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Stochastic Exponential Robust Stability of Delayed Complex-Valued Neural Networks With Markova Jumping Parameters

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ABSTRACT This paper deals with the problem on stochastic exponential robust stability for a class of complex-valued interval neural networks with Markova jumping parameters and mixed delays, including both time-varying delays and continuously distributed delays. By applying the M-matrix theory and coupling with the vector Lyapunov function method, some sufficient conditions are derived to guarantee the existence, uniqueness, and stochastic exponential robust stability of the equilibrium point of the addressed system. The obtained results not only are easy to judge the dynamical behavior of the addressed system, but also are with lower level conservatism in comparison with some existing results. Finally, two numerical examples with simulation results are given to illustrate the effectiveness of the proposed results.

INDEX TERMS Interval neural networks, complex value, Markova jumping parameters, mixed delays, stochastic exponential robust stability, vector Lyapunov function.

I. INTRODUCTION

As an extension of real-valued neural networks, complex-valued neural networks with complex-valued state, input, connection weight and activation function become strongly desired because of their practical applications in a variety of areas dealing with electromagnetic, light, ultrasonic and quantum waves [1], [2]. Therefore, there has been an increasing interest in the research of complex-valued neural networks, and some significant results were obtained in both theory and applications, see [2]–[20].

In the aspect of model for describing complex-valued neural networks, Hu and Wang [9] firstly studied a class of complex-valued neural networks with constant delays and proposed some sufficient conditions for guaranteeing the stability of the equilibrium point of the addressed system. Considering that the time delay is variable and unbounded in the most situations, it is necessary to introduce mixed time delays including both time-varying delay and continuously distributed delay into the model of complex-valued neural networks [16]. Xu *et al.* [16], [21] and Song *et al.* [22] introduced mixed delays into models of complex-valued neural networks and proposed some sufficient conditions for judging the exponential stability of the equilibrium point of the addressed systems. Besides, impulsive disturbances also widely exist in physical dynamical systems [21]–[24]. Therefore, some significant results have been obtained for analyzing the dynamical behavior of complex-valued neural networks with both impulsive effects and time delays. Xu *et al.* [21] investigated the stability problem for a class of impulsive complex-valued neural networks with both time-varying and continuously distributed delays. In [23], some sufficient conditions were obtained for guaranteeing the global asymptotic stability of a class of fractional-order complex-valued neural networks with impulsive effect and time delay by employing contraction mapping principle, comparison theorem and inequality scaling skills. Other significant results concerning delayed complex-valued neural networks with impulsive effect can be referred in [4], [22], and [24].

In the aspect of assumption conditions given for complex-valued neural networks, it is well known that the main challenge is the choice of activation function in complex number domain. By separating and explicitly expressing complex-valued activation functions into their real and imaginary parts, some results on stability and synchronization were given for various complex-valued neural networks. It can be referred in [2], [3], [16], [21], and [23]. As pointed by Xu *et al.* [24] and Pan *et al.* [25], this separation is not always expressible in an analytical form. When the activation functions are not separated into their real parts and imaginary parts, some stability criteria of complex-valued neural networks were also obtained under activation functions satisfying the Lipschitz continuity condition in the complex number domain, see references [22], [24], [25]. The existence, continuity and boundedness of partial derivatives of complex-valued activation function with respect to its real part and imaginary were removed in the mentioned references. The obtained results in [22], [24], and [25] are with lower level of conservatism. Reducing the restriction on the activation functions in both real number domain and complex number domain is a topic of continuous improvement.

In the aspect of results for judging dynamic behavior of complex-valued neural networks, there are two acceptable categories. When a neural network is designed to function as an associative memory, it is desired that the neural network should have multiple equilibrium points. The corresponding subject is to discuss Lagrange stability or Multistability of the system. The related research can be referred in [2]–[5], [19], [26], and [27]. On the other hand, for the problem of optimal computation, the neural network is designed to have a unique equilibrium point with global stability. That is, the existence and uniqueness of the equilibrium point should be discussed, which is the stability in Lyapunov sense. There are main three methods for proving the existence and uniqueness of the equilibrium point of neural networks, which are respectively homeomorphism mapping principle [16], contracting mapping principle [23] and Brouwer's fixed point theorem [25]. In [15], [16], [21], and [23]–[25], scholars established some sufficient conditions for assuring the existence, uniqueness and stability of the equilibrium point of various complex-valued neural networks.

To the best of our knowledge, in the published literatures concerning complex-valued neural networks scholars did not consider Markov jumping parameters in models of the complex-valued systems. The phenomenon of information latching sometimes happens in the physical system. A widely used approach to deal with the information latching problem is to extract finite state representations. In other words, the system may have finite modes which switch from one to another at different times. The switching among different system modes can be governed by a Markov chain. In past two decades, although various complex systems including neural networks with Markov jumping parameters have been studied and some significant results concerning dynamical behaviors analysis and controller design for systems were obtained in [28]–[41] and references therein, there are few literatures concerning the research for complex-valued neural networks with Markov jumping parameters and interval interconnected matrices. From the view of both theory and application, it is necessary to analyze the stochastic stability of the equilibrium point of complex-valued neural networks with Markov jumping parameters.

Based on the above discussion, in this paper we will investigate the dynamical behavior for a class of complex-valued neural networks with mixed delays and Markov jumping parameters. In this paper, advantages and contributions are listed as follows: (1) Both Markov jumping parameters and mixed delays including time-varying delays and continuously distributed delays are introduced in the model of the addressed system, which have not been considered in existing literatures. (2) The complex-valued activation functions are supposed to satisfy the Lipschitz condition without separating into real parts and imaginary parts, which remove the restriction for the choice of complex-valued activation functions. (3) The existence and uniqueness of the equilibrium point of the system are analyzed by using theory of the homeomorphism mapping instead of contracting mapping principle and Brouwer's fixed point theorem. (4) Some sufficient conditions for assuring stochastic exponential robust stability in Lyapunov sense are established in terms of simple forms of matrix, which are easy to be checked in application. (5) Two numerical examples with simulation results and remarks concerning the choice of the complex-valued activations are given to illustrate the obtained results.

The rest of this paper is organized as follows. In Section 2, model description and preliminaries including assumption and lemmas are given. In Section 3, a theorem and several corollaries are proposed for the stability of unique equilibrium of complex-valued neural networks with mixed delays and Markov jumping parameters. In Section 4, two numerical examples with simulation results are given to illustrate the effectiveness of the main results. Finally, in Section 5, the conclusion and future work are drawn.

II. MODEL DESCRIPTION AND PRELIMINARIES

The following notations are used throughout this paper. Let $\mathbf{C}$ denote complex number set, $\mathbf{N}$ denote natural number set and $\mathbf{R}$ denote real number set. Let $|z| = \sqrt{(\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2}$ be the module of complex number z, where $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are the real part and the imaginary part of complex number z. For complex number vector $z \in \mathbf{C}^n$, let $|z| = (|z_1|, |z_2|, \ldots, |z_n|)^{\mathrm{T}}$ be the module of the vector z, here $(.)^{\mathrm{T}}$

denotes the transpose of vector. Let $||z||_\infty = \max_{1\le i\le n}\{|z_i|\}$ and $||z||_1 = \sum_{i=1}^{n}|z_i|$ be the ∞-norm and 1-norm of the vector z respectively. Let (F, P) be a complete probability space with a filtration $\{F_t\}_{t\ge 0}$ satisfying the usual conditions, i.e. the filtration contains all P-null sets and is right continuous.

