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Adaptive Synchronization of Fractional-Order Nonlinearly Coupled Complex Networks With Time Delay and External Disturbances

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ABSTRACT This paper considers both local and global synchronizations of fractional-order nonlinearly-coupled complex networks with time delay and unknown external disturbances. Here, neither delayed or non-delayed configuration matrices are necessarily irreducible or symmetric. Combined with the fractional order compare theorem and fractional order stability theory, some novel sufficient conditions are obtained that guarantee the realization of local and global asymptotic synchronization via adaptive control and pinning control. The network model and conclusion in this paper are more practical and general than those in the existing literature. The experimental results show the feasibility of our theoretical analysis.

INDEX TERMS Complex networks, fractional-order, nonlinearly coupled, time delay, external disturbances.

I. INTRODUCTION

Work and life are closely connected with various complex networks, such as the Internet, gene networks, power grid networks, neural networks, and so on [1]–[5]. Among diverse dynamical behaviors of complex networks, synchronization is one of the most significant collective phenomena, and has many practical applications(such as image encryption, digital communication, secure communication [6], [7]). Network synchronization has garnered increasing research attention. Some networks can be synchronized through information exchange among local connections, but in reality, most complex networks cannot achieve a synchronization state while depending only on their internal structures without using some external driving force [8]. Therefore, according to the characteristics of different network models, various types of control techniques have been used to facilitate achieving synchronization, including feedback control [9], [10], pinning control [11]–[13], adaptive control [14]–[16] and impulsive control [17], [18], etc.

In contrast, fractional order calculus is an important branch of traditional calculus. It was first proposed by generalizing integer order calculus to an arbitrary fractional order [19], [20]. However, in science and engineering research, fractional order calculus has been neglected due to a lack of suitable application history. Fortunately, the gradual evolution of fractional calculus operations has now attracted considerable research attention in various fields [21]–[24]. The main advantage of fractional-order differential equations is its non-locality. This shows the state of fraction-order systems depends on not only the current moment, but also depends on the past state. Many real systems, such as power systems and physical systems, are more likely to be the fractional-order systems. Moreover, compared with integer-order complex networks, the fractional-order ones add a degree of freedom by using fractional derivative, which is more practical in the fields of secure communication and neural networks. Based on the peculiar superiority of memory and hereditary nature of fractional calculus, it has become a useful tool for accurately describing various actual complex issues [25]–[29].

Recently, the Caputo fractional order derivative operator was introduced to complex networks because its initial conditions are the same as integer-order conditions and reflect well-understood physical truths. As this operator has developed, the problem of synchronizing fractional order complex networks has gradually become a hot research point [30]–[40]. Ma and Zhang [34] studied the hybrid synchronization issue of two general fractional-order complex networks with non-delayed coupling by using a feedback control strategy. Wu and Lu [35] obtained some valuable theoretical results for outer synchronization of fractional-order systems with non-delayed coupling by designing a nonlinear feedback controller. They concluded that the synchronization effect depends on both the feedback control gain and fractional order. Wu *et al.* investigated generalized synchronization of fractional-order weighted chaotic networks by using a nonlinear control scheme in [36]. To reduce the control costs, pinning control was used in [37]. Some novel criteria for cluster synchronization of fractional-order linear-coupling complex networks were obtained, and the number of pinned nodes was effectively estimated. Wang *et al.* [38] considered exponential synchronization of fractional-order dynamical systems by using pinning impulsive control, and proposed an estimation method for the system parameters. Note that the fractional order models in [34]–[38] did not consider delayed coupling or any external disturbances.

To simulate networks more realistically, time delay was considered in [39] and [40] and an external disturbance was addressed in [30]–[33] and [41]–[48]. For example, adaptive synchronization of fractional-order uncertain systems with delayed linear coupling was considered in [39]. Projective synchronization of fractional-order undirected dynamical networks with coupling delay was studied by using an integer-order Lyapunov scheme in [40]. Stability analysis for fractional-order financial systems under a disturbance was investigated in [30], in which the unknown parameter can be estimated by a corresponding adaptive law. Finite time synchronization of fractional-order systems with an unknown disturbance was studied in [33] using sliding mode control. Based on sliding control and adaptive control techniques, synchronization of general fractional-order systems with unknown disturbances was studied in [41]. To obtain more useful stability condition, a new T-S fuzzy control method was proposed for the stabilization of fractional-order nonlinear systems with external disturbances [43]. However, many methods and strategies for integer-order complex networks can not be directly extended to fractional-order networks. Therefore, designing a novel fractional-order adaptive controller to synchronize fractional-order nonlinearly-coupled systems with non-delayed and delayed couplings as well as unknown external disturbances is a valuable task.

Most of the research results for fractional-order networks have been based on the following three simple cases: (i) The signal transmission from one node to another node is considered at the time instant t or $t-\tau$. (ii) The coupling function of systems is linear and no external noise disturbance occurs. (iii) The interaction topology of networks is bidirectional (the coupling configuration matrix should be symmetrical or irreducible). However, to the best of our knowledge, few research results exist concerning adaptive pinning synchronization for fractional-order nonlinearly-coupled directed networks with external disturbances nor for fractional-order nonlinear networks with non-delayed and delayed couplings in addition to disturbances, which is a more practical approach considering real-world conditions. For example, time delays often occur in communication and neural networks because of finite signal transmission speeds. Thus, information is communicated between different units not only at time t but also at time $t-\tau$. Meanwhile, external disturbances often exist in communication processes because of interference from various types of noise. Moreover, due to conditions of limited visibility and instrument precision, the state variables $x_i(t)$ of some neural networks may be difficult to observe, but we can easily observe their nonlinear-coupling states. Furthermore, directional networks are widely used for information transmission in modern life, such as research paper citation networks and broadcasting networks.

For the above reasons, this paper investigates locally and globally asymptotic synchronization of fractional-order nonlinearly-coupled complex networks with non-delayed and delayed couplings as well as external disturbances based on fractional-order stability theory. It is worth noting that the coupling configuration matrices considered in this paper can be asymmetric as well as reducible and that the external unknown disturbances are nonlinear. Our work's major contributions include four aspects as follows: (1) The fractional-order model is more general and practical than those in the existing literature and all the results are obtained based on fractional-order stability theory. (2) Few works study adaptive synchronization of fractional-order nonlinear systems with time-delay and disturbances by using fractional-order compare theorem. This paper tries to fill this gap. (3) This paper designs a novel fractional-order adaptive pinning controller and it has the following characteristics. First, its adaptive law contains a free parameter q, which is more practical than the integer-order adaptive controller. Second, it simultaneously introduces the sign function and 1-norm of the synchronization error. This is the main reason that our controller can overcome disturbances in the synchronization process. Our controller inherits the advantages of the pinning controller and the adaptive controller. Pinning control is a very useful technique that controls only a subset of the nodes rather than all the nodes. Moreover, the controllers' gain adjusts adaptively to a suitable strength until synchronization is achieved. (4) Based on the linear matrix inequality and matrix decomposition technique, the lower bound of the control strength can be precisely evaluated.

This paper is organized into five sections. Section II provides important preliminary knowledge and introduces a general fractional-order nonlinearly-coupled system. Section III investigates local and global synchronization conditions of fractional-order systems by designing suitable controllers. Section IV presents two typical numerical examples to support the theoretical analysis and Section V concludes the paper.

