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Nonlinear Modal Decoupling of Multi-Oscillator Systems With Applications to Power Systems

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ABSTRACT Many natural and manmade dynamical systems that are modeled as large nonlinear multi-oscillator systems like power systems are hard to analyze. For such systems, we propose a nonlinear modal decoupling (NMD) approach inversely constructing as many decoupled nonlinear oscillators as the system's oscillation modes so that individual decoupled oscillators can be easily analyzed to infer dynamics and stability of the original system. The NMD follows a similar idea to the normal form except that we eliminate inter-modal terms but allow intra-modal terms of desired nonlinearities in decoupled systems, so decoupled systems can flexibly be shaped into desired forms of nonlinear oscillators. The NMD is then applied to power systems toward two types of nonlinear oscillators, i.e. the single-machine-infinite-bus (SMIB) systems and a proposed non-SMIB oscillator. Numerical studies on a 3-machine 9-bus system and New England 10-machine 39-bus system show that: 1) decoupled oscillators keep a majority of the original system's modal nonlinearities and the NMD can provide a bigger validity region than the normal form and 2) decoupled non-SMIB oscillators may keep more authentic dynamics of the original system than decoupled SMIB systems.

INDEX TERMS Nonlinear modal decoupling (NMD), inter-modal terms, intra-modal terms, oscillator systems, normal form, power systems, nonlinear dynamics.

NOMENCLATURE

A. TERMINOLOGIES

Decoupled k-jet system	Definition 5
Desired modal nonlinearity	Definition 1
Intra- and inter-modal terms	Definition 2
Inverse coordinate transformation	Corollary 3
k-jet equivalence	Definition 4
k-th order NMD	Corollary 1
Mode-decoupled system	Definition 3
Multi-oscillator system	Eq. (1)
μ-coefficients	Definition 1
Nonlinear modal decoupling (NMD)	Theorem 1
Real-valued decoupled k-jet	Theorem 2
SMIB assumption	Section IV-B
Small transfer (ST) assumption	Section IV-C
Validity region of decoupled k-jet system	Eq. (31)
Validity region of NMD transformation	Corollary 2

B. NOTATIONS

$\mathbf{C}_j(\mathbf{z}^{(p)})$	Vector function of $\mathbf{z}^{(p)}$ which only contains inter-modal homogeneous terms of degree j about $\mathbf{z}^{(p)}$
$\mathbf{D}_j(\mathbf{z}^{(p)})$	Vector function of $\mathbf{z}^{(p)}$ which only contains intra-modal homogeneous terms of degree j about $\mathbf{z}^{(p)}$
$e(t,\mathbf{x}_0)$	Response error of the decoupled k-jet system under initial condition $\mathbf{x}_0$ as defined in Eq. (28)
$\mathbf{f}(\mathbf{x})$	Vector field of the system in Eq. (1)
$\mathbf{H}^{(k)}$	k-th order decoupling transformation defined in Eq. (19), which is a composite of $\mathbf{H}_1, \ldots, \mathbf{H}_k$
$\mathbf{H}_{p+1}$	$(p+1)$-th decoupling transformation defined in Eq. (4), which is involved in NMD
$\mathbf{h}_{p+1}$	Nonlinear part of $\mathbf{H}_{p+1}$ defined in Eq. (7)
$h_{p+1,*}$	Coefficients of monomials in $\mathbf{h}_{p+1}$
Λ	Diagonal matrix with eigenvalues of $\mathbf{A}$ on the main diagonal and zeros elsewhere
μ	Desired modal nonlinearity used in Eq. (2)
Ω_ε	Validity region of the decoupled k-jet system for a given error tolerance ε as defined in Eq. (31)
$\Omega^{(k)}$	Validity region of $\mathbf{H}^{(k)}$ defined in Eq. (20)
Ω_{p+1}	Validity region of $\mathbf{H}_{p+1}$, which is a set of $\mathbf{x}$
$\mathbf{U}$	Modal matrix of $\mathbf{A}$, whose columns are right eigenvectors of $\mathbf{A}$

V_i	Energy function of the real-valued decoupled k-jet system about mode i defined in Eq. (55)
w_i	i-th element in the state vector of the real-valued decoupled k-jet system defined in Eq. (24)-(27)
$\mathbf{x}$	N-dimensional state vector of the system in (1)
$\mathbf{z}$	Complex-valued state vector (element: z_i) of the system with desired modal nonlinearity
$\mathbf{z}^{(k)}$	Complex-valued state vector (element: $z_i^{(k)}$) after the k-th decoupling transformation in Eq. (4)
$\mathbf{z}_{\text{jet}}^{(k)}$	Complex-valued state vector (element: $z_{\text{jet},i}^{(k)}$) in the decoupled k-jet system defined in Eq. (23)

I. INTRODUCTION

Oscillator systems, i.e. a system with a number of oscillators interacting with each other, are ubiquitous in both natural systems and manmade systems. In biological systems, low-frequency oscillations in metabolic processes can be observed at intracellular, tissue and entire organism levels and they have a deterministic nonlinear causality [1]. In electric power grids, which are among the largest manmade physical networks, oscillations are continuously presented during both normal operating conditions and disturbed conditions [2]. In some fields of both natural science and social science, the Kuramoto model is built based upon a large set of coupled oscillators modeling periodic, self-oscillating phenomena in, e.g., reaction-diffusion systems in ecology [3] and opinion formation in sociphysics [4]. For all these oscillator systems, the common underlying mathematical model is actually a set of interactive governing differential equations, linear or nonlinear.

An ideal way to study dynamics of a multi-oscillator system from an initial state is to find an analytical solution of its differential equation models and use the solution for further prediction and control. However, even finding an approximate solution of a high-dimensional nonlinear multi-oscillator system has been a challenge for a long time to mathematicians, physicists and engineers [5]. Analytical efforts have been made in broader topics, like dynamical systems [6], [7], nonlinear oscillations [8] and complex networks [9], to better understand, predict and even control the oscillator systems, and some well-known theories are such as the perturbation theory and Kolmogorov-Arnold-Moser theory. Most of these efforts attempt to directly analyze an oscillator system as a whole and extract desired information, e.g. approximate solutions and stability criteria, from the governing differential equations. Especially, extensive attentions recently have been paid to using the theory of synchronization to analyze the interactions among oscillators in a system [10]–[15]. In addition, numerical studies can provide dynamical behaviors of high-dimensional oscillation systems with desired accuracy. However, simulating a high-dimensional oscillator system like a power grid could be very slow if oscillators are coupled through a complex network and interact nonlinearly [16].

In this paper, we aim at *inversely constructing* a set of decoupled, independent oscillators for a given complex high-dimensional multi-oscillator system. Each of those decoupled oscillators is a fictitious 2^{nd}-order nonlinear system that is virtually islanded from the others and corresponds to a single oscillation mode of the original system. By such a transformation, analysis on the dynamics and stability related to each mode can be performed on the corresponding oscillator, which will be easier than on the original complex system. For some real-life oscillator networks such as a power grid networking synchronous generators, those real oscillators themselves often have strong couplings and interactions in dynamics. However, the modal dynamics with respect to different oscillation modes may have relatively weak couplings or interferences unless significant resonances happen between oscillation modes. Thus, the fictitious oscillators that are inversely constructed to represent different oscillation modes are independent, or in other words naturally decoupled, to some extent, and hence can be more easily understood and analyzed to gain insights on the dynamical behaviors, stability and control of the whole original system. In this paper, we define such a process as *nonlinear modal decoupling* (NMD), i.e. the inverse construction from the original nonlinear multi-oscillator system to a set of decoupled fictitious nonlinear oscillators.

Finding the modal decoupling transformation even for general linear dynamical systems has been studied for more than two hundred years, and massive papers aimed at decoupling linear dynamical systems with non-uniform damping. In the 1960s, Caughey and O'Kelly [17] found a necessary and sufficient conditions for a class of damped second-order linear differential equations to be transformed into decoupled linear differential equations based on early mathematical works by Whittaker [18] in the 1850s. In just the last decade, the decoupling of general linear dynamical systems with non-uniform damping was achieved [19]–[21].

For nonlinear oscillator systems, the modal decoupling has not been studied well despite its importance in simplification of stability analysis and control on such complex systems. Some related efforts have focused on the linear/nonlinear transformation of a given nonlinear oscillator system towards an equivalent linear system. One approach is the feedback linearization that introduces additional controllers to decouple the relationship between outputs and inputs in order to control one or some specific outputs of an oscillator system [22]–[25]. Another approach is the normal form [26]–[30] that applies a series of coordinate transformations to eliminate nonlinear terms starting from the 2^{nd}-order until the simplest possible form. If regardless of resonances, such a simplest form is usually taken as a linear oscillator system, whose explicit solution together with the involved series of transformations are then used to study the behavior of the original nonlinear oscillator system. To summarize, many efforts in the present literature on analysis of high dimensional nonlinear oscillator systems tend to generate an approximate or equivalent linear system of the original system by means of linear/nonlinear transformations so as to utilize available linear analysis methods. Based on these efforts, it is quite intuitive to move one step forward to achieve a set of decoupled nonlinear oscillator systems which are each simple enough for analyzing dynamics and stability. That is the objective of this work.

For real-life nonlinear oscillator systems such as a multi-machine power system, a linear decoupling transformation may map the system into its modal space to help improve the modal estimation [31] and assess the transient stability of the system [32]. The normal form method was introduced to power systems in [33] for analyzing stressed power systems and enables the design of controllers considering partial nonlinearities of the systems. Since nonlinearities are considered, like the 2$^{\text{nd}}$-order nonlinearity in [34] and the 3$^{\text{rd}}$-order nonlinearity in [35], the approximated solution from normal form may have a larger validity region than that of the linearized system [36]. Among these attempts, a first attempt of NMD is reported in [32], which does not provide the nonlinear transformation from the original oscillator system to nonlinear modal decoupled systems.

In this paper, the proposed NMD approach is derived adopting an idea similar to the Poincaré normal form in generating a set of nonlinear homogeneous polynomial transformations [37]. However, unlike the classic theory of normal forms, **the proposed NMD approach only eliminates the inter-modal terms and allows decoupled systems to have intra-modal terms of desired nonlinearities**. The rest of the paper is organized as follows: in Section III, the definitions, derivations and error estimation on the NMD are presented. In Section IV, the NMD approach is applied to multi-machine power systems with a sample application in first-integral based stability analysis. Section V shows numerical studies on the IEEE 3-machine 9-bus power system and IEEE 10-machine 39-bus power system. Conclusions are drawn in Section VI.

II. NONLINEAR MODAL DECOUPLING

We will first introduce several definitions before presenting Theorem 1 on NMD.

Given a multi-oscillator system described by a set of ordinary differential equations below:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{1}$$

where $\mathbf{x}$ is the vector containing N state variables, $\mathbf{f}$ is a smooth vector field and all eigenvalues of the Jacobian matrix of $\mathbf{f}(\mathbf{x})$, say $\mathbf{A}$, appear as conjugate pairs of complex numbers. Also assume that the system in (1) has a locally stable equilibrium point and the equilibrium is at the origin (if not, it can be easily moved to the origin by a coordinate transformation). Each conjugate pair of $\mathbf{A}$'s eigenvalues define a unique *mode* of the system. Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ represent the matrix of $\mathbf{A}$'s eigenvalues, where N is an even number. Without loss of generosity, let λ_{2i-1} and λ_{2i} be the two conjugate eigenvalues corresponding to the mode i.

The above model is able to represent a class of dynamical systems, i.e. multi-oscillator systems whose Jacobian only has complex-valued eigenvalues. Generally speaking, (1) cannot represent a general nonlinear dynamical system whose Jacobian has real eigenvalue(s). However, most of the dynamic networked systems in the real world can be modeled as such a multi-oscillator system. For example, as mentioned in section IV-A of this paper, any m-machine power system with uniform damping, whose generators represented by a second-order model, can be described by (1) having m-1 oscillators [48].