Let $r(t), t \ge 0$ be a continuous-time Markova chain taking values in a finite space $S = \{1, 2, \ldots, N\}$ with transition probability parameters given by

$$P\{r(t+\Delta t) = m|r(t) = k\} = \begin{cases} \pi_{km}\Delta t + O(\Delta t), & k \ne m \\ 1 + \pi_{kk}\Delta t + O(\Delta t), & k = m \end{cases} \quad (1)$$

where $\Delta t > 0$, $\lim_{\Delta t\to 0} O(\Delta t)/\Delta t = 0$. In (1), denote $\pi_{km} > 0$ the transition rate from k to m if $k \ne m$ while $\pi_{kk} = -\sum_{m=1, m\ne k}^{N}\pi_{km}$. Let E{·} stand for the mathematical expectation operator with respect to the given probability measure P.

In this paper, considering a class of complex-valued interval neural networks with time-varying delays, continuously distributed delays and Markov jumping parameters, which can be described as follows:

$$\begin{aligned}\frac{dz_i(t)}{dt} &= -d_i(r(t))z_i(t) + \sum_{j=1}^{n}[a_{ij}(r(t))f_j(z_j(t)) \\ &\quad + b_{ij}(r(t))f_j(z_j(t-\tau_{ij}(t))) \\ &\quad + p_{ij}(r(t))\int_{-\infty}^{t}\theta_{ij}(t-s)f_j(z_j(s))ds] + J_i(t) \quad (2)\end{aligned}$$

where $z_i \in \mathbf{C}$ represents the neuron state; n is the number of neurons; $\boldsymbol{A}(r(t)) = (a_{ij}(r(t)))_{n\times n}$, $\boldsymbol{B}(r(t)) = (b_{ij}(r(t)))_{n\times n}$ and $\boldsymbol{P}(r(t)) = (p_{ij}(r(t)))_{n\times n}$ are the connection weight matrices defined in complex number domain; $\boldsymbol{J}(t) = (J_1(t), J_2(t), \ldots, J_n(t))^T \in \mathbf{C}^n$ is external input vector; $d_i(r(t)) > 0$ denotes the self-feedback coefficient; $f_j(z_j(t))$ represents the activation function, $j = 1, 2, \ldots, n$; $\tau_{ij}(t)$ $(i, j = 1, 2, \ldots, n)$ are bounded functions with $\tau = \max_{1\le i,j\le n}\sup_{t\ge 0}\tau_{ij}(t)$. Let θ_{ij}: $[0, +\infty) \to [0, +\infty)$ be piecewise continuous functions and satisfy

$$\int_0^{+\infty}\exp(\beta s)\theta_{ij}(s)ds = \mu_{ij}(\beta), \quad i, j = 1, 2, \ldots, n \quad (3)$$

where $\mu_{ij}(\beta)$ is continuous on $[0, \delta)$, and $\mu_{ij}(0) = 1$.

For convenience, for arbitrary mode k $(k \in S)$, we define the following notions:

$$\boldsymbol{A}(k) = \boldsymbol{A}(r(t) = k) = [a_{ij}(r(t) = k)]_{n\times n} = (a_{ij}^k)_{n\times n}$$
$$\boldsymbol{B}(k) = \boldsymbol{B}(r(t) = k) = [b_{ij}(r(t) = k)]_{n\times n} = (b_{ij}^k)_{n\times n}$$
$$\boldsymbol{P}(k) = \boldsymbol{P}(r(t) = k) = [p_{ij}(r(t) = k)]_{n\times n} = (p_{ij}^k)_{n\times n}$$
$$\boldsymbol{D}(k) = \boldsymbol{D}(r(t)=k) = \mathrm{diag}[d_i(r(t) = k)]_{n\times n} = \mathrm{diag}(d_i^k)_{n\times n}.$$

Let $|\boldsymbol{A}|$ denote the module of matrix $\boldsymbol{A} = (a_{ij})_{n\times n} \in \mathbf{C}^{n\times n}$, which is given by $|\boldsymbol{A}| = (|a_{ij}|)_{n\times n} \in \mathbf{R}^{n\times n}$, where $|a_{ij}| = \sqrt{[\mathrm{Re}(a_{ij})]^2 + [\mathrm{Im}(a_{ij})]^2}$.

In (2), the interval weight matrices in complex number domain are defined as follows:

$$\boldsymbol{A}_I(k) = \{|\boldsymbol{A}(k)| = (|a_{ij}^k|)_{n\times n} : |\underset{\sim}{\boldsymbol{A}}(k)| \le |\boldsymbol{A}(k)| \le |\tilde{\mathbf{A}}(k)|,$$
$$i.e.\ \underset{\sim}{a}_{ij}^k| \le |a_{ij}^k| \le |\tilde{a}_{ij}^k|, \quad i, j = 1, 2, \ldots, n\}$$
$$\boldsymbol{B}_I(k) = \{|\boldsymbol{B}(k)| = (|b_{ij}^k|)_{n\times n} : |\underset{\sim}{\boldsymbol{B}}(k)| \le |\boldsymbol{B}(k)| \le |\tilde{\boldsymbol{B}}(k)|,$$
$$i.e.\ |\underset{\sim}{b}_{ij}^k| \le |b_{ij}^k| \le |\tilde{b}_{ij}^k|, \quad i, j = 1, 2, \ldots, n\}$$
$$\boldsymbol{P}_I(k) = \{|\boldsymbol{P}(k)| = (|p_{ij}^k|)_{n\times n} : |\underset{\sim}{\boldsymbol{P}}(k)| \le |\boldsymbol{P}(k)| \le |\tilde{\boldsymbol{P}}(k)|,$$
$$i.e.\ \underset{\sim}{p}_{ij}^k| \le |p_{ij}^k| \le |\tilde{p}_{ij}^k|, \quad i, j = 1, 2, \ldots, n\}.$$

For every fixed mode of complex-valued neural networks, $\boldsymbol{A}_I(k), \boldsymbol{B}_I(k)$ and $\boldsymbol{P}_I(k)$ are known constant matrices sets with appropriate dimensions.

It is assumed that initial conditions of (2) are $z_i(s) = \varphi_i(s)$, here $\varphi_i(s)$ are bounded and continuous on $(-\infty, 0]$, $i = 1, 2, \ldots, n$.

Denote $\boldsymbol{z}^{\#} = (z_1^{\#}, z_2^{\#}, \ldots, z_n^{\#})^{\mathrm{T}} \in \mathbf{C}^n$ the equilibrium point of (2).

Definition 1: The equilibrium point $\boldsymbol{z}^{\#}$ of (2) is defined to be stochastically exponentially robustly stable, if for every system mode k, there exist constants $M(k) > 0$ and $\lambda > 0$ such that for all $\boldsymbol{A}(k) \in \boldsymbol{A}_I(k), \boldsymbol{B}(k) \in \boldsymbol{B}_I(k), \boldsymbol{P}(k) \in \boldsymbol{P}_I(k)$, $t \ge 0$ and $k \in S$, the inequality $E\{||z(t) - z^{\#}, r(t) = k||\} \le M(k)E\{\sup_{s\in(-\infty,0]}||\boldsymbol{\varphi}(s) - \boldsymbol{z}^{\#}||\}\exp(-\lambda t)$ holds.

Remark 1: It is well known how to choose activation functions is the main challenge for complex-valued neural networks. The activation functions in neural networks defined in real number domain are usually chosen to be with smoothness and boundedness. However, it follows from Liouville's Theorem [2], [9] that every function with boundedness and analyticity in the entire complex domain must be reduced to a constant. That is to say, the activation functions in complex-valued neural networks cannot be bounded and analytic. In order to remove the restriction conditions for complex-valued activation functions, the following assumption is given.

