,\ k = k_o, k_o+1\dots \tag{24}$$

Relation (24) signifies that the current and prediction impulsive observer estimates coincide at the sampling points. As a result, the associated DT equation representing error dynamics for current impulsive observer can be written as

$$\bar{e}[k+1] = (A[k] - H_p[k]C[k])\bar{e}[k], \tag{25}$$

The property of equality of estimation errors for current and prediction impulsive observers is exploited in the development of proposed continuous sampled-data observer in the following section.

III. CONTINUOUS SAMPLED-DATA RECONSTRUCTION OBSERVER

The concept of continuous reconstruction from sampled observation is based on the notion of circumventing the jumps encountered in conventional impulsive observers. This can be realized by exploiting complementary nature of prediction and current observers with regards to the point in time at which impulsive correction takes place. The respective jumps for the two observers occur at different time in a sampling interval whereas their estimates coincide exactly at the sampling instants. The estimates of the two observers are fused using a fusion function such that estimates of the prediction impulsive observer are weighted maximally in the beginning of a sampling interval, thereby avoiding the jump in the estimates of the current observer. Similarly towards the end of a sampling interval, maximum weightage is assigned to the estimates of current impulsive observer so that the jump in the estimates of prediction impulsive observer is avoided.

The fusion can be accomplished by defining a continuous function which weights the current and prediction impulsive observer estimates during different intervals of time within a sampling interval according to the preceding discussion. As an example, one such function is

$$\psi(t) = \begin{cases} 0 & t \in \Delta_1, \\ 1 & t \in \Delta_3, \\ \dfrac{1}{2} & t = t^*, \\ \bar{\psi}(t) & t \in \Delta_2, \end{cases}$$

where t^* is preferably the midpoint i.e. $\left(k + \frac{1}{2}\right)T$ but may have a different value if required and $\bar{\psi}(t)$ is monotonically increasing continuous function. The above definition implies

$$\dot{\psi}(t) = 0 \quad t \in \Delta_1, \Delta_3$$

An example of the fusion function defined as $\psi(t) = 1 - 0.5\, tanh\,(5\mu(t)) + 0.5$, where $\mu(t)$ is sawtooth function, is shown in Fig. 3. The structure of the fusion function will ensure the continuity properties of the observer discussed as follows.

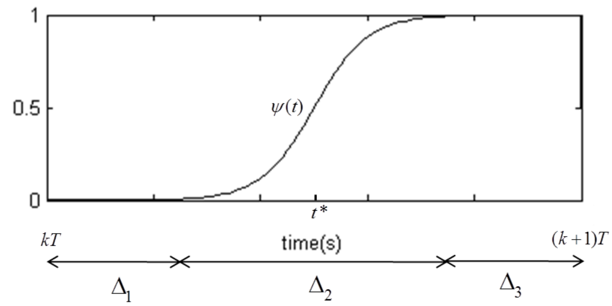

FIGURE 3. Example of Fusion function.

A reconstruction observer with the fusion function defined above, is given by

$$\widehat{x}(t) = \bar{x}(t)\psi(t) + \hat{x}(t)(1 - \psi(t)) \tag{26}$$

The expression (26) suggests that as the output is sampled, maximum weight is given to the prediction observer estimates and the least weight is given to the current observer output. The weight gradually shifts towards the current observer with a reversal of situation at the end of sampling interval.

The reconstruction observer (26) is the proposed novel continuous observer for sampled-data linear systems and the idea is illustrated in Fig. 4.

