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Switching Laws Design for Stability of Finite and Infinite Delayed Switched Systems With Stable and Unstable Modes

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ABSTRACT This paper studies the switching laws designed to maintain the stability of delayed switched nonlinear systems with both stable and unstable modes. The addressed time delays include finite and infinite delays. First, we consider the finitely delayed nonlinear switched systems and establish some delay differential inequalities that play an important role in the design of average dwell time (ADT)-based switching laws. Then, by employing multiple Lyapunov functions and the ADT approach, some delay-dependent switching laws for globally uniform exponential stability are derived. This approach establishes a relationship between time delay, ADT constant, and the ratio of total dwell time between stable and unstable modes. This method can guarantee the stability of finitely delayed switched systems with stable and unstable modes if the divergence rate and total dwell times of unstable modes can be effectively controlled and balanced by an ADT-based switching control with stable modes. Furthermore, based on multiple Lyapunov functions coupled with the Razumikhin technique, we study the infinitely delayed nonlinear switched systems and present some delay-independent switching laws for uniform stability and globally uniformly exponential stability. These can be applied to the cases in which the time delay in stable or unstable modes cannot be exactly observed, and the bound of the time delay may be unknown or infinite. The proposed results in this paper are more general than several recent works. Finally, some numerical examples and their computer simulations are given to demonstrate the effectiveness and advantages of the designed switching laws.

INDEX TERMS Switched systems, stable and unstable modes, average dwell time (ADT), finite/infinite delay, stability.

I. INTRODUCTION

Switched systems have been well-known for their importance in practical applications. Typical examples of switched systems include flight control and management systems [1], [2], computer disc drives [3], stepper motor drives (where only a limited number of gear ratios is available) [4], intelligent vehicle highway systems [5], robotic control systems [6] and networks [7]. One of the important aspects of studying switched systems is to consider the stability problem of the systems. It has been shown that there are three basic problems in the stability and design of switched systems [8]. First, find the conditions for stability under arbitrary switching laws [9]–[11]. Second, identify the limited but useful class of stabilizing switching laws [12], [13]. Third, construct a stabilizing switching law [14]–[16]. To date, much work on those problems has been reported in the literature. In this paper, we mainly consider the third problem for a class of switched systems.

However, as we know, a time delay is often encountered in many real systems such as engineering, biological and economical systems. Its existence is frequently a source of oscillation and instability [17]–[19]. Conversely, a time delay can also be introduced to solve some important problems [20], [21]. For example, Tank and Hopfield [20] designed a neural circuit with distributed delays, which solves a general problem of recognizing patterns in a time-dependent signal. Hence, it is rewarding and important to consider the effect of time delay on the stability of dynamic systems. A delayed switched system is a type of dynamically switched system that includes time delays in individual modes and has wide applications in many fields. These fields include power systems and power electronics [22], [23], Nakagami fading systems [24], time-delay systems with controller or actuator failure [25], and networked control systems [26]. In recent years, many interesting results on the stability of delay switched systems have been presented via different approaches. These approaches include the multiple Lyapunov (combined with Razumikhin techniques) approach [27], [28], average dwell time or dwell time techniques [29], [30], the linear matrix inequality method (LMI) [31], common Lyapunov functions [32], inequalities techniques [33] and others. Generally, those results can be classified into two categories: delay-dependent results [29]–[33] and delay-independent results [27], [28], [34]. The delay-dependent results are considered to be less conservative than the delay-independent ones since they make use of the information on the length of delays, especially for the case when the time delay is small. The delay-independent results are not related to the time-delay information, which is particularly useful for delayed switched systems subject to unknown, infinite, or inestimable value time delays. However, it should be noted that, to date, the switching laws designed to maintain the stability of delayed switched nonlinear systems have not been adequately addressed, which remains an interesting research topic.

In addition, one may observe that most recent studies on the stability of switched systems have focused on the case where all individual modes are stable and then consider the stability of switched systems under given or designed switching signals [10], [12]–[16], [27]–[34]. Note a remarkable fact that the switching between stable and unstable modes or between unstable modes may also lead to the stability (even exponential stability) of switched systems. This indicates that the design of switching laws for the stability of switched systems (including unstable modes) is theoretically feasible. In fact, in many applications, it has been shown that switched systems with stable and unstable modes cannot be avoided. For example, in supervisory control systems [35], [36], the multi-controllers are sequentially applied to the plant and switched until the stabilizing one is found. Before the appropriate control law is found, there exists a certain time lag in which several destabilizing control laws may be applied and render the plant unstable. This process is described by a switched system with unstable modes. Another example, in asynchronous networked control [37], is when the controller and the plant are connected by an unreliable communication link. The closed-loop system can be described by two modes. One mode is related to the healthy, stable link and another one is related to the broken, unstable link. More applications of switched systems, including unstable modes, can be found in multi-agent systems with switching topology, complex dynamical networks and control systems with intermittent faults [38], [39]. Hence, it is necessary and important to study the stability and design of switching laws for switched systems in which stable and unstable modes co-exist. However, when unstable modes exist in a switched system, it is possible that the states of the system will trend towards infinity if the dwell time of unstable modes is too long. Thus, we have a great theoretical challenge. In recent years, increasing attention has been paid to the study of stability and controller design of switched systems with unstable modes. Some interesting results have been reported in the literature. In particular, [40] developed the ADT approach to the stability analysis of switched systems with stable and unstable modes. It was shown that the stability of switched linear systems can be guaranteed if the total activating period of unstable modes is relatively small compared to that of stable modes. Reference [41] improved those results and derived some new ones on stability based on a common Lyapunov function. Reference [42] proposed a new approach to the stabilization problem of switched nonlinear systems with some unstable modes. The developed approach relied on the trade-off between the gains of functions in continuous modes and dropped the required constant ratio condition from [40]. Reference [43] studied the exponential stability and asynchronous stabilization of switched nonlinear systems with stable and unstable modes via T̍CS fuzzy model by minimum dwell time and piecewise Lyapunov-like functions methods. For other related, interesting results, see references [44]–[48]. However, it should be noted that, although those results are very useful for switched systems with unstable modes, they cannot be applied to delayed switched systems. As a class of infinite dimensional systems, delayed switched systems with unstable modes have more complicated structures that lead to complex dynamics. In recent years, switching control for the stability of switched systems with time delays has been extensively studied. For example, [28], [32], and [52]–[54] addressed switched systems with constant delays, and [25], [31], and [33] treated the time-varying delays. However, most of these studies are based on the assumption that time delays are constants or time-varying and differentiable. Moreover, they mainly considered the switched systems that only included stable modes. More recently, [49] studied the switched time-varying delay systems with unstable modes and some ADT-based exponential stability results. They fully considered the effects of mixed modes derived by employing an LK functional and an estimation of the quadrature. However, the approach developed in [49] is only valid for special linear switched delayed systems. Moreover, the assumptions on differentiability and an exact bound of the time-varying delay are imposed. Hence, more methods and tools should be explored and developed on this topic.

With the above motivations, the purpose of this paper is to consider a class of nonlinear delayed switched systems with stable and unstable modes and finite and infinite time delays. In this paper, the assumption on the differentiability of the time-varying delays, such as those in [25], [31], [33], and [49], is completely removed. Inspired by the idea that the stability of switched linear systems can be guaranteed if the total activating period of unstable modes is relatively small compared to that of stable modes [40], first, we developed the idea of finitely delayed switched systems with stable and unstable modes. To this end, some delayed differential inequalities are presented, which play an important role in the design of ADT-based switching laws. Then, based on multiple Lyapunov functions and the ADT approach, we present some delay-dependent switched laws. Although a time-delay effect exists, we do not impose a strict restriction on the dwell time for individual stable and unstable modes. However, the ratio of the total dwell time between them is needed. Then, based on multiple Lyapunov functions coupled with the Razumikhin technique, some delay-independent switched laws for infinitely delayed switched systems are presented, which are also applicable for finite delays or unknown delays. The remainder of this paper is organized as follows. In Section II, we shall introduce some preliminary knowledge. In Section III, some switched laws for finitely delayed switched systems are presented. In Section IV, we present some switched laws for infinitely delayed switched systems. Some numerical examples showing the effectiveness and advantages of the proposed approach are given in Section V. Finally, we shall make concluding remarks in Section VI.

Notations: Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of positive real numbers, $\mathbb{Z}_+$ the set of positive integers and $\mathbb{R}^n$ the n-dimensional real space equipped with the Euclidean norm $|\bullet|$. The notation $\mathcal{A}^T$ and $\mathcal{A}^{-1}$ mean the transpose of $\mathcal{A}$ and the inverse of a square matrix. $\mathcal{A} > 0$ or $\mathcal{A} < 0$ denotes that the matrix $\mathcal{A}$ is a symmetric and positive definite or negative definite matrix. If $\mathcal{A}, \mathcal{B}$ are symmetric matrices, $\mathcal{A} > \mathcal{B}$ means that $\mathcal{A} - \mathcal{B}$ is positive definite matrix. The notation $\star$ always denotes the symmetric block in one symmetric matrix. $[\bullet]^*$ denotes the integer function. For any interval $J \subseteq \mathbb{R}$, set $S \subseteq \mathbb{R}^k (1 \le k \le n)$, $C(J, S) = \{\varphi : J \to S \text{ is continuous}\}$ and $PC(J, S) = \{\varphi : J \to S$ is continuous everywhere except at finite number of points t, at which $\varphi(t^+)$, $\varphi(t^-)$ exist and $\varphi(t^+) = \varphi(t)\}$. Let $\mathscr{T} = \{\varphi : [t_0, \infty) \to \mathscr{P} = \{1, 2, \cdots, m\}, m \in \mathbb{Z}_+$, is a piecewise constant function $\}$. Define $\mathbb{C}_\tau = C([-\tau, 0], \mathbb{R}^n)$ and $\mathbb{BC}_\infty = BC((-\infty, 0], \mathbb{R}^n)$. $\mathcal{K} = \{a \in C(\mathbb{R}_+, \mathbb{R}_+) |\ a(0) = 0$ and $a(s) > 0$ for $s > 0$ and a is strictly increasing in $s\}$.

II. PRELIMINARIES

Consider the following switched nonlinear delayed system:

$$\begin{cases} \dot{x}(t) = f_{\sigma(t)}(t, x_t), & t \ge t_0, \\ x_{t_0} = \phi, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $\dot{x}(t)$ denotes the right-hand derivative of $x(t)$. We assume that the state of the system (1) does not jump at the switching instances, i.e., the trajectory x is everywhere continuous. $\sigma(t) \in \mathscr{T}$ denotes a piecewise constant signal that called a switching signal, which will be determined later. When $\sigma(t) = l, 1 \le l \le m$, we say that the mode $\dot{x}(t) = f_l$ is activated. $f_{\sigma(t)} \in C([t_0, \infty) \times \mathbb{C}_\tau, \mathbb{R}^n)$ and $f_{\sigma(t)}(t, 0) \equiv 0$ for all $t \ge t_0$ and $\sigma \in \mathscr{T}$. For each $t \ge t_0$, $x_t \in \mathbb{C}_\tau$ is defined by $x_t(s) = x(t + s)$, $s \in [-\tau, 0]$ and $||x_t|| = \sup\{|x(t + s)| : -\tau \le s \le 0\}$. Denote by $(x(t, t_0, \phi), \sigma(t))$ (abbr. (x, σ)) the solution of switched system (1) with switched signal $\sigma \in \mathscr{T}$ and initial value (t_0, ϕ), where $\phi \in \mathbb{C}_\tau$.

We make the following preliminary assumptions:

(H_1) $\inf_{k \in \mathbb{Z}_+}\{t_k - t_{k-1}\} > 0$, where t_k denotes the switching instance, $k \in \mathbb{Z}_+$;

(H_2) The switching signal is minimal, i.e., $\sigma(t_k) \neq \sigma(t_{k+1})$ for all $k \in \mathbb{Z}_+$;

(H_3) $\tau \in [0, \infty]$. In the case when $\tau = +\infty$, the interval $[t - \tau, t]$ is understood to be replaced by $(-\infty, t]$, for any $t \in \mathbb{R}_+$ and $\mathbb{C}_\tau$ is understood to be replaced by $\mathbb{BC}_\infty$;

(H_4) $f_\sigma(t, \varphi)$ is Lipschitzian in φ in each compact set in $\mathbb{C}_\tau$.

Note that assumption (H_1) implies that the switching signals have at most finite switching times over a finite time interval (exclude Zeno behavior [1]) and we denote such kind of switching signals by set $\mathscr{T}_0$ for later use. Assumption (H_2) implies that the switched modes are different between any consecutive time intervals $[t_{k-1}, t_k)$ and $[t_k, t_{k+1})$. In fact, if the switched mode is the same one between two consecutive time intervals, then it can be understood that there is no switch happened and the intervals can be merged. Assumption (H_3) implies that the time delay considered in this paper may be non-differentiable, unknown, even unbounded. To this point, our assumption is weaker than those in [27]–[34]. While assumption (H_4) is given to ensure the existence and uniqueness of solutions of (1). Note that $f_{\sigma(t)}(t, 0) \equiv 0$ for any $t \ge t_0$ and $\sigma \in \mathscr{T}$, it implies that $(0, \sigma)$ is a solution of (1), which is called the trivial solution. For a detailed discussion on the existence problem, we refer the reader to the books by Gu *et al.* [18] and Hino *et al.* [50].