Assumption 1: Each function $f_i(.)$ is globally Lipschitz with Lipschitz constant $l_i > 0$, i.e. the inequality $|f_i(u_i(t)) - f_i(v_i(t))| \le l_i|u_i(t) - v_i(t)|$ holds for all $u_i(t)$, $v_i(t) \in \mathbf{C}$, $i = 1, 2, \ldots, n$. Let $\mathbf{L} = \mathrm{diag}(l_1, l_2, \ldots, l_n)$.

Remark 2: The complex-valued activation functions were supposed to need an explicit separation into their real parts and imaginary parts in references [3], [9], [16], and [21]. However, the separation is not always expressible in an analytical form. Activation functions in complex number domain satisfying Assumption 1 are actually the extension of the real-valued functions satisfying the Lipschitz continuity condition.

The following lemmas will be used to obtain results in this paper.

Lemma 1 [16]: Let $\boldsymbol{A} = (a_{ij})_{n\times n} \in \mathbf{R}^{n\times n}$ be a matrix with $a_{ij} \le 0, (i, j = 1, 2, \cdots, n,\ i \ne j)$. The following statements are equivalent.

(i) $\boldsymbol{A} = (a_{ij})_{n\times n}$ is a M-matrix.
(ii) The real parts of all eigenvalues of $\boldsymbol{A}$ are positive.
(iii) There exists a positive vector $\boldsymbol{\xi} \in \mathbf{R}^n$ such that $\boldsymbol{H}\boldsymbol{\xi} > 0$.

In this paper, we will use homeomorphism mapping principle to analyze the existence and uniqueness of the equilibrium point of neural networks. An important lemma concerning homeomorphism mapping principle is given as follows.

Lemma 2 [16]: If $\boldsymbol{H}(z)$ is a continuous function on $\mathbf{C}^n$, and satisfies the following conditions:

(i) $\boldsymbol{H}(z)$ is univalent injective on $\mathbf{C}^n$;
(ii) $\lim_{||z||\to\infty} ||\boldsymbol{H}(z)|| \to \infty$.

Then $\boldsymbol{H}(z)$ is a homeomorphism of $\mathbf{C}^n$ into itself.

III. MAIN RESULTS

In this section, we will give some sufficient conditions for judging the dynamical behavior of the equilibrium point of (2).

Theorem 1: Suppose that Assumption 1 is satisfied for arbitrary input $\boldsymbol{J} \in \mathbf{C}^n$, then the equilibrium point $z^\#$ of (2) is with existence and uniqueness if there exists a positive number $T \geq 1$ such that every matrix $\boldsymbol{Q}(k) = (q_{ij}^k)_{n\times n}$ is a M-matrix, where

$$q_{ii}^k = 2d_i^k - \left(\sum_{m=1,m\neq k}^{N} \pi_{km}T + \pi_{kk}\right)$$

$$q_{ij}^k = -2\sum_{j=1}^{n} \sqrt{T} l_j \left(|\tilde{a}_{ij}^k| + |\tilde{b}_{ij}^k| + |\tilde{p}_{ij}^k|\right)$$

where $k \in S$, $i, j = 1, 2, \ldots, n$. Besides, the equilibrium point $z^\#$ of (2) is stochastically exponentially robustly stable.

Proof: Let

$$V_i(t, z_i(t) - z_i^\#, r(t) = k) = \omega_i^k \exp(\lambda t)|z_i(t) - z_i^\#|^2$$

where ω_i^k are a sequence of positive numbers, $i = 1, 2, \ldots, n$, $k \in S$. Assume that $w^k \leq \omega_i^k \leq W^k$, $i = 1, 2, \ldots, n$. Define $w_{\min} = \min_{1\leq k\leq N}\{w^k\}$ and $W_{\max} = \max_{1\leq k\leq N}\{W^k\}$, choose $T = W_{\max}/w_{\min} \geq 1$.

Firstly, the existence and uniqueness of the equilibrium point $z^\#$ of (2) will be proved for every fixed mode by using the corresponding properties of homeomorphism and M-matrix.

Defining a map $\boldsymbol{H}^k(z) = [H_1^k(z),\ H_2^k(z),\ \ldots\ , H_n^k(z)]^T$ associated with (2) with the following forms:

$$H_i^k(z) = -d_i^k z_i + \sum_{j=1}^{n} \left(a_{ij}^k + b_{ij}^k + p_{ij}^k\right) f_j(z_j) + J_i \tag{4}$$

where $i = 1, 2, \ldots, n$, $k \in S$.

It is well known that if $\boldsymbol{H}^k(z)$ is a homeomorphism on $\mathbf{C}^n$, then (2) has a unique equilibrium point $z^\#$ obviously.

(i) We prove that the map $\boldsymbol{H}^k(z)$ is univalent injective on $\mathbf{C}^n$ under the Assumption 1.

Because the matrix $\boldsymbol{Q}(k)$ is a M-matrix, it is obvious that the matrix $\underline{Q}(k) = (\underline{q}_{ij}^k)_{n\times n}$ is a M-matrix, where

$$\underline{q}_{ii}^k = d_i^k; \quad \underline{q}_{ij}^k = -\sum_{j=1}^{n} l_j \left(|\tilde{a}_{ij}^k| + |\tilde{b}_{ij}^k| + |\tilde{p}_{ij}^k|\right), \quad i = 1, 2, \ldots, n.$$

From Lemma 1 it can be concluded that there exists a positive vector $\boldsymbol{\xi}(k)$ such that the following inequalities hold for $i = 1, 2, \ldots, n$, $k \in S$:

$$-d_i^k \xi_i^k + \sum_{j=1}^{n} \xi_j^k \cdot l_j \left(|\tilde{a}_{ij}^k| + |\tilde{b}_{ij}^k| + |\tilde{p}_{ij}^k|\right) < 0. \tag{5}$$

Moreover, it follows from (5) that there exists a sufficient small positive number $\varepsilon > 0$ such that the following inequalities hold for all $i = 1, 2, \ldots, n$, $k \in S$:

$$d_i^k \xi_i^k - \sum_{j=1}^{n} \xi_j^k \cdot l_j \left(|\tilde{a}_{ij}^k| + |\tilde{b}_{ij}^k| + |\tilde{p}_{ij}^k|\right) \geq \varepsilon > 0. \tag{6}$$

It is assumed that there exist $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{C}^n$ with $\boldsymbol{u} \neq \boldsymbol{v}$, such that $H_i^k(\boldsymbol{u}) = H_i^k(\boldsymbol{v})$, $i = 1, 2, \ldots, n$, $k \in S$, i.e.,

$$-d_i^k u_i + \sum_{j=1}^{n} \left(a_{ij}^k + b_{ij}^k + p_{ij}^k\right) f_j(u_j) = -d_i^k v_i + \sum_{j=1}^{n} \left(a_{ij}^k + b_{ij}^k + p_{ij}^k\right) f_j(v_j). \tag{7}$$

Considering Assumption 1, for $i = 1, 2, \ldots, n$ and $k \in S$, we get that

$$d_i^k |u_i - v_i| \leq \sum_{j=1}^{n} l_j \left(|\tilde{a}_{ij}^k| + |\tilde{b}_{ij}^k| + |\tilde{p}_{ij}^k|\right) |u_j - v_j|. \tag{8}$$

Furthermore, (8) can be rewritten as $\underline{Q}(k)|\boldsymbol{u} - \boldsymbol{v}| \leq 0$. Because $\underline{Q}(k)$ is a M-matrix, we know that $\det \underline{Q}(k) > 0$ holds and $\underline{Q}(k)^{-1}$ exists. Furthermore, it can be concluded that $|\boldsymbol{u} - \boldsymbol{v}| = 0$, i.e. $\boldsymbol{u} = \boldsymbol{v}$. It is a contradiction with the assumption $\boldsymbol{u} \neq \boldsymbol{v}$. Hence the map $\boldsymbol{H}^k(z)$ is univalent injective on $\mathbf{C}^n$, $k \in S$, $i = 1, 2, \ldots, n$.