Definition 1 (Desired Modal Nonlinearity): If the multi-oscillator system (1) can mathematically be transformed into the form (2) below and the two governing differential equations in (2) regarding mode i have μ-coefficients of desired values, then the i-th mode is said to have the desired modal nonlinearity:

$$\begin{aligned}\dot{z}_{2i-1} &= \lambda_{2i-1} z_{2i-1} + \sum_{\alpha=1}^{N}\sum_{\beta=\alpha}^{N} \mu_{2i-1,\alpha\beta} z_\alpha z_\beta + \cdots \\ &\quad + \sum_{\alpha=1}^{N}\sum_{\beta=\alpha}^{N}\cdots\sum_{\rho=\gamma}^{N} \mu_{2i-1,\alpha\beta\cdots\rho} \underbrace{z_\alpha z_\beta \cdots z_\rho}_{k\text{ terms in total}} \\ &\quad + \cdots \stackrel{\text{def}}{=} g_{2i-1}(\mathbf{z}) \\ \dot{z}_{2i} &= g_{2i}(\mathbf{z}) = \bar{g}_{2i-1}(\mathbf{z})\end{aligned} \tag{2}$$

where $\mathbf{z} = [z_1 \ldots z_N]^{\text{T}}$ is the vector of state variables, $k > 1$ and the variable or function having a bar represents its complex conjugate.

In the traditional normal form method, only the modal nonlinearities that cannot be eliminated due to resonance, to be defined in the next paragraph, are retained, which is equivalent to making as many μ-coefficients be zero as possible in (2). Regardless of the resonance, the advantage of the standard normal form is to generate a truncated dynamical system that is linear and has an analytical solution.

The n-triple $\mathbf{\Lambda} = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ of eigenvalues is said to be *resonant* if among the eigenvalues there exists an integral relation $\lambda_s = \sum_k m_k \lambda_k$, where s and $k = 1, \ldots, N$, $m_k \geq 0$ are integers and $\sum_k m_k \geq 2$. Such a relation is called a *resonance*. The number $\sum_k m_k$ is called the *order of the resonance* [40].

However, it is not always true that a linear system is the most desired. For instance, in power systems, power engineers and researchers prefer to assume that the underlying low-dimensional system dominating each nonlinear oscillatory mode satisfies the nonlinearity of a single-machine-infinite-bus (SMIB) power system [32], [38]–[42], i.e. the simplest single-degree-of-freedom power system, where the nonlinearities are represented by terms with non-zero μ-coefficients. Thus, this paper is motivated to keep specific nonlinear terms for the desired modal nonlinearity by following either the SMIB assumption, as shown in Section IV-B, or another assumption to be proposed in Section IV-C.

For the normal form method, the truncated linear system cannot be used for estimating the boundary of stability, which is however important for a nonlinear system. As a comparison, the NMD to be proposed provides a possibility to estimate the boundary of stability using the nonlinearities intentionally kept in the model, although the estimation of the stability boundary even for a truncated nonlinear system model is a long-standing problem. A sample application of NMD on the study of the stability boundary will be presented in Section IV-D.

For the convenience of statements, the following definitions are adopted to introduce which nonlinear terms should be kept or eliminated.

Definition 2 (Intra-Modal Term and Inter-Modal Term): Given the desired modal nonlinearity (2) for mode i of the multi-oscillator system (1), the *intra-modal terms* are the nonlinear terms in the form of $\mu_{j,\alpha\beta\cdots\rho}z_\alpha z_\beta\cdots z_\rho$ (for $k = 2, 3, \ldots$) which involve state variable(s) only corresponding to mode i, i.e. indices $j, \alpha, \beta, \cdots, \rho \in \{2i-1, 2i\}$. All the other nonlinear terms are called the *inter-modal terms*, which involve state variables corresponding to other modes.

Definition 3 (Mode-Decoupled System): If the form (2) with desired modal nonlinearity regarding the i-th mode also makes (3) satisfied, then (2) is called a *mode-decoupled system* w.r.t. mode i.

$$\mu_{j,\alpha\beta\cdots} = \begin{cases} \mu_{j,\text{intra},\alpha\beta\cdots} = \text{desired value} \\ \qquad\qquad \text{if } j, \alpha, \beta, \cdots, \in \{2i-1, 2i\} \\ \mu_{j,\text{inter},\alpha\beta\cdots} = 0 \quad \text{otherwise} \end{cases} \tag{3}$$

Now, we present the Theorem on NMD:

Theorem 1 (Nonlinear Modal Decoupling): Given a multi-oscillator system in (1) and a desired modal nonlinearity without inter-modal terms, if resonance does not exist for any order and all eigenvalues of the Jacobian matrix of system (1) belong to the Poincaré domain [43], then (1) can be transformed into (2) by a nonlinear transformation, denoted as $\mathbf{H}$, around some neighborhood Ω of its equilibrium.

Remark: The rest of this section will focus on giving a constructive proof of the theorem, in which we derive the transformation $\mathbf{H}$ and its inverse that can be numerically computed. Unlike the normal form, the NMD requires elimination of only inter-modal terms so as to decouple the dynamics regarding different modes while leaving room for intra-modal terms to be designed for desired characteristics with each mode-decoupled system. For simplicity, we respectively call the intra- and inter-modal term coefficients μ_{intra} and μ_{inter}. In this section, we assume the desired modal nonlinearity for each mode to be known. The NMD on a real-life high-dimensional multi-oscillator system like a power system might intentionally make each mode-decoupled system have the same modal nonlinearity as a one-degree-of-freedom, single-oscillator system of the same type, e.g. an SMIB system for power systems, for the convenience of using existing analysis methods on the same type of systems. However, for the purpose of stability analysis and control, decoupling a real-life system into a different type of oscillators might also be an option. In the next section, two ways (i.e. the same type and a different type) to choose the desired modal nonlinearity will be illustrated on power systems.

The detailed derivation of the transformation $\mathbf{H}$ used for NMD will be presented in the constructive proof of Theorem 1, where $\mathbf{H}$ will be a composition of a sequence of transformations, denoted as $\mathbf{H}_1, \mathbf{H}_2, \ldots, \mathbf{H}_k, \ldots$, which are polynomial transformations. The relationship between the state variables of the mode-decoupled system, say $\mathbf{z}$, and the state variables after the k-th transformation are shown in (4) based on $\mathbf{H}_1, \mathbf{H}_2, \ldots, \mathbf{H}_k, \ldots$, where we use $\mathbf{z}^{(k)}$ to represent the vector of state variables after the k-th transformation.

$$\begin{aligned} \mathbf{x} = \mathbf{H}(\mathbf{z}) = \cdots &= (\mathbf{H}_1 \circ \mathbf{H}_2 \circ \cdots \circ \mathbf{H}_k \circ \cdots)(\mathbf{z}) \\ \mathbf{z}^{(1)} &= (\mathbf{H}_2 \circ \cdots \circ \mathbf{H}_k \circ \cdots)(\mathbf{z}) \\ &\cdots \\ \mathbf{z}^{(k)} &= (\mathbf{H}_{k+1} \circ \cdots)(\mathbf{z}) \\ &\cdots \end{aligned} \tag{4}$$

We first introduce a lemma before presenting the proof of Theorem 1.

Lemma 1: Given one transformed form (5) of a multi-oscillator system, where $\mathbf{D}_j(\mathbf{z}^{(p)})$ only contains intra-modal terms and $\mathbf{C}_j(\mathbf{z}^{(p)})$ only contains inter-modal terms and they are vectors of polynomials of degree j about $\mathbf{z}^{(p)}$. If resonance does not exist up to the order $p+1$, then in a certain neighborhood of the origin, denoted as Ω_{p+1}, the inter-modal terms of degree $p+1$ can be completely eliminated to give (6) by a polynomial transformation of degree $p+1$ in (7), i.e. $\mathbf{H}_{p+1}$.

$$\dot{\mathbf{z}}^{(p)} = \mathbf{\Lambda}\mathbf{z}^{(p)} + \sum_{j=2}^{p}\mathbf{D}_j(\mathbf{z}^{(p)}) + \sum_{j=p+1}^{\infty}\left(\mathbf{D}_j(\mathbf{z}^{(p)}) + \mathbf{C}_j(\mathbf{z}^{(p)})\right) \tag{5}$$

$$\begin{aligned} \dot{\mathbf{z}}^{(p+1)} &= \mathbf{\Lambda}\mathbf{z}^{(p+1)} + \sum_{j=2}^{p+1}\mathbf{D'}_j(\mathbf{z}^{(p+1)}) \\ &\quad + \sum_{j=p+2}^{\infty}\left(\mathbf{D'}_j(\mathbf{z}^{(p+1)}) + \mathbf{C'}_j(\mathbf{z}^{(p+1)})\right) \end{aligned} \tag{6}$$

$$\mathbf{z}^{(p)} = \mathbf{H}_{p+1}(\mathbf{z}^{(p+1)}) = \mathbf{z}^{(p+1)} + \mathbf{h}_{p+1}(\mathbf{z}^{(p+1)}) \tag{7}$$

Proof of Lemma 1: Consider the transformation in (7), where $\mathbf{h}_{p+1}(\mathbf{z}^{(p+1)})$ is a column vector made of homogeneous polynomials of degree $p+1$ in $\mathbf{z}^{(p+1)}$. Its $(2i\text{-}1)$-th and $2i$-th elements are shown in (8).

$$\begin{cases} h_{p+1,2i-1}(\mathbf{z}^{(p+1)}) = \displaystyle\sum_{\alpha=1}^{N} \\ \qquad \cdots \displaystyle\sum_{\gamma=\eta}^{N} h_{p+1,2i-1,\alpha\cdots\eta\gamma}\underbrace{z_\alpha^{(p+1)}\cdots z_\gamma^{(p+1)}}_{p+1\text{ terms in total}} \\ h_{p+1,2i} = \bar{h}_{p+1,2i-1}(\mathbf{z}^{(p+1)}) \end{cases} \tag{8}$$

Substitute (7) and (8) into (5) and obtain (9).

$$\begin{aligned}\dot{z}_{2i-1}^{(p+1)} &= \lambda_{2i-1}z_{2i-1}^{(p+1)} + \sum_{\alpha=1}^{N}\sum_{\beta=\alpha}^{N}\mu_{2i-1,\alpha\beta}z_{\alpha}^{(p+1)}z_{\beta}^{(p+1)} + \cdots \\ &\quad + \sum_{\alpha=1}^{N}\cdots\sum_{\eta=\chi}^{N}\mu_{2i-1,\alpha\cdots\eta}\underbrace{z_{\alpha}^{(p+1)}\cdots z_{\eta}^{(p+1)}}_{p\text{ terms in total}} \\ &\quad + \sum_{\alpha=1}^{N}\cdots\sum_{\gamma=\eta}^{N}c_{p+1,2i-1,\alpha\cdots\eta\gamma}\underbrace{z_{\alpha}^{(p+1)}\cdots z_{\gamma}^{(p+1)}}_{p+1\text{ terms in total}} + \cdots \\ &\stackrel{\text{def}}{=} f_{2i-1}^{(p+1)}(\mathbf{z}^{(p+1)}) \\ \dot{z}_{2i}^{(p+1)} &= f_{2i}^{(p+1)}(\mathbf{z}^{(p+1)}) = \bar{f}_{2i-1}^{(p+1)}(\mathbf{z}^{(p+1)}) \\ & \qquad i = 1,\cdots,N/2 \end{aligned} \tag{9}$$

where

$$c_{p+1,2i-1,\alpha\cdots\gamma} = \mu_{2i-1,\alpha\cdots\gamma} + h_{p+1,2i-1,\alpha\cdots\gamma} \cdot(\lambda_{2i-1} - \lambda_{\alpha} - \cdots - \lambda_{\gamma}) \tag{10}$$

Let the coefficients of terms of degree $p+1$ in (9) satisfy (3), i.e. (11) where $j = 2i-1$ or $2i$, and then we can obtain (6).

$$\begin{cases} h_{p+1,j,\text{inter},\alpha\cdots\gamma} = \dfrac{c_{p+1,j,\text{inter},\alpha\cdots\gamma}}{\underbrace{\lambda_{\alpha}+\cdots+\lambda_{\gamma}}_{p+1\text{ terms in total}}-\lambda_j} \\ h_{p+1,j,\text{intra},\alpha\cdots\gamma} = \dfrac{c_{p+1,j,\text{intra},\alpha\cdots\gamma}-\mu_{j,\text{intra},\alpha\cdots\gamma}}{\underbrace{\lambda_{\alpha}+\cdots+\lambda_{\gamma}}_{p+1\text{ terms in total}}-\lambda_j} \end{cases} \tag{11}$$

Note that when transformation in (7) is used to transform (5) into (9), calculation of the inverse of the coefficient matrix, e.g. the left-hand side of eq. (6) in paper [44], is implicitly required. This coefficient matrix is actually a function of all state variables $\mathbf{z}^{(p+1)}$, which is a near-identity matrix when the state is close to the origin. However, if the system state is far away from the origin, that matrix may become singular such that (9) cannot be obtained any more. An upper bound of the validity limit may exist [44], indicating that (9) can be obtained only when the system state is close enough to the origin. ■

Remark: Given the transformation (7) obtained in Lemma 1, its *validity region*, denoted as Ω_{p+1}, is defined as the connected set of system states $\mathbf{x}$, where any point in the set does not lead to a singular Jacobian of (7).

