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Structural Properties and t/s-Diagnosis for Star Networks Based on the PMC Model

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ABSTRACT Diagnosability is a key factor in the analysis of reliability for a network system. t/s-diagnosability is a novel measurement for evaluating the reliability of a system. In this paper, we derive some properties, which have not been reported by previous literatures, for a star network. By using these properties, we prove that an n-dimensional star graph (denoted by S_n) is $\lfloor ln - (((l+2)^2)/3)\rfloor/(\lfloor ln - (((l+2)^2)/3)\rfloor + l - 2)$-diagnosable, where $(n \geqslant 5)$, $2 \leqslant l \leqslant n-2$. Furthermore, we prove that given an integer $n(n \geqslant 5)$, and another integer $l(2 \leqslant l \leqslant n-2)$, for some positive integer $\beta \in (\lfloor (l-1)n - (((l+1)^2)/3)\rfloor, \lfloor ln - (((l+2)^2)/3)\rfloor]$, S_n is $\beta/(\beta + l - 2)$-diagnosable. In the last part of this paper, we propose an isolation-fast algorithm for $S_n(n \geqslant 5)$, and its time complexity is only $O(N\log_2 N)$, where $N = n!$.

INDEX TERMS t/s-diagnosable system, t/s-diagnosability, n-dimensional star network, isolation-fast algorithm, PMC model.

I. INTRODUCTION

The reliability of multiprocessor system is a key index to reflect the service quality of the system. Nowadays, the scale of processors(units, nodes or vertices) in a multiprocessor system is often very large. Sometimes, a multiprocessor system has thousands of units. It is very possible that such a system produces some faulty units in working. Hence, the identification of faults in a system is necessary and significant in consideration of reliable computing. In general, there are two methods to identify faulty processors: logic-circuit-level and system-level [1]–[4]. Because of the potentially large number of interconnected units in a network, solutions to the fault identification have tended to emphasize a system-level rather than a logic-circuit-level approach [1].

In 1967, a called Preparata, Metze, and Chien model (in brief, the PMC model), a system-level diagnosis model [6], was presented by [5]. In the PMC model, a graph $G = (V, E)$ is used to denote a multiprocessor system, where each element of $V(G)$ represents a processor, each element of $E(G)$ represents a link between two processors . For $x, y \in V(G)$, $(x, y) \in E(G)$ denotes that x tests y. After performing a round testing of the system, each link $(x, y) \in E(G)$ will be given a testing outcome 1 or 0, the collection of outcomes of all edges in the system is called a syndrome, denoted by ω. $\omega(x, y)$ can be employed to represent the outcome of x testing y. The PMC model thinks that if unit x judges unit y to be faulty (respectively, fault-free), then $\omega(x, y) = 1$(respectively, 0), and thinks that if the tester x is free-fault, then the outcome of x testing the tested unit y is reliable, otherwise unreliable. There are a lot of the studied results being relative to the PMC model (see [4], [7]–[17]).

Under the PMC model, [5] proposed two fault diagnosable systems. One of them is called one-step t-diagnosable system. the merit of such a system is that by one-off diagnosis the system can determinate the set of faulty units, but its shortcoming is that it has a small diagnosability. The results being relative to the t-diagnosable system have been proverbially reported (see [7], [8], [12], [13], [19]–[24]). Another one of them is called a sequentially t-diagnosable system. It is possible that a sequentially t-diagnosable system can not determinate all faulty units by one-off diagnosis, but its diagnosability is usually more than that of the t-diagnosable system. Reference [18] extended the concept of sequentially t-diagnosable system by introducing the two concepts: the t/t-diagnosable system and the t/s-diagnosable system($t \leqslant s$). It is worth mentioning that for the same system, the diagnosability of the later is usually larger than that of the former. The reason is that if a system is t/t-diagnosable, then it must be t/s-diagnosable ($t \leqslant s$). It is a pity that few results on t/s-diagnosable systems have been obtained so far as far as we know.

Many regular network topologies are often employed to model multiprocessor systems. Among them, star graph of n dimension has been paid close attention for its merits such as its regularity, its high recursiveness, and that its diameter and degree are sub logarithmic. Many results on a star graphs have been widely reported (see [14], [23], [30]–[33]). However, there are few papers to report the results on the t/s-diagnosability of star graph as far as we know. In this article, a few new properties on S_n are proposed by us, by means of them, the t/s-diagnosability of S_n is given by us.