(ii) Next we will prove that $\lim_{||z||\to\infty} ||\boldsymbol{H}^k(z)|| \to \infty$.

Let $\tilde{H}_i^k(z) = H_i^k(z) - H_i^k(0)$, $k \in S$, $i = 1, 2, \ldots, n$, i.e.,

$$\tilde{H}_i^k(z) = -d_i^k z_i + \sum_{j=1}^{n} (a_{ij}^k + b_{ij}^k + p_{ij}^k)[f_j(z_j) - f_j(0)]. \tag{9}$$

Multiplying by the conjugate complex number $\bar{z}_i$ of z_i on the both sides of (9), for all $k \in S$, $i = 1, 2, \ldots, n$, we get

$$\tilde{H}_i^k(z)\bar{z}_i = -d_i^k z_i \bar{z}_i + \bar{z}_i \sum_{j=1}^{n} (a_{ij}^k + b_{ij}^k + p_{ij}^k)[f_j(z_j) - f_j(0)]. \tag{10}$$

Taking the conjugate operation on the both sides of (10), for all $k \in S$, $i = 1, 2, \ldots, n$, we have

$$\bar{\tilde{H}}_i^k(z) z_i = -d_i^k \bar{z}_i z_i + z_i \sum_{j=1}^{n} (\bar{a}_{ij}^k + +\bar{b}_{ij}^k + \bar{p}_{ij}^k)[\bar{f}_j(z_j) - \bar{f}_j(0)]. \tag{11}$$

Combining (10) and (11), and considering the Assumption 1, we obtain

$$\begin{aligned}
&\mathrm{Re}[\tilde{H}_i^k(z)\bar{z}_i] \\
&\quad = -d_i^k|z_i|^2 + \mathrm{Re}\{\bar{z}_i\sum_{j=1}^{n}(a_{ij}^k + b_{ij}^k + p_{ij}^k)[f_j(z_j) - f_j(0)]\} \\
&\quad \le -d_i^k|z_i|^2 + |\bar{z}_i|\sum_{j=1}^{n} l_j(|\tilde{a}_{ij}^k| + |\tilde{b}_{ij}^k| + |\tilde{p}_{ij}^k|)|z_j| \qquad (12)
\end{aligned}$$

where $i = 1, 2, \ldots, n, k \in S$.

Multiplying by ξ_i^k ($i = 1, 2, \ldots, n$, $k \in S$) on the both sides of (12) and taking the summation operation, we get

$$\sum_{i=1}^{n}\xi_i^k \mathrm{Re}[\tilde{H}_i^k(z)\bar{z}_i] \le \sum_{i=1}^{n}\xi_i^k|z_i|[-d_i^k|z_i| + \sum_{j=1}^{n} l_j(|\tilde{a}_{ij}^k| + |\tilde{b}_{ij}^k| + |\tilde{p}_{ij}^k|)]|z_j|.$$

Considering (6), we have

$$\sum_{i=1}^{n}\xi_i^k \mathrm{Re}[\tilde{H}_i^k(z)\bar{z}_i] \le -\varepsilon\sum_{i=1}^{n}|z_i|\cdot\sum_{j=1}^{n}|z_j|, \quad k \in S.$$

Furthermore, it can be concluded that

$$\begin{aligned}
\varepsilon\sum_{i=1}^{n}|z_i|\cdot\sum_{j=1}^{n}|z_j| &< -\sum_{i=1}^{n}\xi_i^k \mathrm{Re}[\tilde{H}_i^k(z)\bar{z}_i] \\
&\le \max_{1\le i\le n}\{\xi_i^k\}\sum_{i=1}^{n}|\tilde{H}_i^k(z)|\sum_{i=1}^{n}|z_i|.
\end{aligned}$$

Namely, $\varepsilon||z||_1 \le \max_{1\le i\le n}\{\xi_i^k\}||\tilde{\boldsymbol{H}}^k(z)||_1$. That is to say $||z||_1 \le \varepsilon^{-1}\max_{1\le i\le n}\{\xi_i^k\}||\tilde{\boldsymbol{H}}^k(z)||_1$ holds. Obviously, according to the equivalent principle of norm, we have that $||\tilde{\boldsymbol{H}}^k(z)|| \to \infty$ as $||z|| \to \infty$. It means that $||\boldsymbol{H}^k(z)|| \to \infty$ as $||z|| \to \infty$.

Combining part (i) and part (ii) above, it follows from Lemma 2 that $\boldsymbol{H}^k(z)$ is a homeomorphism on $\mathbf{C}^n$, $k \in S$. Therefore (2) has a unique equilibrium point $z^\#$.

In what follows, the stochastic exponential robust stability of the equilibrium point $z^\#$ will be proved by applying the vector Lyapunov function method.

For analysis convenience, we translate the coordinate of (2). Let $\tilde{\mathbf{z}}(t) = z(t) - z^\#$. By translation, the system (2) is changed into the following forms:

$$\begin{aligned}
\frac{d\tilde{z}_i(t)}{dt} = -d_i(r(t))\tilde{z}_i(t) + \sum_{j=1}^{n}[a_{ij}(r(t))g_j(\tilde{z}_j(t)) \\
+ b_{ij}(r(t))g_j(\tilde{z}_j(t-\tau_{ij}(t))) \\
+ p_{ij}(r(t))\int_{-\infty}^{t}\theta_{ij}(t-s)g_j(\tilde{z}_j(s))ds] \qquad (13)
\end{aligned}$$

where $g_j(\tilde{z}_j(t)) = f_j(\tilde{z}_j(t) + z_j^\#) - f_j(z_j^\#)$, $i, j = 1, 2, \ldots, n$.

Let the initial conditions of (13) be with forms $\psi_i(s) = \varphi_i(s) - z_i^\#$, $i = 1, 2, \ldots, n$, $-\infty < s \le 0$.

Obviously, if the zero solution of (13) is stochastically exponentially robustly stable, the equilibrium point $z^\#$ of (2) is also stochastically exponentially robustly stable.

As $\boldsymbol{Q}(k) = (q_{ij}^k)_{n\times n}$ is a M-matrix, it follows from Lemma 1 that there exists a positive vector $\zeta(k) = (\zeta_1^k, \zeta_2^k, \ldots, \zeta_n^k)^T$ such that for all $k \in S$, $i = 1, 2, \ldots, n$, we have

$$\left[-2d_i^k + \left(\sum_{m=1,m\ne k}^{N}\pi_{km}T + \pi_{kk}\right)\right]\zeta_i^k + 2\sum_{j=1}^{n}\sqrt{T}l_j\zeta_j^k\left(|\tilde{a}_{ij}^k| + |\tilde{b}_{ij}^k| + |\tilde{p}_{ij}^k|\right) < 0.$$

Constructing functions $F_i(\alpha, k)$ as follows:

$$\begin{aligned}
&F_i(\alpha, k) \\
&= \left[-2d_i^k + \alpha + \left(\sum_{m=1,m\ne k}^{N}\pi_{km}T + \pi_{kk}\right)\right]\zeta_i^k \\
&\quad + 2\sum_{j=1}^{n}\sqrt{T}l_j\zeta_j^k\left[|\tilde{a}_{ij}^k| + \exp(0.5\alpha\tau)|\tilde{b}_{ij}^k| + \mu_{ij}(0.5\alpha)|\tilde{p}_{ij}^k|\right].
\end{aligned}$$