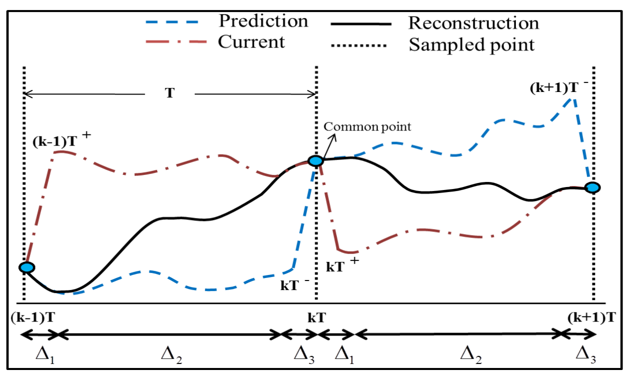

FIGURE 4. Continuous reconstruction observer.

The smoothness properties of the proposed continuous sampled-data reconstruction observer with necessary conditions are discussed in the following two Lemmas. Continuity of the estimated states is established in Lemma 1. Conditions for continuous differentiability of the state estimates are summarized in Lemma 2.

Lemma 1: Under the stated assumptions, the state estimates using the proposed continuous sampled-data reconstruction observer (26) are continuous.

Proof: The continuity of the state estimates outside the neighborhood of sampling points is trivial under the assumptions. At the sample points, the current impulsive observer estimates are continuous from left, that is $\bar{x}(t) = \bar{x}(t^-)$ whereas, prediction impulsive observer estimates are continuous from right, that is $\hat{x}(t) = \hat{x}(t^+)$. From (24), $\bar{x}(t^-) = \hat{x}(t^+)$, which proves the continuity of reconstruction observer at sampled point and its neighborhood.

Lemma 2: The state estimates (26) are continuously differentiable under the additional assumption of continuous differentiability of $u(t)$ and $\psi(t)$. Δ_1 and Δ_3 are assumed to be sufficiently large to contain the jump.

Proof: The continuous differentiability of state estimates of continuous reconstruction observer outside the intervals Δ_1 and Δ_3, again follows trivially under the assumptions of continuous differentiability of $u(t)$ and $\psi(t)$. In the proximity of a sample point, the dynamics of the continuous reconstruction observer are represented by the following differential equations.

Immediately prior to sample point $t = kT$ in the interval $((k-1)T, kT)$

$$\begin{aligned} \dot{\widehat{x}}(t) &= A(t)\bar{x}(t) + B(t)u(t) \quad t \in \Delta_3, \\ D^-\widehat{x}(t) &= A(t)\bar{x}(t) + B(t)u(t) \quad t = kT, \end{aligned} \tag{27}$$

and immediately following the sample point $t = kT$ the dynamics in the interval $(kT, (k+1)T)$

$$\begin{aligned} D^+\widehat{x}(t) &= A(t)\hat{x}(t) + B(t)u(t) \quad t = kT, \\ \dot{\widehat{x}}(t) &= A(t)\hat{x}(t) + B(t)u(t) \quad t \in \Delta_1. \end{aligned} \tag{28}$$

From (24), 2^{nd} equation of (27) and 1^{st} equation of (28)

$$D^-\widehat{x}(t) = D^+\widehat{x}(t) \quad t = kT,$$

which guarantees the existence of $\frac{d}{dt}\widehat{x}$ and proves the assertion.

IV. STABILITY ANALYSIS

The convergence of the continuous sampled-data reconstruction observer can only be guaranteed if a continuous exponential bound is established for estimation error of the constituent current and prediction impulsive observers. Contrary to the prevalent notion, the UES of the associated DT equation of impulsive observers does not guarantee the convergence of the inter-sample open loop estimates as illustrated by the following motivating example.