The follows are the arrangements on the remainder of this article: In Section 2, the preliminaries are presented. Section 3 introduces some new properties of star graph. In Section 4, the t/s-diagnosability of star graph is discussed. Section 5 gives a diagnosis algorithm for S_n ($n \geqslant 5$). And Section 6 is a conclusion.

II. PRELIMINARIES

A multiprocessor system can be modeled by a graph $G = (V, E)$, each node $x \in V$ represents one processor, each edge $(x, y) \in E$ represents the communication link between x and y. In this paper, all considered graphs are undirected graphs without loops. For convenience, in this paper, we doesn't distinguish the four terminologies: unit, node, vertex and processor. At the same time, the other three terminologies: network, system and graph are also not distinguished.

For a given graph $G = (V, E)$, we use $V(G)$ to represent the set of vertices of graph G, $E(G)$ the set of edges of graph. A connected subgraph of G, say X, is described as a component of G if there doesn't exist an edge $(x, y) \in E(G)$ such that either $x \in V(X)$ and $y \in V(G) - V(X)$ or $y \in V(X)$ and $x \in V(G) - V(X)$. We use $C_{sub}(G)$ to denote the set of all components of G. If G has k components, then $C_{sub}(G) = \{C_i | C_i \text{ is a component of } G, 1 \leqslant i \leqslant k\}$. Particularly, for a connected graph G we have that $C_{sub}(G) = \{G\}$.

Let $Y \subset V$, the induced subgraph of Y can be represented by $G_{induced}(Y, G) = (V_1, E_1)$, where $V_1 = Y$ and $E_1 = \{(v_1, v_2) \in E | v_1, v_2 \in Y\}$.

Let $Card_k(C_{sub}(G)) = \{Z \in C_{sub}(G) : |V(Z)| = k\}$. To explain the two notations $G_{induced}(Y, G)$ and $Card_k(C_{sub}(G))$, we consider a graph of 7-node G shown by Figure 1. In Figure 1, we have that $C_{sub}(G) = \{C_1, C_2\}$ where $C_1 = G_{induced}(\{v_2, v_3, v_6\}, G)$ and $C_2 = G_{induced}(\{v_1, v_4, v_5, v_7\}, G)$. $Card_3(C_{sub}(G)) = \{C_1\}$ and $Card_4(C_{sub}(G)) = \{C_2\}$.

Let $v \in V$, $X \subset V$. We call a node $x \in V - X$ to be an out-neighbor of X if X has such a node $y \in X$ satisfying $(x, y) \in E$. We employ $N_G(v)$ (in brief, $N(v)$, when no any confusion) to represent the collection of neighbors of v, $N(v) = \{z \in V | (z, v) \in E \text{ or } (v, z) \in E\}$, $N_G(X)$ (in brief, $N(X)$, when no any confusion) to represent the collection of out-neighbors of X, $N(X) = \cup_{v \in X} N(v) - X$.

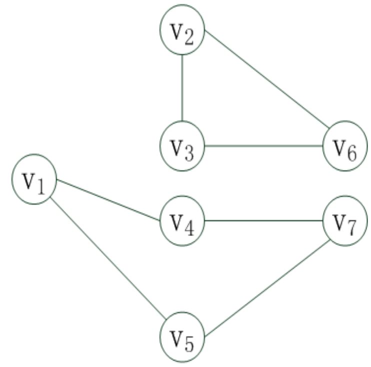

FIGURE 1. A graph of 7-node.

we follow [34] for definitions and notations not mentioned above.

Definition 1: Let $G = (V, E)$ represent a system, $Y \subset V$. Y is said to be a fault-free link of the system if the following conditions hold:

1. Each node in Y is fault-free; and
2. For any two nodes $v_1, v_2 \in Y$, there exists a path between v_1 and v_2 satisfying each node of it belongs to Y.

Lemma 2: For a system $G = (V, E)$ and a syndrome ω, assume that the number of faulty units in G is less than or equal to t. For a subset $X = \{x_1, x_2, \cdots, x_k\} \subset V$ with $|X| = k \geqslant t + 1$, X is a fault-free link of the system if the two conditions described as follows hold:

i) $(x_j, x_{j+1}) \in E$ where $1 \leqslant j \leqslant k - 1$.

ii) $\omega(x_j, x_{j+1}) = \omega(x_{j+1}, x_j) = 0$, $1 \leqslant j \leqslant k - 1$.