Since for all $k \in S$, $i = 1, 2, \ldots, n$,

$$\begin{aligned}
F_i(0, k) &= \left[-2d_i^k + \left(\sum_{m=1,m\ne k}^{N}\pi_{km}T + \pi_{kk}\right)\right]\zeta_i^k \\
&\quad + 2\sum_{j=1}^{n}\sqrt{T}l_j\zeta_j^k\left(|\tilde{a}_{ij}^k| + |\tilde{b}_{ij}^k| + |\tilde{p}_{ij}^k|\right) \\
&< 0.
\end{aligned}$$

It is obvious that there exists $\lambda > 0$ such that

$$\begin{aligned}
&F_i(\lambda, k) \\
&= \left[-2d_i^k + \lambda + \left(\sum_{m=1,m\ne k}^{N}\pi_{km}T + \pi_{kk}\right)\right]\zeta_i^k \\
&\quad + 2\sum_{j=1}^{n}\sqrt{T}l_j\zeta_j^k\left[|\tilde{a}_{ij}^k| + \exp(0.5\lambda\tau)|\tilde{b}_{ij}^k| + \mu_{ij}(0.5\lambda)|\tilde{p}_{ij}^k|\right] \\
&< 0, \quad i = 1, 2, \ldots, n, \ k \in S. \qquad (14)
\end{aligned}$$

For writing simplification, and considering that $\tilde{z}_i(t) = z_i(t) - z_i^\#$, let $V_i(t, \tilde{z}_i(t), r(t) = k) = \omega_i^k\exp(\lambda t)|\tilde{z}_i(t)|^2$ be $V_i(t, k)$ if there is no confusion, here $i = 1, 2, \ldots, n$, $k \in S$.

Let $L_{(13)}V_i(t, k)$ be the weak infinitesimal operator along the solution of Eq. (13). Defining $L_{(13)}V_i(t, k)$, $i = 1, 2, \ldots, n$, $k \in S$, as follows:

$$\begin{aligned}
&L_{(13)}V_i(t, k) \\
&\quad = \lim_{\Delta\to 0^+}\frac{1}{\Delta}\{\mathrm{E}\{V_i(t+\Delta,\ r(t+\Delta))\,|t,\ r(t) = k\} \\
&\quad\quad - \mathrm{E}\{V_i(t,\ r(t) = k)\}\}. \qquad (15)
\end{aligned}$$

With the properties of Markov stochastic process and transition probability defined by (1), (15) can be described as

$$L_{(13)}V_i(t,k) = \sum_{m=1}^{N} \pi_{km} V_i(t,m) + D^+ V_i(t,k), \quad i = 1,2,\ldots,n,\ k \in S.$$

Calculating the right upper derivation $D^+V_i(t,\ k)$ along the zero solution of (13), we have that

$$\begin{aligned}
&L_{(13)}V_i(t,k)\\
&= \sum_{m=1}^{N} \pi_{km} V_i(t,m) + D^+ V_i(t,\ k)\\
&= \exp(\lambda t)\Bigg\{\sum_{m=1}^{N} \omega_i^m (\omega_i^k)^{-1} \pi_{km} \omega_i^k |\tilde{z}_i(t)|^2\\
&\quad + \omega_i^k \Big[\lambda |\tilde{z}_i(t)|^2 + 2\bar{\tilde{z}}_i(t)\dot{\tilde{z}}_i(t)\Big]\Bigg\}\\
&\le \exp(\lambda t)\Bigg\{\sum_{m=1}^{N} T\pi_{km}\omega_i^k |\tilde{z}_i(t)|^2 + \lambda \omega_i^k |\tilde{z}_i(t)|^2\\
&\quad + 2|\bar{\tilde{z}}_i(t)|\{-d_i^k \omega_i^k |\tilde{z}_i(t)|\\
&\quad + \sum_{j=1}^{n} \omega_i^k l_j [|\tilde{a}_{ij}^k| \tilde{z}_j(t)| + |\tilde{b}_{ij}^k||\tilde{z}_j(t-\tau_{ij}(t))|\\
&\quad + |\tilde{p}_{ij}^k| \int_{-\infty}^{t} \theta_{ij}(t-s)|\tilde{z}_j(s)|ds]\}\Bigg\}\\
&\le \sqrt{\omega_i^k}\exp(\lambda t)|\tilde{z}_i(t)| \Bigg\{\sum_{m=1}^{N} T\pi_{km}\sqrt{\omega_i^k}|\tilde{z}_i(t)|\\
&\quad + \lambda\sqrt{\omega_i^k}|\tilde{z}_i(t)| + \{-2d_i^k\sqrt{\omega_i^k}|\tilde{z}_i(t)|\\
&\quad + 2\sum_{j=1}^{n}\sqrt{T}\sqrt{\omega_j^k} l_j [|\tilde{a}_{ij}^k|\tilde{z}_j(t)| + |\,\tilde{b}_{ij}^k||\tilde{z}_j(t-\tau_{ij}(t))|\\
&\quad + |\tilde{p}_{ij}^k| \int_{-\infty}^{t} \theta_{ij}(t-s)|\tilde{z}_j(s)|ds]\}\Bigg\}\\
&\le \sqrt{V_i(t)}\Bigg\{[-2d_i^k + \sum_{m=1}^{N} T\pi_{km} + \lambda]\sqrt{V_i(t)}\\
&\quad + 2\sum_{j=1}^{n} l_j\sqrt{T}[|\tilde{a}_{ij}^k|\sqrt{V_j(t)} + \tilde{b}_{ij}^k|\exp(0.5\lambda\tau)\sqrt{V_j(t-\tau_{ij}(t))}\\
&\quad + |\tilde{p}_{ij}^k| \int_{-\infty}^{t} \theta_{ij}(t-s)\exp(0.5\lambda(t-s))\sqrt{V_j(s)}ds]\Bigg\}. \quad (16)
\end{aligned}$$

Computing the expectation on both sides of (16), we have

$$\begin{aligned}
&\mathrm{E}\left\{L_{(13)}V_i(t,k)\right\}\\
&\le \mathrm{E}\{\sqrt{V_i(t)}\}\{[-2d_i^k + \sum_{m=1}^{N} T\pi_{km} + \lambda]\mathrm{E}\{\sqrt{V_i(t)}\}\\
&\quad + 2\sum_{j=1}^{n} l_j\sqrt{T}[|\tilde{a}_{ij}^k|\mathrm{E}\{\sqrt{V_j(t)}\}\\
&\quad + |\tilde{b}_{ij}^k|\exp(0.5\lambda\tau)E\{\sqrt{V_j(t-\tau_{ij}(t))}\}\\
&\quad + |\tilde{p}_{ij}^k| \int_{-\infty}^{t} \theta_{ij}(t-s)\exp(0.5\lambda(t-s))E\{\sqrt{V_j(s)}\}ds]\}. \quad (17)
\end{aligned}$$

Defining the curve

$$\zeta(k) = \{\gamma(\chi,k) : \gamma_i = (\zeta_i^k)^2\chi, \quad \chi > 0, k \in S, i = 1,2,\ldots,n\}$$

and the set

$$\mathbf{\Omega}(\gamma) = \{y : 0 \le y \le \gamma, \gamma \in \zeta(k), k \in S\}.$$

When $\chi > \chi'$, it is obvious that $\mathbf{\Omega}(\gamma(\chi)) \supset \mathbf{\Omega}(\gamma(\chi'))$. Let

$$\begin{aligned}
\zeta_{\max}^k &= \max_{1\le i\le n}\{\zeta_i^k\}\\
\zeta_{\min}^k &= \min_{1\le i\le n}\{\zeta_i^k\}\\
\chi_0(k) &= \frac{W^k E\{||\psi(s)||^2\}}{(\zeta_{\min}^k)^2}.
\end{aligned}$$

Then

$$\begin{aligned}
&\{\mathrm{E}\{V(k)\} : \mathrm{E}\{V_i(s,k)\}\\
&\quad = e^{\lambda s}\omega_i^k \mathrm{E}\{|\varphi_i(s)|^2\},\\
&\qquad -\infty < s \le 0, 1 \le i \le n, k \in S\} \subset \mathbf{\Omega}(\gamma(\chi_0(k),k)).
\end{aligned}$$

Namely, the inequality $e^{\lambda s}\omega_i^k \mathrm{E}\{|\varphi_i(s)|^2\} < (\zeta_i^k)^2\chi_0(k)$ hold for all $-\infty < s \le 0, i = 1,2,\ldots,n, k \in S$.