A. MOTIVATING EXAMPLE

Consider the following LTV system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + Bu(t) \quad x(t_o) = x_o, \\ y[k] &= C[k]x[k], \end{aligned}$$

where $A(t) = \begin{bmatrix} 2\varepsilon t & 0 \\ 0 & -2\kappa t \end{bmatrix}$, $B = [0\ 1]^T$, $C = [1\ 2]$ and $u(t) = sin(2\pi t)$ with initial conditions $x_o = [1\ 0.5]^T$. The system is unstable for $\varepsilon = \frac{1}{3}$ and $\kappa = -\frac{1}{30}$. Prediction impulsive observer is designed for the system with following observer gain

$$\begin{aligned} H_p[k] = [&\Phi^T[k-l+1, k+1]M_{\tilde{\lambda}}[k-l+1, k+1] \\ &\times \Phi[k-l+1, k+1]]^{-1}A^{-T}[k]C^T[k], \end{aligned}$$

where $M_{\tilde{\lambda}}(k_o, k_f)$ is the DT observability Gramian defined as

$$M_{\tilde{\lambda}}(k_o, k_f) = \sum_{j=k_0}^{k_f-1} \tilde{\lambda}^{4(j-k_f+1)}\Phi^T(j, k_o)C^T(j)C(j)\Phi(j, k_o),$$

with discrete convergence rate $\tilde{\lambda} > 1$, which guarantees the convergence of associated DT estimation error dynamics (19) [21, p. 548].

State estimation error with two different convergence rates $\tilde{\lambda}_1 = 1.11$ and $\tilde{\lambda}_2 = 1.60$ for prediction impulsive observers are plotted in Fig. 5 and Fig. 6 respectively. Fig 5(a) shows convergence of estimation error at discrete points in time for $\tilde{\lambda}_1 = 1.11$, however Fig 5(b) shows that prediction estimation error diverges within the sampling points described by (11) and (12). In Fig 6, estimation error for convergence rate $\tilde{\lambda}_2 > \tilde{\lambda}_1$ is plotted for $\tilde{\lambda}_2 = 1.60$. In this case estimation error is convergent at discrete points in time and also within the sampling interval.

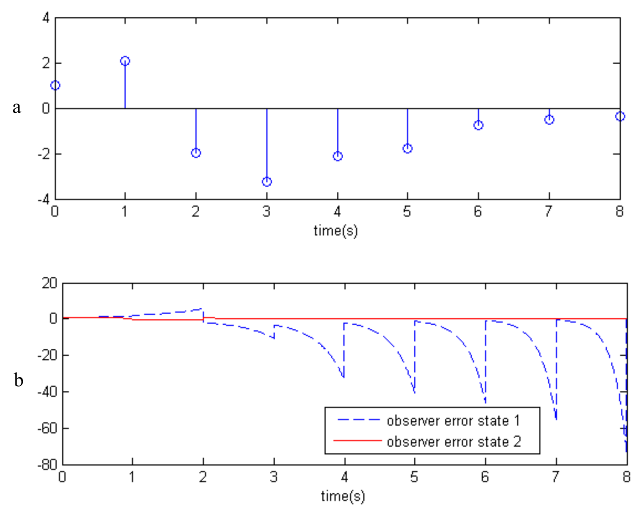

FIGURE 5. Estimation error with prediction impulsive observer, $\tilde{\lambda}_1 = 1.11$ (a). Convergence at discrete points (b). Divergent inter sample behavior.

It turns out that for unstable systems, divergent inter-sample behavior and convergent impulsive corrections occurs simultaneously. For the overall asymptotic stability of estimation error, convergence must dominate divergence. This warrants convergence analysis for determination of conditions that guarantee convergence of CT estimation error of current and prediction impulsive observers.

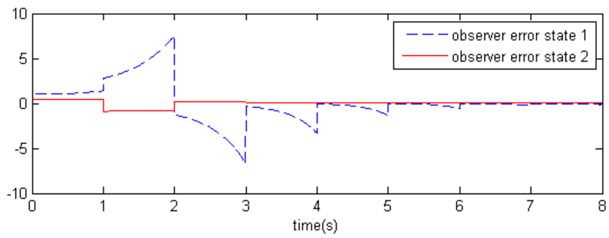

FIGURE 6. **Prediction impulsive observer with convergent inter-sample behavior $\tilde{\lambda}_2 = 1.60$.**