Now we introduce some of the notations [40], [46], [51]. For any $\delta > 0$, let $\mathscr{T}_1(\delta)$ denote the set of all switching signals satisfying

$$\mathscr{T}_1(\delta) = \Big\{\sigma \in \mathscr{T}_0 : \sup_{\sigma(t_{k-1}) \in \mathscr{P}_u,\ k \in \mathbb{Z}_+} \{t_k - t_{k-1}\} \le \delta\Big\},$$

where $\mathscr{P}_u \subseteq \mathscr{P}$ denote the unstable modes of system (1) and denote by $\mathscr{P}_s \subseteq \mathscr{P}$ the stable ones. Clearly, $\mathscr{P}_u \cap \mathscr{P}_s = \emptyset$, $\mathscr{P}_u \cup \mathscr{P}_s = \mathscr{P}$. Let $\pi_s(t)$ denotes the total activation time of the stable modes and $\pi_u(t)$ the total activation time of the unstable modes on the time interval $[t_0, t)$, i.e.,

$$\pi_u(t) = \sum_{\substack{\sigma(t_{k-1}) \in \mathscr{P}_u, \\ 1 \le k \le N_\sigma(t_0, t)}} (t_k - t_{k-1}),$$

$$\pi_s(t) = \sum_{\substack{\sigma(t_{k-1}) \in \mathscr{P}_s, \\ 1 \le k \le N_\sigma(t_0, t)}} (t_k - t_{k-1}),$$

where $N_\sigma(t_0, t)$ denotes the number of switching times of σ over the interval $[t_0, t)$.

For given $N_0 \geq 0$ and $\tau_D > 0$, let $\mathcal{S}_{ave}[\tau_D, N_0]$ be the set of all switching signals satisfying

$$\mathcal{S}_{ave}[\tau_D, N_0] = \left\{\sigma \in \mathscr{T}_0 : N_\sigma(\beta, t) \leq N_0 + \frac{t-\beta}{\tau_D}\right\},$$

for $\forall t \geq \beta \geq t_0$, where the constant τ_D is called the "average dwell time" and N_0 the "chatter bound". The idea behind it is that there may exist some consecutive switchings separated by less than τ_D, but the average interval between consecutive switchings is no less than τ_D. In addition, to control the relation between unstable modes and stable ones, the definition of set $\mathscr{T}_2(r, T)$ is introduced as follows:

$$\mathscr{T}_2(r, T) = \left\{\sigma \in \mathscr{T}_0 : \frac{\pi_u(t)}{\pi_s(t)} \leq r,\ \forall t \geq T \geq t_0\right\},$$

where $r \geq 0$ and $T \geq t_0$ are two constants.

Let $\{V_p, p \in \mathscr{P}\}$ be a family of continuous functions from from $\mathbb{R}_+ \times \mathbb{R}^n$ to $\mathbb{R}_+$. Denote by $\mathbb{V}_0$ the class of all radially unbounded, infinitesimal upper bound, and positive definite continuous functions $V_p, p \in \mathscr{P}$. Denote by $\mathbb{V}_1$ the class of functions $V_p \in \mathbb{V}_0$ satisfy $\omega_1(|x|) \leq V_p(t, x) \leq \omega_2(|x|)$, uniformly for all $p \in \mathscr{P}$ and $(t, x) \in [t_0 - \tau, \infty) \times \mathbb{R}^n$, where $\omega_1, \omega_2 \in \mathcal{K}$. Denote by $\mathbb{V}_2$ the class of functions $V_p \in \mathbb{V}_1$ satisfy $c_1|x|^m \leq V_p(t, x) \leq c_2|x|^m$, uniformly for all $p \in \mathscr{P}$ and $(t, x) \in [t_0 - \tau, \infty) \times \mathbb{R}^n$, where c_1, c_2 and m are some positive constants. For given $V_p \in \mathbb{V}_0$, the upper right-hand derivative of V_p along the solution of switched system (1) is defined by

$$D^+V_p = \limsup_{h \to 0^+} \frac{1}{h}\{V(t+h, \varphi(0) + hf_p) - V(t, \varphi(0))\}$$

for $(t, \varphi) \in \mathbb{R}_+ \times \mathbb{C}_\tau$.

Definition 1: Given some family of piecewise switching signals $\mathscr{T}$, the switched system (1) is said to be

(P_1) *uniformly stable (US)* over $\mathscr{T}$, if for any $t_0 \geq 0$ and $\varepsilon > 0$, there exists some $\delta_0 = \delta_0(\varepsilon) > 0$, independent of t_0 and $\sigma \in \mathscr{T}$, such that $\phi \in \mathbb{C}_\tau$ and $\|\phi\| \leq \delta_0$ implies that for each $\sigma \in \mathscr{T}$, $|x(t)| \leq \varepsilon, t \geq t_0$.

(P_2) *globally uniformly exponentially stable (GUES)* over $\mathscr{T}$, if there exist two constants $\gamma > 0$ and $\mathbb{M} \geq 1$, independent of t_0 and $\sigma \in \mathscr{T}$, such that for any given initial data $\phi \in \mathbb{C}_\tau$ and each $\sigma \in \mathscr{T}$,

$$|x(t)| \leq \mathbb{M}\|\phi\| \exp\big(-\gamma(t - t_0)\big), t \geq t_0.$$

III. SWITCHED SYSTEMS WITH FINITE DELAY

In this section, based on the ADT approach, we shall design some delay-dependent switching laws to guarantee the GUES of system (1) with finite delay (i.e.,$\tau < \infty$).

First, for any $T \geq t_0$, let $\mathcal{U}(t) \in PC([t_0, T), \mathbb{R}_+)$ with $t_1, t_2, \cdots, t_{N(t_0,T)}$ as the finite discontinuous points, where $N(t_0, T)$ denotes the number of discontinuous points over the interval $[t_0, T)$. Clearly, $\mathcal{U}(t) \in C([t_0, T), \mathbb{R}_+)$ when $N(t_0, T) = 0$. Define

$$\pi(T) = \Big\{[t_0, t_1), [t_1, t_2), \cdots, [t_{N(t_0,T)}, T)\Big\}.$$

Set $\pi_{\bar{u}}(T), \pi_{\bar{s}}(T) \subseteq \pi(T)$ and satisfy

$$\pi_{\bar{u}}(T) \cap \pi_{\bar{s}}(T) = \emptyset, \quad \pi_{\bar{u}}(T) \cup \pi_{\bar{s}}(T) = \pi(T).$$

Denote

$$|\pi_{\bar{u}}(T)| = \sum_{I \in \pi_{\bar{u}}(T)} |I|, \quad |\pi_{\bar{s}}(T)| = \sum_{I \in \pi_{\bar{s}}(T)} |I|,$$

where $|I|$ denotes the length of interval I. Obviously, it holds that $|\pi_{\bar{u}}(T)| + |\pi_{\bar{s}}(T)| = T - t_0$.

Lemma 1: Assume that there exist constants $\lambda \geq 0$, $\bar{\lambda} \geq 0$, and function $\mathcal{U}(t) \in PC([t_0, T), \mathbb{R}_+)$ such that

$$D^+\mathcal{U}(t) \leq \lambda\mathcal{U}(t) + \bar{\lambda}\bar{\mathcal{U}}(t), \quad t \in [t_{k-1}, t_k), \tag{2}$$

for every $1 \leq k \leq N(t_0, T)$. Then

$$\mathcal{U}(t) \leq \bar{\mathcal{U}}(t_{k-1}) \exp\big((\lambda + \bar{\lambda})(t - t_{k-1})\big), \tag{3}$$

for all $t \in [t_{k-1}, t_k)$, where $\bar{\mathcal{U}}(t) = \sup_{s \in [t-\tau, t]} \mathcal{U}(s)$. In particular, the interval $[t_{k-1}, t_k)$ is understood to be replaced by $[t_0, T)$ when $N(t_0, T) = 0$.

Proof: For any $\varepsilon > 0$, we establish an auxiliary function:

$$\Gamma_\varepsilon(t) = \mathcal{U}(t) \exp\big(-(\lambda + \bar{\lambda} + \varepsilon)(t - t_{k-1})\big), \quad t \in [t_{k-1}, t_k).$$

To show that (2) holds, we first claim that

$$\Gamma_\varepsilon(t) \leq \bar{\mathcal{U}}(t_{k-1}), \quad t \in [t_{k-1}, t_k). \tag{4}$$

It is clear that $\Gamma_\varepsilon(t_{k-1}) = \mathcal{U}(t_{k-1}) \leq \bar{\mathcal{U}}(t_{k-1})$. If there exists some $t \in (t_{k-1}, t_k)$ such that $\Gamma_\varepsilon(t) > \bar{\mathcal{U}}(t_{k-1})$, then one may choose a $t^* \in [t_{k-1}, t_k)$ such that $\Gamma_\varepsilon(t^*) = \bar{\mathcal{U}}(t_{k-1})$, $\Gamma_\varepsilon(t) \leq \bar{\mathcal{U}}(t_{k-1})$, $t \in [t_{k-1}, t^*]$ and $D^+\Gamma_\varepsilon(t^*) \geq 0$. In this case, it holds that

$$\begin{aligned}
\mathcal{U}(t) &= \Gamma_\varepsilon(t) \exp\big((\lambda + \bar{\lambda} + \varepsilon)(t - t_{k-1})\big) \\
&\leq \bar{\mathcal{U}}(t_{k-1}) \exp\big((\lambda + \bar{\lambda} + \varepsilon)(t - t_{k-1})\big) \\
&= \Gamma_\varepsilon(t^*) \exp\big((\lambda + \bar{\lambda} + \varepsilon)(t - t_{k-1})\big) \\
&= \mathcal{U}(t^*) \exp\big((\lambda + \bar{\lambda} + \varepsilon)(t - t^*)\big) \\
&\leq \mathcal{U}(t^*), \quad t \in [t_{k-1}, t^*].
\end{aligned}$$

Note that $t^* \in [t_{k-1}, t_k)$ and $\Gamma_\varepsilon(t^*) \leq \mathcal{U}(t^*)$. It then can be deduced that

$$\bar{\mathcal{U}}(t^*) \leq \max\{\mathcal{U}(t^*), \bar{\mathcal{U}}(t_{k-1})\} = \max\{\mathcal{U}(t^*), \Gamma_\varepsilon(t^*)\} \leq \mathcal{U}(t^*),$$

which together (2) yields

$$\begin{aligned}
D^+\Gamma_\varepsilon(t^*) &= D^+\mathcal{U}(t^*) \exp\big(-(\lambda + \bar{\lambda} + \varepsilon)(t^* - t_{k-1})\big) \\
&\quad - (\lambda + \bar{\lambda} + \varepsilon)\mathcal{U}(t^*) \\
&\quad \times \exp\big(-(\lambda + \bar{\lambda} + \varepsilon)(t^* - t_{k-1})\big) \\
&\leq \exp\big(-(\lambda + \bar{\lambda} + \varepsilon)(t^* - t_{k-1})\big) \\
&\quad \times \big\{\lambda\mathcal{U}(t^*) + \bar{\lambda}\bar{\mathcal{U}}(t^*) - (\lambda + \bar{\lambda} + \varepsilon)\mathcal{U}(t^*)\big\} \\
&\leq -\varepsilon \exp\big(-(\lambda + \bar{\lambda} + \varepsilon)(t^* - t_{k-1})\big)\mathcal{U}(t^*) < 0,
\end{aligned}$$

which is a contradiction with $D^+\Gamma_\varepsilon(t^*) \geq 0$ and thus (4) holds for any $\varepsilon > 0$. Let $\varepsilon \to 0^+$, then inequality (3) can be directly obtained. This completes the proof. ■

Lemma 2: Assume that there exist constants $\lambda^\star > \bar{\lambda}^\star \geq 0$ and function $\mathcal{U}(t) \in PC([t_0, T), \mathbb{R}_+)$ such that

$$D^+\mathcal{U}(t) \leq -\lambda^\star \mathcal{U}(t) + \bar{\lambda}^\star \bar{\mathcal{U}}(t), \quad t \in [t_{k-1}, t_k), \tag{5}$$

for every $1 \leq k \leq N(t_0, T)$. Then

$$\mathcal{U}(t) \leq \bar{\mathcal{U}}(t_{k-1}) \exp(-h(t - t_{k-1})), \quad t \in [t_{k-1}, t_k), \tag{6}$$

where $h > 0$ is a constant and satisfies $\bar{\lambda}^\star \exp(h\tau) + h \leq \lambda^\star$. In particular, the interval $[t_{k-1}, t_k)$ is understood to be replaced by $[t_0, T)$ when $N(t_0, T) = 0$.