Notations: In this paper, $min\{\}$ and $max\{\}$ denote the smallest and largest element in the set $\{\}$, respectively, such as $\bar{a} = min\{a_1, a_2, \ldots, a_n\}$, $\hat{a} = max\{a_1, a_2, \ldots, a_n\}$, $\bar{\vartheta} = min\{\vartheta_1, \vartheta_2, \ldots, \vartheta_n\}$ and $\hat{\vartheta} = max\{\vartheta_1, \vartheta_2, \ldots, \vartheta_n\}$. For a real matrix B,$\lambda_{max}(B)$ denotes its maximum eigenvalue, B^T represents its transpose, and $B^s = (B+B^T)/2$. $\|\cdot\|_1$, $\|\cdot\|_\infty$ and $\|\cdot\|$ denote the 1-norm,infinite norm and Euclidean norm of a matrix or vector, respectively. For a real function $x(t)$, $I_{t_0}^q x(t)$ and $D_{t_0}^q x(t)$ denote the q order integral and Caputo derivative whose initial value is t_0, respectively.

II. SYSTEMS AND PRELIMINARIES

Consider a general fractional-order nonlinear complex network with N nodes, which can be described as follows:

$$D_0^q x_i(t) = f(x_i(t)) + c_1 \sum_{j=1}^{N} \phi_{ij} A \Pi(x_j(t)) + c_2 \sum_{j=1}^{N} \psi_{ij} A \Theta(x_j(t-\tau)) + \Delta_i(t) + u_i(t) \quad (1)$$

where $0 < q < 1$, $x_i(t) = [x_{1i}(t), x_{2i}(t), \cdots, x_{ni}(t)]^T \in R^n$ is the state variable of node i and $f : R^n \to R^n$ is a smooth vector-valued function. The constants $c_1 > 0$ and $c_2 > 0$ denote coupling strengths, and $\tau > 0$ is the coupling delay. A =diag$\{a_1, a_2, \ldots, a_n\} \in R^{n\times n}$ is a diagonal matrix that satisfies $a_i > 0, i = 1, 2, \ldots, n$. $\Phi = (\phi_{ij}) \in R^{N\times N}$ and $\Psi = (\psi_{ij}) \in R^{N\times N}$ are the coupling configuration matrices that satisfy the conditions $\phi_{ii} = -\sum_{j=1, j\neq i}^{N} \phi_{ij}$ and $\psi_{ii} = -\sum_{j=1, j\neq i}^{N} \psi_{ij}$, respectively. If there is a connection from node j to node $i (i \neq j)$, then $\phi_{ij} \neq 0$ and $\psi_{ij} \neq 0$; otherwise, $\phi_{ij} = 0$ and $\psi_{ij} = 0$. The continuous nonlinearly-coupled functions $\Pi(\cdot) : R^n \to R^n$ and $\Theta(\cdot) : R^n \to R^n$ have the forms: $\Pi(x_i(t)) = [\pi_1(x_{1i}(t)), \pi_2(x_{2i}(t)), \ldots, \pi_n(x_{ni}(t))]^T$ and $\Theta(x_i(t-\tau)) = [\theta_1(x_{1i}(t-\tau)), \theta_2(x_{2i}(t-\tau)), \ldots, \theta_n(x_{ni}(t-\tau))]^T$, respectively. The initial conditions associated with (1) are $x_i^0(t) = [x_{1i}^0(t), x_{2i}^0(t), \cdots, x_{ni}^0(t)] \in C([t_0 - \tau, t_0], R^n)$. Here, $\Delta_i(t) \in R^n$ are external time-varying disturbance vectors.

Suppose that $s(t)$ is a solution of an isolated node that satisfies

$$D_0^q s(t) = f(s(t)). \quad (2)$$

Our goal is to design some appropriate controllers $u_i(t)$ to synchronize the solutions of network (1) to the given solution $s(t)$. Define the error vector as $e_i(t) = x_i(t) - s(t)$, $1 \leq i \leq N$. Because the coupling matrices Φ and Ψ satisfy the zero-sum-row conditions, the error system is

$$D_0^q e_i(t) = F(e_i(t)) + c_1 \sum_{j=1}^{N} \phi_{ij} A[\Pi(x_j(t)) - \Pi(s(t))] + c_2 \sum_{j=1}^{N} \psi_{ij} A[\Theta(x_j(t-\tau)) - \Theta(s(t-\tau))] + \Delta_i(t) + u_i(t) \quad (3)$$

where $F(e_i(t)) = f(x_i(t)) - f(s(t))$, $i = 1, 2, \ldots, N$. As we all know, the dynamical system (1) asymptotically synchronizes to the goal trajectory $s(t)$ if the zero solution of the error system (3) is asymptotically stable.

Remark 1: The non-delayed and delayed coupling matrices are not necessarily symmetric or irreducible in this paper. This shows that the fractional-order networks (1) can be undirected or directed, and may also include some isolated nodes.

Remark 2: Complex network (1) is a general fractional-order model in which external unknown disturbances, directed communications, and non-delayed and delayed nonlinear couplings exist. Note that the delayed coupling was not considered in [30]–[38]. Although the fractional-order systems in [40] considered both delayed and non-delayed couplings, the coupling function was linear and external disturbance was neglected. Moreover, the control scheme in [40] was based on integer-order stability theory, not on fractional-order stability theory.

Definition 1 [38]: The dynamical network (1) is said to realize synchronization if

$$\sum_{i=1}^{N} \lim_{t\to\infty} \|x_i(t) - s(t)\| = 0.$$

Definition 2 [28]: The Caputo fractional derivative for a function $x(t)$ is defined as

$$D_{t_0}^q x(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-\tau)^{n-q-1} x^{(n)}(\tau) d\tau$$

where $0 \leq n-1 < q < n$ and $\Gamma(\cdot)$ is the Gamma function.

Lemma 1 [49]: Let a vector-value function $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T$ is differentiable. Then, for $\forall t \geq t_0$, one has

$$D_{t_0}^q [y^T(t) y(t)] \leq 2 y^T(t) D_{t_0}^q y(t), 0 < q < 1.$$

Lemma 2 [39]: Suppose that $W(t) \in R^1 (W(t) \geq 0)$ is a continuously differentiable function, and satisfies

$$\begin{cases} D_{t_0}^q W(t) \leq -\kappa_1 W(t) + \kappa_2 W(t-\tau), & 0 < q < 1 \\ W(t) = \kappa(t) \geq 0, & t \in [-\tau, 0] \end{cases}$$

where $t \geq 0$ and $\kappa_1 > \kappa_2 > 0$. Then, $\lim_{t\to+\infty} W(t) = 0$ for all $\kappa(t) \geq 0$ and $\tau > 0$.

Lemma 3 [10]: Assume that $1_n = (1, 1, \ldots, 1)^T$, $Q = I_n - \frac{1}{n} 1_n . 1_n^T$, $\vartheta > 0$ and $H \in R^{m\times n}$ satisfies zero-row-sum condition, one has

$$x^T H y \leq \frac{1}{2} (\frac{1}{\vartheta} x^T H H^T x + \vartheta y^T Q y).$$

Lemma 4 [50]: let a symmetrical matrix $Q \in R^{n\times n}$ satisfies $q_{ii} = -\sum_{j=1, i\neq j}^{n} q_{ij}$, $i, j = 1, 2, \ldots, n$, then for all vectors $\mu = (\mu_1, \mu_2, \ldots, \mu_n)^T$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_n)^T$, we have

$$\mu^T Q \nu = \sum_{i=1}^{n} \sum_{j=1}^{n} \mu_i q_{ij} \nu_j = -\sum_{j>i} q_{ij} (\mu_i - \mu_j)(\nu_i - \nu_j).$$

Lemma 5 [8]: The following linear matrix inequality (LMI)

$$\begin{pmatrix} Q(x) & M(x) \\ M(x)^T & R(x) \end{pmatrix} < 0$$

is equivalent to $R(x) < 0$ and $Q(x) - M(x) R(x)^{-1} M(x)^T < 0$, where $Q(x)$ and $R(x)$ satisfy $Q(x) = Q(x)^T$ and $R(x) = R(x)^T$, respectively.