Proof: By the definition of the PMC model and Definition 1, the result is true. ∎

Definition 3: Let $G = (V, E)$ represent a system, ω a syndrome obtained after performing a test, X a subset of V. For ω, X is called an allowable fault set (in brief, AFS) if the following conditions are true,

i) For $(u, v) \in E$, if $u, v \in V - X$, then $\omega(u, v) = 0$, and

ii) For $(u, v) \in E$, if $u \in V - X$ and $v \in X$, then $\omega(u, v) = 1$.

Lemma 4: Let $G = (V, E)$ represent a system, ω a syndrome. If $S_1, S_2 \subseteq V$ are AFSs for ω, then $S_1 \cup S_2$ is also an AFS for ω.

Proof: Assume that $S_1 \cup S_2$ is not an AFS for ω, then at least one of conditions of Definition 2.2 is not true.

If i) of Definition 3 is not true, then $V - (S_1 \cup S_2)$ has two nodes, say u and v, with $(u, v) \in E$ such that $\omega(u, v) = 1$, which implies that each one of S_1 and S_2 is not an AFS for ω, a contradiction.

If ii) of Definition 3 is not true, there exists an edge $(u, v) \in E$ satisfying $u \in V - S_1 \cup S_2$, $v \in S_1 \cup S_2$ and $\omega(u, v) = 0$. If $v \in S_1$, then S_1 is not an AFS for ω, a contradiction. If $v \in S_2$, then S_2 is not an AFS for ω, a contradiction. ∎

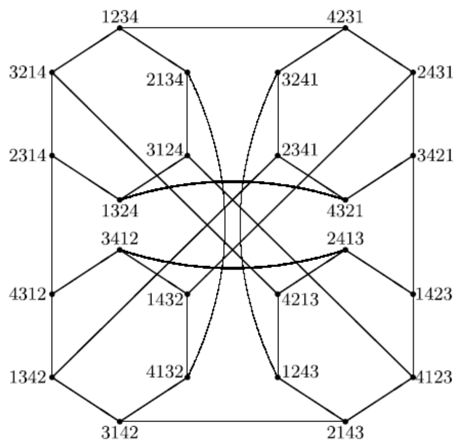

FIGURE 2. A 4-dimensional star graph.

III. PROPERTIES OF STAR GRAPH

A star network of n dimension, denoted by S_n, is a graph $(V(S_n), E(S_n))$, where $V(S_n) = \{s_1s_2s_3\cdots s_n | s_i \in \{1, 2, 3, \cdots, n\}, s_i \neq s_j (i \neq j)\}$ and $E(S_n) = \{(s_1s_2s_3\cdots s_n, s_is_2s_3\cdots s_{i-1}s_1s_{i+1}\cdots s_n) | 2 \leqslant i \leqslant n\}$. Suppose that $x = x_1x_2\cdots x_n, y = y_1y_2\cdots y_n \in V(S_n)$, then $(x, y) \in E(S_n)$ if and only if $\exists k \in \{2, 3, \cdots, n\}$ such that $y_1 = x_k, y_k = x_1, y_i = x_i (2 \leqslant i \leqslant n, i \neq k)$. For a node $x \in V(S_n)$, we use $add(x)$ to denote it's address.

Figure 2 is a star network of 4 dimension.

Lemma 5 [23]: In S_n, there are no odd cycles and there are even cycles with length $L (6 \leqslant L \leqslant n!)$.

Lemma 6 [30]: In S_n, let $u \in V(S_n)$, $N(u) = \{u_1, u_2, \cdots, u_{n-1}\}$. Then for each pair u_i, u_j there are exactly three nodes, say x, y, z, in $V(S_n) - N(u) - \{u\}$ such that u_i, u_j, u, x, y, z form a 6-node loop.

Lemma 7: In $S_n (n \geqslant 5)$, let $S = \{s_i \in V(S_n) | 1 \leqslant i \leqslant k, |\cap_{i=1}^{k} N(s_i)| = 1, 1 \leqslant k \leqslant n-1\}$. Then $|N(S)| = (n-2)k + 1$.