Proof of Theorem 1: Given a multi-oscillator system (1), its *modal space representation* can be obtained as (12) by the transformation in (13), where $\mathbf{z}^{(1)}$ is the vector of state variables in the modal space and $\mathbf{U}$ is the matrix consisting of the right eigenvectors of $\mathbf{A}$.

$$\dot{\mathbf{z}}^{(1)} = \mathbf{U}^{-1}\mathbf{f}(\mathbf{U}\mathbf{z}^{(1)}) \tag{12}$$

$$\mathbf{x} = \mathbf{U}\mathbf{z}^{(1)} \stackrel{\text{def}}{=} \mathbf{H}_1(\mathbf{z}^{(1)}) \tag{13}$$

Taylor expansion of (12) can be written as

$$\dot{\mathbf{z}}^{(1)} = \mathbf{\Lambda}\cdot\mathbf{z}^{(1)} + \sum_{j=2}^{\infty}\left(\mathbf{D}_j^{<1>}(\mathbf{z}^{(1)}) + \mathbf{C}_j^{<1>}(\mathbf{z}^{(1)})\right) \tag{14}$$

Apply Lemma 1 with $p = 1$, then we can transform (14) into

$$\dot{\mathbf{z}}^{(2)} = \mathbf{\Lambda}\cdot\mathbf{z}^{(2)} + \mathbf{D}_2^{<2>}(\mathbf{z}^{(2)}) + \sum_{j=3}^{\infty}\left(\mathbf{D}_j^{<2>}(\mathbf{z}^{(2)}) + \mathbf{C}_j^{<2>}(\mathbf{z}^{(2)})\right) \tag{15}$$

Apply Lemma 1 for k-2 times respectively with $p = 2, \ldots, k$-1, then we can transform (15) into (16).

$$\begin{aligned}\dot{\mathbf{z}}^{(k)} &= \mathbf{\Lambda}\cdot\mathbf{z}^{(k)} + \sum_{j=2}^{k}\mathbf{D}_j^{<j>}(\mathbf{z}^{(k)}) \\ &\quad + \sum_{j=k+1}^{\infty}\left(\mathbf{D}_j^{<k>}(\mathbf{z}^{(k)}) + \mathbf{C}_j^{<k>}(\mathbf{z}^{(k)})\right)\end{aligned} \tag{16}$$

Since all eigenvalues of the Jacobian matrix of system (1) belong to the Poincaré domain, then Theorem 1.1 in [43] guarantees the convergences of such process when the order k approaches infinity. Finally, (2) will be achieved, i.e. $\mathbf{z} = \mathbf{z}^{(\infty)}$ and the transformation $\mathbf{H}$ in (4) is composed by $\mathbf{H}_1$ in (13) and $\mathbf{H}_{p+1}$ in (7) with $p = 1, 2, \ldots, \infty$. Since transformation $\mathbf{H}_{p+1}$ is valid when the system state $\mathbf{x}$ belongs to Ω_{p+1}, then the transformation $\mathbf{H}$ and transformed system in (2) are valid when $\mathbf{x}$ belongs to $\Omega = \Omega_2 \cap \Omega_3 \cap \ldots$. ■

Note that in principle the validity region Ω of the NMD transformation is the intersection of an infinite number of open sets and actually converges to a single point, i.e. the equilibrium point. In practice, it is hard to deal with an infinite number of transformations. Still, for any expected order k, we can use the system truncated at that order as an approximation for practical applications. The following gives three corollaries of the NMD for any expected order k with the help of the k-jet concept. Then, the decoupled k-jet system is introduced.

Definition 4 (k-Jet Equivalence [26]): Assume $\mathbf{f}_1(\mathbf{x})$ and $\mathbf{f}_2(\mathbf{x})$ are two vector functions of the same dimension. We say that $\mathbf{f}_1(\mathbf{x})$ and $\mathbf{f}_2(\mathbf{x})$ are k-jet equivalent at $\mathbf{x}_0$, or $\mathbf{f}_1(\mathbf{x})$ is a k-jet equivalence of $\mathbf{f}_2(\mathbf{x})$ and vice versa, *iff* corresponding monomials in the Taylor expansions of $\mathbf{f}_1(\mathbf{x})$ and $\mathbf{f}_2(\mathbf{x})$ at $\mathbf{x}_0$ are identical up to order k.

Then, a k-jet system of (1) can be rewritten in (17), where $\mathbf{A}_j(\mathbf{x})$ is a vector function whose elements are weighted sums of homogeneous polynomials of degree j about elements of $\mathbf{x}$. The differences between the systems (1) and (17) are only truncated terms of orders greater than k.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \sum_{j=2}^{k}\mathbf{A}_j(\mathbf{x}) \tag{17}$$

In the following, we will continue NMD with (17).

Corollary 1 (k-th Order NMD): Given a multi-oscillator system in (17), if the resonance does not exist up to the given order k, then the k-th *order nonlinearly mode-decoupled system* can be achieved as (18) by the k-th *order decoupling transformation* $\mathbf{H}^{(k)}$ in (19).

$$\dot{\mathbf{z}}^{(k)} = \boldsymbol{\Lambda} \cdot \mathbf{z}^{(k)} + \sum_{j=2}^{k} \mathbf{D}_j(\mathbf{z}^{(k)}) + \sum_{j=k+1}^{\infty} \left(\mathbf{D}_j(\mathbf{z}^{(k)}) + \mathbf{C}_j(\mathbf{z}^{(k)})\right) \tag{18}$$

$$\mathbf{x} = \mathbf{H}^{(k)}(\mathbf{z}^{(k)}) = (\mathbf{H}_1 \circ \mathbf{H}_2 \circ \cdots \circ \mathbf{H}_k)(\mathbf{z}^{(k)}) \tag{19}$$

where $\mathbf{z}^{(k)}$ is the vector of state variables in the k-th order mode-decoupled system, $\mathbf{D}_j$ and $\mathbf{C}_j$ are vector functions whose elements are weighted sums of the terms of degree j about $\mathbf{z}^{(k)}$. $\mathbf{D}_j$ only contains intra-modal terms while $\mathbf{C}_j$ only has inter-modal terms.

Corollary 2: The *validity region* of the transformation $\mathbf{H}^{(k)}$, denoted as $\Omega^{(k)}$, is

$$\Omega^{(k)} = \bigcap_{p=2}^{k} \Omega_p \tag{20}$$

Corollary 3: The inverse coordinate transformation, i.e. the inverse of the transformation $\mathbf{H}_{p+1}$ in (7), can be approximated by a power series

$$z_i^{(p+1)} = z_i^{(p)} + \sum_{\alpha=1}^{N}\sum_{\beta=\alpha}^{N} s_{i,\alpha\beta} z_\alpha^{(p)} z_\beta^{(p)} + \cdots + \sum_{\alpha=1}^{N}\sum_{\beta=\alpha}^{N}\cdots\sum_{\gamma=\eta}^{N} s_{i,\alpha\beta\cdots\gamma} z_\alpha^{(p)} z_\beta^{(p)} \cdots z_\gamma^{(p)} + \cdots \tag{21}$$

Remark: Existing literature has not reported any explicit form for such inverse transformation, but proposed to numerically transform the states from $\mathbf{z}^{(p)}$ to $\mathbf{z}^{(p+1)}$ by some iterative approach with an initial guess. It was reported that the effectiveness of such numerical approach largely depends on the quality of the initial guess, which may lead to either divergence or convergence to a different $\mathbf{z}^{(p)}$ [45], [46]. Actually, the inverse of (7) can be written in the form of (22). An approximate analytical expression of $\mathbf{h}_{p+1}^{-1}$ in (22) is provided by Corollary 3. The proof is omitted while the idea is quite straightforward: first, assume that the inverse transformation (22) has a polynomial form, as shown in (21) on the i-th equation of (22) where s-coefficients are unknown; second, substitute (21) into the right side of (7) and equate both sides term by term about $\mathbf{z}^{(p)}$ to formulate equations about s-coefficients; finally, solve these s-coefficients and substitute them back to (21) to obtain the inverse transformation. Note that those formulated equations on s–coefficients are linear and can always be solved recursively due to the merits that vector function $\mathbf{h}_{p+1}$ in (7) only contains homogeneous polynomials of degree $p+1$ in $\mathbf{z}^{(p+1)}$.

$$\mathbf{z}^{(p+1)} = \mathbf{z}^{(p)} + \mathbf{h}_{p+1}^{-1}(\mathbf{z}^{(p)}) \tag{22}$$

Definition 5 (Decoupled k-Jet System): By ignoring terms with orders higher than k in (18), we obtain a special k-jet system of (18), named a *decoupled k-jet system*.

$$\dot{\mathbf{z}}_{\text{jet}}^{(k)} = \boldsymbol{\Lambda} \cdot \mathbf{z}_{\text{jet}}^{(k)} + \sum_{j=2}^{k} \mathbf{D}_j(\mathbf{z}_{\text{jet}}^{(k)}) \tag{23}$$

Remark: Theoretically speaking, the nonlinearities maintained with the decoupled k-jet system (23) are defined by intra-modal terms, i.e. $\mathbf{D}_j$, and can have any desired form, according to which a k-th order nonlinear transformation $\mathbf{H}^{(k)}$ is determined. If we do not let a priori knowledge about nonlinearities with the original system (17) limit the form of (23), there could be an infinite number of ways to design its intra-modal terms so as to generate many decoupled k-jet systems differing in terms and hence the sizes and shapes of their validity regions. Also note that the equations in (23) about one mode are completely independent of the equations of any other mode, while polynomial nonlinearities up to order k regarding each individual mode are still maintained. Next, two theorems about the decoupled k-jet system are introduced.

Theorem 2 (Real-Valued Decoupled k-Jet): The decoupled k-jet in (23) is equivalent to a real-valued system, called a real-valued decoupled k-jet.

Proof of Theorem 2: Since there may be multiple ways to construct a real-valued decoupled k-jet, we only provide the construction leading to two coordinates respectively having physical meanings similar to displacement and velocity.

The differential equations for mode i in the decoupled k-jet are shown in (24), which are the $(2i-1)$-th and $2i$-th equations of all such equations on $\mathbf{z}_{\text{jet}}^{(k)}$.

$$\begin{cases} \dot{z}_{\text{jet},2i-1}^{(k)} = \lambda_{2i-1} z_{\text{jet},2i-1}^{(k)} \\ \quad + \sum_{\alpha=1}^{N}\sum_{\beta=\alpha}^{N} \mu_{2i-1,\alpha\beta} z_{\text{jet},\alpha}^{(k)} z_{\text{jet},\beta}^{(k)} + \cdots \\ \quad + \sum_{\alpha=1}^{N}\cdots\sum_{\rho=\gamma}^{N} \mu_{2i-1,\alpha\cdots\rho} \underbrace{z_{\text{jet},\alpha}^{(k)} \cdots z_{\text{jet},\rho}^{(k)}}_{k \text{ terms in total}} \overset{\text{def}}{=} f_{\text{jet},2i-1}^{(k)}(\mathbf{z}_{\text{jet}}^{(k)}) \\ \dot{z}_{\text{jet},2i}^{(k)} = f_{\text{jet},2i}^{(k)}(\mathbf{z}_{\text{jet}}^{(k)}) = \bar{f}_{\text{jet},2i-1}^{(k)}(\mathbf{z}_{\text{jet}}^{(k)}) \end{cases} \tag{24}$$

Note that the two state variables are a conjugate pair of complex-valued variables, and those μ-coefficients are determined in (3). Denote the right-hand sides of the first and second equations respectively as $a + jb$ and $a - jb$, where a and b are real-valued functions on $\mathbf{z}_{\text{jet}}^{(k)}$, λ and μ. Then, apply the coordinate transformation in (25) and (26) to yield (27), where all parameters and variables are real-valued.

$$\begin{bmatrix} z_{\text{jet},2i-1}^{(k)} \\ z_{\text{jet},2i}^{(k)} \end{bmatrix} = \mathbf{U}_{\text{mode } i}^{-1} \begin{bmatrix} w_{2i-1} \\ w_{2i} \end{bmatrix} \tag{25}$$

$$\mathbf{U}_{\text{mode } i} = \begin{bmatrix} \lambda_{2i-1} & \lambda_{2i} \\ 1 & 1 \end{bmatrix} \tag{26}$$

$$\begin{cases} \dot{w}_{2i-1} = \upsilon_{i10} w_{2i-1} + \sum_{l=1}^{k} \upsilon_{i0l} w_{2i}^{l} \\ \qquad + \sum_{\substack{j \geq 1, l \geq 0 \\ (j,l) \neq (1,0)}}^{j+l \leq k} \upsilon_{ijl} w_{2i-1}^{j} w_{2i}^{l} \\ \dot{w}_{2i} = w_{2i-1} + \sum_{\substack{j \geq 0, l \geq 0}}^{2 \leq j+l \leq k} \upsilon_{ijl} w_{2i-1}^{j} w_{2i}^{l} \end{cases} \quad (27)$$

Remark: Note that the transformation $\mathbf{H}_1$ in (13) needs to be normalized in order to make the new coordinates with (27) have a scale comparable to that with (23). The normalization introduced in [32] is adopted here:

(i) Classify the elements related to the displacement in a left eigenvector (complex-valued) of $\mathbf{A}$ into two opposing groups based on their angles;
(ii) Calculate the sum of the coefficients in one group;
(iii) Divide the entire left eigenvector by that sum;
(iv) Do such normalization for every left eigenvector.