Furthermore, we can claim that $\mathrm{E}\{V_i(t,k)\} \le (\zeta_i^k)^2\chi_0(k)$, $i = 1,2,\ldots,n, k \in S, t \ge 0$. If it is not true, then there exist some $i' \in \{1,2,\ldots,n\}$ and $t^*(t^* > 0)$ such that

$$\mathrm{E}\{V_{i'}(t^*,k)\} = (\zeta_i^k)^2\chi_0(k), \quad \mathrm{E}\left\{L_{(13)}V_{i'}(t^*,k)\right\} \ge 0$$

and

$$\mathrm{E}\{V_j(t,k)\} \le \left(\zeta_j^k\right)^2 \chi_0(k), \quad -\infty < t \le t^*, \quad j = 1,2,\ldots,n,\ k \in S.$$

Substituting them into (17), and considering (14), we get

$$\begin{aligned}
&\mathrm{E}\left\{L_{(13)}V_{i'}(t^*,k)\right\}\\
&\le \zeta_i^k\chi_0(k)\{[-2d_i^k + \sum_{m=1}^{N} T\pi_{km} + \lambda]\zeta_i^k + 2\sum_{j=1}^{n} l_j\zeta_j^k\sqrt{T}[|\tilde{a}_{ij}^k|\\
&\quad + |\tilde{b}_{ij}^k|\exp(0.5\lambda\tau) + \mu_{ij}(0.5\lambda)|\tilde{p}_{ij}^k|]\}\\
&< 0, \quad k \in S,\ i = 1,2,\ldots,n.
\end{aligned}$$

This is a contradiction with the above assumption $\mathrm{E}\left\{L_{(13)}V_{i'}(t^*,k)\right\} \ge 0$. So we have $\mathrm{E}\{V_i(t^*,k)\} < (\zeta_i^k)^2\chi_0(k)$, $i = 1,2,\ldots,n, k \in S$. That is to say

$$\begin{aligned}
|\tilde{z}_i(t)| &< \exp(-0.5\lambda t)\frac{\zeta_i^k}{\sqrt{\omega_i^k}}\sqrt{\chi_0(k)}\\
&= \exp(-0.5\lambda t)\frac{\sqrt{W^k}\zeta_i^k\sqrt{E\{||\psi(s)||^2\}}}{\zeta_{\min}^k\sqrt{\omega_i^k}}.
\end{aligned}$$

Let $M(k) = \frac{\sqrt{W^k}\zeta_i^k}{\zeta_{\min}^k\sqrt{\omega_i^k}}$, $k \in S$. Obviously, $M(k) \ge 1, k \in S$.

According to the Definition 1, we know that the zero solution of (13) is stochastically exponentially robustly stable. That is to say the equilibrium point $z^{\#}$ of (2) is also stochastically exponentially robustly stable. The proof is completed. □

From the Theorem 1, it can be directly obtained corresponding conditions for guaranteeing the existence, uniqueness and stochastic exponential robust stability of the equilibrium point of (2) with only time-varying delays or continuously distributed delays.

When there is only time-varying delay in (2), we obtain the following Corollary.

Corollary 1: Suppose that Assumption 1 is satisfied for arbitrary input $\boldsymbol{J} \in \mathbf{C}^n$, then the equilibrium point $z^{\#}$ of (2) is with existence and uniqueness if there exists a positive number $T \geq 1$ such that every matrix $\boldsymbol{Q}(k) = (q_{ij}^k)_{n\times n}$ is a M-matrix for all $k \in S$ and $i, j = 1, 2, \ldots, n$, where

$$q_{ii}^k = 2d_i^k - \left(\sum_{m=1,m\neq k}^{N} \pi_{km}T + \pi_{kk}\right)$$

$$q_{ij}^k = -2\sum_{j=1}^{n} \sqrt{T}l_j\left(|\tilde{a}_{ij}^k| + |\tilde{b}_{ij}^k|\right).$$

Besides, the equilibrium point $z^{\#}$ of (2) is stochastically exponentially robustly stable.

When there is only continuously distributed delay in (2), we obtain the following Corollary.

Corollary 2: Suppose that Assumption 1 satisfies for arbitrary input $\boldsymbol{J} \in \mathbf{C}^n$, then the equilibrium point $z^{\#}$ of (2) is with existence and uniqueness if there exists a positive number $T \geq 1$ such that every matrix $\boldsymbol{Q}(k) = (q_{ij}^k)_{n\times n}$ is a M-matrix for all $k \in S$, $i, j = 1, 2, \ldots, n$, where

$$q_{ii}^k = 2d_i^k - \left(\sum_{m=1,m\neq k}^{N} \pi_{km}T + \pi_{kk}\right)$$

$$q_{ij}^k = -2\sum_{j=1}^{n} \sqrt{T}l_j\left(|\tilde{a}_{ij}^k| + |\tilde{p}_{ij}^k|\right).$$

Besides, the equilibrium point $z^{\#}$ of (2) is stochastically exponentially robustly stable.

The complex-valued weight matrices in (2) are supposed to be in interval. If they are constants in complex number domain, we can establish the following corollary.

Corollary 3: Suppose that Assumption 1 is satisfied for arbitrary input $\boldsymbol{J} \in \mathbf{C}^n$, then the equilibrium point $z^{\#}$ of (2) is with existence and uniqueness if there exists a positive number $T \geq 1$ such that every matrix $\boldsymbol{Q}(k) = (q_{ij}^k)_{n\times n}$ is a M-matrix for all $k \in S$ and $i, j = 1, 2, \ldots, n$, where

$$q_{ii}^k = 2d_i^k - \left(\sum_{m=1,m\neq k}^{N} \pi_{km}T + \pi_{kk}\right)$$

$$q_{ij}^k = -2\sum_{j=1}^{n} \sqrt{T}l_j\left(|a_{ij}^k| + |b_{ij}^k| + |p_{ij}^k|\right).$$

Besides, the equilibrium point $z^{\#}$ of (2) is stochastically exponentially stable.

When there is no Markova jumping parameters in (2) with constant connected matrices, the corresponding sufficient conditions for judging the dynamical behavior of the equilibrium point are as follows.

Corollary 4: Suppose that Assumption 1 is satisfied for arbitrary input $\boldsymbol{J} \in \mathbf{C}^n$, then the equilibrium point $z^{\#}$ of (2) is with existence and uniqueness if the matrix $\boldsymbol{Q} = (q_{ij})_{n\times n}$ is a M-matrix, where

$$q_{ii} = d_i; \quad q_{ij} = -\sum_{j=1}^{n} l_j\left(|a_{ij}| + |b_{ij}| + |p_{ij}|\right), \qquad i, j = 1, 2, \ldots, n.$$

Besides, the equilibrium point $z^{\#}$ of (2) is exponentially stable.