Proof: Since $\lambda^\star > \bar{\lambda}^\star \geq 0$, one may choose a constant $h > 0$ such that $\bar{\lambda}^\star \exp(h\tau) + h \leq \lambda^\star$. Then choose $\bar{h} \in (0, h)$ such that for any $\varepsilon \in (0, \bar{h}]$,

$$0 < \bar{\lambda}^\star \exp((h - \varepsilon)\tau) + h - \varepsilon < \lambda^\star. \tag{7}$$

Consider the following auxiliary function:

$$\Omega_\varepsilon(t) = \mathcal{U}(t) \exp\big((h - \varepsilon)(t - t_{k-1})\big), \quad t \in [t_{k-1}, t_k).$$

We claim that

$$\Omega_\varepsilon(t) \leq \bar{\mathcal{U}}(t_{k-1}), \quad t \in [t_{k-1}, t_k). \tag{8}$$

Firstly, it is clear that $\Omega_\varepsilon(t_{k-1}) = \mathcal{U}(t_{k-1}) \leq \bar{\mathcal{U}}(t_{k-1})$. If there exists some $t \in (t_{k-1}, t_k)$ such that $\Omega_\varepsilon(t) > \bar{\mathcal{U}}(t_{k-1})$, then one may choose a $t^\star \in [t_{k-1}, t_k)$ such that $\Omega_\varepsilon(t^\star) = \bar{\mathcal{U}}(t_{k-1})$, $\Omega_\varepsilon(t) \leq \bar{\mathcal{U}}(t_{k-1})$, $t \in [t_{k-1}, t^\star]$ and $D^+\Omega_\varepsilon(t^\star) \geq 0$. In this case, it holds that

$$\begin{aligned}
\mathcal{U}(t) &= \Omega_\varepsilon(t) \exp\big(-(h - \varepsilon)(t - t_{k-1})\big) \\
&\leq \bar{\mathcal{U}}(t_{k-1}) \exp\big(-(h - \varepsilon)(t - t_{k-1})\big) \\
&= \Omega_\varepsilon(t^\star) \exp\big(-(h - \varepsilon)(t - t_{k-1})\big) \\
&= \mathcal{U}(t^\star) \exp\big((h - \varepsilon)(t^\star - t)\big), \quad t \in [t_{k-1}, t^\star].
\end{aligned} \tag{9}$$

Then we can show that

$$\mathcal{U}(t) \leq \mathcal{U}(t^\star) \exp((h - \varepsilon)\tau), \quad t \in [t^\star - \tau, t^\star]. \tag{10}$$

In fact, one may prove the above problem from two cases. First, if $t^\star - \tau \geq t_{k-1}$, then the inequality (10) can be directly deduced in view of (9). Second, if $t^\star - \tau < t_{k-1}$, then it follows that

$$\begin{aligned}
\mathcal{U}(t) &\leq \begin{cases} \mathcal{U}(t^\star) \exp\big((h - \varepsilon)(t^\star - t)\big), & t \in [t_{k-1}, t^\star] \\ \bar{\mathcal{U}}(t_{k-1}), & t \in [t^\star - \tau, t_{k-1}] \end{cases} \\
&\leq \max\{\mathcal{U}(t^\star) \exp\big((h - \varepsilon)(t^\star - t_{k-1})\big),\ \bar{\mathcal{U}}(t_{k-1})\} \\
&= \max\{\mathcal{U}(t^\star) \exp\big((h - \varepsilon)(t^\star - t_{k-1})\big),\ \Omega_\varepsilon(t^\star)\} \\
&= \mathcal{U}(t^\star) \exp\big((h - \varepsilon)(t^\star - t_{k-1})\big) \\
&\leq \mathcal{U}(t^\star) \exp((h - \varepsilon)\tau),
\end{aligned}$$

that is, the inequality (10) holds. Hence, it follows from (5), (7) and (10) that

$$\begin{aligned}
D^+\Omega_\varepsilon(t^\star) &= D^+\mathcal{U}(t^\star) \exp\big((h - \varepsilon)(t^\star - t_{k-1})\big) \\
&\quad + (h - \varepsilon)\mathcal{U}(t^\star) \exp\big((h - \varepsilon)(t^\star - t_{k-1})\big) \\
&\leq \exp\big((h - \varepsilon)(t^\star - t_{k-1})\big)\big\{-\lambda^\star \mathcal{U}(t^\star) + \bar{\lambda}^\star \bar{\mathcal{U}}(t^\star) \\
&\quad + (h - \varepsilon)\mathcal{U}(t^\star)\big\} \\
&\leq \exp\big((h - \varepsilon)(t^\star - t_{k-1})\big)\mathcal{U}(t^\star)\big\{-\lambda^\star + h - \varepsilon \\
&\quad + \bar{\lambda}^\star \exp((h - \varepsilon)\tau)\big\} < 0,
\end{aligned}$$

which is contradiction with $D^+\Omega_\varepsilon(t^\star) \geq 0$ and thus (8) holds for any $\varepsilon \in (0, \bar{h}]$. Let $\varepsilon \to 0^+$, then inequality (6) can be derived. This completes the proof. ■

Based on Lemmas 1 and 2, the following result can be derived, which plays an important role in the design of ADT-based switching laws for stability of delayed switched system (1) with unstable modes.

Lemma 3: Assume that there exist constants $\lambda \geq 0$, $\bar{\lambda} \geq 0, \lambda^\star > \bar{\lambda}^\star \geq 0, \rho \geq 1$ and function $\mathcal{U}(t) \in PC([t_0, T), \mathbb{R}_+)$ such that

$$D^+\mathcal{U}(t) \leq \begin{cases} \lambda \mathcal{U}(t) + \bar{\lambda}\bar{\mathcal{U}}(t), & t \in \pi_{\bar{u}}(T), \\ -\lambda^\star \mathcal{U}(t) + \bar{\lambda}^\star \bar{\mathcal{U}}(t), & t \in \pi_{\bar{s}}(T) \end{cases} \tag{11}$$

and when $N(t_0, T) \geq 1$, it holds that

$$\mathcal{U}(t_k) \leq \rho \mathcal{U}(t_k^-), \quad 1 \leq k \leq N(t_0, T). \tag{12}$$

Then

$$\begin{aligned}
\mathcal{U}(T^-) &\leq \bar{\mathcal{U}}(t_0)\Big[\exp(h\tau)\rho\Big]^{N(t_0, T)} \\
&\quad \times \exp\Big((\lambda + \bar{\lambda})|\pi_{\bar{u}}(T)| - h|\pi_{\bar{s}}(T)|\Big),
\end{aligned} \tag{13}$$

where $h > 0$ is a constant and satisfies $\bar{\lambda}^\star \exp(h\tau) + h \leq \lambda^\star$.

Proof: First, if there is no switching point over the interval $[t_0, T)$, i.e., $N(t_0, T) = 0$, then $[t_0, T) \in \pi_{\bar{u}}(T)$ or $\pi_{\bar{s}}(T)$. If $[t_0, T) \in \pi_{\bar{u}}(T)$, then it implies that $\pi_{\bar{s}}(T) = \emptyset$. By (11) and lemma 1, we have

$$\mathcal{U}(t) \leq \bar{\mathcal{U}}(t_0) \exp\big((\lambda + \bar{\lambda})(t - t_0)\big), \quad t \in [t_0, T),$$

which together with the fact that $|\pi_{\bar{u}}(T)| + |\pi_{\bar{s}}(T)| = T - t_0$ yields that

$$\begin{aligned}
\mathcal{U}(T^-) &\leq \bar{\mathcal{U}}(t_0) \exp\big((\lambda + \bar{\lambda})(T - t_0)\big) \\
&= \bar{\mathcal{U}}(t_0) \exp\big((\lambda + \bar{\lambda})|\pi_{\bar{u}}(T)|\big).
\end{aligned}$$

Hence, (13) holds for $[t_0, T) \in \pi_{\bar{u}}(T)$. If $[t_0, T) \in \pi_{\bar{s}}(T)$, then it implies that $\pi_{\bar{u}}(T) = \emptyset$. Similarly, by (11) and lemma 1, it can be easily deduced that (13) holds.

Now we assume that (13) holds for $N(t_0, T) = k$. By induction, next we shall show that (13) holds for $N(t_0, T) = k + 1$. Note that $t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < T$, by inductive assumption, it holds that

$$\begin{aligned}
\mathcal{U}(t_{k+1}^-) &\leq \bar{\mathcal{U}}(t_0)\Big[\exp(h\tau)\rho\Big]^{N(t_0, t_{k+1})} \\
&\quad \times \exp\Big((\lambda + \bar{\lambda})|\pi_{\bar{u}}(t_{k+1})| - h|\pi_{\bar{s}}(t_{k+1})|\Big).
\end{aligned} \tag{14}$$

Since $|\pi_{\bar{u}}(t)|$ and $|\pi_{\bar{s}}(t)|$ are both monotonic nondecreasing for $t \in [t_0, T)$, we can easily obtain that

$$|\pi_{\bar{s}}(t_{k+1} - \tau)| \leq |\pi_{\bar{s}}(t_{k+1})| \leq |\pi_{\bar{s}}(t_{k+1} - \tau)| + \tau,$$

which together with (14) gives

$$
\begin{aligned}
\bar{\mathcal{U}}(t_{k+1}^-) &\le \bar{\mathcal{U}}(t_0)\Big[\exp(h\tau)\rho\Big]^{N(t_0,t_{k+1})}\\
&\quad\times \exp\Big((\lambda+\bar\lambda)|\pi_{\bar u}(t_{k+1})| - h|\pi_{\bar s}(t_{k+1}-\tau)|\Big)\\
&\le \bar{\mathcal{U}}(t_0)\Big[\exp(h\tau)\rho\Big]^{N(t_0,t_{k+1})}\\
&\quad\times \exp\Big((\lambda+\bar\lambda)|\pi_{\bar u}(t_{k+1})| - h|\pi_{\bar s}(t_{k+1})| + h\tau\Big).
\end{aligned}
\tag{15}
$$

Next we consider the interval $[t_{k+1}, T)$. There are two cases. First, assume that $[t_{k+1}, T) \in \pi_{\bar u}(T)$. Then it implies that $\pi_{\bar u}(t_{k+1}) + T - t_{k+1} = \pi_{\bar u}(T)$ and $\pi_{\bar s}(T) = \pi_{\bar s}(t_{k+1})$. By lemma 1 and considering (12) and (15), we get

$$
\begin{aligned}
\mathcal{U}(T^-) &\le \bar{\mathcal{U}}(t_{k+1})\exp\big((\lambda+\bar\lambda)(T-t_{k+1})\big)\\
&\le \rho\bar{\mathcal{U}}(t_0)\Big[\exp(h\tau)\rho\Big]^{N(t_0,t_{k+1})}\exp\big((\lambda+\bar\lambda)(T-t_{k+1})\big)\\
&\quad\times \exp\Big((\lambda+\bar\lambda)|\pi_{\bar u}(t_{k+1})| - h|\pi_{\bar s}(t_{k+1})| + h\tau\Big)\\
&= \bar{\mathcal{U}}(t_0)\exp\Big(h\tau[N(t_0,t_{k+1})+1]\Big)\rho^{N(t_0,t_{k+1})+1}\\
&\quad\times \exp\big((\lambda+\bar\lambda)|\pi_{\bar u}(T)| - h|\pi_{\bar s}(T)|\big)\\
&= \bar{\mathcal{U}}(t_0)\Big[\exp(h\tau)\rho\Big]^{N(t_0,T)}\\
&\quad\times \exp\Big((\lambda+\bar\lambda)|\pi_{\bar u}(T)| - h|\pi_{\bar s}(T)|\Big),
\end{aligned}
$$

where $N(t_0, t_{k+1}) + 1 = N(t_0, T)$. Second, assume that $[t_{k+1}, T) \in \pi_{\bar s}(T)$. Then it implies that $\pi_{\bar s}(t_{k+1}) + T - t_{k+1} = \pi_{\bar s}(T)$ and $\pi_{\bar u}(T) = \pi_{\bar u}(t_{k+1})$. By lemma 2 and considering (12) and (15), we get

$$
\begin{aligned}
\mathcal{U}(T^-) &\le \bar{\mathcal{U}}(t_{k+1})\exp\big(-h(T-t_{k+1})\big)\\
&\le \rho\bar{\mathcal{U}}(t_0)\Big[\exp(h\tau)\rho\Big]^{N(t_0,t_{k+1})}\exp\big((\lambda+\bar\lambda)|\pi_{\bar u}(t_{k+1})|\big)\\
&\quad\times \exp\big(-h|\pi_{\bar s}(t_{k+1})| + h\tau\big)\exp\big(-h(T-t_{k+1})\big)\\
&= \bar{\mathcal{U}}(t_0)\exp\Big(h\tau[N(t_0,t_{k+1})+1]\Big)\rho^{N(t_0,t_{k+1})+1}\\
&\quad\times \exp\Big((\lambda+\bar\lambda)|\pi_{\bar u}(T)| - h|\pi_{\bar s}(T)|\Big)\\
&= \bar{\mathcal{U}}(t_0)\Big[\exp(h\tau)\rho\Big]^{N(t_0,T)}\\
&\quad\times \exp\Big((\lambda+\bar\lambda)|\pi_{\bar u}(T)| - h|\pi_{\bar s}(T)|\Big).
\end{aligned}
$$

Thus either case implies that (13) holds for $N(t_0, T) = k + 1$. The proof is completed. ■

Now we are in a position to establish some sufficient conditions on GUES of systems (1).

Theorem 1: Assume that there exist a family of continuous functions $\{V_p \in \mathbb{V}_2, p \in \mathscr{P}\}$ and constants $\lambda \ge 0$, $\bar\lambda \ge 0$, $\lambda^\star > \bar\lambda^\star \ge 0$, $\rho \ge 1$, $r \ge 0$, $\tau_D > 0$, $h > 0$ such that

(i) for every $t \ge t_0$, it holds that

$$
D^+V_p \le \begin{cases} \lambda V_p(t,x(t)) + \bar\lambda \bar V_p(t,x(t)), & \forall p \in \mathscr{P}_u,\\ -\lambda^\star V_p(t,x(t)) + \bar\lambda^\star \bar V_p(t,x(t)), & \forall p \in \mathscr{P}_s, \end{cases}
$$

where $\bar V_p(t,x(t)) = \sup\{V_p(s,x(s)) : t-\tau \le s \le t\}$;

(ii) $V_{\sigma(t)}(t,x(t)) \le \rho V_{\sigma(t^-)}(t,x(t))$, for all $t > t_0$;

(iii) $\bar\lambda^\star \exp(h\tau) + h \le \lambda^\star$;

(iv) $\dfrac{h\tau + \ln\rho}{\tau_D} + \dfrac{(\lambda+\bar\lambda)r - h}{1+r} < 0$,

where (x, σ) is a solution of system (1) with $\sigma \in \mathscr{T}$. Then the switched system (1) is GUES over $\mathscr{T}$, where $\mathscr{T} \subseteq \mathscr{T}_2(r, T) \cap \mathcal{S}_{ave}[\tau_D, N_0]$ for some $N_0 > 0$ and $T \ge t_0$.