Proof: Let $u \in N(s_1) \cap N(s_2) \cap \cdots \cap N(s_k)$. By Lemma 5, it is true that for any two nodes $s_i, s_j \in S$, u is their unique common neighbor. Since the degree of each vertex in S_n is exactly $(n-1)$, $|N(S)| = (n-2)k + 1$. ∎

Definition 8: Let $G = (V, E)$ denote a system, $L \subseteq V$, $x \in L$. $y \in (V - L) \cap N(x)$ is called a private neighbor of x for L if L has no a node z such that $y \in N(z)$.

For $v \in L$, we employ $PN_L(v)$ to represent the collection of all private neighbors of v for L. Obviously, $PN_L(v) = N(v) - N(L - \{v\}) - L$.

Lemma 9: In $S_n (n \geqslant 5)$, let $S = \{s_i \in V(S_n) | i = 1, 2, 3\}$, then following conditions hold:

i) If any two nodes of S are disconnected, then $|PN_S(s_1)| + |PN_S(s_2)| + |PN_S(s_3)| \geqslant 3n - 9$.

ii) Otherwise, $|PN_S(s_1)| + |PN_S(s_2)| + |PN_S(s_3)| \geqslant 3n - 7$.

Proof: *Condition i):* Since any two nodes of n-dimensional star graph share at most one public neighbor, $|PN_S(s_i)| \geqslant (n-1) - 2 = n - 3$ $(i = 1, 2, 3)$. Hence $|PN_S(s_1)| + |PN_S(s_2)| + |PN_S(s_3)| \geqslant 3n - 9$.

Condition ii): There always exist two nodes, say $s_1, s_2 \in S$ are adjacent, consider the following cases:

Case 1: There is exactly one of s_1 and s_2 being adjacent to s_3.

Without loss of generality, let $(s_1, s_3) \in E(S_n)$, then $|PN_S(s_1)| + |PN_S(s_2)| + |PN_S(s_3)| = (n-3) + (n-2) + (n-2) = 3n - 7$.

Case 2: None of s_1 and s_2 is adjacent to s_3

Case 2.1: There is exactly one in $\{s_1, s_2\}$ sharing one common neighbor with s_3.

Without loss of generality, suppose that s_2 and s_3 share one public neighbor, then by Lemma 5 there is not a node shared by s_1 and s_3. So, $|PN_S(s_1)| + |PN_S(s_2)| + |PN_S(s_3)| = (n-2) + (n-3) + (n-2) = 3n - 7$.

Case 2.2: Each one of $\{s_1, s_2\}$ shares one common neighbor with s_3. By Lemma 5, we have that S_n doesn't have odd cycles. Hence, the case can not take place.

Case 2.3: None of s_1 and s_2 shares one common node with s_3. Then $|PN_S(s_1)| + |PN_S(s_2)| + |PN_S(s_3)| = (n-2) + (n-2) + (n-1) = 3n - 5$. ∎

Lemma 10 [23]: In $S_n (n \geqslant 5)$, let $S = \{s_i \in V(S_n) | i = 1, 2, 3, 4\}$, then $|N(S)| \geqslant 4n - 10$.

Lemma 11: In $S_n (n \geqslant 5)$, there exists no such a subgraph of S_n shown in Figure 3.

Proof: Suppose that, to the contrary, Figure 3 is a subgraph of S_n. Let the address of v_1 be $a_1a_2a_3\cdots a_n$. The address of v_1 can be changed to the address of v_2 by four time changes as follows: $add(u_1) = a_ia_2\cdots a_{i-1}a_1a_{i+1}\cdots a_n$, $add(u_2) = a_ja_2\cdots a_1\cdots a_i\cdots a_n$, $add(u_3) = a_ka_2\cdots a_1\cdots a_i\cdots a_j\cdots a_n$, $add(v_2) = a_la_2\cdots a_1\cdots a_i\cdots a_j\cdots a_k\cdots a_n (2 \leqslant i, j, k, l \leqslant n)$. We claim that at least two of i, j, k, l are identical. Otherwise, any two of i, j, k, l are not identical, we derive a contradiction. By the assumption, we conclude that S_n has an 8-node cycle, say C_8, such that $v_1, v_2 \in V(C_8)$. Without loss of generality, let $add(v_1) = 12345\cdots n$, $add(u_1) = 21345\cdots n$, $add(u_2) = 31245\cdots n$, $add(u_3) = 41235\cdots n$, $add(v_2) = 51234\cdots n$. Since $\{v_1, u_1, u_2, u_3, v_2, w_3, w_2, w_1\}$ can form an 8-node cycle, then $add(w_1) = i2\cdots 1\cdots n$ where $3 \leqslant i \leqslant n$. And the address of w_2 is the one of the following: $2i\cdots 1\cdots n$, $32i\cdots 1\cdots n, \cdots, n23\cdots 1\cdots i$. Since the second position of the address of node v_2 is 1 and the second position from the left of the address of node w_2 is not 1, therefore the first position from the left of the address of node w_3 is 1. Then the address of w_3 is one of the following addresses: $1i\cdots 2\cdots n$, $12i\cdots 3\cdots n, \cdots, 123\cdots n\cdots i$. Therefore, v_2 and w_3 is not adjacent which is a contradiction to the hypothesis.