Remark 3: It is worth noting that conditions in [35] and [38] for stochastic exponential stability of neural networks with Markov jumping parameters are implicit because the positive scalars ω_i^k ($i = 1, 2, \ldots, n, k \in S$) in the stability conditions are difficult to choose for application. The stability conditions of the Theorem 1 established in this paper are with simple form and easy to apply to judge the dynamical behavior of neural networks.

Remark 4: It is well known that the synchronization problem of chaotic neural networks can be translated into the stability problem of the corresponding error system of driving system and driven system. Zhou *et al.* [11] concerned the problem of synchronization control for a class of mixed delays complex-valued neural networks with external uncertain perturbations. The complex-valued activation functions given in [11] needed to be with the existence, boundedness and continuity assumption for the partial derivatives with two variables of the activation functions. Besides, Markova jumping parameters were not considered in the addressed system. In future works, we will investigate the problem of synchronization control for chaotic complex-valued neural networks with mixed delays and uncertain perturbations including Markova jumping parameters and stochastic disturbances by giving lower level of conservatism of assumption conditions. We expect the approach used in this paper will play an important role in our future research

Remark 5: The obtained conditions in this paper are used to guarantee the module stability of states of neural networks in complex number domain. The Assumption I in [40] proposed a generalized assumption condition for activation function of neural networks in real number domain. It only required that activation functions be with lower and upper bounds, which was considered to reduce the possible conservatism than usual sigmoid functions and Lipschitz continuous functions. If we adopt the method of separating the complex-valued system into their real parts and imaginary parts and analyze the stability of their real part system and imaginary part, the Assumption I in [40] could be used to establish some results with less level of conservatism than the existing ones. In future works, we will try to do the corresponding research.

Remark 6: In [41]–[43], authors proposed several discrete-time complex systems defined in real number domain and analyzed the dynamical behavior of the addressed systems including stability, passivity and problem of synchronization control. The model considered in this paper is continuous-time system defined in complex number domain. The method adopted in this paper could be extended to study the discrete-time neural networks with complex number variables.

IV. EXAMPLES

In this section, an example with simulation results is given to illustrate the effectiveness of the sufficient conditions established in the preceding section.

Remark 7: We claim that the activation functions defined in complex domain in this paper means the considered model includes CVNNs whose are not explicitly expressed by separating real and imaginary parts. It can be proved by the following examples. Assume that the activation function is $f(z(t)) = \frac{1}{1+\exp(-\bar{z}(t))}$. By computation, it satisfies Lipschitz condition with Lpschitz constant $l = 0.25$ under the condition $f(z(t))|_{z(t)=0}$. However, the function $f(z(t))$ is unable to separate into its real and imaginary part explicitly. That is to say the conditions proposed on the activation functions are not only easy to calculate, but also weaker than those in [9], [16], and [21].

In order to verify the above analysis, we will give a numerical example as follows.

Example 1: Considering the two order complex-valued neural networks model with three modes described by (2) in section 2.

It is assumed that the interval weight matrices defined in the complex number domain are same in every mode. Assuming the weight matrices are as follows:

$$\boldsymbol{A}_I(r(t)) = \begin{bmatrix} [-0.5, 0.7] + [0.6, 0.9]\mathrm{i} & [-1, 1] + [0.7, 1]\mathrm{i} \\ [0.5, 0.8] + [-0.4, 1]\mathrm{i} & [-0.4, 0.5] + [0.9, 1.2]\mathrm{i} \end{bmatrix}$$

$$\boldsymbol{B}_I(r(t)) = \begin{bmatrix} [0.4, 0.8] + [0.8, 1]\mathrm{i} & [-0.3, 0.6] + [0, 0.8]\mathrm{i} \\ [-0.3, 0] + [0.1, 0.6]\mathrm{i} & [0.7, 0.9] + [-1, -0.2]\mathrm{i} \end{bmatrix}$$

$$\boldsymbol{P}_I(r(t)) = \begin{bmatrix} [0, 0.5] + [0.1, 0.6]\mathrm{i} & [0.2, 0.4] + [-0.1, 0.5]\mathrm{i} \\ [-1, 1] + [-0.3, 0.9]\mathrm{i} & [0, 0.7] + [-0.1, \sqrt{2}]\mathrm{i} \end{bmatrix}.$$

Let $T = 2$, $d_1(1) = 10$, $d_2(1) = 9$, $d_1(2) = 5$, $d_2(2) = 6$, $d_1(3) = 4$, $d_2(3) = 4$. Let jumping transfer parameters be $\pi_{11} = -1/2$, $\pi_{12} = 1/6$, $\pi_{13} = 1/3$, $\pi_{21} = 1/8$, $\pi_{22} = -1/2$, $\pi_{23} = 3/8$, $\pi_{31} = 1/5$, $\pi_{32} = 3/10$, $\pi_{33} = -1/2$. It is assumed that activation functions are

$$f_1(z_1(t)) = 0.5\frac{1 - \exp(-\bar{z}_1(t))}{1 + \exp(-\bar{z}_1(t))}$$
$$f_2(z_2(t)) = \frac{1.5}{1 + \exp(-\bar{z}_2(t))}.$$

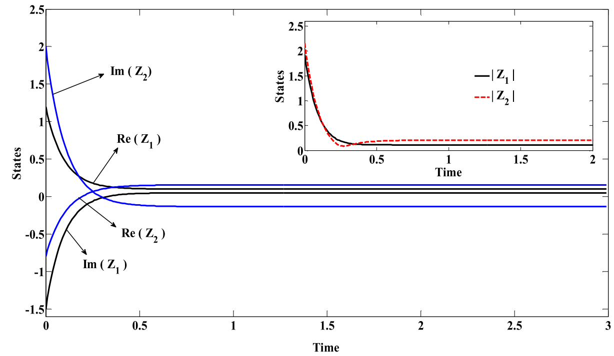

FIGURE 1. State curves of the system with the mode 1.

By calculation, we obtain that

$$l_1 = 0.25, \quad l_2 = 0.375, \quad \left|\tilde{\boldsymbol{A}}\right| = \begin{bmatrix} 1.140 & 1.414 \\ 1.281 & 1.300 \end{bmatrix},$$
$$\left|\tilde{\boldsymbol{B}}\right| = \begin{bmatrix} 1.281 & 1.000 \\ 0.671 & 1.345 \end{bmatrix}, \quad \left|\tilde{\boldsymbol{P}}\right| = \begin{bmatrix} 0.781 & 0.640 \\ 1.345 & 1.578 \end{bmatrix}.$$

Furthermore, we get

$$\boldsymbol{Q}(1) = \begin{bmatrix} 19.500 & -5.504 \\ -6.810 & 17.500 \end{bmatrix}$$
$$\boldsymbol{Q}(2) = \begin{bmatrix} 9.500 & -5.504 \\ -6.810 & 11.500 \end{bmatrix}$$
$$\boldsymbol{Q}(3) = \begin{bmatrix} 7.500 & -5.504 \\ -6.810 & 7.500 \end{bmatrix}.$$

It follows from the Lemma 1 that matrices $\boldsymbol{Q}(1)$, $\boldsymbol{Q}(2)$ and $\boldsymbol{Q}(3)$ are all M-matrix. Obviously, assumption conditions in Theorem 1 are satisfied. According to Theorem 1, the equilibrium point of (2) with above assumptions is with existence, uniqueness and stochastic exponential robust stability.