Proof: Let $(x, \sigma) = (x(t), \sigma(t)) = (x(t, t_0, \phi), \sigma(t))$ be a solution of system (1) with initial value (t_0, ϕ) and switched signal $\sigma \in \mathscr{T}$, where $\mathscr{T} \subseteq \mathscr{T}_2(r, T) \cap \mathcal{S}_{ave}[\tau_D, N_0]$ for any $N_0 > 0$ and $T \ge t_0$. For convenience, define $\mathcal{U}(t) := V_{\sigma(t)}(t, x(t))$. Obviously, $\mathcal{U}$ is a piecewise continuous function on $[t_0, \infty)$. Next we shall prove that

$$
\mathcal{U}(t) \le \bar{\mathcal{U}}(t_0)\mathscr{M}\exp\big(-\Delta(t-t_0)\big), \quad t \ge t_0, \tag{16}
$$

where

$$
\mathscr{M} = \rho[\exp(h\tau)\rho]^c\exp((\lambda+\bar\lambda+\Delta)T) + \exp(h\tau N_0)\rho^{N_0+1},
$$

$$
c = N_0 + \frac{T}{\tau_D}, \quad \Delta = -\frac{h\tau + \ln\rho}{\tau_D} - \frac{(\lambda+\bar\lambda)r - h}{1+r} > 0.
$$

We shall show that (16) holds on the interval $[t_0, T)$ and $[T, \infty)$, respectively. First, we consider the interval $[t_0, T)$. Note that $\sigma \in \mathscr{T} \subseteq \mathcal{S}_{ave}[\tau_D, N_0]$, we get

$$
N_\sigma(t_0, t) \le N_0 + \frac{t - t_0}{\tau_D} \le N_0 + \frac{T}{\tau_D} = c, \quad t \in [t_0, T].
$$

On the other hand, it follows from the definitions of π_u and π_s that for $t \ge t_0$, $|\pi_{\bar u}(t)| = \pi_u(t)$, $|\pi_{\bar s}(t)| = \pi_s(t)$, $\pi_u(t) + \pi_s(t) = t - t_0$ and $N(t_0, t) = N_\sigma(t_0, t)$. Thus by assumptions (i) and (iii), it is easy to check that all conditions in Lemma 3 are satisfied, which leads to

$$
\begin{aligned}
\mathcal{U}(t^-) &\le \bar{\mathcal{U}}(t_0)\Big[\exp(h\tau)\rho\Big]^{N(t_0,t)}\\
&\qquad\times \exp\big((\lambda+\bar\lambda)\pi_u(t) - h\pi_s(t)\big)\\
&\le \bar{\mathcal{U}}(t_0)\Big[\exp(h\tau)\rho\Big]^{c}\exp\big((\lambda+\bar\lambda)\pi_u(t)\big)\\
&\le \bar{\mathcal{U}}(t_0)\Big[\exp(h\tau)\rho\Big]^{c}\exp\big((\lambda+\bar\lambda)T\big),
\end{aligned}
\tag{17}
$$

for every $t \in [t_0, T)$. Then we claim that

$$
\mathcal{U}(t) \le \bar{\mathcal{U}}(t_0)\rho\Big[\exp(h\tau)\rho\Big]^{c}\exp\big((\lambda+\bar\lambda)T\big), \tag{18}
$$

for every $t \in [t_0, T)$. In fact, if $t \in [t_0, T)$ is a continuous point of function $\mathcal{U}$, then (18) can be easily obtained by (17). If not, assume that $t = t_k \in [t_0, T)$, then $\mathcal{U}(t_k) = V_{\sigma(t_k)}(t_k, x(t_k)) \le \rho V_{\sigma(t_k^-)}(t_k, x(t_k)) = \rho\mathcal{U}(t_k^-)$ in view of condition (ii) and the continuity of V_σ for given σ. This together with (17) yields

$$
\mathcal{U}(t_k) \le \rho\mathcal{U}(t_k^-) \le \bar{\mathcal{U}}(t_0)\rho\Big[\exp(h\tau)\rho\Big]^{c}\exp\big((\lambda+\bar\lambda)T\big),
$$

which implies that (18) holds for every $t \in [t_0, T)$. To show that (16) holds on the interval $[t_0, T)$, by (18), we only need to show that

$$\begin{aligned}&\bar{\mathcal{U}}(t_0)\rho\Big[\exp(h\tau)\rho\Big]^c \exp\big((\lambda+\bar{\lambda})T\big)\\ &\qquad\le \bar{\mathcal{U}}(t_0)\mathscr{M}\exp\big(-\Delta(t-t_0)\big),\quad t\in[t_0,T).\end{aligned}$$

For convenience, let

$$\Xi(t) := \bar{\mathcal{U}}(t_0)\mathscr{M}\exp\big(-\Delta(t-t_0)\big).$$

In view of the definition of $\mathscr{M}$ and (18), it is obvious that

$$\begin{aligned}\Xi(t) &\ge \bar{\mathcal{U}}(t_0)\mathscr{M}\exp\big(-\Delta(T-t_0)\big)\\ &\ge \bar{\mathcal{U}}(t_0)\mathscr{M}\exp(-\Delta T)\\ &\ge \bar{\mathcal{U}}(t_0)\rho\Big[\exp(h\tau)\rho\Big]^c\exp\big((\lambda+\bar{\lambda})T\big),\end{aligned}$$

which implies that (16) holds on the interval $[t_0, T)$.

Next we show that (16) holds on the interval $[T, \infty)$. Since $\sigma \in \mathscr{T} \subseteq \mathscr{T}_2(r, T) \cap \mathcal{S}_{ave}[\tau_D, N_0]$, we get

$$\frac{\pi_u(t)}{\pi_s(t)} \le r,\quad t \ge T \text{ and } N_\sigma(t_0,t) \le N_0 + \frac{t-t_0}{\tau_D}, \tag{19}$$

for $t \ge t_0$. Note that $\pi_u(t) + \pi_s(t) = t - t_0$, it follows from (19) that

$$\pi_s(t) \ge \frac{t-t_0}{1+r},\quad t \ge T. \tag{20}$$

It then follows from assumptions (i) and (iii) in Theorem 1 that all conditions in Lemma 3 are satisfied. Thus from (19), (20) and assumption (iv), one derives that

$$\begin{aligned}\mathcal{U}(t^-) &\le \bar{\mathcal{U}}(t_0)\Big[\exp(h\tau)\rho\Big]^{N(t_0,t)}\exp\Big((\lambda+\bar{\lambda})\pi_u(t)-h\pi_s(t)\Big)\\ &\le \bar{\mathcal{U}}(t_0)\Big[\exp(h\tau)\rho\Big]^{N_0+\frac{t-t_0}{\tau_D}}\exp\Big(\big[(\lambda+\bar{\lambda})r-h\big]\pi_s(t)\Big)\\ &\le \bar{\mathcal{U}}(t_0)\Big[\exp(h\tau)\rho\Big]^{N_0+\frac{t-t_0}{\tau_D}}\exp\Big(\frac{(\lambda+\bar{\lambda})r-h}{1+r}(t-t_0)\Big)\\ &\le \bar{\mathcal{U}}(t_0)\exp\big(h\tau N_0\big)\rho^{N_0}\exp\big(-\Delta(t-t_0)\big),\end{aligned}$$

for every $t \ge T$. Then applying the similar argument as the proof of (18), it can be deduced from the above that

$$\begin{aligned}\mathcal{U}(t) &\le \bar{\mathcal{U}}(t_0)\exp\big(h\tau N_0\big)\rho^{N_0+1}\exp\big(-\Delta(t-t_0)\big)\\ &\le \bar{\mathcal{U}}(t_0)\mathscr{M}\exp\big(-\Delta(t-t_0)\big),\quad t\ge T.\end{aligned}$$

This completes the proof of (16) for all $t \ge t_0$.

Note that $V_p \in \mathbb{V}_2, p \in \mathscr{P}$. By (16), one may derive that

$$\begin{aligned}c_1|x(t)|^m &\le \mathcal{U}(t) \le \bar{\mathcal{U}}(t_0)\mathscr{M}\exp\big(-\Delta(t-t_0)\big)\\ &\le c_2\|\phi\|^m\mathscr{M}\exp\big(-\Delta(t-t_0)\big),\quad t\ge t_0,\end{aligned}$$

i.e.,

$$|x(t)| \le \Big(\frac{c_2}{c_1}\Big)^{\frac{1}{m}}\|\phi\|\mathscr{M}^{\frac{1}{m}}\exp\Big(-\frac{\Delta}{m}(t-t_0)\Big),\quad t\ge t_0.$$

Note that $\mathscr{M}$ and Δ are independent of t_0 and σ. It implies that system (1) is GUAS over $\mathscr{T}$, where $\mathscr{T} \subseteq \mathscr{T}_2(r, T) \cap \mathcal{S}_{ave}[\tau_D, N_0]$ for some $N_0 > 0$ and $T \ge t_0$. The proof is completed. ■

Remark 1: In Theorem 1, when $p \in \mathscr{P}_s$, the constant $\lambda^\star$ provides an estimate of the decay rate and $\bar{\lambda}^\star$ reflects the insidious destructive effect of the time delay in stable modes. While $p \in \mathscr{P}_u$, constants λ and $\bar{\lambda}$ provide an estimate for the divergence rate of unstable modes. Generally, (but not always) the unstable modes will destabilize the process. We usually require that they do not diverge too fast and the dwell time in those modes is not too long. Not surprisingly, in this case, conditions (iii) and (iv) in Theorem 1 enforce restrictions on the divergence rate and total dwell time of unstable modes. In fact, it is easy to find that condition (iv) implies that

$$r < \frac{(\tau_D-\tau)h}{h\tau+\tau_D(\lambda+\bar{\lambda})},$$

where constant r denotes the ratio of the total dwell time between unstable and stable modes. Note that the above inequality indicates that the dwell times in some unstable modes may be longer, but the ratio of the total dwell time between unstable and stable modes should be less than the right term of the above inequality. Moreover, it implies that it is possible that $r \ge 1$. This means that the total dwell time on unstable modes is longer than on stable ones if τ_D, τ, h and $\lambda + \bar{\lambda}$ satisfy certain conditions. This inequality establishes a relationship between average dwell time τ_D, time delay τ and the ratio r. It shows that the stability of finitely delayed switched systems with stable and unstable modes can be guaranteed if the divergence rate and total dwell times of unstable modes can be effectively controlled and balanced by the ADT-based switching control with stable modes.

Remark 2: Switching controls for the stability of switched systems with time delays have been extensively studied in recent years. For example, [28], [32], and [52]–[54] addressed switched systems with constant delays, and [25], [31], [33], and [49] examined time-varying delays. However, most of them are based on the assumption that time delays are constant or time-varying and differentiable. Moreover, they mainly considered the switched systems with only stable modes. Recently, [49] studied the switched time-varying delay systems with both stable and unstable modes and derived some interesting results. However, the results are only valid for a special linearly switched delay system. Moreover, the differentiability and exact bound of the time-varying delay is needed. In this paper, the differentiability of the time-varying delays is removed. Moreover, the obtained result can be applied to nonlinear switched systems with unstable modes.

Remark 3: [33] and [55] studied delay switched systems with both stable and unstable modes under the assumption that $t_k - t_{k-1} \ge \tau$, (the dwell time is not less than the maximum time delay). Our results relax that requirement and do not impose direct restrictions between the time delay and dwell time of two consecutive switching signals in which the switched criteria in this section are established via the ADT technique. The advantages will be illustrated in Section V.

In particular, when there is no unstable mode in (1), i.e., $\mathscr{P}_u = \emptyset$ and $\mathscr{P} = \mathscr{P}_s$, it is obvious that $\pi_u(t) \equiv 0$ and $r = 0$. In this case, the following corollary can be obtained by Theorem 1.

Corollary 1: Assume that there exist a family of continuous functions $\{V_p \in \mathbb{V}_2, p \in \mathscr{P}\}$ and constants $\lambda^\star > \bar{\lambda}^\star \geq 0$, $\rho \geq 1, \tau_D > 0, h > 0$ such that
(i) $D^+V_p \leq -\lambda^\star V_p(t,x) + \bar{\lambda}^\star \bar{V}_p(t,x)$, for all $t \geq t_0$ and $p \in \mathscr{P}$;
(ii) $V_{\sigma(t)}(t,x(t)) \leq \rho V_{\sigma(t^-)}(t,x(t))$, for all $t > t_0$;
(iii) $\bar{\lambda}^\star \exp(h\tau) + h \leq \lambda^\star$;
(iv) $\frac{h\tau+\ln\rho}{h} < \tau_D$,
where (x,σ) is a solution of system (1) with $\sigma \in \mathscr{T}$. Then the system (1) is GUES over $\mathscr{T}$, where $\mathscr{T} \subseteq \mathcal{S}_{ave}[\tau_D, N_0]$ for some $N_0 > 0$.

Assume that $\mathscr{T}_s = \{s_1, \cdots, s_k\}$, $\mathscr{T}_u = \{u_1, \cdots, u_l\}$, $m = k + l$. Next, let us consider a class of switched linear delayed systems

$$\dot{x}(t) = A_{s_i}x(t) + B_{s_i}x(t-\tau(t)), \quad \sigma = s_i \in \mathscr{T}_s, \tag{21a}$$
$$\dot{x}(t) = A_{u_j}x(t) + B_{u_j}x(t-\tau(t)), \quad \sigma = u_j \in \mathscr{T}_u, \tag{21b}$$

where $\tau(t) \in [0, \tau]$, $\tau > 0$ is a real constant, modes (21a) are stable and modes (21b) are unstable. $A_{s_i}, B_{s_i}, A_{u_j}$, and B_{u_j} are some $n \times n$ real matrices. For system (21), we have the following result.