For three distinct integers $i, j, k \leqslant n$, without loss of generality, let $i = 2, j = 3, k = 4$, then $add(v_1) = 1234\cdots n$, $add(u_1) = 2134\cdots n$, $add(w_1) = 3214\cdots n$, $add(y_1) = 4231\cdots n$, $add(x_1) = l234\cdots 1\cdots n$ $(l \neq 1, 2, 3, 4)$. Since $\{v_1, u_1, u_2, u_3, v_2, x_3, x_2, x_1\}$ can form an 8-node cycle. Then the address of node x_2 is the one of the following: $2l34\cdots 1\cdots n$, $32l4\cdots 1\cdots n, \cdots, n234\cdots 1\cdots l$. Note that the lth position from the left of the address of

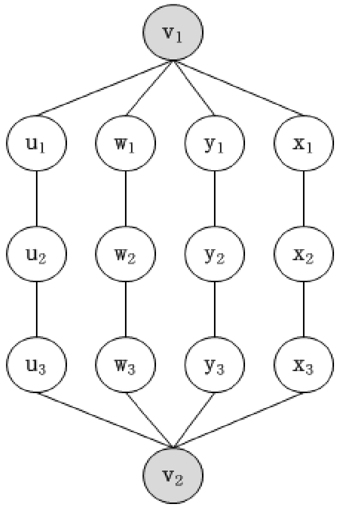

FIGURE 3. A graph of Lemma 11.

node v_2 is l and the lth position from the left of the address of node x_2 is 1. Therefore, the first position from the left of the address x_3 is l, which implies that $add(x_3) = add(x_1)$, a contradiction. ■

By Lemma 11, we get easily the following lemma, and omit its proof.

Lemma 12: In $S_n(n \geqslant 5)$, for any subset $S = \{v_i \in V(S_n)|1 \leqslant i \leqslant 5$ and $v_{j+1} \in N(v_j)(1 \leqslant j \leqslant 4)\} \subseteq V$, let $L(S) = \{L_i = \{x_i, y_i, z_i\} \subset V(S_n)|L_i \cup S$ can form an 8-node cycle and $L_i \cap L_j = \varnothing(i \neq j)\}$. Then $|L(S)| \leqslant 2$.

Lemma 13: In $S_n(n \geqslant 5)$, let $V' \subset V(S_n)$ with $|V'| = k$ $(3 \leqslant k \leqslant 3n-6)$. If there exist three nodes $v_1, v_2, v_3 \in V'$ such that v_1, v_2 share one common neighbor and v_2, v_3 share one common neighbor, then the following conditions hold:

i) If v_1, v_2, v_3 share one common neighbor, then V' has a node, say v, satisfying $|PN_{V'}(v)| \geqslant n-2-\lfloor\frac{k}{3}\rfloor$.

ii) If v_1, v_2, v_3 are in the same 6-node cycle, then V' has a node, say v, satisfying $|PN_{V'}(v)| \geqslant n-2-\lfloor\frac{k}{3}\rfloor$.

iii) If v_1, v_2, v_3 are in a 5-node line but not in the same 6-node cycle, then V' has a node, say v, satisfying $|PN_{V'}(v)| \geqslant n-2-\lfloor\frac{k}{3}\rfloor$.

Proof: Let $S = \{v_1, v_2, v_3\}$. We need only to show that there exist three nodes, say w_1, w_2, w_3, such that $|PN_{V'}(w_1)| + |PN_{V'}(w_2)| + |PN_{V'}(w_3)| \geqslant 3n-6-k$, which implies that there exists a node $x \in \{w_1, w_2, w_3\}$ satisfying $|PN_{V'}(x)| \geqslant n-2-\lfloor\frac{k}{3}\rfloor$.