B. CONTINUOUS EXPONENTIAL BOUND FOR CURRENT IMPULSIVE OBSERVER

We define $\alpha_i \geq 1$ as the following bound for STM over the sampling interval

$$\|\Phi_A(kT, t)\| \leq \alpha_i \quad (k+1)T \geq t > kT. \tag{29}$$

The order of the two time argument of STM indicates integration backward in time, where $i = 0, 1, 2, \ldots$. Such a bound α_i always exists in linear systems on STM over a closed intervals $(k+1)T \geq t > kT$, where k is the discrete time index. A sequence of such bounds termed as State Transition Matrix Backward Sequence (STMBS) over the sampling interval T is given by

$$\{\alpha_0, \alpha_1, \alpha_2, \ldots\}. \tag{30}$$

The STMBS diverges with less than or equal to exponential rate if

$$\alpha_i \leq \eta_c \alpha^i \quad \forall\, i \geq 0, \tag{31}$$

where $\alpha = \frac{\alpha_1}{\eta_c}$ and $\eta_c = \alpha_0$.

With the above definitions, the following theorem establishes sufficient conditions for continuous exponential bound for estimation error of current impulsive observers.

Theorem 1: The CT estimation error for the current impulsive observer is uniformly exponentially bounded as follows

$$\|\bar{e}(t)\| \leq \gamma_c e^{-\lambda_c(t-t_0)} \|e_o\|,$$

with positive constant γ_c and $\lambda_c = \left(\frac{-1}{T}\right) \ln\left(\frac{\alpha}{\tilde{\lambda}}\right)$, if the observer gain $H_c[k]$ is designed such that the DT observer error dynamics (25) exponentially convergence with convergence rate $\tilde{\lambda}$ satisfying $\alpha < \tilde{\lambda}$.

Proof: The exponential convergence of (25) with convergence rate $\tilde{\lambda}$ implies

$$\left\|\hat{e}[k]\right\| \leq \tilde{\gamma} \left(\frac{1}{\tilde{\lambda}}\right)^{k-k_0} \left\|\hat{e}[k_0]\right\|. \tag{32}$$

where $\tilde{\gamma} > 1$ [21, p. 526] and $\tilde{\lambda} \geq 1$, both independent of k_0. The maximum current observer error in between the sampled intervals is expressed as

$$\|\bar{e}(t)\| \leq \alpha_i \|\bar{e}[k]\| \quad (k+1)T \geq t > kT, \tag{33}$$

iteratively (32) and (33) are written while considering the left continuity of states for current observer i.e. $\Phi(t, kt) = \Phi(t, kt^+)$

$$\begin{aligned}
\|\bar{e}[k_1]\| \leq \frac{\tilde{\gamma}}{\tilde{\lambda}} \|\bar{e}[k_0]\| \quad & \|\bar{e}(t)\| \leq \|\Phi(t,1)\| \, \|\bar{e}[k_1]\| \;\forall k_1T \geq t > k_0T, \\
& \leq \frac{\tilde{\gamma}\alpha_0}{\tilde{\lambda}} \|\bar{e}[k_0]\|, \\
\|\bar{e}[k_2]\| \leq \frac{1}{\tilde{\lambda}} \|\bar{e}[k_1]\| \quad & \|\bar{e}(t)\| \leq \alpha_1 \|\bar{e}[k_1]\| \;\forall k_2T \geq t > k_1T, \\
& \leq \frac{\tilde{\gamma}\alpha_1}{\tilde{\lambda}^2} \|\bar{e}[k_0]\|, \\
\vdots \qquad\qquad & \qquad \vdots \qquad\qquad\qquad \vdots \\
\|\bar{e}[k]\| \leq \frac{1}{\tilde{\lambda}} \|\bar{e}[k-1]\| \quad & \|\bar{e}(t)\| \leq \alpha_k \|\bar{e}[k-1]\| \;\forall (k+1)T \geq t > kT, \\
& \leq \frac{\tilde{\gamma}\alpha_k}{\tilde{\lambda}^{k-k_0}} \|\bar{e}[k_0]\|.
\end{aligned} \tag{34}$$