Theorem 2: Assume that there exist some constants $\lambda \geq 0, \bar{\lambda} \geq 0, \lambda^\star > \bar{\lambda}^\star \geq 0, \rho \geq 1, r \geq 0, \tau_D > 0, h > 0$, and some $n \times n$ matrices $P_{s_i} > 0, Q_{u_j} > 0, i = 1, ..., k$, $j = 1, ..., l$, such that

$$\begin{cases} \Xi_\sigma \leq 0, & \sigma \in \mathscr{T}, \\ \bar{\lambda}^\star \exp(h\tau) + h \leq \lambda^\star, & \dfrac{h\tau + \ln\rho}{\tau_D} + \dfrac{(\lambda+\bar{\lambda})r - h}{1+r} < 0, \\ Q_{s_i} \leq \bar{\lambda}^\star P_{s_i}, & i = 1, ..., k, \\ Q_{u_j} \leq \bar{\lambda} P_{u_j}, & j = 1, ..., l, \\ P_\sigma \leq \rho P_{\sigma^-}, & \sigma \in \mathscr{T}, \end{cases} \tag{22}$$

where

$$\Xi_{s_i} = \begin{bmatrix} P_{s_i}A_{s_i} + A_{s_i}^TP_{s_i} + \lambda^\star P_{s_i} & P_{s_i}B_{s_i} \\ \star & -Q_{s_i} \end{bmatrix},$$
$$\Xi_{u_j} = \begin{bmatrix} P_{u_j}A_{u_j} + A_{u_j}^TP_{u_j} - \lambda P_{u_j} & P_{u_j}B_{u_j} \\ \star & -Q_{u_j} \end{bmatrix}.$$

Then the system (21) is GUES over $\mathscr{T}$, where $\mathscr{T} \subseteq \mathscr{T}_2(r,T) \cap \mathcal{S}_{ave}[\tau_D, N_0]$ for some $N_0 > 0$ and $T \geq 0$.

Proof: Let $V_\sigma(t) = x^T(t)P_\sigma x(t), \sigma \in \mathscr{T}$, then we can compute that

$$\begin{aligned} D^+V_\sigma &\leq 2x^T(t)P_\sigma\Big[A_\sigma x(t) + B_\sigma x(t-\tau)\Big] \\ &\leq 2x^T(t)P_\sigma A_\sigma x(t) + 2x^T(t)P_\sigma B_\sigma x(t-\tau) \\ &\leq 2x^T(t)P_\sigma A_\sigma x(t) + x^T(t)P_\sigma B_\sigma Q_\sigma^{-1}B_\sigma^TP_\sigma x(t) \\ &\quad + x^T(t-\tau)Q_\sigma x(t-\tau). \end{aligned}$$

When $\sigma = s_i \in \mathscr{T}_s$, it follows from the assumption (30) that

$$\begin{aligned} D^+V_{s_i} &\leq x^T\Big[P_{s_i}A_{s_i} + A_{s_i}^TP_{s_i} + P_{s_i}B_{s_i}Q_{s_i}^{-1}B_{s_i}^TP_{s_i}\Big]x \\ &\quad + x^T(t-\tau)Q_{s_i}x(t-\tau) \\ &\leq -\lambda^\star x^T(t)P_{s_i}x(t) + \bar{\lambda}^\star x^T(t-\tau)P_{s_i}x(t-\tau) \\ &= -\lambda^\star V_{s_i}(t) + \bar{\lambda}^\star V_{s_i}(t-\tau). \end{aligned}$$

While $\sigma = u_j \in \mathscr{T}_u$, it leads to

$$\begin{aligned} D^+V_{u_j} &\leq x^T\Big[P_{u_j}A_{u_j} + A_{u_j}^TP_{u_j} + P_{u_j}B_{u_j}Q_{u_j}^{-1}B_{u_j}^TP_{u_j}\Big]x \\ &\quad + x^T(t-\tau)Q_{u_j}x(t-\tau) \\ &\leq \lambda x^T(t)P_{u_j}x(t) + \bar{\lambda}x^T(t-\tau)P_{u_j}x(t-\tau) \\ &= \lambda V_{u_j}(t) + \bar{\lambda}V_{u_j}(t-\tau). \end{aligned}$$

Then it is easy to check that all conditions in Theorem 1 are satisfied. Hence, we obtain that the switched system (21) is GES over $\mathscr{T}$, where $\mathscr{T} \subseteq \mathscr{T}_2(r,T) \cap \mathcal{S}_{ave}[\tau_D, N_0]$ for some $N_0 > 0$ and $T \geq 0$. The proof is completed. ■

In Theorem 2, it is shown that if the parameters of the switched system (21) satisfy those inequalities in assumption (22), then it is GES over $\mathscr{T}$, where $\mathscr{T} \subseteq \mathscr{T}_2(r,T) \cap \mathcal{S}_{ave}[\tau_D, N_0]$. However, it should be pointed out that those inequalities in assumption (22) can only be partially solved by Matlab's LMI toolbox since there exist some nonlinear terms such as $\mu_i P_{\sigma_i}$ and ρP_{σ_i}. In practical applications, we have to give those parameters (μ_i and ρ) a priori such that the those inequalities work via the LMI toolbox.

IV. SWITCHED SYSTEMS WITH INFINITE DELAY

In this section, we focus on the study of infinite delay ($\tau = \infty$) for switched systems (1). We suppose that the term x_t in system (1) is still given as follows: $x_t = x(t+s), s \in [-\tau, 0]$, where $\tau = \infty$. Some delay-independent switching laws which guarantee the US and GUES will be presented via multiple Lyapunov functions and Razumikhin technique.

Theorem 3: Assume that there exist a family of continuous functions $\{V_p \in \mathbb{V}_1, p \in \mathscr{P}\}$ and constants $\lambda \geq 0, \lambda^\star > 0, \eta > 0, \delta > 0, q > 1$ such that
(i) $D^+V_p \leq \lambda V_p(t,x(t)), t \geq t_0$, whenever $V_p(s,x(s)) \leq qV_p(t,x(t))$ for all $s \in [t-\tau, t]$ and $p \in \mathscr{P}_u$;
(ii) $D^+V_p \leq 0, t \geq t_0$, whenever $V_p(s,x(s)) \leq qV_p(t,x(t))$ for all $s \in [t-\tau, t]$ and $p \in \mathscr{P}_s$;
(iii) $\sigma(t_b^-) \in \mathscr{P}_u$, for some $b \in \mathbb{Z}_+$, implies that $\sigma(t_b) \in \mathscr{P}_s$ and $t_{b+1} - t_b \geq \eta$, and moreover, $D^+V_{\sigma(t_b)} \leq -\lambda^\star V_{\sigma(t_b)}(t,x(t)), t \in [t_b, t_{b+1})$, whenever $V_{\sigma(t_b)}(s,x(s)) \leq qV_{\sigma(t_b)}(t,x(t))$ for all $s \in [t-\tau, t]$;
(iv) $V_{\sigma(t)}(t,x(t)) \leq V_{\sigma(t^-)}(t,x(t)), \ \forall t > t_0$;
(v) $\lambda^\star\eta \geq \ln q > \delta\lambda$;
where (x,σ) is a solution of system (1) with $\sigma \in \mathscr{T}$. Then the switched system (1) with $\tau = \infty$ is US over $\mathscr{T}$, where $\mathscr{T} \in \mathscr{T}_1(\delta)$.

Proof: Note that $V_p \in \mathbb{V}_1, p \in \mathscr{P}$. There exist $\omega_1, \omega_2 \in \mathcal{K}$ such that $\omega_1(|x|) \leq V_p(t,x) \leq \omega_2(|x|)$, uniformly for all

$p \in \mathscr{P}$ and $(t, x) \in [t_0 - \tau, \infty) \times \mathbb{R}^n$. For any $\varepsilon > 0$, choose $\delta_0 > 0$ such that $qw_2(\delta_0) \le w_1(\varepsilon)$. Let $(x, \sigma) = (x(t), \sigma(t)) = (x(t, t_0, \phi), \sigma(t))$ be the solution of system (1) with initial value (t_0, ϕ) and switched signal $\sigma \in \mathscr{T}$, where $\mathscr{T} \in \mathscr{T}_1(\delta)$. Next we shall prove the uniform stability, that is, for any $t_0 \ge 0$ and $\phi \in \mathbb{C}_\tau$, $\|\phi\| \le \delta_0$ implies that for each $\sigma \in \mathscr{T}$, $|x(t)| \le \varepsilon, t \ge t_0$.

For convenience, let $\mathcal{U}(t) = V_{\sigma(t)}(t, x(t))$. If we can prove that for any $T^* \ge t_0, \mathcal{U}(T^{*-}) \le qw_2(\delta_0)$, then it follows from assumption (iv) and the continuity of V_σ for given σ that

$$V_{\sigma(T^*)}(T^*, x(T^*)) \le V_{\sigma(T^{*-})}(T^*, x(T^*)) = V_{\sigma(T^{*-})}(T^{*-}, x(T^{*-})) \le qw_2(\delta_0).$$

In view of the fact that $qw_2(\delta_0) \le w_1(\varepsilon)$, the uniform stability can be directly obtained. Hence, in the following our main work is to show that for any given $T^* \ge t_0$, it holds that $\mathcal{U}(T^{*-}) \le qw_2(\delta_0)$. For later use, denote by $t_1 < t_2 < \cdots < t_{N_\sigma(t_0, T^*)}$ the switching times of σ over the interval $[t_0, T^*)$ and $N_\sigma(t_0, T^*)$ the number of switching times of σ over the interval $[t_0, T^*)$.

First, we consider the case that $N_\sigma(t_0, T^*) = 0$, i.e., there is no switching signal on the interval $[t_0, T^*)$. One may analyze the problem from two cases: $\sigma(t_0) \in \mathscr{P}_s$ or $\sigma(t_0) \in \mathscr{P}_u$.

Case 1: When $\sigma(t_0) \in \mathscr{P}_s$, we claim that $\mathcal{U}(t) \le w_2(\delta_0), t \in [t_0, T^*)$. For any $\kappa > 0$, let $\mathcal{U}_\kappa(t) = \mathcal{U}(t)\exp(-\kappa(t - t_0)), t \in [t_0, T^*)$ and $\mathcal{U}_\kappa(t) = \mathcal{U}(t), t \in [t_0 - \tau, t_0]$. We first prove that $\mathcal{U}_\kappa(t) \le w_2(\delta_0), t \in [t_0, T^*)$. Suppose on the contrary, then there exists a $t^* \in [t_0, T^*)$ such that $\mathcal{U}_\kappa(t^*) = w_2(\delta_0), \mathcal{U}_\kappa(t) \le w_2(\delta_0), t \in [t_0, t^*]$ and $D^+\mathcal{U}_\kappa(t^*) \ge 0$. Since $\|\phi\| \le \delta_0$, it can be deduced that $q\mathcal{U}_\kappa(t^*) = qw_2(\delta_0) \ge q\mathcal{U}_\kappa(s) \ge \mathcal{U}_\kappa(s), s \in [t^* - \tau, t^*]$. Note that if $t^* - \tau \ge t_0$ and $s \in [t^* - \tau, t^*]$, $q\mathcal{U}_\kappa(t^*) \ge \mathcal{U}_\kappa(s) = \mathcal{U}(s)\exp(-\kappa(s - t_0))$, which implies that $q\mathcal{U}(t^*) \ge q\mathcal{U}(t^*)\exp(-\kappa(t^* - s)) \ge \mathcal{U}(s)$; While if $t^* - \tau < t_0$ and $s \in [t^* - \tau, t_0)$, $q\mathcal{U}(t^*) \ge q\mathcal{U}_\kappa(t^*) \ge \mathcal{U}_\kappa(s) = \mathcal{U}(s)$, which implies that $q\mathcal{U}(t^*) \ge \mathcal{U}(s)$ for all $s \in [t^* - \tau, t^*]$. By assumption (ii), it holds that $D^+\mathcal{U}(t^*) \le 0$, which implies that $D^+\mathcal{U}_\kappa(t^*) = D^+\mathcal{U}(t^*)\exp(-\kappa(t - t_0)) - \kappa\mathcal{U}(t^*)\exp(-\kappa(t - t_0)) \le -\kappa\mathcal{U}(t^*)\exp(-\kappa(t - t_0)) < 0$, which is contradiction with $D^+\mathcal{U}_\kappa(t^*) \ge 0$. Hence, we obtain that for any $\kappa > 0$, it holds that $\mathcal{U}_\kappa(t) \le w_2(\delta_0), t \in [t_0, T^*)$, which leads to $\mathcal{U}(t) \le w_2(\delta_0)\exp(\kappa(T^* - t_0)), t \in [t_0, T^*)$. By the arbitrary of constant $\kappa > 0$, it can be deduced that $\mathcal{U}(t) \le w_2(\delta_0), t \in [t_0, T^*)$. This completes the proof of Case 1.

Case 2: When $\sigma(t_0) \in \mathscr{P}_u$, we claim that $\mathcal{U}(t) \le qw_2(\delta_0), t \in [t_0, T^*)$. Suppose on the contrary, then there exists a $t^\star \in [t_0, T^*)$ such that $\mathcal{U}(t^\star) = qw_2(\delta_0), \mathcal{U}(t) \le qw_2(\delta_0), t \in [t_0, t^\star]$. Moreover, it follows from $\mathcal{U}(t_0) \le w_2(\delta_0)$ that $t^\star > t_0$. In this case, we note that $\mathcal{U}(t^\star) = qw_2(\delta_0) > w_2(\delta_0)$ and $\mathcal{U}(t_0) \le w_2(\delta_0)$. Thus there must exist a $t^{\star\star} \in [t_0, t^\star)$ such that $\mathcal{U}(t^{\star\star}) = w_2(\delta_0)$ and $\mathcal{U}(t) > w_2(\delta_0), t \in (t^{\star\star}, t^\star]$. Hence, we get $q\mathcal{U}(t) \ge qw_2(\delta_0) \ge \mathcal{U}(s), s \in [t-\tau, t], t \in [t^{\star\star}, t^\star]$. By assumption (i), $D^+\mathcal{U}(t) \le \lambda\mathcal{U}(t), t \in [t^{\star\star}, t^\star]$. Integrating it from $t^{\star\star}$ to $t^\star$, it holds from $\mathscr{T} \in \mathscr{T}_1(\delta)$ that $qw_2(\delta_0) = \mathcal{U}(t^\star) \le \mathcal{U}(t^{\star\star})e^{\lambda(t^\star - t^{\star\star})} \le w_2(\delta_0)e^{\lambda\delta}$, which contradicts $\ln q > \lambda\delta$ in assumption (v). This completes the proof of Case 2.