Assumption 1: Assume that there exists a constant $\eta \geq 0$ such that $\|Df(s)\| \leq \eta$, where $Df(s)$ is the Jacobian of f evaluated at $x = s$.

Assumption 2: Assume that there exists a constant $L \geq 0$ such that $\|f(x_i(t)) - f(s(t))\| \leq L\|x_i(t) - s(t)\|$.

Assumption 3: [10] For each nonlinear function $\pi_k(\cdot)$ and $\theta_k(\cdot)$ in system (1), there exist constants $\beta \geq 0$ and $\epsilon \geq 0$ such that $\lambda_k(x) = \pi_k(x) - \beta x$ and $r_k(x) = \theta_k(x) - \beta x$ satisfy the following conditions

$$|\pi_k(x_1) - \pi_k(x_2) - \beta(x_1 - x_2)| \leq \epsilon|x_1 - x_2|$$
$$|\theta_k(x_1) - \theta_k(x_2) - \beta(x_1 - x_2)| \leq \epsilon|x_1 - x_2|$$

respectively, for $\forall x_1, x_2 \in R$, $k = 1, 2, \cdots, n$.

Assumption 4: The norm of the time-varying disturbance $\Delta_i(t)$ is bounded, i.e., $\|\Delta_i(t)\| \leq w_i$.

Remark 3: In fact, it has been verified that many chaotic systems, such as Lorenz system, Chen system, Chua's circuit system and Lü system satisfy Assumption 2.

Remark 4: Nonlinear coupling exists widely in the actual complex network, such as the coupling between different neurons in the neural network. The nonlinear function can be decomposed into the linear part βx and the oscillatory part by using projection method.

III. MAIN RESULTS

A. LOCALLY ASYMPTOTIC SYNCHRONIZATION VIA ADAPTIVE CONTROL

To reduce the enormous difference of control strength between theoretical value and practical need, the adaptive controllers $u_i(t)$ of the error system (3) can be designed as follows:

$$\begin{cases} u_i(t) = -(d_i(t) + d_i^*)Ae_i(t) - \tilde{d}_1 sgn(e_i(t))/\|e_i(t)\|_1, \\ D_0^q d_i(t) = b_i\|e_i(t)\|^2, \end{cases} \tag{4}$$

for $1 \leq i \leq N$, where $d_i(t)$, d_i^* and $\tilde{d}_1 = \frac{1}{2N}\sum_{i=1}^N w_i^2$ are the control gains ($d_i(t) \geq 0$, $d_i^* > 0$, $b_i > 0$), and $sgn(e_i(t)) = (sign(e_{1i}(t)), sign(e_{2i}(t)), \ldots, sign(e_{ni}(t)))^T$ are signum vectors. Substituting equation (4) into (3), the error system becomes

$$\begin{cases} D_0^q e_i(t) = F(e_i(t)) + c_1\sum\limits_{j=1}^N \phi_{ij}A[\Pi(x_j(x)) - \Pi(s(t))] \\ \qquad + c_2\sum\limits_{j=1}^N \psi_{ij}A[\Theta(x_j(t-\tau)) - \Theta(s(t-\tau))] + \Delta_i \\ \qquad - (d_i(t) + d_i^*)Ae_i(t) - \tilde{d}_1 sgn(e_i(t))/\|e_i(t)\|_1 \\ D_0^q d_i(t) = b_i\|e_i(t)\|^2, \ 1 \leq i \leq N. \end{cases} \tag{5}$$

Remark 5: Let $d_i(0) \geq 0$, clearly, $d_i(t) = d_i(0) + I_0^q(b_i\|e_i(t)\|^2) \geq d_i(0)$; therefore, one can easily obtain $d_i(t) \geq 0$.

Theorem 1: Under Assumptions 1 and 3–4, the locally asymptotic synchronization of the network (1) can be achieved via the adaptive controllers (4) if there exist positive constants $\bar{\vartheta}$, $\hat{\vartheta}$, ρ, c_1, c_2 and $d_i^*(1 \leq i \leq N)$, such that

$$2\eta + 1 + \bar{a}\lambda_{max}(\Omega_1) < -c_2\hat{a}\big(\rho\beta\|\Psi\|_1 + 2\epsilon^2\hat{\vartheta}(1 - \frac{1}{N})\big) \tag{6}$$

where $\Omega_1 = \beta(2c_1\Phi^s + \frac{c_2}{\rho}\|\Psi\|_\infty I_N) + \frac{1}{\bar{\vartheta}}(c_1\Phi\Phi^T + c_2\Psi\Psi^T) + 2\big(c_1\epsilon^2\hat{\vartheta}(1 - \frac{1}{N})I_N - D_1\big)$ and $D_1 = diag\{d_1^*, d_2^*, \ldots, d_N^*\}$.

Proof: Consider the following function:

$$V(t) = \sum_{i=1}^N e_i^T(t)e_i(t).$$

Since $d_i(t) \geq 0$ and $D_0^q d_i(t) = b_i\|e_i(t)\|^2 \geq 0$, one has (by Lemma 1)

$$\begin{aligned}
&D_0^q V(t) \\
&\leq D_0^q V(t) + \sum_{i=1}^N \frac{2\bar{a}}{b_i} d_i(t) D_0^q d_i(t) \\
&\leq 2\sum_{i=1}^N e_i^T(t) D_0^q e_i(t) + \sum_{i=1}^N \frac{2\bar{a}}{b_i} d_i(t) D_0^q d_i(t) \\
&= 2\Big\{\sum_{i=1}^N e_i^T(t) D(f(s)) e_i(t) \\
&\quad + c_1\sum_{i=1}^N\sum_{j=1}^N e_i^T(t)\phi_{ij}A[\Pi(x_j(x)) - \Pi(s(t))] \\
&\quad + c_2\sum_{i=1}^N\sum_{j=1}^N e_i^T(t)\psi_{ij}A[\Theta(x_j(t-\tau)) - \Theta(s(t-\tau))] \\
&\quad - \sum_{i=1}^N e_i^T(t)\tilde{d}_1 sgn(e_i(t))/\|e_i(t)\|_1 \\
&\quad + \sum_{i=1}^N e_i^T(t)\Delta_i(t) - \sum_{i=1}^N e_i^T(t) d_i(t) A e_i(t) \\
&\quad - \sum_{i=1}^N e_i^T(t) d_i^* A e_i(t) + \sum_{i=1}^N e_i^T(t) d_i(t)\bar{a} e_i(t)\Big\} \\
&\leq 2\Big\{\sum_{i=1}^N e_i^T(t) D(f(s)) e_i(t) + \sum_{i=1}^N e_i^T(t)\Delta_i(t) \\
&\quad - \sum_{i=1}^N e_i^T(t)\tilde{d}_1 sgn(e_i(t))/\|e_i(t)\|_1 - \sum_{i=1}^N e_i^T(t) d_i^* A e_i(t) \\
&\quad + c_1\sum_{i=1}^N\sum_{j=1}^N e_i^T(t)\phi_{ij}A[\Pi(x_j(x)) - \Pi(s(t))] \\
&\quad + c_2\sum_{i=1}^N\sum_{j=1}^N e_i^T(t)\psi_{ij}A[\Theta(x_j(t-\tau)) - \Theta(s(t-\tau))]\Big\} \\
&= V_1(t) + V_2(t) + V_3(t).
\end{aligned} \tag{7}$$