Incorporating (31), considering the starting point of the STMBS as k_0, we get

$$\|\bar{e}(t)\| \leq \frac{\tilde{\gamma}\alpha^{k-k_0}}{\tilde{\lambda}^{k-k_0}} \|\bar{e}[k_0]\|, \tag{35}$$

hence from (35), convergence of (34) is guaranteed only if

$$\alpha < \tilde{\lambda}. \tag{36}$$

Following uniform exponential bound is defined by choosing $\gamma_c = \frac{\tilde{\gamma}\eta_c}{\alpha}$ and $\lambda_c = \left(\frac{-1}{T}\right) \ln\left(\frac{\alpha}{\tilde{\lambda}}\right)$ for any t_o and e_o for the solution of linear state equation (6) and (7)

$$\|\bar{e}(t)\| \leq \gamma_c e^{-\lambda_c(t-t_0)} \|e_o\|. \tag{37}$$

□□□□

A special case of the above result if a supremum of (30) exists, with α defined as

$$\alpha = \sup\{\alpha_0, \alpha_1, \alpha_2, \ldots\}, \tag{38}$$

is presented in the following corollary.

Corollary 1: The CT state estimation error for current impulsive observer is uniformly exponentially bounded with convergence rate $\lambda_c = \left(\frac{1}{T}\right) \ln(\tilde{\lambda})$ if there exists α defined in (38) and observer gain $H_c[k]$ designed such that the DT estimation error (25) exponentially converges with rate $\tilde{\lambda}$.

Proof: Under these condition, $\alpha = 1$ in (31). The rest of the proof is trivial.

Corollary 2: UES of DT estimation error (25) for linear time invariant (LTI) systems implies UES of current observer.

Proof: Supremum of (38) always exists for LTI systems.

C. CONTINUOUS EXPONENTIAL BOUND FOR PREDICTION IMPULSIVE OBSERVER

We define $\beta_i \geq 1$ as the following bound on the STM

$$\|\Phi(t, kT)\| \leq \beta_i \quad kT \leq t < (k+1)T. \tag{39}$$

A sequence of such bounds termed as State Transition Matrix Forward Sequence (STMFS) over the sampling interval T is given by

$$\{\beta_0, \beta_1, \beta_2, \ldots\}. \tag{40}$$

The sequence STMFS diverges with less than or equal to exponential rate, then

$$\beta_i \leq \eta_p \beta^i, \tag{41}$$

where $\eta_p = \beta_0$ and $\beta = \frac{\beta_1}{\eta_p}$.

With the above definitions, we present the following theorem which establishes sufficient conditions for exponential convergence of CT state estimation error of the prediction impulsive observer.

Theorem 2: The CT estimation error for the prediction impulsive observer is uniformly exponentially bounded as follows

$$\left\|\hat{e}(t)\right\| \leq \gamma_p e^{-\lambda_p(t-t_0)} \left\|e_o\right\|,$$

with positive constants γ_p and $\lambda_p = \left(\frac{-1}{T}\right) \ln\left(\frac{\beta}{\tilde{\lambda}}\right)$. If the observer gain $H_p[k]$ is designed such that the DT estimation error dynamics (19) exponentially converge with convergence rate $\tilde{\lambda}$ satisfying $\beta < \tilde{\lambda}$.