Thus either Case 1 or Case 2, we arrive at the assertion that $\mathcal{U}(t) \le qw_2(\delta_0), t \in [t_0, T^*)$, which implies that $\mathcal{U}(T^{*-}) \le qw_2(\delta_0)$.

Now we consider the case that $N_\sigma(t_0, T^*) \ge 1$ (abbr. N_σ), i.e., there exists at least a switching point on the interval $[t_0, T^*)$. For this case, we will prove that

$$\sigma(t_j) \in \mathscr{P}_s \Rightarrow \begin{cases} \mathcal{U}(t) \le qw_2(\delta_0), & t \in [t_j, t_{j+1}), \\ \mathcal{U}(t_{j+1}^-) \le w_2(\delta_0), & 0 \le j < N_\sigma. \end{cases}$$
$$\sigma(t_j) \in \mathscr{P}_u \Rightarrow \mathcal{U}(t) \le qw_2(\delta_0), \quad t \in [t_j, t_{j+1}),\ 0 \le j < N_\sigma. \tag{23}$$

The above assertions can be shown by induction. First, it is obvious that (23) holds when $N_\sigma = 1$ from the proof of the case that $N_\sigma = 0$. Now we assume that (23) holds when $N_\sigma = k$. Then it holds that

$$\sigma(t_j) \in \mathscr{P}_s \Rightarrow \begin{cases} \mathcal{U}(t) \le qw_2(\delta_0), & t \in [t_j, t_{j+1}), \\ \mathcal{U}(t_{j+1}^-) \le w_2(\delta_0), & 0 \le j < k. \end{cases}$$
$$\sigma(t_j) \in \mathscr{P}_u \Rightarrow \mathcal{U}(t) \le qw_2(\delta_0), \quad t \in [t_j, t_{j+1}),\ 0 \le j < k. \tag{24}$$

which implies that

$$\mathcal{U}(t) \le qw_2(\delta_0), \quad t \in [t_0, t_k). \tag{25}$$

Next we shall show that (23) holds when $N_\sigma = k + 1$. To do this, we begin it with $\sigma(t_{k-1}) \in \mathscr{P}_s$ or $\sigma(t_{k-1}) \in \mathscr{P}_u$. If $\sigma(t_{k-1}) \in \mathscr{P}_s$, then there are two cases: $\sigma(t_k) \in \mathscr{P}_s$ or $\sigma(t_k) \in \mathscr{P}_u$.

Case 1**:* When $\sigma(t_k) \in \mathscr{P}_s$, we claim that $\mathcal{U}(t) \le w_2(\delta_0), t \in [t_k, t_{k+1})$. Since $\sigma(t_{k-1}) \in \mathscr{P}_s$, it holds from (24) that $\mathcal{U}(t_k^-) \le w_2(\delta_0)$, which together with assumption (iv) yields that $\mathcal{U}(t_k) \le \mathcal{U}(t_k^-) \le w_2(\delta_0)$. Thus the assertion $\mathcal{U}(t) \le w_2(\delta_0)$ holds for $t = t_k$. Next we only need to prove that $\mathcal{U}(t) \le w_2(\delta_0), t \in (t_k, t_{k+1})$. For any $\kappa > 0$, let $\mathcal{U}_\kappa(t) = \mathcal{U}(t)\exp(-\kappa(t - t_k)), t \in [t_k, t_{k+1})$ and $\mathcal{U}_\kappa(t) = \mathcal{U}(t), t \in [t_0, t_k)$. We firstly prove that $\mathcal{U}_\kappa(t) \le w_2(\delta_0), t \in [t_k, t_{k+1})$. If it is false, then there exists a $\hat{t} \in [t_k, t_{k+1})$ such that $\mathcal{U}_\kappa(\hat{t}) = w_2(\delta_0), \mathcal{U}_\kappa(t) \le w_2(\delta_0), t \in [t_k, \hat{t}]$ and $D^+\mathcal{U}_\kappa(\hat{t}) \ge 0$. In this case, it can be deduced from (25) that $q\mathcal{U}(\hat{t}) \ge q\mathcal{U}_\kappa(\hat{t}) = qw_2(\delta_0) \ge \mathcal{U}(s), s \in [\hat{t} - \tau, \hat{t}]$. By assumption (ii), $D^+\mathcal{U}_\kappa(\hat{t}) \le -\kappa\mathcal{U}(\hat{t})\exp(-\kappa(\hat{t} - t_0)) < 0$, which is contradiction. Then by the arbitrary of constant $\kappa > 0$, it can be deduced that $\mathcal{U}(t) \le w_2(\delta_0), t \in [t_k, t_{k+1})$. This completes the proof of Case 1*.

Case 2**:* When $\sigma(t_k) \in \mathscr{P}_u$, we claim that $\mathcal{U}(t) \le qw_2(\delta_0), t \in [t_k, t_{k+1})$. The proof process is similar to Case 2, where $N_\sigma(t_0, T^*) = 0$, and thus is omitted here.

If $\sigma(t_{k-1}) \in \mathscr{P}_u$, then $\sigma(t_k) \in \mathscr{P}_s$ in view of assumption (iii).

In this case, we can show that

$$\mathcal{U}(t) \le qw_2(\delta_0), \quad t \in [t_k, t_{k+1}) \tag{26}$$

and

$$\mathcal{U}(t_{k+1}^-) \leq w_2(\delta_0). \tag{27}$$

The proof of (26) is similar to Case $1^\star$, where $\sigma(t_k) \in \mathscr{P}_s$ and is also omitted here. Next we only show that (27) holds. If this assertion is false, then $\mathcal{U}(t_{k+1}^-) > w_2(\delta_0)$. There are two cases: (I). $\mathcal{U}(t) > w_2(\delta_0)$ for all $t \in [t_k, t_{k+1})$; (II) There exists some $t \in [t_k, t_{k+1})$ such that $\mathcal{U}(t) \leq w_2(\delta_0)$. For case (I), it can be deduced from (25) and (26) that $q\mathcal{U}(t) \geq qw_2(\delta_0) \geq \mathcal{U}(s), s \in [t-\tau, t], t \in [t_k, t_{k+1})$. By assumption (iii) and (26), we get $w_2(\delta_0) < \mathcal{U}(t_{k+1}^-) \leq \mathcal{U}(t_k)e^{-\lambda^\star(t_{k+1}-t_k)} \leq qw_2(\delta_0)e^{-\lambda^\star\eta}$, which is a contradiction with $e^{\lambda^\star\eta} \geq q$. Thus case (I) is impossible. For case (II), one may choose a $\bar{t} \in [t_k, t_{k+1})$ such that $\mathcal{U}(\bar{t}) = w_2(\delta_0)$ and $\mathcal{U}(t) > w_2(\delta_0), t \in (\bar{t}, t_{k+1}]$. It is obvious that $\bar{t} < t_{k+1}$ in view of the assumption that $w_2(\delta_0) < \mathcal{U}(t_{k+1}^-)$. Thus it can be deduced from (25) and (26) that $q\mathcal{U}(t) \geq qw_2(\delta_0) \geq \mathcal{U}(s), s \in [t-\tau, t], t \in [\bar{t}, t_{k+1})$. By assumption (iii), we know that $\mathcal{U}$ is monotonic nonincreasing on $[t_k, t_{k+1})$, which yields that $w_2(\delta_0) < \mathcal{U}(t_{k+1}^-) \leq \mathcal{U}(\bar{t}) = w_2(\delta_0)$. Obviously, this is a contradiction and case (II) is also impossible. Thus (27) holds.

From the above discussion, we obtain that

$$\begin{aligned} &\sigma(t_k) \in \mathscr{P}_s \Rightarrow \begin{cases} \mathcal{U}(t) \leq qw_2(\delta_0), & t \in [t_k, t_{k+1}), \\ \mathcal{U}(t_{k+1}^-) \leq w_2(\delta_0), \end{cases} \\ &\sigma(t_k) \in \mathscr{P}_u \Rightarrow \mathcal{U}(t) \leq qw_2(\delta_0), \quad t \in [t_k, t_{k+1}). \end{aligned} \tag{28}$$

By induction, we know that (23) holds for any $N_\sigma \geq 1$. Based on (23), we can show that $\mathcal{U}(t) \leq qw_2(\delta_0), t \in [t_{N_\sigma(t_0,T^*)}, T^*)$, which implies that $\mathcal{U}(T^{*-}) \leq qw_2(\delta_0)$. The proof process is similar to the proof of (28). We need only to notice two points here. First, we should begin the proof with $\sigma(t_{N_\sigma(t_0,T^*)-1}) \in \mathscr{P}_s$ or $\sigma(t_{N_\sigma(t_0,T^*)-1}) \in \mathscr{P}_u$. Second, it follows from (23) that $\mathcal{U}(t) \leq qw_2(\delta_0), t \in [t_0, t_{N_\sigma(t_0,T^*)})$. Combining all the assertions obtained, the proof is completed. ■

In particular, if we strengthen some conditions in Theorem 3, then the following immediate results can be derived.

Corollary 2: Assume that there exist a family of continuous functions $\{V_p \in \mathbb{V}_1, p \in \mathscr{P}\}$ and constants $\lambda \geq 0, \lambda^\star > 0, \eta > 0, \delta > 0, q > 1$ such that
(i) $D^+V_p \leq \lambda V_p(t, x(t)), t \geq t_0$, whenever $V_p(s, x(s)) \leq qV_p(t, x(t))$ for all $s \in [t-\tau, t]$ and $p \in \mathscr{P}_u$;
(ii) $D^+V_p \leq -\lambda^\star V_p(t, x(t)), t \geq t_0$, whenever $V_p(s, x(s)) \leq qV_p(t, x(t))$ for all $s \in [t-\tau, t]$ and $p \in \mathscr{P}_s$;
(iii) $\sigma(t_b^-) \in \mathscr{P}_u$, for some $b \in \mathbb{Z}_+$, implies that $\sigma(t_b) \in \mathscr{P}_s$ and $t_{b+1} - t_b \geq \eta$;
(iv) $V_{\sigma(t)}(t, x(t)) \leq V_{\sigma(t^-)}(t, x(t)), \forall t > t_0$;
(v) $\lambda^\star\eta \geq \ln q > \delta\lambda$;
where (x, σ) is a solution of system (1) with $\sigma \in \mathscr{T}$. Then the switched system (1) with $\tau = \infty$ is US over $\mathscr{T}$, where $\mathscr{T} \in \mathscr{T}_1(\delta)$.

Corollary 3: Assume that there exist a family of continuous functions $\{V_p \in \mathbb{V}_1, p \in \mathscr{P}\}$ and constants $\lambda \geq 0, \lambda^\star > 0, \eta > 0, \delta > 0$ such that
(i) $D^+V_p \leq \lambda V_p(t, x(t)), t \geq t_0$, whenever $V_p(s, x(s)) \leq e^{\lambda^\star\eta} V_p(t, x(t))$ for all $s \in [t-\tau, t]$ and $p \in \mathscr{P}_u$;
(ii) $D^+V_p \leq -\lambda^\star V_p(t, x(t)), t \geq t_0$, whenever $V_p(s, x(s)) \leq e^{\lambda^\star\eta} V_p(t, x(t))$ for all $s \in [t-\tau, t]$ and $p \in \mathscr{P}_s$;
(iii) $\sigma(t_b^-) \in \mathscr{P}_u$, for some $b \in \mathbb{Z}_+$, implies that $\sigma(t_b) \in \mathscr{P}_s$ and $t_{b+1} - t_b \geq \eta$;
(iv) $V_{\sigma(t)}(t, x(t)) \leq V_{\sigma(t^-)}(t, x(t)), \forall t > t_0$;
(vi) $\lambda^\star\eta - \delta\lambda > 0$,
where (x, σ) is a solution of system (1) with $\sigma \in \mathscr{T}$. Then the switched system (1) with $\tau = \infty$ is US over $\mathscr{T}$, where $\mathscr{T} \in \mathscr{T}_1(\delta)$.

Furthermore, when there is no unstable mode in (1), i.e., $\mathscr{P}_u = \emptyset$ and $\mathscr{P} = \mathscr{P}_s$, the following result can be obtained by Theorem 3.

Corollary 4: Assume that there exist a family of continuous functions $\{V_p \in \mathbb{V}_1, p \in \mathscr{P}\}$ and constant $q > 1$ such that
(i) $D^+V_p \leq 0, t \geq t_0$, whenever $V_p(s, x(s)) \leq qV_p(t, x(t))$ for all $s \in [t-\tau, t]$ and $p \in \mathscr{P}_s$;
(ii) $V_{\sigma(t)}(t, x(t)) \leq V_{\sigma(t^-)}(t, x(t)), \forall t > t_0$;
where (x, σ) is a solution of system (1) with $\sigma \in \mathscr{T}$. Then the switched system (1) with $\tau = \infty$ is US over $\mathscr{T}$, where $\mathscr{T} \in \mathscr{T}_0$.