The proposed NMD method gives a number of decoupled 2nd-order systems in (23) as an approximate of the original high-dimensional nonlinearly coupled multi-oscillator system. Theorem 3 below will show that the error in the response from (23), if inside the validity region $\Omega^{(k)}$, decreases with a higher decoupling order.

Theorem 3 (Error Estimation): Given the multi-oscillator system in (17), its k-th order nonlinearly mode-decoupled system satisfies (18) and the corresponding decoupled k-jet satisfies (23) through the transformation $\mathbf{H}^{(k)}$ in (19) for any system dynamics belonging to $\Omega^{(k)}$ defined in (20). Assume that all eigenvalues of the Jacobian matrix $\mathbf{A}$ of system (1) belong to the Poincaré domain and have real parts less than a negative number $\alpha < 0$. Let $\mathbf{x}(t,\mathbf{x}_0)$, $\mathbf{z}^{(k)}(t,\mathbf{x}_0)$ and $\mathbf{z}_{\text{jet}}^{(k)}(t,\mathbf{x}_0)$ be the solutions of (17), (18) and (23), respectively, under the equivalent initial conditions $\mathbf{x}_0$, $\mathbf{z}_0^{(k)}$ and $\mathbf{z}_{\text{jet0}}^{(k)}$, i.e. satisfying $\mathbf{x}_0 = \mathbf{H}^{(k)}(\mathbf{z}_0^{(k)}) = \mathbf{H}^{(k)}(\mathbf{z}_{\text{jet},0}^{(k)})$. Then, when $\mathbf{x}(t, \mathbf{x}_0)$, $\mathbf{H}^{(k)}\left(\mathbf{z}^{(k)}(t,\mathbf{x}_0)\right)$ and $\mathbf{H}^{(k)}\left(\mathbf{z}_{\text{jet}}^{(k)}(t,\mathbf{x}_0)\right)$ belong to $\Omega^{(k)}$, there exists positive numbers ε_1 and c such that for any ε satisfying $0 \leq \varepsilon \leq \varepsilon_1$, $\|\mathbf{x}_0\| \leq \varepsilon$ implies

$$e_k(t,\mathbf{x}_0) \stackrel{\text{def}}{=} \left\| \mathbf{x}(t,\mathbf{x}_0) - \mathbf{H}^{(k)}\left(\mathbf{z}_{\text{jet}}^{(k)}(t,\mathbf{x}_0)\right) \right\| \leq c\varepsilon^{k+1} e^{\alpha t/2} \quad (28)$$

Here, the norm $||\cdot||$ can be of any type. This theorem indicates that if the initial condition is close enough to the origin, the response of the decoupled k-jet at any finite t will become closer to that of the original system when k increases and ε takes a value smaller than one.

Proof of Theorem 3: It is easy to see that the two systems in (17) and (18) are equivalent in $\Omega^{(k)}$ over the transformation $\mathbf{H}^{(k)}$ in (19). To show that the error $e_k(t,\mathbf{x}_0)$ approaches zero, we only need to show that error defined in (29), or equivalently (30), approaches zero:

$$\hat{e}_k(t,\mathbf{x}_0) \stackrel{\text{def}}{=} \left\| \mathbf{H}^{(k)}\left(\mathbf{z}^{(k)}(t,\mathbf{x}_0)\right) - \mathbf{H}^{(k)}\left(\mathbf{z}_{\text{jet}}^{(k)}(t,\mathbf{x}_0)\right) \right\| \quad (29)$$

$$\tilde{\hat{e}}_k(t,\mathbf{x}_0) \stackrel{\text{def}}{=} \left\| \mathbf{z}^{(k)}(t,\mathbf{x}_0) - \mathbf{z}_{\text{jet}}^{(k)}(t,\mathbf{x}_0) \right\| \quad (30)$$

The rest of the proof is omitted since it is similar to the Theorem 5.3.4 in [47]. □

For any given $\varepsilon > 0$, the *validity region* Ω_ε *of the decoupled k-jet* system (23) is defined by (31), which relies on the selection of ε: the larger ε the larger validity region.

$$\Omega_\varepsilon = \left\{ \mathbf{x}_0 \,\middle|\, \left\| \mathbf{x}(t,\mathbf{x}_0) - \mathbf{H}^{(k)}\left(\mathbf{z}_{\text{jet}}^{(k)}(t,\mathbf{x}_0)\right) \right\| < \varepsilon \right\} \quad (31)$$

Remark: This section presents the NMD of an N-oscillator system in the absence of resonance. In case of the existence of resonance, the modes that involved in the resonance cannot be decoupled, so the resulting decoupled sub-systems by NMD will include decoupled k-jets fewer than N and one or more sub-systems of orders higher than 2 representing the part of the system that cannot be decupled. For instance, if the original system has a second-order resonance involving its modes i and j, the NMD approach will give a 4th-order sub-system dominating the dynamics of modes i and j and N-2 decoupled 2-jets respectively dominating dynamics with the other N-2 modes. All these N-1 sub-systems have polynomial nonlinearities up to the second order. Note that for cases in the absence of resonance, that 4th-order sub-system can further be decomposed into two decoupled 2-jets. In this paper, only non-resonance condition is considered while the NMD under resonance condition will be investigated in future.

Note that although the validity region Ω converges to a single point, the validity region $\Omega^{(k)}$ can still be nontrivially large. A higher decoupling order can reduce the error of the analysis on decoupled k-jet system, but may result in smaller validity region. So there exists a trade-off between the size of validity region and accuracy. In the following section IV, we will elaborate the proposed NMD on nonlinear decoupling of power systems and then in section V, we will compare to normal form and demonstrate its accuracy. As mentioned in the literature, there are other approaches such as a modal series based approach [36] that can find an approximate but more accurate solution of a nonlinear dynamical system than normal form. Comparisons of NMD with other approaches will be conducted in future.

III. NONLINEAR MODAL DECOUPLING OF POWER SYSTEMS

This section will apply the proposed NMD analysis to power systems. Firstly, the nonlinear differential equations of a multi-machine power system and its equivalent multi-oscillator system are introduced. Then, two forms of desired modal nonlinearity are proposed and their corresponding decoupled k-jet systems are derived. Finally, a first-integral based method is applied to the decoupled k-jet systems to demonstrate a sample application of NMD in power system stability analysis.

A. POWER SYSTEM MODEL

Consider an m-machine power system model (32), where each generator is represented by a 2nd-order classic model and all loads are modeled by constant impedances, while it should be emphasized that this model considers the losses from network parameters and loads:

$$\ddot{\delta}_i + \frac{\varsigma_i}{M_i}\dot{\delta}_i + \frac{\omega_s}{M_i}(P_{mi} - P_{ei}) = 0$$
$$P_{ei} = E_i^2 g_i + \sum_{j=1, j\neq i}^{m}\left[a_{ij}\sin(\delta_i - \delta_j) + b_{ij}\cos(\delta_i - \delta_j)\right] \quad (32)$$

where $i \in \{1, 2, \ldots, m\}$, δ_i, P_{mi}, P_{ei}, E_i, M_i and ς_i respectively represent the absolute rotor angle, mechanical power, electrical power, electromotive force, the inertia constant and damping constant of machine i, and g_i, a_{ij}, and b_{ij} represent network parameters including all loads.

Assume that the system has a stable equilibrium point and the system has a uniform damping, i.e. ς_i/M_i is a common constant for all i. It is shown in [48] that (i) the oscillatory dynamics and stability of the system are dominated by the relative motions [49], [50] among different machines; and (ii) those relative motions can always be represented by an $(m-1)$-oscillator system.

Denote $\Delta = [\delta^T \ \dot{\delta}^T]^T$ as the state vector. Then the first-order representation of the system (32) has this form

$$\dot{\Delta} = \mathbf{f}_0(\mathbf{\Delta}) \quad (33)$$

Generate a transformation matrix $\mathbf{R}$, whose columns are right eigenvectors of $\mathbf{f}_0$'s Jacobian matrix. Apply $\mathbf{R}$ to both sides of (33) and its modal space representation (34) can be obtained, where $\mathbf{y}= [y_1 y_2 \ldots y_{2m}]^T$ is the state vector in the modal space.

$$\dot{\mathbf{y}} = \mathbf{R}^{-1}\mathbf{f}_0(\mathbf{R}\mathbf{y}), \text{ where } \mathbf{y} = \mathbf{R}^{-1}\mathbf{\Delta} \quad (34)$$

Without loss of generality, let $y_1, y_2, \ldots, y_{2m-2}$ represent the relative motions of the system and that y_{2m-1} and y_{2m} represent the mean motion. It has been proved in [48] that the relative motions can be represented by the $(m-1)$-oscillator system consisting of different equations about $y_1, y_2, \ldots, y_{2m-2}$, to which the proposed NMD will be applied.

Next, we present two ways to choose the desired modal nonlinearity respectively under an SMIB assumption and another small transfer (ST) assumption proposed below.

B. NMD UNDER THE SMIB ASSUMPTION

1) SMIB ASSUMPTION

The nonlinearity associated with each oscillatory mode has the same form as an SMIB system. Under the SMIB assumption, the proposed NMD will result in a number of fictitious SMIB systems each of which is associated with a specific 2-way partition of all generators. The dynamics of each fictitious SMIB system correspond to one oscillation mode at which the two partitioned groups of generators oscillate. Note: "fictitious" means that the single generator in each resulting SMIB system does not solely linked to or dominated by any physical generator; rather its behaviors depend on every generator of the system as long as the system keeps its integrity.

In practice, this assumption is intuitive to power system scholars and engineers and hence has been widely used in power system studies [32], [38]–[42]. For instance, a power system that consists of two areas being weakly interconnected is often simplified to an SMIB system for stability studies regarding the inter-area oscillation mode. In the following, we study a general multi-machine power system and our goal is to intentionally manipulate the nonlinear terms of each decoupled system following a certain fictitious SMIB.

2) DESIRED DECOUPLED SYSTEM FOR MODE i

Under the SMIB assumption, the desired decoupled system about mode i, associated with λ_{2i-1} and λ_{2i}, can be written as (35) [51].

$$\ddot{y}_i + \alpha_i\dot{y}_i + \beta_i(\sin(y_i + y_{is}) - \sin y_{is}) = 0 \quad (35)$$

where y_i is called a generalized angle of fititious generator i (i.e. mode i), while α_i, β_i and y_{is} are constants that can be uniquely determined by (36).

$$\begin{cases} y_{is} = \sum_{j=1}^{m}\tau_{ij}\delta_{js} \\ \alpha_i = -2\mathrm{Re}\{\lambda_{2i-1}\} \\ \beta_i = \dfrac{\lambda_{2i-1}\lambda_{2i}}{\cos y_{is}} \end{cases} \quad (36)$$

where τ_{ij} is the i-th row j-th column element from the matrix of the left eigenvectors of the Jacobian of the third term in (35) [32] and δ_{js} is the steady-state value of δ_j.