For convenience use later, the following notations are introduced for $k = 1, 2, \ldots, n$:

$$\begin{aligned}
e^k(t) &= [e_{k1}(t), e_{k2}(t), \ldots, e_{kN}(t)]^T \\
\widetilde{\Pi}_k(x^k(t)) &= [\pi_k(x_{k1}(t)), \pi_k(x_{k2}(t)), \ldots, \pi_k(x_{kN}(t))]^T \\
\widetilde{\lambda}_k(x^k(t)) &= [\lambda_k(x_{k1}(t)), \lambda_k(x_{k2}(t)), \ldots, \lambda_k(x_{kN}(t))]^T \\
\widetilde{\Theta}_k(x^k(t-\tau)) &= [\theta_k(x_{k1}(t-\tau)), \theta_k(x_{k2}(t-\tau)), \ldots, \\
&\qquad \theta_k(x_{kN}(t-\tau))]^T \\
\widetilde{r}_k(x^k(t-\tau)) &= [r_k(x_{k1}(t-\tau)), r_k(x_{k2}(t-\tau)), \ldots, \\
&\qquad r_k(x_{kN}(t-\tau))]^T.
\end{aligned}$$

Using Assumptions 1 and 4, we obtain

$$\begin{aligned}
V_1(t) &= 2\Big\{\sum_{i=1}^{N} e_i^T(t)D(f(s))e_i(t) + \sum_{i=1}^{N} e_i^T(t)\Delta_i(t) \\
&\quad - \sum_{i=1}^{N} e_i^T(t)\tilde{d}_1 sgn(e_i(t))/\|e_i(t)\|_1 \\
&\quad - \sum_{i=1}^{N} e_i^T(t)d_i^* A e_i(t)\Big\} \\
&\leq 2\eta \sum_{i=1}^{N} e_i^T(t)e_i(t) + \sum_{i=1}^{N} e_i^T(t)e_i(t) + \sum_{i=1}^{N} \|\Delta_i(t)\|^2 \\
&\quad - 2N\tilde{d}_1 - 2\sum_{i=1}^{N} e_i^T(t)d_i^* A e_i(t) \\
&\leq (2\eta+1)\sum_{i=1}^{N} e_i^T(t)e_i(t) - 2\sum_{i=1}^{N} e_i^T(t)d_i^* A e_i(t) \\
&= \sum_{k=1}^{n} e^k(t)^T \big((2\eta+1)I_N - 2a_k D_1\big)e^k(t). \qquad (8)
\end{aligned}$$

Using Assumption 3 and Lemma 3, we have

$$\begin{aligned}
V_2(t) &= 2c_1 \sum_{i=1}^{N}\sum_{j=1}^{N} e_i^T(t)\phi_{ij} A[\Pi(x_j(t)) - \Pi(s(t))] \\
&= 2c_1 \sum_{k=1}^{n} a_k e^k(t)^T \Phi[\widetilde{\Pi}_k(x^k(t)) - \widetilde{\Pi}_k(s^k(t))] \\
&= 2c_1\beta \sum_{k=1}^{n} a_k e^k(t)^T \Phi e^k(t) \\
&\quad + 2c_1 \sum_{k=1}^{n} a_k e^k(t)^T \Phi[\widetilde{\lambda}_k(x^k(t)) - \widetilde{\lambda}_k(s^k(t))] \\
&\leq 2c_1\beta \sum_{k=1}^{n} a_k e^k(t)^T \Phi e^k(t) \\
&\quad + c_1 \sum_{k=1}^{n} \frac{a_k}{\vartheta_k} e^k(t)^T \Phi\Phi^T e^k(t) + c_1 \sum_{k=1}^{n} a_k\vartheta_k \\
&\quad [\widetilde{\lambda}_k(x^k(t)) - \widetilde{\lambda}_k(s^k(t))]^T Q[\widetilde{\lambda}_k(x^k(t)) - \widetilde{\lambda}_k(s^k(t))]. \qquad (9)
\end{aligned}$$

Note that Φ is a zero-row-sum matrix. Using Assumption 3 and Lemma 4, we have

$$\begin{aligned}
&\sum_{k=1}^{n} a_k\vartheta_k[\widetilde{\lambda}_k(x^k(t)) - \widetilde{\lambda}_k(s^k(t))]^T Q[\widetilde{\lambda}_k(x^k(t)) - \widetilde{\lambda}_k(s^k(t))] \\
&= -\sum_{k=1}^{n} a_k\vartheta_k \sum_{j>i} q_{ij}\Big\{[\lambda_k(x_{kj}(t)) - \lambda_k(s_{kj}(t))] \\
&\quad - [\lambda_k(x_{ki}(t)) - \lambda_k(s_{ki}(t))]\Big\}^2 \\
&\leq -2\sum_{k=1}^{n} a_k\vartheta_k \sum_{j>i} q_{ij}\Big\{[\lambda_k(x_{kj}(t)) - \lambda_k(s_{kj}(t))]^2 \\
&\quad + [\lambda_k(x_{ki}(t)) - \lambda_k(s_{ki}(t))]^2\Big\} \\
&\leq -2\epsilon^2 \sum_{k=1}^{n} a_k\vartheta_k \sum_{j>i} q_{ij}(e_{kj}(t)^2 + e_{ki}(t)^2) \\
&= 2\epsilon^2 \sum_{k=1}^{n} a_k\vartheta_k(1 - \frac{1}{N})e^k(t)^T e^k(t). \qquad (10)
\end{aligned}$$

Using Assumption 3 and Lemma 3, one has

$$\begin{aligned}
V_3(t) &= 2c_2 \sum_{i=1}^{N}\sum_{j=1}^{N} e_i^T(t)\psi_{ij} A[\Theta(x_j(t-\tau)) - \Theta(s(t-\tau))] \\
&= 2c_2 \sum_{k=1}^{n} a_k e^k(t)^T \Psi[\widetilde{\Theta}_k(x^k(t-\tau)) - \widetilde{\Theta}_k(s^k(t-\tau))] \\
&= 2c_2\beta \sum_{k=1}^{n} a_k e^k(t)^T \Psi e^k(t-\tau) \\
&\quad + 2c_2 \sum_{k=1}^{n} a_k e^k(t)^T \Psi[\widetilde{r}_k(x^k(t-\tau)) - \widetilde{r}_k(s^k(t-\tau))] \\
&\leq \frac{c_2\beta}{\rho} \sum_{k=1}^{n} a_k\|\Psi\|_\infty e^k(t)^T e^k(t) + c_2\rho\beta \sum_{k=1}^{n} a_k\|\Psi\|_1 \\
&\quad e^k(t-\tau)^T e^k(t-\tau) + c_2 \sum_{k=1}^{n} \frac{a_k}{\vartheta_k} e^k(t)^T \Psi\Psi^T e^k(t) \\
&\quad + c_2 \sum_{k=1}^{n} a_k\vartheta_k[\widetilde{r}_k(x^k(t-\tau)) - \widetilde{r}_k(s^k(t-\tau))]^T \\
&\quad Q[\widetilde{r}_k(x^k(t-\tau)) - \widetilde{r}_k(s^k(t-\tau))]. \qquad (11)
\end{aligned}$$