In switched systems, a significant control problem is to design a set of controllers for the unforced system and find admissible switching signals such that the closed-loop system is stable and satisfies certain performance. If we ignore the unstable modes and focus on the design of controllers. Then let us consider a class of linear delayed switched systems with control input, which is given in the form

$$\dot{x}(t) = A_{\sigma(t)}x(t-\tau(t)) + B_{\sigma(t)}u(t), \quad \sigma(t) \in \mathscr{T}, \tag{29}$$

where $u(t) = \mathcal{K}_{\sigma(t)}x(t)$ is the control input and $\mathcal{K}_{\sigma(t)}$ is control gain that will be designed to achieve the stability. Assume that the time delay $\tau(t) > 0$ in switched system (29) can not be exactly observed, that is, it is possible that $\tau(t)$ is mall or large, even unbounded. Also it is possible that $\tau(t)$ is not differentiable or discontinuous. In this case, all those results in [27]–[34], [49], and [52]–[55] cannot be applied to the design of controller u such that the switched system (29) is stable. By Corollary 4, we can derive the following result.

Theorem 4: Assume that there exist $n \times n$ real symmetric matrices $U_{\sigma(t)}$, positive definite matrices $P_{\sigma(t)} > 0, \sigma \in \mathscr{T}$, and constant $q > 1$ such that $P_{\sigma(t)} \leq P_{\sigma(t^-)}$ and

$$B_{\sigma(t)}U_{\sigma(t)} + U_{\sigma(t)}B_{\sigma(t)}^T + A_{\sigma(t)}P_{\sigma(t)}A_{\sigma(t)}^T + qP_{\sigma(t)} \leq 0, \tag{30}$$

for all $\sigma \in \mathscr{T}$. Then the switched system (29) is US over $\mathscr{T}$, where $\mathscr{T} \in \mathscr{T}_0$ and the input gains are designed by

$$\mathcal{K}_{\sigma(t)} = U_{\sigma(t)}P_{\sigma(t)}^{-1}, \quad \sigma \in \mathscr{T}.$$

Proof: Define

$$\begin{aligned} \Xi := P_{\sigma(t)}^{-1}B_{\sigma(t)}\mathcal{K}_{\sigma(t)} + \mathcal{K}_{\sigma(t)}^T B_{\sigma(t)}^T P_{\sigma(t)}^{-1} + qP_{\sigma(t)}^{-1} \\ + P_{\sigma(t)}^{-1}A_{\sigma(t)}P_{\sigma(t)}A_{\sigma(t)}^T P_{\sigma(t)}^{-1} \end{aligned}$$

Note that $\mathcal{K}_{\sigma(t)} = U_{\sigma(t)}P^{-1}_{\sigma(t)}, \sigma \in \mathscr{T}$. It implies that

$$\Xi \leq 0 \Leftrightarrow (30) \text{ holds.}$$

Consider Lyapunov function $V_\sigma(t) = x^T(t)P^{-1}_{\sigma(t)}x(t)$. When $V_p(s, x(s)) \leq qV_p(t, x(t))$ for all $s \in [t-\tau, t]$, we can compute that

$$\begin{aligned}
D^+V_\sigma &= 2\,x^T(t)P^{-1}_{\sigma(t)}\{A_{\sigma(t)}x(t-\tau(t)) + B_{\sigma(t)}\mathcal{K}_{\sigma(t)}x(t)\} \\
&\leq 2\,x^T(t)P^{-1}_{\sigma(t)}B_{\sigma(t)}\mathcal{K}_{\sigma(t)}x(t) \\
&\quad + x^T(t)P^{-1}_{\sigma(t)}A_{\sigma(t)}P_{\sigma(t)}A^T_{\sigma(t)}P^{-1}_{\sigma(t)}x(t) \\
&\quad + x^T(t-\tau(t))P^{-1}_{\sigma(t)}x(t-\tau(t)) \\
&\leq x^T(t)\{P^{-1}_{\sigma(t)}B_{\sigma(t)}\mathcal{K}_{\sigma(t)} + \mathcal{K}^T_{\sigma(t)}B^T_{\sigma(t)}P^{-1}_{\sigma(t)} \\
&\quad + P^{-1}_{\sigma(t)}A_{\sigma(t)}P_{\sigma(t)}A^T_{\sigma(t)}P^{-1}_{\sigma(t)} + qP^{-1}_{\sigma(t)}\}x(t) \\
&= x^T(t)\Xi x(t) \leq 0.
\end{aligned}$$

Based on Corollary 4, the switched system (29) is US over $\mathscr{T}$, where $\mathscr{T} \in \mathscr{T}_0$. The proof is completed. ∎

By establishing another auxiliary function, the result for GUES of the switched systems (1) can be derived as follows.

Theorem 5: Assume that there exist a family of continuous functions $\{V_p \in \mathbb{V}_2, p \in \mathscr{P}\}$ and constants $\lambda \geq 0, \lambda^\star > 0, \eta > 0, \delta > 0, q > 1, \gamma > 0$ such that

(i) $D^+V_p \leq \lambda V_p(t, x(t)), t \geq t_0$, whenever $V_p(s, x(s)) \leq qe^{\gamma(t-s)}V_p(t, x(t))$ for all $s \in [t-\tau, t]$ and $p \in \mathscr{P}_u$;

(ii) $D^+V_p < -\lambda^\star V_p(t, x(t)), t \geq t_0$, whenever $V_p(s, x(s)) \leq qe^{\gamma(t-s)}V_p(t, x(t))$ for all $s \in [t-\tau, t]$ and $p \in \mathscr{P}_s$;

(iii) $\sigma(t_b^-) \in \mathscr{P}_u$, for some $b \in \mathbb{Z}_+$, implies that $\sigma(t_b) \in \mathscr{P}_s$ and $t_{b+1} - t_b \geq \eta$;

(iv) $V_{\sigma(t)}(t, x(t)) \leq V_{\sigma(t^-)}(t, x(t)), \forall t > t_0$;

(v) $\lambda^\star\eta > \ln q > \delta\lambda$;

where (x, σ) is a solution of system (1) with $\sigma \in \mathscr{T}$. Then the switched system (1) with $\tau = \infty$ is GUES over $\mathscr{T}$, where $\mathscr{T} \in \mathscr{T}_1(\delta)$.

Proof: Let $(x, \sigma) = (x(t), \sigma(t)) = (x(t, t_0, \phi), \sigma(t))$ be the solution of system (1) with initial value (t_0, ϕ) and switched signal $\sigma \in \mathscr{T}$, where $\mathscr{T} \in \mathscr{T}_1(\delta)$. Since $V_p \in \mathbb{V}_2, p \in \mathscr{P}$. There exist constants c_1, c_2 and m such that $c_1|x|^m \leq V_p(t, x) \leq c_2|x|^m$, uniformly for all $p \in \mathscr{P}$ and $(t, x) \in [t_0 - \tau, \infty) \times \mathbb{R}^n$. Then we can show that for any $t_0 \geq 0$ and $\phi \in \mathbb{C}_\tau$, the following inequality holds for each $\sigma \in \mathscr{T}$,

$$|x(t)| \leq \left(\frac{qc_2}{c_1}\right)^{\frac{1}{m}}\|\phi\|e^{-\frac{\epsilon}{m}(t-t_0)}, \quad t \geq t_0, \tag{31}$$

where $\epsilon = \frac{1}{2}\min\{\frac{\ln q - \delta\lambda}{\delta}, \frac{\lambda^\star\eta - \ln q}{\eta}\}$ and $\mathscr{T} \in \mathscr{T}_1(\delta)$.

In fact, define an auxiliary function:

$$\mathcal{U}(t) = \begin{cases} V_{\sigma(t)}(t, x(t))e^{\epsilon(t-t_0)}, & t \geq t_0, \\ V_{\sigma(t)}(t, x(t)), & t_0 - \tau \leq t \leq t_0. \end{cases}$$

Then applying exactly the same argument as Theorem 3, one may prove that for any $T^* \geq t_0$, $\mathcal{U}(T^{*-}) \leq qw_2(\|\phi\|)$, where $w_2(\|\phi\|) = c_2\|\phi\|^m$, which will lead to (31). The proof process is repetitive and thus is omitted here. ∎

As an application, we consider the delayed switched system (1) with

$$f_{s_i} = A_{s_i}x(t) + B_{s_i}\int_0^\infty x(t-v)h(v)dv, \quad s_i \in \mathscr{T}_s, \tag{32a}$$

$$f_{u_j} = A_{u_j}x(t) + B_{u_j}\int_0^\infty x(t-v)h(v)dv, \quad u_j \in \mathscr{T}_u, \tag{32b}$$

where $A_{s_i}, B_{s_i}, A_{u_j}$, and B_{u_j} are some $n \times n$ real matrices, $\mathscr{T}_s = \{s_1, \cdots, s_k\}$, $\mathscr{T}_u = \{u_1, \cdots, u_l\}$, $m = k + l$, modes (32a) are stable and modes (32b) are unstable. $h \in C([0, \infty), \mathbb{R}_+)$ satisfying

$$\int_0^\infty h(v)dv = 1 \text{ and } \int_0^\infty h(v)\exp(\omega v)dv := h^\star < \infty,$$

in which $\omega > 0$ is a given constant.

Theorem 6: Assume that there exist $n \times n$ positive definite matrices $P_{s_i} > 0, P_{u_j} > 0, s_i \in \mathscr{T}_s, u_j \in \mathscr{T}_u$, and constants $q > 1, \eta > 0, \lambda \geq 0, \lambda^\star > 0$, and $\delta > 0$ satisfying $\lambda^\star\eta > \ln q > \delta\lambda$,

$$A_{s_i}P_{s_i} + P_{s_i}A^T_{s_i} + B_{s_i}P_{s_i}B^T_{s_i} + [qh^\star + \lambda^\star]P_{s_i} \leq 0, \tag{33}$$

$$A_{u_j}P_{u_j} + P_{u_j}A^T_{u_j} + B_{u_j}P_{u_j}B^T_{u_j} + [qh^\star - \lambda]P_{u_j} \leq 0, \tag{34}$$

for all $s_i \in \mathscr{T}_s, u_j \in \mathscr{T}_u$. Moreover, $\sigma(t_b^-) \in \mathscr{P}_u$, for some $b \in \mathbb{R}_+$, implies that $\sigma(t_b) \in \mathscr{P}_s$, $t_{b+1} - t_b \geq \eta$. Then the switched system (32) is GUES over $\mathscr{T}$, where $\mathscr{T} \in \mathscr{T}_1(\delta)$.

Proof: Define

$$\begin{aligned}
\Sigma_1 &:= P^{-1}_{s_i}A_{s_i} + A^T_{s_i}P^{-1}_{s_i} + [qh^\star + \lambda^\star]P^{-1}_{s_i} \\
&\quad + P^{-1}_{s_i}B_{s_i}P_{s_i}B^T_{s_i}P^{-1}_{s_i}, \quad s_i \in \mathscr{T}_s. \\
\Sigma_2 &:= P^{-1}_{u_j}A_{u_j} + A^T_{u_j}P^{-1}_{u_j} + [qh^\star - \lambda]P^{-1}_{u_j} \\
&\quad + P^{-1}_{u_j}B_{u_j}P_{u_j}B^T_{u_j}P^{-1}_{u_j}, \quad u_j \in \mathscr{T}_u.
\end{aligned}$$

It is easy to find that

$$\begin{aligned}
&\Sigma_1 \leq 0 \Leftrightarrow (33) \text{ holds,} \\
&\Sigma_2 \leq 0 \Leftrightarrow (34) \text{ holds.}
\end{aligned}$$

Then considering Lyapunov function $V_\sigma(t) = x^T(t)P^{-1}_{\sigma(t)}x(t)$, $\sigma(t) = s_i$ or u_j, and applying the similar discussion in Theorem 4, we can obtain the above result. The proof is omitted here. ∎

Remark 4: In this section, we present some results for the uniform stability and globally uniformly exponential stability of delayed switched systems with stable and unstable modes in which the time delay may be unknown or infinite. To overcome the difficulties caused by the fact that the exact time delay cannot be observed, all those results are based on the assumption that $\sigma(t_b^-) \in \mathscr{P}_u$, for some $b \in \mathbb{R}_+$, implying that $\sigma(t_b) \in \mathscr{P}_s$, $t_{b+1} - t_b \geq \eta$. This indicates that when an unstable mode is working, the next mode to be activated should be a stable one and the dwell time at that mode should be not less than η. This assumption is slightly stronger than the results for the finitely delayed switched systems in Section III. Hence, the stability and switching signal design for infinitely delayed switched systems requires further study.

V. NUMERICAL EXAMPLES

In this section, we shall present some numerical examples and their simulations to demonstrate the validity and advantages of the designed switched laws.

Example 1: Consider a 1D switched delayed system (1) with $\mathscr{P} = \{1, 2\}$ and

$$f_1(t, x_t) = x(t) + 0.2 \sin\big(x(t - \tau(t))\big), \quad t \geq 0, \tag{35a}$$

$$f_2(t, x_t) = -2x(t) + 0.1 \tanh\big(x(t - \tau(t))\big), \quad t \geq 0. \tag{35b}$$

where $\tau(t) \in [0, \tau]$, $\tau \leq \infty$. We have the following assertions for $\tau < \infty$ and $\tau = \infty$, respectively.

Assertion 1 (Case: $\tau < \infty$): Assume that there exist constants $h > 0$, $\tau_D > 0$ and $r \geq 0$ such that $0.1 \exp(h\tau) + h \leq 2$ and

$$\frac{h\tau}{\tau_D} + \frac{1.2r - h}{1 + r} < 0.$$

Then the switched system (35) is GUES over $\mathscr{T}$, where $\mathscr{T} \subseteq \mathscr{T}_2(r, T) \cap \mathcal{S}_{ave}[\tau_D, N_0]$ for some $N_0 > 0$ and $T \geq 0$.