3) NMD TRANSFORMATION

Assume each complex-valued decoupled system to be

$$\begin{cases} \dot{z}_{2i-1} = \lambda_{2i-1}z_{2i-1} + \sum_{j=2}^{\infty}\sum_{l=0}^{j}\mu_{i,l,j-l}z_{2i-1}^{l}z_{2i}^{j-l} \\ \dot{z}_{2i} = \lambda_{2i}z_{2i} + \sum_{j=2}^{\infty}\sum_{l=0}^{j}\bar{\mu}_{i,l,j-l}z_{2i}^{l}z_{2i-1}^{j-l} \end{cases} \quad (37)$$

Toward the real-valued desired form of the decoupled system in (35), μ-coefficients of intra-modal terms have to be determined. Apply the following coordinate transformation to (37) to obtain (40) where all quantities are real-valued.

$$\begin{bmatrix} z_{2i-1} \\ z_{2i} \end{bmatrix} = \mathbf{V}_{\text{mode } i}\begin{bmatrix} \dot{y}_i \\ y_i \end{bmatrix} \quad (38)$$

where

$$\mathbf{V}_{\text{mode } i} = \frac{2}{\lambda_{2i-1} - \lambda_{2i}}\begin{bmatrix} 1 & -\lambda_{2i} \\ -1 & \lambda_{2i-1} \end{bmatrix} \quad (39)$$

$$\begin{cases} \dfrac{d\dot{y}_i}{dt} + \alpha_i\dot{y}_i + \sum_{n=1}^{\infty} r_{in}y_i^n = 0 \\ \dfrac{dy_i}{dt} = \dot{y}_i \end{cases} \quad (40)$$

Coefficient r_{in} is determined by (41) to make (35) and (40) have identical nonlinearities up to the k-th order. Finally, the μ-coefficients can be determined by transforming (40) back to the form of (37) using transformation defined in (38) and (39).

$$r_{in} = \frac{\beta_i \cos\left(y_{is} + \frac{(n-1)\pi}{2}\right)}{n!} \tag{41}$$

C. NMD UNDER THE ST ASSUMPTION

We also propose the following alternative assumption for each desired mode-decoupled system and compare its result with that from the SMIB assumption.

1) SMALL TRANSFER (ST) ASSUMPTION

Assume that the second equation of (11) to be zero, i.e. $h_{p+1,j,\text{intra},\alpha\cdots\gamma}=0$.

2) NMD TRANSFORMATION

Under the ST assumption, the desired modal nonlinearity, i.e. μ-coefficients, is calculated by (11) as

$$\mu_{j,\text{intra},\alpha\cdots\gamma} = c_{p+1,j,\text{intra},\alpha\cdots\gamma} \tag{42}$$

Remark: The μ-coefficients determined by (42) is the desired modal nonlinearities under the ST assumption, which may give NMD transformation a relatively large validity region such that the resulting decoupled k-jet systems can be accurate for representing the dynamics of the original system under relatively larger disturbances. The reasoning for this will be shown below and supported by the numerical studies in Section V. In addition, the form of each mode-decoupled system under the ST assumption might not be as clear as that of an SMIB system before the entire NMD process is finished. However, the implementation is quite simple since we just need to let the second equation in (11) be zero. The physical insight behind this ST assumption is that we want to limit the propagation of nonlinear terms to higher orders over the transformation in (7), which can be seen in the example below.

Without loss of generality, consider the system (43) represented by two first-order differential equations with polynomial nonlinearities up to the 2$^{\text{nd}}$-order. λ_1 and λ_2 represent two different modes. Note that the observations below also apply to large systems having more state variables.

$$\begin{cases} \dot{x}_1 = \lambda_1 x_1 + b_{1,11}x_1^2 + b_{1,12}x_1x_2 + b_{1,22}x_2^2 \\ \dot{x}_2 = \lambda_2 x_2 + b_{2,11}x_1^2 + b_{2,12}x_1x_2 + b_{2,22}x_2^2 \end{cases} \tag{43}$$

Intra-modal terms and inter-modal terms for these two equations in (43) are respectively listed in (44) and (45).

$$\left\{b_{1,11}x_1^2,\ b_{2,22}x_2^2\right\} \tag{44}$$

$$\left\{b_{1,12}x_1x_2,\ b_{1,22}x_2^2,\ b_{2,11}x_1^2,\ b_{2,12}x_1x_2\right\} \tag{45}$$

Then, apply a 2$^{\text{nd}}$-order coordinate transformation (46): Substitute (46) into (43) and obtain a new system about z_1 and z_2, where intra-modal terms and inter-modal terms are similar to those in (44) and (45) in form. In the new system, utilize the first equation of (11) to find coefficients h to cancel its inter-modal terms as shown in (47) and obtain (48), where P, Q, R and $\mathbf{S}$ are polynomial functions, the vector function $\mathbf{S}$ satisfies (50), and the coefficients of the intra-modal terms $h_{1,11}$ and $h_{2,22}$, denoted by $\mathbf{h}_{\text{intra}} = [h_{1,11}, h_{2,22}]^{\text{T}}$, are yet to be determined.

$$\begin{cases} x_1 = z_1 + h_{1,11}z_1^2 + h_{1,12}z_1z_2 + h_{1,22}z_2^2 \\ x_2 = z_2 + h_{2,11}z_1^2 + h_{2,12}z_1z_2 + h_{2,22}z_2^2 \end{cases} \tag{46}$$

$$\begin{aligned} h_{1,12} &= \frac{b_{1,12}}{\lambda_2}, \quad h_{1,22} = \frac{b_{1,22}}{2\lambda_2 - \lambda_1}, \\ h_{2,12} &= \frac{b_{2,12}}{\lambda_1}, \quad h_{2,11} = \frac{b_{2,11}}{2\lambda_1 - \lambda_2} \end{aligned} \tag{47}$$

$$\begin{cases} \dot{z}_1 = \lambda_1 z_1 + (b_{1,11} - h_{1,11}\lambda_1)z_1^2 \\ \qquad + \sum_{i=3}^{\infty} \sum_{\substack{0\le j_1,j_2\le i \\ j_1+j_2=i}} T_{1ij_1j_2} z_1^{j_1} z_2^{j_2} \\ \dot{z}_2 = \lambda_2 z_2 + (b_{2,22} - h_{2,22}\lambda_2)z_2^2 \\ \qquad + \sum_{i=3}^{\infty} \sum_{\substack{0\le j_1,j_2\le i \\ j_1+j_2=i}} T_{2ij_1j_2} z_1^{j_1} z_2^{j_2} \end{cases} \tag{48}$$

$$\begin{cases} T_{1ij_1j_2} = \frac{P_{10ij_1j_2}(\lambda)R_{10ij_1j_2}(b)}{Q_{10ij_1j_2}(\lambda)} \\ \qquad + \frac{P_{11ij_1j_2}(\lambda)R_{11ij_1j_2}(b)}{Q_{11ij_1j_2}(\lambda)} \cdot S_{1ij_1j_2}(\mathbf{h}_{intra}) \\ T_{2ij_1j_2} = \frac{P_{20ij_1j_2}(\lambda)R_{20ij_1j_2}(b)}{Q_{20ij_1j_2}(\lambda)} \\ \qquad + \frac{P_{21ij_1j_2}(\lambda)R_{21ij_1j_2}(b)}{Q_{21ij_1j_2}(\lambda)} \cdot S_{2ij_1j_2}(\mathbf{h}_{intra}) \end{cases} \tag{49}$$

$$\mathbf{S}(\mathbf{0}) = \mathbf{0} \tag{50}$$

Note that the information corresponding to the 2$^{\text{nd}}$-order nonlinearities in system (43) spreads out to all higher order nonlinear terms in system (48) through the transformation in (46). Theoretically speaking, such a transfer of nonlinearities does not impact the process of NMD if infinite nonlinear terms are kept in (48). However, the implementation of NMD in practice can only keep nonlinear terms up to a finite order, say k. Thus, it may be desired in NMD to keep as few nonlinearities transferred to higher-order terms as possible. For specific systems, it might be possible that there exists a way to determine those non-zero $\mathbf{h}_{\text{intra}}$ guaranteeing a minimum transfer, e.g. (51) where D is a set of points (z_1, z_2) containing all concerned dynamics of system (48). In general, without a priori knowledge about the selection of $\mathbf{h}_{\text{intra}}$, it might be preferred to let $\mathbf{h}_{\text{intra}} = 0$ in order to limit the transfer of nonlinearities as in (50). Thus, this is called the

ST assumption.

$$\min_{h_{intra}}\left\{\max_{(u_1,u_2)\in D}\left[\left(\sum_{i=3}^{\infty}\sum_{\substack{0\le j_1,j_2\le i\\ j_1+j_2=i}} T_{1ij_1j_2}z_1^{j_1}z_2^{j_2}\right)^2 + \left(\sum_{i=3}^{\infty}\sum_{\substack{0\le j_1,j_2\le i\\ j_1+j_2=i}} T_{2ij_1j_2}z_1^{j_1}z_2^{j_2}\right)^2\right]\right\} \tag{51}$$

Remark: The coefficients h_{ij} in (46) (or μ in Eq. (2)) represent the extent and distribution of system nonlinearities, especially the strength of nonlinear interactions between different modes (or states) [52].

D. NMD BASED STABILITY ANALYSIS

Application of the proposed NMD to a multi-machine power system brings a new method for power system analysis, i.e. analysis of a set of independent 2^{nd}-order nonlinear dynamic systems instead of the original system. This subsection adopts the closest unstable equilibrium point method [53] to perform the conservative stability analysis individually on the decoupled 2^{nd}-order systems. Note that the purpose is just to demonstrate how NMD enables new analysis methods while other methods with less conservativeness, e.g. BCU method [54], will be investigated in future.

The following assumption is adopted to create a first-integral based transient energy function for stability analysis: (52) and (53) holds for coefficients in (27).

$$\begin{cases} \upsilon_{ijl}=0 & \text{for all } j\ge 1,\quad l\ge 1,\ j+l\le k,\ (j,l)\ne(1,0)\\ \nu_{ijl}=0 & \text{for all } j\ge 0,\quad l\ge 0,\ 2\le j+l\le k \end{cases} \tag{52}$$

$$\upsilon_{i10}=0 \tag{53}$$

Then, (27) becomes (54).

$$\begin{cases} \dot{w}_{2i-1}=\sum_{j=1}^{k}\upsilon_{ij}w_{2i}^{j}\\ \dot{w}_{2i}=w_{2i-1} \end{cases} \tag{54}$$

A transient energy function of the system in (54) is

$$\begin{aligned} V_i(w_{2i-1},w_{2i}) &= \frac{w_{2i-1}^2}{2}+\int_0^{z_{2i}}\sum_{j=1}^{k}\upsilon_{ij}s^j ds\\ &= \frac{w_{2i-1}^2}{2}+\sum_{j=1}^{k}\frac{\upsilon_{ij}}{j+1}w_{2i}^{j+1} \end{aligned} \tag{55}$$

The stability analysis of disturbed power systems is studied by introducing the *region of attraction* (ROA). The ROA of a dynamical system at its *stable equilibrium point* (SEP) is defined as the region that if initializing the system with any point in the region, the system trajectory will eventually approach the SEP. The power system stability analysis is actually to determine whether a given initial condition belongs to the ROA or not. Note that the system in (54) has one SEP at the origin. The ROA with this SEP will be approximated by using the closest *unstable equilibrium point* (closest UEP) method [53]. The closest UEP can be obtained by letting the right hand side of (54) be zero and solving the resulting algebraic equations. Denote the closest UEP by $w_{2i,\mathrm{UEP}}$. Then, detailed procedures are:

(i) Simulate the disturbed power system until the disturbance is cleared, transform the final system condition, i.e. the initial condition of the post-disturbance system, to the real-valued decoupled coordinates and denote as ($w_{2i-1}(0)$, $w_{2i}(0)$).

(ii) Compute the initial energy $V_i(w_{2i-1}(0), w_{2i}(0))$ and the critical energy $V_i(0, w_{2i,\mathrm{UEP}})$ using (55).

(iii) If $V_i(w_{2i-1}(0), w_{2i}(0)) < V_i(0, w_{2i,\mathrm{UEP}})$, then the initial condition ($w_{2i-1}(0)$, $w_{2i}(0)$) belongs the ROA, i.e. the post-disturbance power system is stable; Otherwise, the post-disturbance power system is unstable.

Remark: Since the proposed NMD method is essentially not a global method, similar to normal form in this sense, only dynamics within the validity region of all transformations can be analyzed. Thus, the above NMD based power system stability analysis is not always achievable for a general power system since the validity region might be smaller than the stability region of the system. In this case, the dynamics under a severe disturbance, e.g. leading to loss of stability, might not be transformed between the original coordinates and the coordinates of the decoupled k-jet. Investigations on the size of the validity region and the comparison to the stability region of a power system will be our future work.