Proof: The exponential convergence of (19) with convergence rate $\tilde{\lambda}$ implies (32). Maximum prediction observer error in between the sampled intervals is expressed as

$$\left\|\hat{e}(t)\right\| \leq \beta_i \left\|\hat{e}[k]\right\| \quad kT \leq t < (k+1)T, \tag{42}$$

Iteratively (32) and (42) can be combined as

$$\begin{aligned}
&\left\|\hat{e}(t)\right\| \leq \beta_0 \left\|\hat{e}[k_0]\right\| \,\forall\, k_0T \leq t < k_1T,\\
&\left\|\hat{e}(t)\right\| \leq \beta_1 \left\|\hat{e}[k_1]\right\| \,\forall k_1T \leq t < k_2T \; \left\|\hat{e}[k_1]\right\| \leq \frac{\tilde{\gamma}}{\tilde{\lambda}} \left\|\hat{e}[k_0]\right\|,\\
&\qquad \leq \frac{\tilde{\gamma}\beta_1}{\tilde{\lambda}} \left\|\hat{e}[k_0]\right\|,\\
&\quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
&\left\|\hat{e}(t)\right\| \leq \beta_k \left\|\hat{e}[k]\right\| \,\forall kT \leq t < (k+1)\,T \left\|\hat{e}[k]\right\| \leq \frac{\tilde{\gamma}}{\tilde{\lambda}^{k-k_o}} \left\|\hat{e}[k_0]\right\|,\\
&\qquad \leq \frac{\tilde{\gamma}\beta_k}{\tilde{\lambda}^{k-k_o}} \left\|\hat{e}[k_0]\right\|.
\end{aligned} \tag{43}$$

Incorporating (41) by considering the starting point for the STMFS sequence as k_0, we get

$$\left\|\hat{e}(t)\right\| \leq \frac{\tilde{\gamma}\eta_p\beta^{k-k_0}}{\tilde{\lambda}^{k-k_o}} \left\|\hat{e}[k_0]\right\|$$

It can be seen that the convergence of the above is guaranteed if

$$\beta < \tilde{\lambda}. \tag{44}$$

Consequently, the following uniform exponential bound is defined by choosing $\gamma_p = \frac{\tilde{\gamma}\eta_p\tilde{\lambda}}{\beta}$ and $\lambda_p = \left(\frac{-1}{T}\right) \ln\left(\frac{\beta}{\tilde{\lambda}}\right)$ for any t_o and e_o for the solution of linear state equation (11) and (12)

$$\left\|\hat{e}\,(t)\right\| \leq \gamma_p e^{-\lambda_p(t-t_0)} \left\|e_o\right\|. \tag{45}$$

□□□□

Analogous to the case of current impulsive observers if the supremum of (40) exists, which is defined as

$$\beta = \sup\{\beta_0, \beta_1, \beta_2, \ldots.\}. \tag{46}$$

Then a special case of Theorem 2 for prediction impulsive observers is stated as follows

Corollary 3: The CT state estimation error for prediction impulsive observer is uniformly exponentially bounded with convergence rate $\lambda_p = \left(\frac{1}{T}\right) \ln(\tilde{\lambda})$, if there exists β defined in (46) and observer gain $H_p[k]$ designed such that the DT estimation error (19) exponentially converges with rate $\tilde{\lambda}$.

Proof: Under these condition, $\beta = 1$ in (46). The rest of the proof is trivial.

Corollary 4: UES of DT estimation error (19) for LTI systems implies UES of prediction observer.

Proof: Supremum of (46) always exists in LTI systems.

D. EXPONENTIAL BOUND FOR CONTINUOUS SAMPLED-DATA RECONSTRUCTION OBSERVER

The estimation error for the continuous reconstruction observer is defined as

$$\widehat{e}(t) = \bar{e}(t)\psi(t) + \hat{e}(t)(1 - \psi(t)). \tag{47}$$

The exponential convergence of the estimation error (47) follows naturally if the continuous estimation errors of the current and prediction impulsive observers are exponentially bounded. The result is summarized in the following Lemma.

Lemma 3: The estimation error of the continuous reconstruction observer (47) is exponentially bounded if the CT estimation errors of current and prediction impulsive observers are exponentially bounded.