Similarly, (by the inequality (10)), we have

$$\begin{aligned}
&\sum_{k=1}^{n} a_k\vartheta_k[\widetilde{r}_k(x^k(t-\tau)) - \widetilde{r}_k(s^k(t-\tau))]^T \\
&\quad Q[\widetilde{r}_k(x^k(t-\tau)) - \widetilde{r}_k(s^k(t-\tau))] \\
&\quad \leq 2\epsilon^2 \sum_{k=1}^{n} a_k\vartheta_k(1 - \frac{1}{N})e^k(t-\tau)^T e^k(t-\tau). \qquad (12)
\end{aligned}$$

Substituting inequalities (8)-(12) into (7), we have

$$\begin{aligned} D_0^q V(t) \le & \sum_{k=1}^{n} e^k(t)^T \Big\{(2\eta+1)I_N + a_k\big[\beta(2c_1\Phi^s \\ & + \frac{c_2}{\rho}\|\Psi\|_\infty I_N) + \frac{1}{\bar{\vartheta}}(c_1\Phi\Phi^T + c_2\Psi\Psi^T) \\ & + 2\big(c_1\epsilon^2\hat{\vartheta}(1-\frac{1}{N})I_N - D_1\big)\big]\Big\} e^k(t) \\ & + \sum_{k=1}^{n} c_2 a_k \Big(\rho\beta\|\Psi\|_1 + 2\epsilon^2\hat{\vartheta}(1-\frac{1}{N})\Big) \\ & e^k(t-\tau)^T e^k(t-\tau). \end{aligned} \tag{13}$$

Let $\Omega_1 = \beta(2c_1\Phi^s + \frac{c_2}{\rho}\|\Psi\|_\infty I_N) + \frac{1}{\bar{\vartheta}}(c_1\Phi\Phi^T + c_2\Psi\Psi^T) + 2\big(c_1\epsilon^2\hat{\vartheta}(1-\frac{1}{N})I_N - D_1\big)$, clearly, Ω_1 is a symmetric matrix. If $2\eta+1+\bar{a}\lambda_{max}(\Omega_1)+c_2\hat{a}\Big(\rho\beta\|\Psi\|_1 + 2\epsilon^2\hat{\vartheta}(1-\frac{1}{N})\Big) < 0$, it means $\lambda_{max}(\Omega_1) < 0$. Apply it into (13), one has

$$\begin{aligned} D_0^q V(t) \le & \Big(2\eta + 1 + \bar{a}\lambda_{max}(\Omega_1)\Big)V(t) \\ & + c_2\hat{a}\Big(\rho\beta\|\Psi\|_1 + 2\epsilon^2\hat{\vartheta}(1-\frac{1}{N})\Big)V(t-\tau). \end{aligned} \tag{14}$$

From lemma 2, when $-\big(2\eta + 1 + \bar{a}\lambda_{max}(\Omega_1)\big) > c_2\hat{a}\big(\rho\beta\|\Psi\|_1 + 2\epsilon^2\hat{\vartheta}(1-\frac{1}{N})\big)$, the local synchronization of the network (1) is achieved under the adaptive controllers (4).

The following corollary is given when $\theta_k(x) = x$ is a linear function and $c_1 = 0$. In this case, the system (1) becomes a linearly-coupled system with no non-delayed coupling, which can be described as

$$D_0^q x_i(t) = f(x_i(t)) + c_2\sum_{j=1}^{N}\psi_{ij}Ax_j(t-\tau) + \Delta_i(t) + u_i(t) \tag{15}$$

where $i = 1, 2, \ldots, N$ and $u_i(t)$ is the same as in (4).

Corollary 1: Under Assumption 1, the locally asymptotic synchronization of the linearly-coupled network (15) can be achieved via the adaptive controllers (4) if there exist positive constants $\bar{\vartheta}$, ρ, c_2 and $d_i^*(1 \le i \le N)$ such that

$$2\eta + 1 + \bar{a}\lambda_{max}(\Omega_1^*) < -c_2\hat{a}\rho\|\Psi\|_1 \tag{16}$$

holds, where $\Omega_1^* = c_2(\frac{1}{\rho}\|\Psi\|_\infty I_N + \frac{1}{\bar{\theta}}\Psi\Psi^T) - 2D_1$ and $D_1 = diag\{d_1^*, d_2^*, \ldots, d_N^*\}$.

Remark 6: Although the synchronization of integer-order complex networks with non-delayed and delayed couplings has been investigated widely [9], [12], [14], their control schemes are not suitable for fractional-order nonlinearly-coupled delayed complex networks. Note that most of results on synchronizing integer-order complex network (such as the obtained results in [9], [12]–[14], [17], and [18]) cannot be directly extended to fractional-order networks.

B. GLOBALLY ASYMPTOTIC SYNCHRONIZATION VIA ADAPTIVE CONTROL AND PINNING CONTROL

Without loss of generality, we select the first l nodes as pinned nodes. Then, the adaptive pinning controllers can be designed as follows:

$$\begin{cases} u_i(t) = -(d_i(t) + d_i^*)Ae_i(t) - \tilde{d}_2 sgn(e_i(t))/\|e_i(t)\|_1, \\ \qquad D_0^q d_i(t) = b_i\|e_i(t)\|^2,\ 1 \le i \le l \\ u_i(t) = 0,\ l+1 \le i \le N \end{cases} \tag{17}$$

where $d_i(t)$, d_i^* and $\tilde{d}_2 = \frac{1}{2l}\sum_{i=1}^{N} w_i^2$ are the control gains ($d_i(t) \ge 0$, $d_i^* > 0$, $b_i > 0$). Thus the controlled error system (3) can be rewritten as follows:

$$\begin{cases} D_0^q e_i(t) = F(e_i(t)) + c_1\sum_{j=1}^{N}\phi_{ij}A[\Pi(x_j(t)) - \Pi(s(t))] \\ \qquad + c_2\sum_{j=1}^{N}\psi_{ij}A[\Theta(x_j(t-\tau)) - \Theta(s(t-\tau))] \\ \qquad + \Delta_i(t) - (d_i(t) + d_i^*)Ae_i(t) \\ \qquad - \tilde{d}_2 sgn(e_i(t))/\|e_i(t)\|_1,\ 1 \le i \le l \\ D_0^q e_i(t) = F(e_i(t)) + c_1\sum_{j=1}^{N}\phi_{ij}A[\Pi(x_j(t)) - \Pi(s(t))] \\ \qquad + c_2\sum_{j=1}^{N}\psi_{ij}A[\Theta(x_j(t-\tau)) - \Theta(s(t-\tau))] \\ \qquad + \Delta_i(t),\ l+1 \le i \le N. \end{cases} \tag{18}$$

Theorem 2: Under Assumptions 2–4, the globally asymptotic synchronization of the network (1) can be achieved via the controllers (17) if there exist positive constants $\bar{\vartheta}$, $\hat{\vartheta}$, ρ, c_1, c_2 and $d_i^*(1 \le i \le l)$, such that

$$(2L+1)I_N + \bar{a}\Omega_2 + c_2\hat{a}\big(\beta + 2\epsilon^2\hat{\vartheta}(1-\frac{1}{N})\big)I_N < 0 \tag{19}$$

holds, where $\Omega_2 = \beta(2c_1\Phi^s + \frac{c_2}{\rho}\|\Psi\|_\infty I_N) + \frac{1}{\bar{\vartheta}}(c_1\Phi\Phi^T + c_2\Psi\Psi^T) + 2\big(c_1\epsilon^2\hat{\vartheta}(1-\frac{1}{N})I_N - D_2\big)$ and $D_2 = diag\{d_1^*, d_2^*, \ldots, d_l^*, 0, \ldots, 0\}$.