IV. NUMERICAL STUDY

This section will present the numerical studies of the proposed NMD on two test power systems: the IEEE 3-machine 9-bus system [55] and the New England 10-machine 39-bus system [56]. Both are modeled by (32).

In the IEEE 9-bus power system, the detailed results from the proposed NMD will be presented: 1) two sets of decoupled system equations are respectively derived under the SMIB and the ST assumptions; 2) numerical simulation results on the decoupled 3-jet systems are created and compared to that from the normal form method; 3) stability on the original system is analyzed by means of analysis on the decoupled 3-jet system. The New England 39-bus power system is then used to demonstrate the applicability of the proposed NMD method on a high-dimensional dynamical system.

A. TEST ON THE IEEE 9-BUS SYSTEM

The IEEE 9-bus system is shown in Fig. 1. The following disturbance is considered: a temporary three-phase fault is added on bus 5 and cleared by disconnection of line 5-7 after a fault duration time. The critical clearing time (CCT)

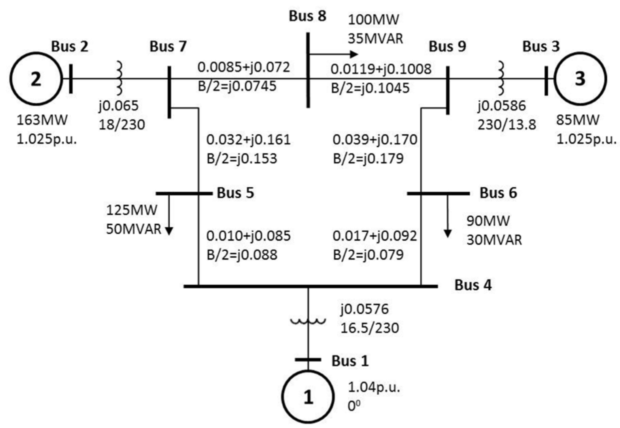

FIGURE 1. IEEE 3-machine, 9-bus power system.

TABLE 1. Time domain errors of simulated responses.

FD	NDSMIB		NDST		NF	
	E[$e(t)$]	Std[$e(t)$]	E[$e(t)$]	Std[$e(t)$]	E[$e(t)$]	Std[$e(t)$]
0.01s	0.96	1.07	0.007	0.008	0.12	0.11
0.05s	1.20	1.36	0.013	0.014	0.34	0.34
0.10s	3.87	4.28	0.13	0.16	2.26	2.39
0.15s	15.34	13.99	1.98	2.56	16.30	17.87

of this disturbance, i.e. the longest fault duration without causing instability, is found to be 0.17s. The post-disturbance system corresponding to the model in (33) is represented by differential equations in (56). A 3rd-order Taylor expansion of (56) gives an estimate of CCT equal to 0.16s, which has been very close to the accurate 0.17s, so the 3rd-order Taylor expansion can credibly keep the stability information of the original system and is used below as the basis for deriving decoupled systems as well as the benchmark for comparison. The 2-oscillator system derived from the 3rd-order Taylor expansion of (56) is shown in (57) according to [48].

Following the NMD respectively under the SMIB and the ST assumptions, the complex-valued decoupled 3-jet systems are respectively shown in (58) and (59). As a comparison, the counterpart from the 3rd-order normal form gives (60). The time cost is investigated on an Inter Core^{TM} i7-6700 3.4GHz desktop computer, where the derivation of NMD using the Symbolic Math Toolbox in Matlab takes about 4.5 seconds.

Systems (58), (59) and (60) are respectively named NDSMIB, NDST and NF. Their dynamical performances are compared under the same initial conditions from the post-disturbance period. The error of each simulated system response is calculated and compared to the "true" response, i.e. the response of the 3rd-order Taylor expansion of (56).

The errors $e(t)$ of these responses in the time domain are calculated by (28) and shown in Table 1 for four disturbances with increasing fault duration times from 0.01s to 0.15s. The last one is a marginally stable case. The simulated responses from these systems and their time domain errors are shown in Fig. 2 to Fig. 5. From those figures and Table 1, the NDST has the smallest error and the NDSMIB has the largest error.

Then, the stability analysis of the original system is studied using the NDST system (59). Transform (59) into real-valued

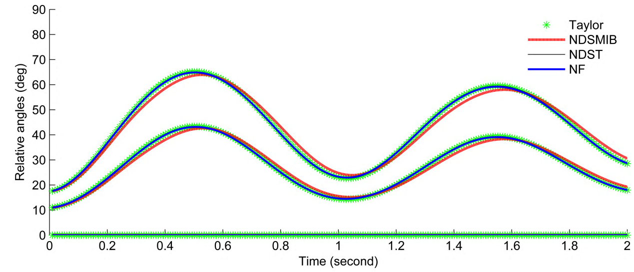

FIGURE 2. Simulated system responses under the disturbance (fault duration = 0.01s) respectively by 3rd-order Taylor expansion, NDSMIB, NDST and NF.

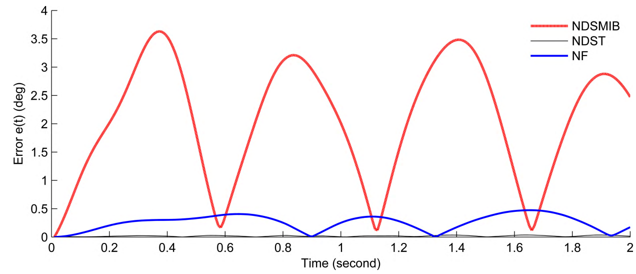

FIGURE 3. Time domain error of the simulated system responses under the disturbance (fault duration = 0.01s) respectively by NDSMIB, NDST and NF.

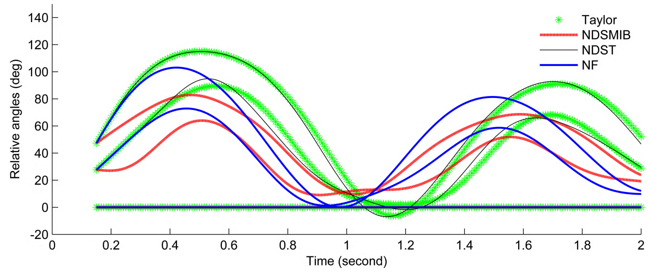

FIGURE 4. Simulated system responses under the disturbance (fault duration = 0.15s) respectively by 3rd-order Taylor expansion, NDSMIB, NDST and NF.

equations by (25) and obtain (61). The four transformations for transforming (56) to (61) are shown in Appendix.

E[e(t)] and Std[$e(t)$] are the expectation and standard deviation of the error signal $e(t)$, which are in degrees.

$$\begin{cases} \dot{x}_1 = x_2 + 3.12 \\ \dot{x}_2 = -0.5x_2 - 1.14\cos(x_{13} - 0.728) \\ \qquad - 6.25\sin(x_{13} - 0.728) - 1.56\cos(x_{15} - 0.463) \\ \qquad - 9.11\sin(x_{15} - 0.463) - 5.98 \\ \dot{x}_3 = x_4 + 3.12 \\ \dot{x}_4 = -0.5x_4 - 4.22\cos(x_{13} - 0.728) \\ \qquad + 23.1\sin(x_{13} - 0.728) - 6.04\cos(x_{35} + 0.265) \\ \qquad - 38.0\sin(x_{35} + 0.265) - 5.98 \\ \dot{x}_5 = x_6 + 3.12 \\ \dot{x}_6 = -0.5x_6 - 12.3\cos(x_{15} - 0.463) \\ \qquad + 71.6\sin(x_{15} - 0.463) - 12.8\cos(x_{35} + 0.265) \\ \qquad + 80.7\sin(x_{35} + 0.265) - 5.98 \end{cases} \tag{56}$$

where $x_{ij} = x_i - x_j$

$$
\begin{cases}
\dot{y}_1 = (-0.25 + j12.9)y_1 - (0.002 + j0.10)y_1^2 \\
\quad + (0.004 - j0.20)y_1 y_2 \\
\quad + (0.006 - j0.10)y_2^2 - (0.01 + j0.27)y_1^3 - j0.81 y_1^2 y_2 \\
\quad + (0.03 - j0.81)y_1 y_2^2 + (0.02 - j0.27)y_2^3 \\
\quad - (0.03 + j0.72)y_1 y_3 + (0.03 - j0.72)y_1 y_4 \\
\quad - (0.002 + j0.72)y_2 y_3 \\
\quad + (0.06 - j0.71)y_2 y_4 + (0.002 + j0.04)y_1^2 y_3 \\
\quad - (0.001 - j0.04)y_1^2 y_4 \\
\quad - (0.02 + j0.21)y_1 y_3^2 + (0.02 - j0.21)y_1 y_4^2 \\
\quad - (0.001 - j0.04)y_2^2 y_3 \\
\quad - (0.004 - j0.04)y_2^2 y_4 - (0.009 + j0.21)y_2 y_3^2 \\
\quad + (0.02 - j0.21)y_2 y_4^2 \\
\quad + (0.002 + j0.08)y_1 y_2 y_3 - (0.005 - j0.08)y_1 y_2 y_4 \\
\quad - j0.41 y_1 y_3 y_4 \\
\quad + (0.02 - j0.41)y_2 y_3 y_4 - (0.005 + j0.07)y_3^2 \\
\quad + (0.003 - j0.15)y_3 y_4 \\
\quad + (0.008 - j0.07)y_4^2 - (0.002 + j0.02)y_3^3 \\
\quad - (0.001 + j0.05)y_3^2 y_4 \\
\quad + (0.003 - j0.05)y_3 y_4^2 + (0.002 - j0.02)y_4^3 \overset{\text{def}}{=} \mathbf{f}_1^{(0)}(\mathbf{y}) \\
\dot{y}_2 = \mathbf{f}_2^{(0)}(\mathbf{y}) = \bar{\mathbf{f}}_1^{(0)}(\mathbf{y}) \\
\dot{y}_3 = (-0.25 + j6.08)y_3 - (0.02 + j0.57)y_3^2 \\
\quad + (0.05 - j1.14)y_3 y_4 \\
\quad + (0.07 - j0.57)y_4^2 - (0.008 + j0.10)y_3^3 - j0.30 y_3^2 y_4 \\
\quad + (0.02 - j0.30)y_3 y_4^2 + (0.02 - j0.10)y_4^3 \\
\quad + (0.001 - j0.44)y_1^2 + (0.03 - j0.88)y_1 y_2 \\
\quad + (0.04 - j0.44)y_2^2 \\
\quad + (0.001 + j0.05)y_1 y_3 - (0.003 - j0.05)y_1 y_4 \\
\quad - (0.001 - j0.05)y_2 y_3 \\
\quad - (0.005 - j0.05)y_2 y_4 + (2\times 10^{-4} + j0.01)y_1^3 \\
\quad - (0.001 - j0.01)y_2^3 \\
\quad - (0.001 - j0.03)y_1^2 y_2 \\
\quad - (0.007 + j0.17)y_1^2 y_3 + (0.007 - j0.17)y_1^2 y_4 \\
\quad + (0.007 - j0.17)y_2^2 y_3 \\
\quad + (0.02 - j0.17)y_2^2 y_4 - (0.002 - j0.03)y_1 y_2^2 \\
\quad - (0.002 + j0.04)y_1 y_3^2 + (0.004 - j0.04)y_1 y_4^2 \\
\quad - (0.001 + j0.04)y_2 y_3^2 \\
\quad + (0.005 - j0.04)y_2 y_4^2 - j0.34 y_1 y_2 y_3 \\
\quad + (0.03 - j0.34)y_1 y_2 y_4 \\
\quad + (0.002 - j0.07)y_1 y_3 y_4 + (0.004 - j0.07) \\
\quad y_2 y_3 y_4 \overset{\text{def}}{=} \mathbf{f}_3^{(0)}(\mathbf{y}) \\
\dot{y}_4 = \mathbf{f}_4^{(0)}(\mathbf{y}) = \bar{\mathbf{f}}_3^{(0)}(\mathbf{y})
\end{cases}
\tag{57}
$$