The proof of the Lemma trivially follows from Theorem 1 and Theorem 2.

V. EXAMPLE

The prediction, current and reconstruction observers are illustrated for state estimation with following LTV system. The peculiar reason for selecting simple and non-trivial second order sampled-data system example is to clearly demonstrate the discussion results of unstable system

$$\dot{x}(t) = \begin{bmatrix} 2\varepsilon t & 0 \\ 0 & -2\kappa t \end{bmatrix} + Bu(t), \tag{48}$$

where $B = [0\ 1]^T$ and $C = [12]$. The constant $\varepsilon = 0.05$ and $\kappa = 0.05$ results in a unstable (divergent) system. The simulation results with $T = 1$ sec and sinusoidal input $u(t) = sin(2\pi t)$ are discussed with initial conditions $x_o = [1\ 0.5]^T$.

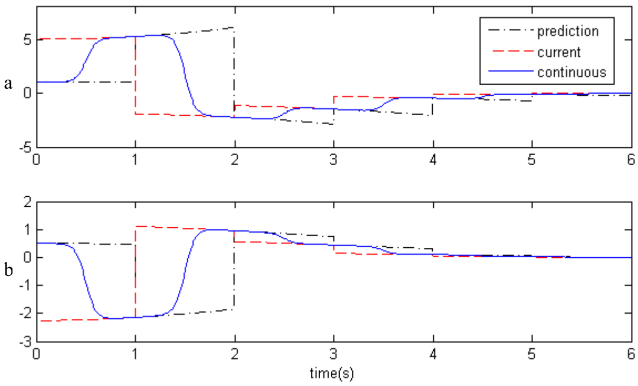

FIGURE 7. Estimation Error for divergent system for $x_0 = [1\ 0.5]^T$ and $\tilde{\lambda} = 1.11$ (a) Observer error for x_1 (b) Observer error for x_2.

The estimation error for second order system with prediction, current and reconstruction observer is illustrated in Fig. 7 for discrete convergence rate $\tilde{\lambda} = 1.11$ and l step observability for $l = 3$. Prediction estimate is the continuation of current estimate from the common point of both observers. Reconstruction observer utilizes current and prediction impulsive estimates to provide continuous (non-impulsive) output for sampled-data system. In this example $t^* = (k + \frac{1}{2})T$ is considered for simulation results with $\psi(t)$ discussed in previous section.

Impulsive jumps can be observed at sampling time instants in prediction and current estimated error states. The reconstruction observer error states are available at sampling instants without impulsive jumps. Reconstruction observer error output follows prediction observer for Δ_1 duration and current observer for Δ_3 duration. Reconstruction observer error for Δ_2 duration is measured by using (47) with current, prediction and fusion function outputs.

VI. CONCLUSION

A continuous (non-impulsive) reconstruction observer is proposed for sampled-data system. Its construction is based on the fusion of two proposed impulsive observers. Prediction and current impulsive observer designs are based on associated DT dynamical equation for estimation error. For a current observer, impulsive correction is applied immediately after receiving the output sample, whereas the same is done just before the next observation for a prediction observer. The relationship between the current and prediction impulsive observers exhibits a commonality of estimation at the sampled point. This relationship is exploited to develop a continuous reconstruction observer without jumps by fusing the estimates of the two impulsive observers.

The analyses carried out in this paper deal with convergence properties of the discussed observers in a comprehensive manner. It covers for both stable and unstable systems. Certain bounds on STM play an important role in this context. As a future recommendation, this paper is a pre-sequel to sampled-data LTV regulator problem. The philosophy introduced here can also be extended to the nonlinear realm. Dead beat observer has not been discussed in this paper intentionally, as a detailed discussion on receding horizon observer for sampled-data LTV demands a separate paper. The simulation example demonstrates the performance of the reconstruction observer in relation to the functioning of current and the prediction impulsive observers.

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