Proof: Consider the following function:

$$V(t) = \sum_{i=1}^{N} e_i^T(t)e_i(t).$$

Since $d_i(t) \ge 0$ and $D_0^q d_i(t) = b_i\|e_i(t)\|^2 \ge 0$, one has

$$\begin{aligned} & D_0^q V(t) \\ & \le D_0^q V(t) + \sum_{i=1}^{l}\frac{2\bar{a}}{b_i}d_i(t)D_0^q d_i(t) \\ & \le 2\sum_{i=1}^{N}e_i^T(t)D_0^q e_i(t) + \sum_{i=1}^{l}\frac{2\bar{a}}{b_i}d_i(t)D_0^q d_i(t) \\ & = 2\Big\{\sum_{i=1}^{N}e_i^T(t)F(e_i(t)) \\ & \quad + c_1\sum_{i=1}^{N}\sum_{j=1}^{N}e_i^T(t)\phi_{ij}A[\Pi(x_j(x)) - \Pi(s(t))] \end{aligned}$$

$$
\begin{aligned}
&+c_2\sum_{i=1}^{N}\sum_{j=1}^{N}e_i^T(t)\psi_{ij}A[\Theta(x_j(t-\tau))-\Theta(s(t-\tau))]\\
&-\sum_{i=1}^{l}e_i^T(t)\tilde{d}_2sgn(e_i(t))/\|e_i(t)\|_1\\
&+\sum_{i=1}^{N}e_i^T(t)\Delta_i(t)-\sum_{i=1}^{l}e_i^T(t)d_i(t)Ae_i(t)\\
&-\sum_{i=1}^{l}e_i^T(t)d_i^*Ae_i(t)+\sum_{i=1}^{l}e_i^T(t)d_i(t)\bar{a}e_i(t)\Big\}\\
\le{}& 2\Big\{\sum_{i=1}^{N}e_i^T(t)F(e_i(t))+\sum_{i=1}^{N}e_i^T(t)\Delta_i(t)\\
&-l\tilde{d}_2-\sum_{i=1}^{l}e_i^T(t)d_i^*Ae_i(t)\\
&+c_1\sum_{i=1}^{N}\sum_{j=1}^{N}e_i^T(t)\phi_{ij}A[\Pi(x_j(x))-\Pi(s(t))]\\
&+c_2\sum_{i=1}^{N}\sum_{j=1}^{N}e_i^T(t)\psi_{ij}A[\Theta(x_j(t-\tau))-\Theta(s(t-\tau))]\Big\}
\end{aligned}
\tag{20}
$$

According to Assumption 1, we have

$$
\begin{aligned}
2\Big\{&\sum_{i=1}^{N}e_i^T(t)F(e_i(t))+\sum_{i=1}^{N}e_i^T(t)\Delta_i(t)\\
&-l\tilde{d}_2-\sum_{i=1}^{l}e_i^T(t)d_i^*Ae_i(t)\Big\}\\
\le{}& 2L\sum_{i=1}^{N}e_i^T(t)e_i(t)+\sum_{i=1}^{N}e_i^T(t)e_i(t)+\sum_{i=1}^{N}\|\Delta_i(t)\|^2\\
&-\sum_{i=1}^{N}w_i^2-2\sum_{i=1}^{l}e_i^T(t)d_i^*Ae_i(t)\\
\le{}& \sum_{k=1}^{n}e^k(t)^T\big((2L+1)I_N-a_kD_2\big)e^k(t).
\end{aligned}
\tag{21}
$$

Similarly, we have(by Theorem 1)

$$
\begin{aligned}
D_0^qV(t)\le{}& \sum_{k=1}^{n}e^k(t)^T\Big\{(2L+1)I_N+a_k\big[\beta(2c_1\Phi^s\\
&+\frac{c_2}{\rho}\|\Psi\|_\infty I_N)+\frac{1}{\vartheta}(c_1\Phi\Phi^T+c_2\Psi\Psi^T)\\
&+2\big(c_1\epsilon^2\hat{\vartheta}(1-\frac{1}{N})I_N-D_2\big)\big]\Big\}e^k(t)\\
&+\sum_{k=1}^{n}c_2a_k\Big(\rho\beta\|\Psi\|_1+2\epsilon^2\hat{\vartheta}(1-\frac{1}{N})\Big)\\
&e^k(t-\tau)^Te^k(t-\tau).
\end{aligned}
\tag{22}
$$

Let $\Omega_2=\beta(2c_1\Phi^s+\frac{c_2}{\rho}\|\Psi\|_\infty I_N)+\frac{1}{\vartheta}(c_1\Phi\Phi^T+c_2\Psi\Psi^T)+2\big(c_1\epsilon^2\hat{\vartheta}(1-\frac{1}{N})I_N-D_2\big)$, obviously, Ω_2 is a symmetric matrix. If $(2L+1)I_N+\bar{a}\Omega_2+c_2\hat{a}\Big(\rho\beta\|\Psi\|_1+2\epsilon^2\hat{\vartheta}(1-\frac{1}{N})\Big)I_N<0$, it means $\lambda_{max}(\Omega_2)<0$. Apply it into (22), one has

$$
\begin{aligned}
D_0^qV(t)\le{}& \Big(2L+1+\bar{a}\lambda_{max}(\Omega_2)\Big)V(t)\\
&+c_2\hat{a}\Big(\rho\beta\|\Psi\|_1+2\epsilon^2\hat{\vartheta}(1-\frac{1}{N})\Big)V(t-\tau).
\end{aligned}
\tag{23}
$$

From Lemma 2, when $(2L+1)I_N+\bar{a}\Omega_2+c_2\hat{a}\big(\rho\beta\|\Psi\|_1+2\epsilon^2\hat{\vartheta}(1-\frac{1}{N})\big)I_N<0$, the global synchronization of network (1) is achieved under the adaptive pinning controllers (17).

Corollary 2: Under the condition of Theorem 2, the lower bound of $d_i^*(i=1,2,\ldots l)$ can be estimated by the following inequality

$$
\bar{d}>\frac{1}{2\bar{a}}\lambda_{max}(M_1-M_2Q_l^{-1}M_2^T)
\tag{24}
$$

where $\bar{d}=min\{d_1^*,d_2^*,\ldots,d_l^*\}$, $Q=(2L+1)I_N+\bar{a}\Big(\beta(2c_1\Phi^s+\frac{c_2}{\rho}\|\Psi\|_\infty)+\frac{1}{\vartheta}(c_1\Phi\Phi^T+c_2\Psi\Psi^T)+2c_1\epsilon^2\hat{\vartheta}(1-\frac{1}{N})I_N\Big)+c_2\hat{a}\big(\rho\beta\|\Psi\|_1+2\epsilon^2\vartheta(1-\frac{1}{N})\big)I_N=\begin{bmatrix}M_1 & M_2\\ M_2^T & Q_l\end{bmatrix}$, and $Q_l\in R^{(N-l)\times(N-l)}$ is the minor matrix of Q by removing its first $l(1\le l\le N)$ row-column pairs.

Proof: Let $Q-2\bar{a}D_2=\begin{bmatrix}M_1-2\bar{a}D_2^* & M_2\\ M_2^T & Q_l\end{bmatrix}$, $D_2^*=diag\{d_1^*,d_2^*,\ldots,d_l^*\}$ and $D_2=diag\{d_1^*,d_2^*,\ldots,d_l^*,0\ldots,0\}$. Using the linear matrix inequality(Lemma 5), it is easy to get that $Q-2\bar{a}D_2<0$ is equivalent to $Q_l<0$ and $M_1-2\bar{a}D_2^*-M_2Q_l^{-1}M_2^T<0$. Consequently, $\bar{d}>\frac{1}{2\bar{a}}\lambda_{max}\{M_1-M_2Q_l^{-1}M_2^T\}$, which yields an estimated lower bound of d_i^*, $i=1,2,\ldots,l$.