$$
\begin{cases}
\dot{z}_1^{(3)} = (-0.25 + j12.9)z_1^{(3)} - j2.83 z_1^{(3)} z_{(3)2} - j1.08\left(z_1^{(3)}\right)^3 \\
\quad - j1.42\left(z_1^{(3)}\right)^2 - j1.08\left(z_2^{(3)}\right)^3 - j1.42\left(z_2^{(3)}\right)^2 \\
\quad - j3.23\left(z_1^{(3)}\right)^2 z_2^{(3)} - j3.23 z_1^{(3)}\left(z_2^{(3)}\right)^2 \\
\quad + \mathrm{O}\left(\mathbf{z}^{(3)}\right)^4 \overset{\text{def}}{=} f_{1,\text{smib}}^{(3)}(\mathbf{z}^{(3)}) \\
\dot{z}_2^{(3)} = \mathbf{f}_{2,\text{smib}}^{(3)}(\mathbf{z}^{(3)}) = \bar{\mathbf{f}}_{1,\text{smib}}^{(3)}(\mathbf{z}^{(3)}) \\
\dot{z}_3^{(3)} = (-0.25 + j6.08)z_3^{(3)} - j1.52\left(z_3^{(3)}\right)^2 z_4^{(3)} j0.51\left(z_3^{(3)}\right)^3 \\
\quad - j0.86\left(z_3^{(3)}\right)^2 - j0.51\left(z_4^{(3)}\right)^3 - j0.86\left(z_4^{(3)}\right)^2 \\
\quad - j1.72 z_3^{(3)} z_4^{(3)} - j1.52 z_3^{(3)}\left(z_4^{(3)}\right)^2 \\
\quad + \mathrm{O}\left(\mathbf{z}^{(3)}\right)^4 \overset{\text{def}}{=} f_{3,\text{smib}}^{(3)}(\mathbf{z}^{(3)}) \\
\dot{z}_4^{(3)} = \mathbf{f}_{4,\text{smib}}^{(3)}(\mathbf{z}^{(3)}) = \bar{\mathbf{f}}_{3,\text{smib}}^{(3)}(\mathbf{z}^{(3)})
\end{cases}
\tag{58}
$$

$$
\begin{cases}
\dot{z}_1^{(3)} = (-0.25 + j12.9)z_1^{(3)} + \mathrm{O}\left(\mathbf{z}^{(3)}\right)^4 \\
\quad - (0.0019 + j0.0975)\left(z_1^{(3)}\right)^2 \\
\quad + (0.0057 - j0.097)\left(z_2^{(3)}\right)^2 \\
\quad + (0.0038 - j0.195) z_1^{(3)} z_2^{(3)} \\
\quad + (0.02 - j0.262)\left(z_2^{(3)}\right)^3 \\
\quad - (0.0101 + j0.262)\left(z_1^{(3)}\right)^3 \\
\quad + (1.3\times 10^{-4} - j1.01)\left(z_1^{(3)}\right)^2 z_2^{(3)} \\
+ (0.0389 - j1.01) z_1^{(3)}\left(z_2^{(3)}\right)^2 \overset{\text{def}}{=} \mathbf{f}_{1,\text{st}}^{(3)}(\mathbf{z}^{(3)}) \\
\dot{z}_2^{(3)} = \mathbf{f}_{2,\text{st}}^{(3)}(\mathbf{z}^{(3)}) = \bar{\mathbf{f}}_{1,\text{st}}^{(3)}(\mathbf{z}^{(3)}) \\
\dot{z}_3^{(3)} = (-0.25 + j6.08)z_3^{(3)} + \mathrm{O}\left(\mathbf{z}^{(3)}\right)^4 \\
\quad - (0.023 + j0.57)\left(z_3^{(3)}\right)^2 + (0.07 - j0.566)\left(z_4^{(3)}\right)^2 \\
\quad + (0.047 - j1.14) z_3^{(3)} z_4^{(3)} + (0.017 - j0.092)\left(z_4^{(3)}\right)^3 \\
\quad - (0.009 + j0.093)\left(z_3^{(3)}\right)^3 \\
\quad - (0.002 + j0.29)\left(z_3^{(3)}\right)^2 z_4^{(3)} \\
\quad + (0.025 - j0.289) z_3^{(3)}\left(z_4^{(3)}\right)^2 \overset{\text{def}}{=} \mathbf{f}_{3,\text{st}}^{(3)}(\mathbf{z}^{(3)}) \\
\dot{z}_4^{(3)} = \mathbf{f}_{4,\text{st}}^{(3)}(\mathbf{z}^{(3)}) = \bar{\mathbf{f}}_{3,\text{st}}^{(3)}(\mathbf{z}^{(3)})
\end{cases}
\tag{59}
$$

$$
\begin{cases}
\dot{z}_1^{(3)} = (-0.25 + j12.9)z_1^{(3)} + \mathrm{O}\left(\mathbf{z}^{(3)}\right)^4 \\
\dot{z}_2^{(3)} = (-0.25 - j12.9)z_2^{(3)} + \mathrm{O}\left(\mathbf{z}^{(3)}\right)^4 \\
\dot{z}_3^{(3)} = (-0.25 + j6.08)z_3^{(3)} + \mathrm{O}\left(\mathbf{z}^{(3)}\right)^4 \\
\dot{z}_4^{(3)} = (-0.25 - j6.08)z_4^{(3)} + \mathrm{O}\left(\mathbf{z}^{(3)}\right)^4
\end{cases}
\tag{60}
$$

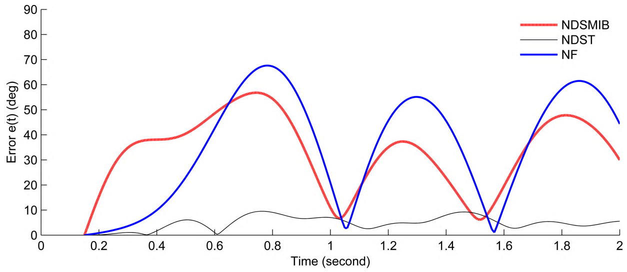

FIGURE 5. Time domain error of the simulated system responses under the disturbance (fault duration = 0.15s) respectively by NDSMIB, NDST and NF.

$$
\begin{cases}
\dot{w}_1 = -0.5w_1 - 166.0w_2 + 5.0w_2^2 + 32.7w_2^3 \\
\quad - 0.015w_1w_2 - 0.112w_1w_2^2 + 0.034w_1^2w_2 \\
\quad + (1\times10^{-5})\cdot w_1^2 - (5\times10^{-5})\cdot w_1^3 \\
\dot{w}_2 = w_1 + (2\times10^{-8})\cdot w_1^2 + 0.008w_2^2 \\
\quad - (2\times10^{-5})w_1w_2 \\
\quad - (8\times10^{-8})\cdot w_1^3 + 0.05w_2^3 + (5\times10^{-5})w_1^2w_2 \\
\quad - (2\times10^{-4})\cdot w_1w_2^2 \\
\dot{w}_3 = -0.5w_3 - 37.1w_4 + 13.8w_4^2 + 4.62w_4^3 \\
\quad - 0.186w_3w_4 - 0.103w_3w_4^2 + 0.003w_3^2w_4 \\
\quad + (6\times10^{-4})\cdot w_3^2 - (2\times10^{-5})\cdot w_3^3 \\
\dot{w}_4 = w_3 + (4\times10^{-6})\cdot w_3^2 + 0.093w_4^2 - 0.0013w_3w_4 \\
\quad - (1\times10^{-7})\cdot w_3^3 + 0.03w_4^3 + (2\times10^{-5})w_3^2w_4 \\
\quad - (7\times10^{-4})\cdot w_3w_4^2
\end{cases}
\tag{61}
$$

Simplify (61) to (62) under assumption in (52) and (53):

$$
\begin{cases}
\dot{w}_1 = -166.0w_2 + 5.0w_2^2 + 32.7w_2^3 \\
\dot{w}_2 = w_1 \\
\dot{w}_3 = -37.1w_4 + 13.8w_4^2 + 4.62w_4^3 \\
\dot{w}_4 = w_3
\end{cases}
\tag{62}
$$

Compare (62) with (61), the terms ignored according to (53)-(54) are actually either small or related to the damping effects. Thus, the stability analysis results on (62) are expected to be conservative for systems in (61). Then, the first-integral based energy functions for the two modes are calculated to be (63).

$$
\begin{cases}
V_1(w_1, w_2) = \dfrac{w_1^2}{2} - 83w_2^2 + 1.6667w_2^3 + 8.175w_2^4 \\
V_2(w_3, w_4) = \dfrac{w_3^2}{2} - 18.55w_4^2 + 4.6w_4^3 + 1.155w_4^4
\end{cases}
\tag{63}
$$

Let the right hand side of (62) be zeros and solve for the closest UEPs and get $w_{2,\mathrm{UEP}} = 2.181$ and $w_{4,\mathrm{UEP}} = 1.711$. The critical energy for the two modes are

TABLE 2. Initial energy of NDST systems under different fault durations.

Fault Duration (s)	$V_1(w_1(0), w_2(0))$	$V_2(w_3(0), w_4(0))$
0.01	0.0037	3.2085
0.05	0.0193	4.6490
0.10	0.1208	9.6184
0.15	0.4283	19.502
0.16	0.4982	22.244

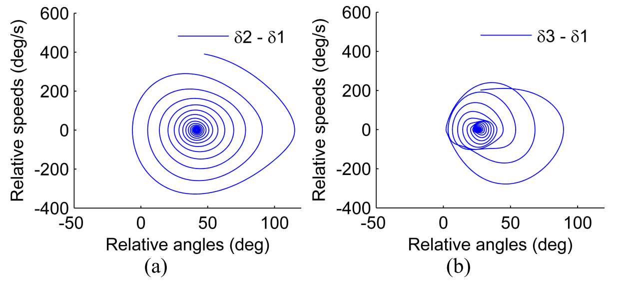

FIGURE 6. Angle-speed trajectories of relative coordinates.

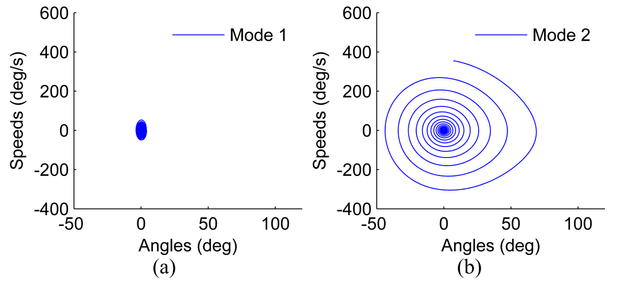

FIGURE 7. Angle-speed trajectories in decoupled systems.

$V_1(0, w_{2,\mathrm{UEP}}) = 193.671$ and $V_2(0, w_{4,\mathrm{UEP}}) = 21.402$. Under different fault durations, the initial energy of the decoupled systems is shown in Table II, which tells that the initial energy of the system corresponding to the second mode first exceeds its critical energy when the fault duration reaches 0.16s, while the initial energy corresponding to the first mode is always much smaller than its critical energy. Table II also shows that the CCT found by this analysis is 0.15s, which is fairly accurate when compared to 0.16s, i.e. the "true" CCT from the 3$^{\mathrm{rd}}$-order Taylor expansion of the original system in (56).

Another benefit of the NMD is that the responses of each decoupled system can be drawn in the corresponding coordinates as a trajectory only about one mode. In that sense, the original system's trajectories regarding different modes are also nonlinearly decoupled. For the marginally stable case with fault duration = 0.15s, Fig. 6 plots the trajectories of the original system in different coordinates while Fig. 7 visualizes the modal trajectories in the coordinates about each decoupled mode. In this case, both oscillatory modes of the system are excited, so trajectories in the original coordinates may be tangled. However, the trajectory of each decoupled system is clean and easier to analyze.

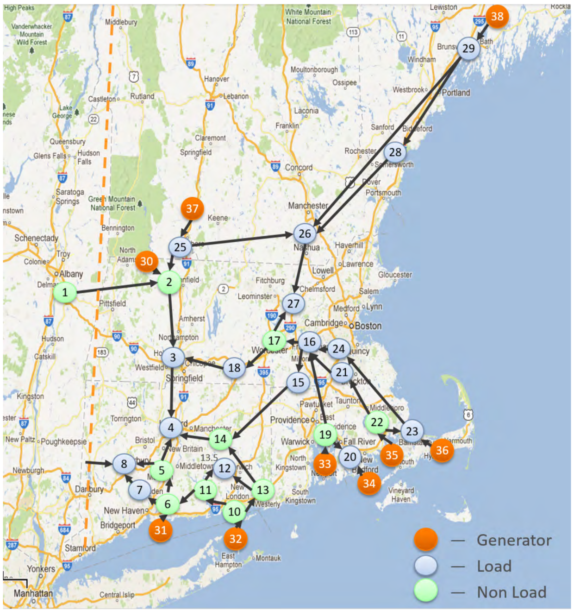

FIGURE 8. New England 39-bus power system.