Assertion 2 (Case: $\tau = \infty$): Assume that there exist constants $q > 1$, $\eta > 0$ and $\delta > 0$ such that

$$\big(2 - \frac{q}{10}\big)\eta \geq \ln q > \delta\big(1 + \frac{q}{5}\big).$$

Moreover, $\sigma(t_b^-) \in \mathscr{P}_u$, for some $b \in \mathbb{R}_+$, implies that $\sigma(t_b) \in \mathscr{P}_s$ and $t_{b+1} - t_b \geq \eta$. Then the switched system (35) is US over $\mathscr{T}$, where $\mathscr{T} \in \mathscr{T}_1(\delta)$.

Remark 5: Consider Lyapunov functions $V_1 = V_2 = |x|$. Then it is easy to derive that

$$D^+V_1(t) \leq V_1(t) + 0.2V_1(t - \tau(t)),$$
$$D^+V_2(t) \leq -2V_2(t) + 0.1V_2(t - \tau(t)).$$

By Theorem 1 and Theorem 3, it is easy to derive the above two assertions. In particular, if $\tau = 0.2$ and $\tau_D = 1$, then we take $h = 1.85$, and $r = 0.9$ such that all conditions in Assertion 1 hold. We obtain that the switched system (35) with $\tau = 0.2$ is GES over $\mathscr{T}$, where $\mathscr{T} \subseteq \mathscr{T}_2(0.9, T) \cap \mathcal{S}_{ave}[1, N_0]$ for some $N_0 > 0$ and $T \geq 0$. In simulations, we choose the switching signal as follows

$$\bar{\sigma}(t) = \begin{cases} 2, & t \in [t_{2k-2}, t_{2k-1}),\ k \in \mathbb{Z}_+, \\ 1, & t \in [t_{2k-1}, t_{2k}),\ k \in \mathbb{Z}_+, \end{cases} \tag{36}$$

where the switching points $t_k, k \in \mathbb{Z}_+$, satisfy

$$t_{4k} - t_{4k-1} = 0.88, \quad t_{4k-1} - t_{4k-2} = 2,$$
$$t_{4k-2} - t_{4k-3} = 1, \quad t_{4k-3} - t_{4k-4} = 0.12. \tag{37}$$

In this case, it is easy to check that one may choose $N_0 = 3$ and $T = 40$ such that $N_\sigma(\beta, t) \leq 3 + t - \beta$ for all $t \geq \beta \geq 0$ and $\pi_u/\pi_s \leq 0.9$ for all $t \geq 40$. Then we know that $\bar{\sigma} \in \mathscr{T}_2(0.9, 40) \cap \mathcal{S}_{ave}[1, 3]$. Fig. 1.(a) shows the trajectories of switched system (35) with $\tau(t) = 0.1 + 0.1[\sin t]^*$. Under the same conditions, if we exchange the dwell times of $[t_{4k-3}, t_{4k-2}]$ and $[t_{4k-2}, t_{4k-1}]$, i.e., $t_{4k-1} - t_{4k-2} = 1$ and $t_{4k-2} - t_{4k-3} = 2$. Then it will go against the Assertion 1 and in this case, Fig. 1.(b) tells us that switched system (35) is unstable under such undesigned switching law.

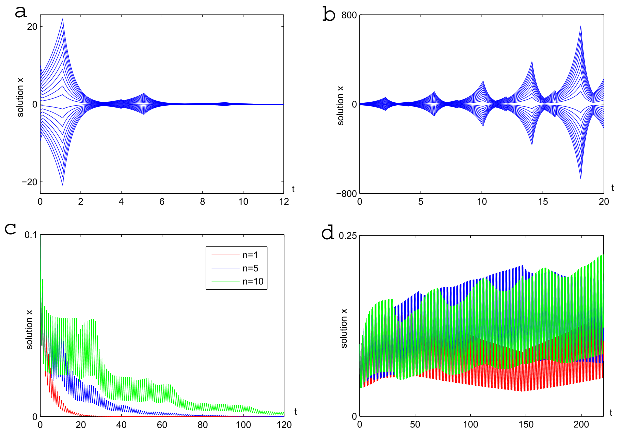

FIGURE 1. (a). Trajectories of switched system (35) with switching law (36)-(37) and time delay $\tau(t) = 0.1 + 0.1[\sin t]^*$. (b) Trajectories of switched system (35) with the undesigned switching law. (c) Trajectories of switched system (35) with $\tau(t) = n[\ln(1 + t)]^*$, $n = 1, 5$, and 10, respectively. (d). Trajectories of switched system (35) with the undesigned switching law.

On the other hand, if $\tau = \infty$ and we take $q = 2.4$, $\eta = 0.5$ and $\delta = 0.59$. It then follows from Assertion 2 that if $\sigma(t_b^-) \in \mathscr{P}_u$ implies that $\sigma(t_b) \in \mathscr{P}_s$ and $t_{b+1} - t_b \geq 0.5$, then the switched system (35) with $\tau = \infty$ is US over $\mathscr{T}$, where $\mathscr{T} \in \mathscr{T}_1(0.59)$. In simulations, we choose the same switching signal (36) but the switching points $t_k, k \in \mathbb{Z}_+$, is updated by

$$t_{2k} - t_{2k-1} = 0.55, \quad t_{2k-1} - t_{2k-2} = 0.5.$$

Obviously, it satisfies that $\sigma(t_b^-) \in \mathscr{P}_u$ implies that $\sigma(t_b) \in \mathscr{P}_s$ and $t_{b+1} - t_b \geq 0.5$. Fig. 1.(c) shows the trajectories of switched system (35) with $\tau(t) = n[\ln(1 + t)]^*$, $n = 1, 5$, and 10, respectively. Under the same conditions, if the dwell time on the interval $[t_{2k-1}, t_{2k}]$ is extended by $t_{2k} - t_{2k-1} = 0.8$, which goes against the Assertion 2. In this case, the switched system (35) is unstable under the same initial value, which is shown in Fig. 1.(d). In above simulations, we choose the time step $s = 0.001$ and the initial values $\phi = 0.5l(-1)^l, l = 1, \cdots, 20$, in Fig. 1.$(a, b)$, and $\phi = 0.1$ in Fig. 1.(c, d).

Example 2: Consider the switched delay system (21) with $\mathscr{P} = \{1, 2\}$, $\tau(t) \in [0, \tau]$, $\tau = 0.1$, and

$$A_1 = \begin{bmatrix} 2 & -0.3 & 0.1 \\ 0.2 & 1 & 0 \\ 0 & -0.2 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0.5 & -0.2 \\ 0.1 & 0.1 & 0.1 \\ -0.2 & 0 & 0.3 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -4 & 0.1 & 0.13 \\ 0.1 & -5 & 0.4 \\ 0 & 0.3 & -4 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.2 & -0.3 & 0.3 \\ 0.1 & 0.3 & 0.2 \\ 0 & -0.2 & 0.1 \end{bmatrix}.$$

When $\sigma = 1$, the mode is unstable; while $\sigma = 2$, the mode is stable. Based on Theorem 2, we have the following assertion.

Assertion 3: Assume that there exist two constants τ_D and r such that

$$\frac{0.7523}{\tau_D} + \frac{3r - 5.7}{1 + r} < 0. \tag{38}$$

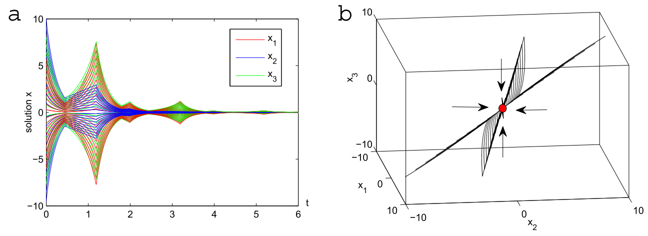

FIGURE 2. **(a). Trajectories of switched system (21) with $\tau(t) = 0.08 - 0.02[\cos t]^*$. (b) Phase plots of switched system (21).**

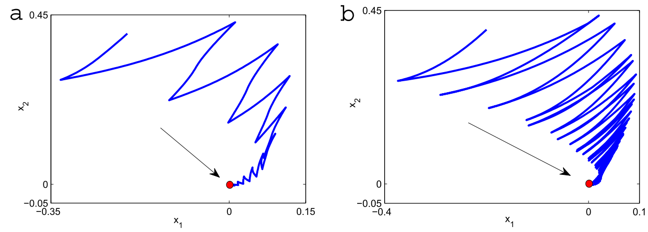

FIGURE 3. **(a). Phase plot of switched system (29) with input gains $\mathcal{K}_1, \mathcal{K}_2$ and time delay $\tau = 0.2$ (b) Phase plot of switched system (29) with input gains $\mathcal{K}_1, \mathcal{K}_2$ and time delay $\tau = [\sqrt{t}]^*$.**

Then the system (21) is GUES over $\mathscr{T}$, where $\mathscr{T} \subseteq \mathscr{T}_2(r, T) \cap \mathcal{S}_{ave}[\tau_D, N_0]$ for some $N_0 > 0$ and $T \geq 0$.

In fact, choose $\lambda = 2, \bar{\lambda} = \bar{\lambda}^\star = 1, \lambda^\star = 7.5, \rho = 1.2, \bar{\mu}_2 = 1$. Via the Matlab's LMI toolbox, we can obtain that LMIs in Theorem 2 have feasible solutions. Then one may choose $h = 5.7$ such that all conditions in Theorem 2 are satisfied and thus the above assertion can be obtained. In particular, we take $\tau_D = 0.5$ and $r = 0.93$. It then follows from (38) that the stability can be guaranteed when $\mathscr{T} \subseteq \mathscr{T}_2(0.93, T) \cap \mathcal{S}_{ave}[0.5, N_0]$ for some $N_0 > 0$ and $T \geq 0$. In simulations, we still consider the switching signal (36) but the switching points $t_k, k \in \mathbb{Z}_+$, is replaced by

$$t_{4k} - t_{4k-1} = 0.45, \quad t_{4k-1} - t_{4k-2} = 0.75,$$
$$t_{4k-2} - t_{4k-3} = 0.6, \quad t_{4k-3} - t_{4k-4} = 0.2.$$

In this case, one may choose $N_0 = 3$ and $T = 50$ such that $N_\sigma(\beta, t) \leq 3 + 2(t - \beta)$ for all $t \geq \beta \geq 0$ and $\pi_u/\pi_s \leq 0.93$ for all $t \geq 50$. Then it holds that $\bar{\sigma} \in \mathscr{T}_2(0.93, 50) \cap \mathcal{S}_{ave}[0.5, 3]$. Fig. 2. shows the trajectories and phase plots of switched system (21) with $\tau(t) = 0.08 - 0.02[\cos t]^*$.

In above simulations, we choose the time step $s = 0.001$ and the initial values $\phi = (-1)^l(-0.3l, 0.5l, 0.4l)^T, l = 1, \cdots, 20$.

Example 3: Consider the closed-loop switched system (29) with $\mathscr{P} = \{1, 2\}$, $\tau(t) : \mathbb{R}_+ \to \mathbb{R}_+$ is unknown, and

$$A_1 = \begin{bmatrix} 2 & 0.1 \\ 0.2 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0.5 \\ 2 & -1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 1 & 0.2 \\ 0.1 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & -0.5 \\ 0.1 & 0.3 \end{bmatrix}.$$

Based on Theorem 4, we have the following assertion.

Assertion 4: The switched system (29) is US over $\mathscr{T}$, where $\mathscr{T} \in \mathscr{T}_0$ and the input gains are designed by

$$\mathcal{K}_1 = \begin{bmatrix} -1.2359 & -1.2401 \\ -1.2401 & 0.3111 \end{bmatrix},$$
$$\mathcal{K}_2 = \begin{bmatrix} -1.7548 & 1.2286 \\ 1.2286 & -0.1889 \end{bmatrix}$$

In fact, choose $q = 1.2$ and it is easy to check that the LMIs in (30) have feasible solution and then the above input gains $\mathcal{K}_1$ and $\mathcal{K}_2$ can be derived. The corresponding numerical simulations are shown in Fig. 3, where Fig. 3.(a) deals with $\tau = 0.2$ and Fig. 3.(b) deals with $\tau = [\sqrt{t}]^*$. In simulations, we choose the time step $s = 0.001$ and the initial values $\phi = (-0.2, 0.4)^T$.

Remark 6: In this section, we present three numerical examples in which the time delays are finitely/infinitely time-varying but not differentiable. Existing results (such as [40]–[42] and [44]–[48]) from studies on switched systems with unstable modes cannot be utilized to design the stable switching laws. Moreover, our designed switching law ensures that the total dwell time on unstable modes is longer than stable ones, such as with the switching signals in Assertion 2 or Assertion 3 with $\tau_D \geq 0.5573$.

VI. CONCLUSION

This paper was dedicated to the problem of designing stable switching laws for delay switched systems with stable and unstable modes, in which the modes may be finite delay, infinite delay or unknown delay modes. Different approaches (including inequality techniques, average dwell time techniques, multiple Lyapunov functions, and the Razumikhin method with both delay-dependent and delay-independent switching laws) have been presented to guarantee uniform stability and globally uniform exponential stability. Our results show that the stability of finitely delayed switched systems with stable and unstable modes can be guaranteed if the divergence rate and total dwell times of unstable modes can be effectively controlled and balanced by the ADT-based switching control with stable modes. Our results for infinitely delayed switched systems are very useful for the design of switching laws for stable or unstable modes that are subject to unknown, infinite, or inestimable value time delays. However, it should be mentioned that the designed switching law for infinitely delayed switched systems is not applicable to switched controls that include two consecutive unstable modes, and it requires further study.

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