Remark 7: Adaptive control [41], fuzzy control [43], sliding control [44] and feedback control [48] have been used to overcome the effects of disturbances for network synchronization. However, in this paper, we adaptively control only a small subset of the nodes to realize the global synchronization of general fractional-order networks with external disturbances. Moreover, our adaptive law contains a free parameter q, which is more practical than the integer-order adaptive controller. Different from the controllers in [30]–[33], the second term of the controller($\tilde{d}_2sgn(e_i(t))/\|e_i(t)\|_1$, $1\le i\le l<N$) uses the sign function and 1-norm of the synchronization error. The controller (4) is a fractional-order adaptive controller and the controller (17) is a fractional-order adaptive pinning controller.

Remark 8: Generally speaking, to reduce control cost, all the pinned nodes should satisfy a condition that is in-degree is smaller than their out-degree. In fact, the network nodes can be rearranged in descending order based on the out-degree and in-degree and the first l nodes can be selected as pinned candidates. The proof of Corollary 2 gives us a scheme to determine the least number l_0 of pinned nodes, which satisfies $\lambda_{max}(Q_{l_0})<0$ and $\lambda_{max}(Q_{l_0-1})\ge 0$ to reach network synchronization.

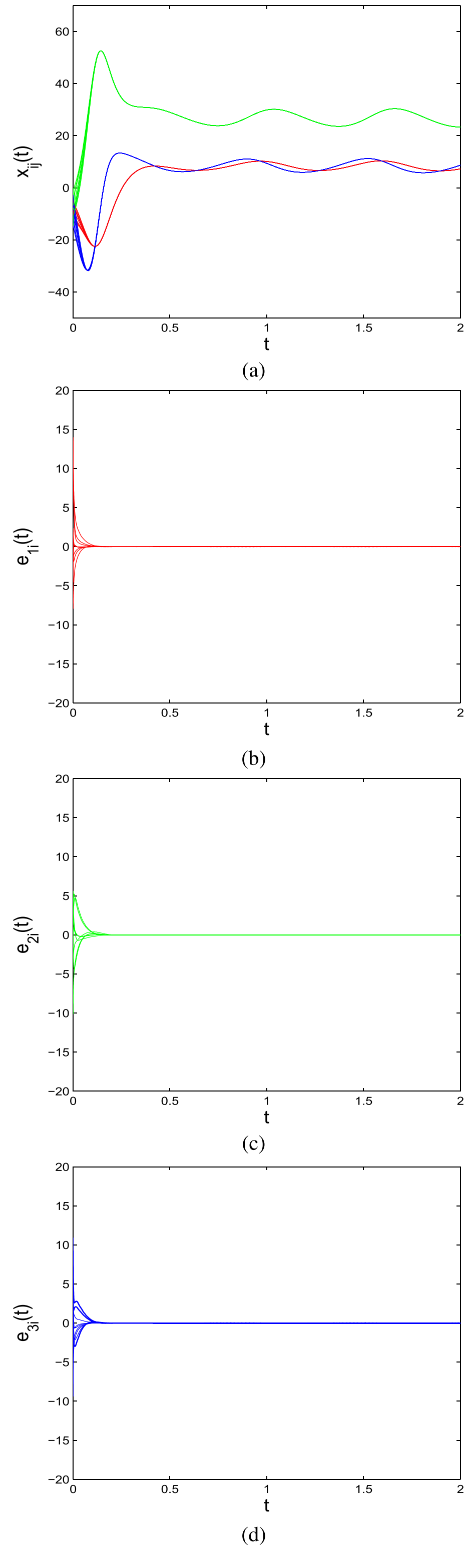

FIGURE 1. **(a) The behavior of $x(t)$ for the controlled network; (b)(c)(d) The time-evolution of local synchronization errors $e_{1i}(t)$, $e_{2i}(t)$ and $e_{3i}(t)$ for $i = 1, 2, \ldots, 9$.**

Remark 9: Obviously, $\beta, \epsilon, \vartheta$ and L can be calculated when the node dynamic f, the coupling matrices Φ, Ψ, A, the coupling functions Π, Θ and c_1, c_2 are known. Based on Corollary 2 and [8], we need to select only the pinned nodes and d_i^* to satisfy condition (19) of Theorem 2. Similarly, we can select suitable parameters for Theorem 1.

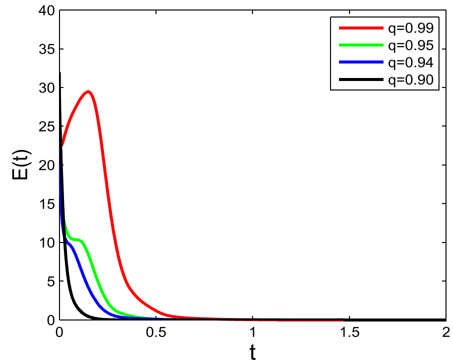

FIGURE 2. **The time-evolution of total error under different q.**

IV. NUMERICAL SIMULATIONS

This section presents two examples to verify Theorems 1 and 2, respectively.

Example 1: Local Synchronization via adaptive control.

Consider a fractional-order nonlinear dynamical network consisting of 9 nodes, which is described by

$$D_0^q x_i(t) = f(x_i(t)) + c_1 \sum_{j=1}^{9} \phi_{ij} A \Pi(x_j(t)) + c_2 \sum_{j=1}^{9} \psi_{ij} A \Theta(x_j(t-\tau)) + u_i(t) \qquad (25)$$

where $i = 1, 2, \ldots, 9$ and $f(x_i(t)) = [a(x_{2i} - x_{1i}), (c - a)x_{1i} - x_{1i}x_{3i} + cx_{2i}, x_{1i}x_{2i} - bx_{3i}]^T$. Then, $D_0^q x_i(t) = f(x_i(t))$ is actually a fractional-order chaotic Chen system if we set $q = 0.9, a = 35, b = 3$ and $c = 28$.

For simplicity, the inner-linking matrix A is taken as a diagonal matrix, i.e., $A = diag\{5, 5, 5\}$, and the coupling configuration matrix Φ is

$$\Phi = \begin{bmatrix} -3 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -3 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -3 \end{bmatrix}.$$

Let $\Psi = 0.5\Phi$, $c_1 = 0.5, c_2 = 1$, $\bar{\vartheta} = \hat{\vartheta} = 5$, $\rho = 1, \tau = 0.2, \pi_1(x) = \pi_2(x) = 2x + 0.5sinx, \pi_3(x) = 2x + 0.5cosx, \theta_1(x) = \theta_2(x) = 2x + 0.5sinx, \theta_3(x) = 2x + 0.5cosx$ and $\Delta_i(t) = (0.2sintcost, 0.3cost, 0.5sint)^T$. The initial values are $x_{0i} = ((15rand - 15), (15rand - 15), (15rand - 15))^T$. The parameters of the controllers (4) are set to $d_i^* = 20, b_i = 1$ and $\tilde{d}_1 = 0.5$. Note that the Chen system has bounded trajectories (a detailed analysis can be found in [15]). From calculations,