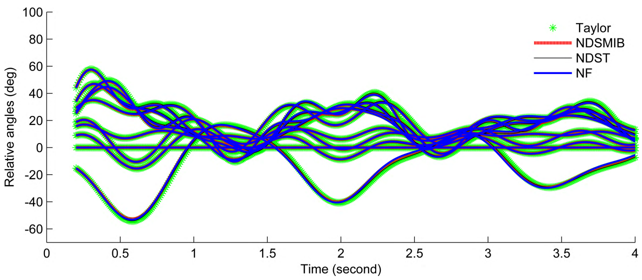

FIGURE 9. Simulated system responses under the disturbance (fault duration = 0.2s) respectively by 2^{nd}-order Taylor expansion, NDST, NF and NFSMIB.

B. TEST ON THE NEW ENGLAND 39-BUS SYSTEM

This subsection will test the proposed NMD on the New England 10-machine, 39-bus power system as shown in Fig. 8 [56]. Using the 2^{nd}-order Taylor expansion of the 20 nonlinear differential equations to formulate a 9-oscillator system and two sets of decoupled 2-jets can be obtained respectively under the two assumptions in Section IV. With the same desktop computer used above, the time cost of NMD is 1810 seconds, i.e. about 0.5 hour. For applications involving analyzing system dynamic behaviors around a specific stable equilibrium point, the NMD can be offline derived for once and such time cost is not an issue. However, for real-time applications requiring very fast algorithms, the current time cost is too high and needs to be reduced. The current implementation of NMD using the Symbolic Math Toolbox treats all expressions as symbolic variables/functions and thereby is not efficient. A computationally efficient implementation only handling the coefficients of polynomials should be achievable and will be the future work.

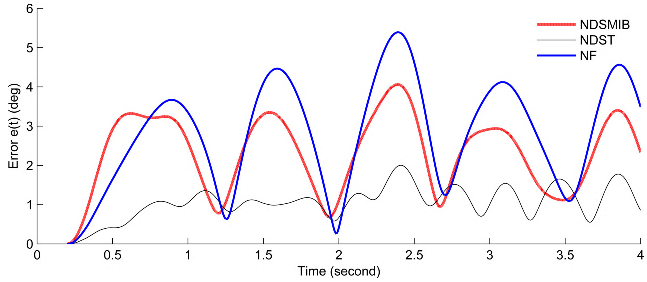

FIGURE 10. Time domain error of the simulated system responses under the disturbance (fault duration = 0.2s) respectively by NDSMIB, NDST and NF.

A three-phase fault is added on bus 16 and cleared after 0.2 second by disconnecting the line 15-16. With the same initial condition under this fault, the 2^{nd}-order Taylor expansion of the original system, the two decoupled 2-jets, and the 2^{nd}-order normal form are simulated and compared in the original space, as shown in Fig. 9 and Fig. 10. Similar to the case study on the IEEE 9-bus system, the error of NDST is the smallest among the three.

V. CONCLUSION

This paper proposes the nonlinear modal decoupling (NMD) analysis to transform a general multi-oscillator system into a set of decoupled 2^{nd}-order nonlinear single oscillator systems with polynomial nonlinearities up to a given order. Since the decoupled systems are low-dimensional and independent with each other up to the given order, they can be easier analyzed compared to the original system.

The derivation of the NMD adopts an idea similar to the idea of the normal form method, and the decoupling transformation turns out to be the composition of a set of nonlinear homogeneous polynomial transformations. The key step in deriving the NMD is the elimination of the inter-modal terms and the retention of nonlinearities only related to the intra-modal terms. The elimination of inter-modal terms can be achieved uniquely while the intra-modal terms could be maintained in an infinite number of ways such that a desired form has to be specified.

Then, the NMD analysis is applied to power systems toward two desired forms of decoupled systems: (i) the single-machine-infinite-bus (SMIB) assumption; (ii) the small transfer (ST) assumption. Note that the ST assumption does not limit mode-decoupled systems to the power system models; rather, they can be any other type of oscillator systems if a priori knowledge or preference on the form of mode-decoupled systems is not available. Numerical studies on both a small IEEE 3-machine 9-bus system and a larger New England 10-machine 39-bus system show that the decoupled system under the ST assumption has a larger validity region than the decoupled systems under the SMIB assumption and the transformed linear system from the normal form method. It is also demonstrated that the decoupled systems enable easier and fairly accurate analyses, e.g. on stability of the original system.

Appendix

The linear transformation for transforming (56) to (57) is

$$\left\{\begin{aligned}
x_1 &= (-0.020 + j4\times 10^{-4})y_1 - (0.020 + j4\times 10^{-4})y_2\\
&\quad - 0.577y_5 - (0.164 - j0.007)y_3\\
&\quad - (0.164 + j0.007)y_4 - 0.516y_6\\
x_2 &= -j0.253y_1 + j0.253y_2 - jy_3 + jy_4 + 0.258y_6\\
x_3 &= (-0.121 + j0.002)y_1 - (0.121 + j0.002)y_2\\
&\quad - 0.577y_5 + (0.368 - j0.015)y_3\\
&\quad + (0.368 + j0.015)y_4 - 0.516y_6\\
x_4 &= -j1.56y_1 + j1.56y_2 + j2.24y_3 - j2.24y_4 + 0.258y_6\\
x_5 &= (0.412 - j0.008)y_1 + (0.412 + j0.008)y_2 - 0.577y_5\\
&\quad + (0.198 - j0.008)y_3 + (0.198 + j0.008)y_4\\
&\quad - 0.516y_6\\
x_6 &= j5.31y_1 - j5.31y_2 + j1.2y_3 - j1.2y_4 + 0.258y_6
\end{aligned}\right. \tag{A1}$$

The two nonlinear transformations for transforming (57) to (59) are

$$\left\{\begin{aligned}
y_1 &= z_1^{(2)} + (-0.117 + j0.010)z_1^{(2)}z_3^{(2)}\\
&\quad + (0.117 + j0.010)z_1^{(2)}z_4^{(2)}\\
&\quad + (0.036 + j4\times 10^{-4})z_2^{(2)}z_3^{(2)}\\
&\quad + (0.022 + j0.002)z_2^{(2)}z_4^{(2)}\\
&\quad + (0.012 + j4\times 10^{-5})z_3^{(2)}z_4^{(2)}\\
&\quad + (0.093 + j0.025)\left(z_3^{(2)}\right)^2\\
&\quad + (0.003 + j3\times 10^{-4})\left(z_4^{(2)}\right)^2\\
y_2 &= \bar{y}_1\\
y_3 &= z_3^{(2)} + (0.144 + j0.012)z_1^{(2)}z_2^{(2)}\\
&\quad + (0.004 - j1\times 10^{-4})z_1^{(2)}z_3^{(2)}\\
&\quad + (0.062 - j0.017)z_1^{(2)}z_4^{(2)}\\
&\quad - (0.004 + j1\times 10^{-4})z_2^{(2)}z_3^{(2)}\\
&\quad - (0.002 + j2\times 10^{-4})z_2^{(2)}z_4^{(2)}\\
&\quad - (0.022 - j2\times 10^{-4})\left(z_1^{(2)}\right)^2\\
&\quad + (0.014 + j0.001)\left(z_2^{(2)}\right)^2\\
y_4 &= \bar{y}_3
\end{aligned}\right. \tag{A2}$$

$$\left\{\begin{aligned}
z_1^{(2)} &= z_1^{(3)} + (0.002 - j2\times 10^{-4})\left(z_1^{(3)}\right)^2 z_3^{(3)}\\
&\quad - (0.003 - j0.002)\left(z_1^{(3)}\right)^2 z_4^{(3)}\\
&\quad + (0.007 + j0.002)z_1^{(3)}z_2^{(3)}z_3^{(3)}\\
&\quad + (1\times 10^{-4} - j8\times 10^{-4})z_1^{(3)}z_2^{(3)}z_4^{(3)}\\
&\quad - (0.024 - j0.002)z_1^{(3)}\left(z_3^{(3)}\right)^2\\
&\quad - (0.007 - j1.47)z_1^{(3)}z_3^{(3)}z_4^{(3)}\\
&\quad + (0.039 + j0.004)z_1^{(3)}\left(z_4^{(3)}\right)^2\\
&\quad + (0.001 + j5\times 10^{-4})\left(z_2^{(3)}\right)^2 z_3^{(3)}\\
&\quad - (0.001 + j2\times 10^{-4})\left(z_2^{(3)}\right)^2 z_4^{(3)}\\
&\quad + (0.025 + j3\times 10^{-4})z_2^{(3)}\left(z_3^{(3)}\right)^2\\
&\quad + (0.018 + j9\times 10^{-4})z_2^{(3)}z_3^{(3)}z_4^{(3)}\\
&\quad + (0.004 + j3\times 10^{-4})z_2^{(3)}\left(z_4^{(3)}\right)^2\\
&\quad + (0.003 + j0.002)\left(z_3^{(3)}\right)^3\\
&\quad - (0.010 + j0.008)\left(z_3^{(3)}\right)^2 z_4^{(3)}\\
&\quad + (0.002 - j0.003)z_3^{(3)}\left(z_4^{(3)}\right)^2\\
&\quad + (0.003 - j3\times 10^{-4})\left(z_4^{(3)}\right)^3\\
z_2^{(2)} &= \bar{z}_1^{(2)}\\
z_3^{(2)} &= z_3^{(3)} - (3\times 10^{-4} + j7\times 10^{-6})\left(z_1^{(3)}\right)^3\\
&\quad + (0.003 - j2\times 10^{-4})\left(z_1^{(3)}\right)^2 z_2^{(3)}\\
&\quad - (0.003 - j4\times 10^{-6})\left(z_1^{(3)}\right)^2 z_3^{(3)}\\
&\quad - (0.021 + j7\times 10^{-4})\left(z_1^{(3)}\right)^2 z_4^{(3)}\\
&\quad - (0.003 + j3\times 10^{-4})z_1^{(3)}\left(z_2^{(3)}\right)^2\\
&\quad + (0.002 + j1.43)z_1^{(3)}z_2^{(3)}z_3^{(3)}\\
&\quad + (0.057 + j0.008)z_1^{(3)}z_2^{(3)}z_4^{(3)}\\
&\quad - (0.008 + j8\times 10^{-5})z_1^{(3)}\left(z_3^{(3)}\right)^2\\
&\quad - (0.032 - j0.009)z_1^{(3)}z_3^{(3)}z_4^{(3)}\\
&\quad + (0.040 - j0.004)z_1^{(3)}\left(z_4^{(3)}\right)^2\\
&\quad - (5\times 10^{-4} + j6\times 10^{-5})\left(z_2^{(3)}\right)^3\\
&\quad + (0.011 + j5\times 10^{-4})\left(z_2^{(3)}\right)^2 z_3^{(3)}\\
&\quad + (0.002 + j2\times 10^{-4})\left(z_2^{(3)}\right)^2 z_4^{(3)}\\
&\quad + (0.026 + j0.009)z_2^{(3)}\left(z_3^{(3)}\right)^2\\
&\quad + (0.008 + j0.002)z_2^{(3)}z_3^{(3)}z_4^{(3)}\\
&\quad + (0.004 - j2\times 10^{-4})z_2^{(3)}\left(z_4^{(3)}\right)^2\\
z_4^{(2)} &= \bar{z}_3^{(2)}
\end{aligned}\right. \tag{A3}$$

The linear transformation for transforming (59) to (61) is

$$\left\{\begin{aligned}
z_1^{(3)} &= -j0.078w_1 + (1 - j0.019)w_2\\
z_2^{(3)} &= j0.078w_1 + (1 + j0.019)w_2\\
z_3^{(3)} &= -j0.164w_3 + (1 - j0.041)w_4\\
z_4^{(3)} &= j0.164w_3 + (1 + j0.041)w_4
\end{aligned}\right. \tag{A4}$$

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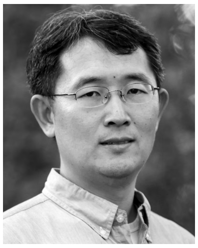

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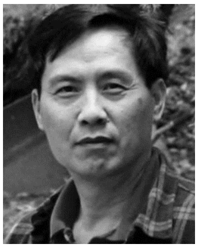

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