In order to finish the numerical simulations of (2), it is supposed that initial conditions of (2) are $z_1(s) = 1.2 - 1.5i$, $z_2(s) = -0.8 + 2i$, $s \in (-\infty, 0]$. Assume that external inputs are $J_1(t) = 1 + 0.5i$, $J_2(t) = 1.4 + 2i$. Let time-varying delays be $\tau_{1j} = 0.7|\sin t|$, $\tau_{2j} = 0.6|\cos t|$, $j = 1, 2$, $t \geq 0$. Let $\theta_{ij}(t - s) = \exp(-(t - s))$, $i, j = 1, 2$.

The state curves of system for three modes are respectively given as representative in Fig.1∼Fig.3. From simulation results, we can conclude that the equilibrium point of (2) is existent, unique and stable, which verifies the correctness of Theorem 1.

Remark 8: It should be pointed out the established results in this paper can also be used to study complex-valued neural networks whose activation functions need to be expressed by separating their real and imaginary parts. For example, it is assumed the activation function is $f(z(t)) = l(|x| + \mathrm{i}|y|)$, where $l > 0$ is Lipschitz constant. However, only the Lipschitz condition on the activation functions needs to be satisfied. The existence, continuity and boundedness of partial derivatives $f(.)$ with respect to its real part x and imaginary part y are removed, here let $z = x + yi$.

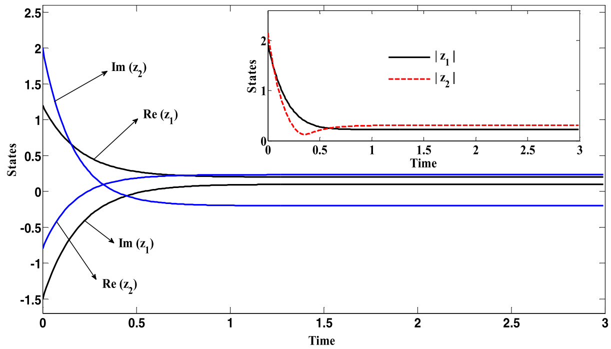

FIGURE 2. State curves of the system with the mode 2.

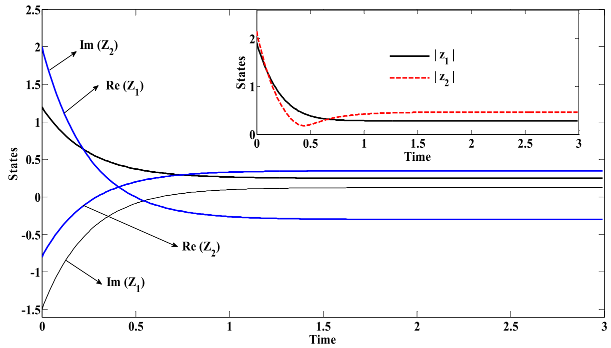

FIGURE 3. State curves of the system with the mode 3.

In order to verify the above analysis, we will give another numerical example as follows. For simplification, the Markova jumping parameters are not considered in the Example 2.

Example 2: Considering the two order complex-valued neural networks model described by

$$\frac{dz_i(t)}{dt} = -d_i z_i(t) + \sum_{j=1}^{2} [a_{ij} f_j(z_j(t)) + b_{ij} f_j(z_j(t-\tau_{ij}(t))) + p_{ij} \int_{-\infty}^{t} \theta_{ij}(t-s) f_j(z_j(s)) ds] + J_i(t), \quad i = 1, 2. \tag{18}$$

Let $d_1 = 5$ and $d_2 = 4$. Assuming that the activation functions are $f_1(z_1(t)) = 0.8|x_1| + |0.6y_1|\text{i}$ and $f_2(z_2(t)) = 0.6|x_2| + 0.5|y_2|\text{i}$. It is easy to get $l_1 = 0.8$, $l_2 = 0.6$.

Assuming that the interconnected matrices are as follows:

$$\boldsymbol{A} = \begin{bmatrix} -1+0.6\text{i} & 0.8-0.1\text{i} \\ -0.3\text{i} & -0.1\text{i} \end{bmatrix}$$
$$\boldsymbol{B} = \begin{bmatrix} -0.5+0.6\text{i} & -1+0.7\text{i} \\ 0.5-0.4\text{i} & -0.4+0.9\text{i} \end{bmatrix}$$
$$\boldsymbol{P} = \begin{bmatrix} 0.5+0.3\text{i} & -0.8+0.5\text{i} \\ 0.4+0.6\text{i} & 0.7 \end{bmatrix}.$$

By calculation, we get $\boldsymbol{Q} = \begin{bmatrix} 5.000 & -3.806 \\ -2.400 & 4.000 \end{bmatrix}$. It follows from the Lemma 1 that matrices is M-matrix. Obviously, assumption conditions in Corollary 4 are satisfied. According to Corollary 4, the equilibrium point of (18) with above assumptions is with existence, uniqueness and exponential stability.

In order to do the numerical simulations of (18), it is supposed that initial conditions of (18) are $z_1(s) = 2 + 3i$, $z_2(s) = -1 - 2i$, $s \in (-\infty, 0]$. Assuming that external inputs are $J_1(t) = 3i$, $J_2(t) = -2 + 1.5i$. Let time-varying delays be $\tau_{1j} = 0.9|\sin t|$, $\tau_{2j} = 0.8|\cos t|$, $j = 1, 2$, $t \geq 0$. Let $\theta_{ij}(t-s) = \exp(-(t-s))$, $i, j = 1, 2$.

The state curve of (18) is given in Fig.4. From simulation results, we can conclude that the equilibrium point of (18) is existent, unique and stable, which verifies the correctness of Corollary 4.

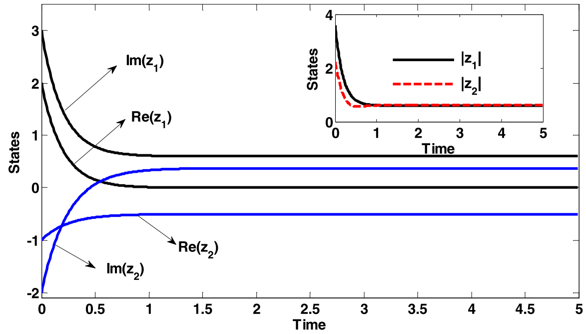

FIGURE 4. State curves of the system (18).

V. CONCLUSION AND FUTURE WORK

Existence, uniqueness and stochastic exponential robust stability of the equilibrium point of a class of complex-valued neural networks with mixed delays and Markova jumping parameters are investigated in this paper. By employing homeomorphism mapping principle and M-matrix theory, the existence and uniqueness of the equilibrium point for the considered neural networks are guaranteed. Some sufficient conditions for stochastic exponential robust stability of delayed complex-valued neural networks with Markova jumping parameters are established by constructing the appropriate vector Lyapunov function. Finally, two numerical examples with simulation results illustrate the validity and feasibility of the obtained results. We expect the method applied in this paper will play a significant role in the future research. Based on the work in this paper, we will attempt to study the stability analysis and synchronization control for delayed complex-valued neural networks with multiple uncertain factors, such as impulsive effect, Markova jumping parameters and stochastic disturbance, etc.

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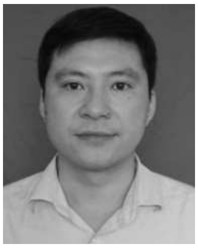

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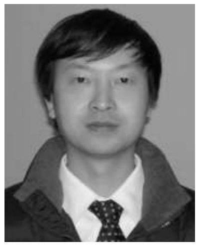

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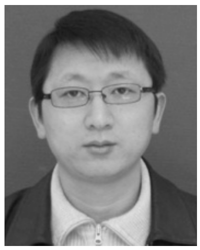

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