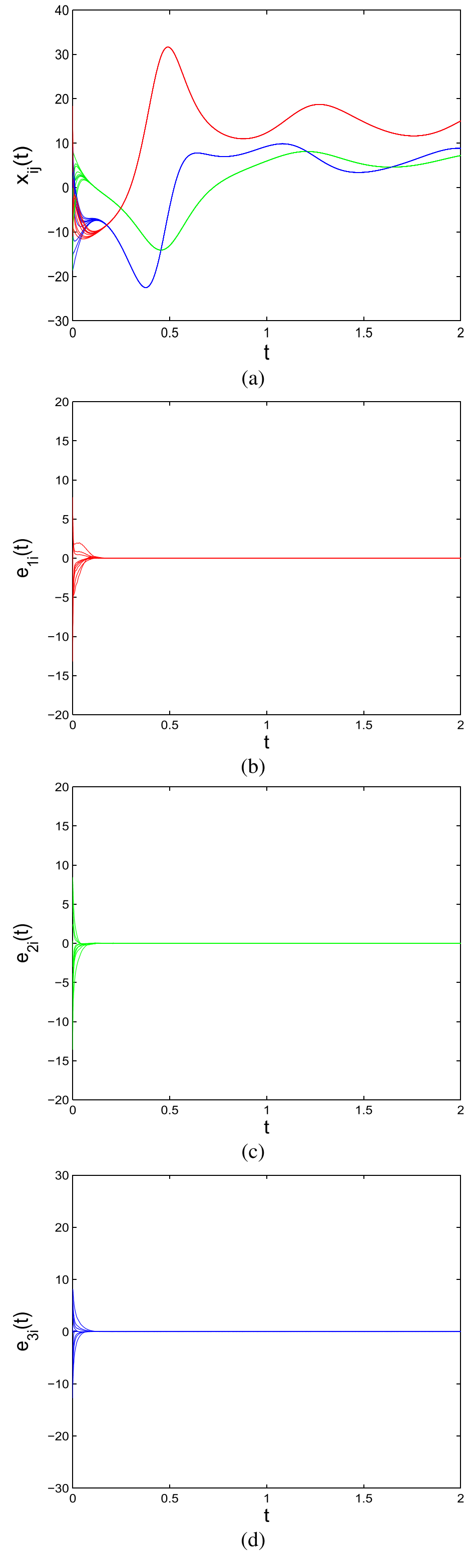

FIGURE 3. **(a) The behavior of $x(t)$ for the controlled network; (b)(c)(d) The time-evolution of global synchronization errors $e_{1i}(t)$, $e_{2i}(t)$ and $e_{3i}(t)$ for $i = 1, 2, \ldots, 9$.**

the eigenvalues of matrix Ω_1 are -38.8593, -38.0000, -37.6099, -37.2982, -36.7173, -36.3547, -35.8695, -34.7722 and -32.5190. Moreover, $2\eta + 1 + \bar{a}\lambda_{max}(\Omega_1) =$

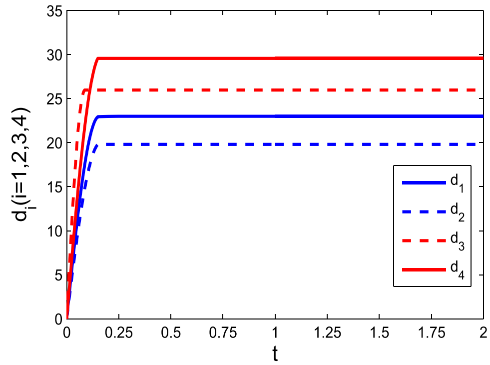

FIGURE 4. **The control input responses $d_i(t)(i = 1, 2, 3, 4)$ in Example 2.**

$-47.5950, -c_2\hat{a}\big(\rho\beta\|\Psi\|_1 + 2\epsilon^2\hat{\vartheta}(1-\frac{1}{N})\big) = -41.1111$. It is easy to verify that the inequality (6) of Theorem 1 holds under these parameters. The time evolution of the states trajectories and the synchronization errors under adaptive control are shown in Figure 1. Clearly, local synchronization is achieved. Define total synchronization error $E(t)$ for the complex networks as $E(t) = \sqrt{\frac{1}{9}\sum_{i=1}^{9}\|x_i(t) - s(t)\|^2}$. Fig 2 shows that q can effect synchronization process through impacting node dynamics.

Example 2: Global synchronization via adaptive pinning control.

We also consider a nonlinear fractional-order complex network with 9 nodes. The fractional-order Chua's chaotic circuit with system parameters can be given as follows:

$$D_0^q x_i(t) = f(x_i(t)) = B_1 x_i(t) + g(x_i(t)) \tag{26}$$

where

$$B_1 = \begin{bmatrix} -a_{11} & a_{11} & 0 \\ 1 & -1 & 1 \\ 0 & -b_{11} & 0 \end{bmatrix}, g(x_i(t)) = \begin{bmatrix} -a_{11}g(x_{1i}) \\ 0 \\ 0 \end{bmatrix}$$

$q = 0.99$, $a_{11} = 10$, $b_{11} = 14.87$, $g(x_{1i}) = m_2 x_{1i} + 0.5(m_1 - m_2)(|x_{1i} + 1| - |x_{1i} - 1|)$, $m_1 = -1.27$ and $m_2 = -0.68$. By calculation, one can set $L = 36$ such that Assumption 2 holds. Here, the inner-linking matrix is $A = diag\{10, 10, 10\}$ and the coupling configuration matrix Φ is chosen as

$$\Phi = \begin{bmatrix} -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

Let $\Psi = 0.5\Phi$, $c_1 = 10$, $c_2 = 0.5$, $\bar{\vartheta} = \hat{\vartheta} = 6$, $\rho = 4$, $\tau = 0.1$, $x_{0i} = ((30rand - 20), (30rand - 20), (30rand - 20))^T$, $\pi_1(x) = \pi_2(x) = 2x + 0.2cosx$, $\pi_3(x) = 2x + 0.2sinx$, $\theta_1(x) = \theta_2(x) = 2x + 0.2cosx$, $\theta_3(x) = 2x + 0.2sinx$ and $\Delta_i(t) =$

$(0.3sintcost, 0.5sint, 0.2cost)^T$. Using the adaptive pinning controllers described in (17), the nodes 1,2,3,4 are selected as the pinned candidates and the corresponding parameters are set to $d_i^* = 40(i = 1, 2, 3, 4)$, $d_i^* = 0(i = 5, 6, 7, 8, 9)$, $b_i = 1, \tilde{d}_2 = 9/8$. By performing simple calculations, the eigenvalues of matrix Ω_2 are -179.6161, -161.7836, -147.9985, -119.6539, -77.2655, -44.1281, -30.9913, -26.3891, and -18.3989. One can easily verify that these parameters conform to the requirement of Theorem 2. Figure 3 shows that global synchronization is achieved under the adaptive pinning controllers. Fig 4 gives control input responses $d_i(t)(i = 1, 2, 3, 4)$ of global synchronization.

V. CONCLUSION

This paper studied the synchronization problem for fractional order nonlinearly coupled networks with non-delayed and delayed couplings as well as unknown disturbances. Based on fractional-order stability theory, we converted the network synchronization problem into one involving the stability of its error system. By using adaptive control and adaptive pinning control schemes, some sufficient local and global asymptotic synchronization conditions were derived. Note that the external time-varying disturbances of fractional-order systems had an impact on each node; however, existing feedback control methods often controlled all the nodes to achieve global synchronization of such systems. In contrast, this paper adaptively controlled only a small fraction of the nodes to realize this goal. Finally, two numerical examples were presented to illustrate the theoretical results.

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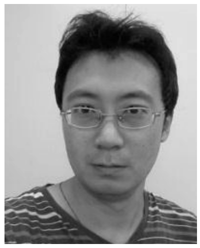

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