Condition i): By Figure 4, we have that $|PN_S(v_1)| + |PN_S(v_2)| + |PN_S(v_3)| = 3n-6$. Let $v = N(v_1) \cap N(v_2) \cap N(v_3)$, according to Lemma 6, for $\{v_1, v_2\}$, V' has at most one node, say u, such that v_1, v, v_2, u can form a 6-node cycle with two other nodes and $|(N(u) \cap (PN_S(v_1) \cup PN_S(v_2))| = 2$. Similarly, for $\{v_1, v_3\}$ and $\{v_2, v_3\}$, we have similar results. Therefore, $|PN_{V'}(v_1)| + |PN_{V'}(v_2)| + |PN_{V'}(v_3)| \geqslant 3n-6-(k-3)-3 = 3n-6-k$.

Condition ii): $|PN_S(v_1)| + |PN_S(v_2)| + |PN_S(v_3)| = 3n-9$. Therefore, $|PN_{V'}(v_1)| + |PN_{V'}(v_2)| + |PN_{V'}(v_3)| \geqslant 3n-9-(k-3) = 3n-6-k$.

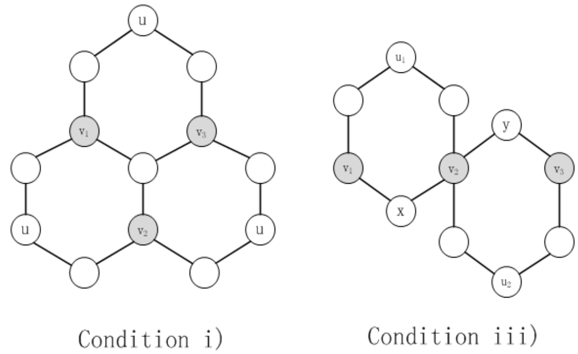

FIGURE 4. An illustration of Lemma 12.

Condition iii): According to Lemma 6, there exists a subgraph shown as Figure 4, $|PN_S(v_1)| + |PN_S(v_2)| + |PN_S(v_3)| = 3n-7$. If $u_1 \in V'(u_2 \in V')$, then for the set $\{v_1, v_2, u_1\}(\{v_2, v_3, u_2\})$, a similar argument to condition ii) can be used. If there exists an 8-node cycle, say C_8, such that $v_1, x, v_2, y, v_3 \in V(C_8)$, according to Lemma 12, there exist at most two nodes in V', say u, which satisfies that $|N(u) \cap [PN_S(v_1) \cup PN_S(v_3)]| = 2$. Therefore, $|PN_{V'}(v_1)| + |PN_{V'}(v_2)| + |PN_{V'}(v_3)| \geqslant 3n-7-(k-3)-2 = 3n-6-k$. ■

Lemma 14: In $S_n(n \geqslant 5)$, let $V' \subset V(S_n)$ with $|V'| = k$ $(3 \leqslant k \leqslant 3n-6)$, then V' has always a node, say v, satisfying $|PN_{V'}(v)| \geqslant n-2-\lfloor\frac{k}{3}\rfloor$.

Proof:

Case 1: $k = 1$ or $k = 2$.

When $k = 1$, since for each node $v \in V'$ $|PN_{V'}(v)| = n-1$, the claim is true. When $k = 2$, since $|PN_{V'}(v)| \geqslant n-2$, the claim holds.

Case 2: $k \geqslant 3$.

We prove the claim by showing that there exist three nodes, say w_1, w_2, w_3, such that $|PN_{V'}(w_1)| + |PN_{V'}(w_2)| + |PN_{V'}(w_3)| \geqslant 3n-6-k$, which implies that $\{w_1, w_2, w_3\}$ has a node v satisfying $|PN_{V'}(v)| \geqslant n-2-\lfloor\frac{k}{3}\rfloor$. Let $S = \{v_1, v_2, v_3\} \subseteq V'$. We will discuss these cases described as follows:

Case 2.1: There exist two nodes, say $v_1, v_2 \in S$, such that v_1, v_2 are adjacent (see Figure 5):

Case 2.1.1: v_2 and v_3 are adjacent.

For S, $|PN_S(v_1)| + |PN_S(v_2)| + |PN_S(v_3)| = 3n-7$. According to Lemma 6, we get that V' has a node u such that $|N(u) \cap [PN_S(v_1) \cup PN_S(v_3)]| = 2$. Therefore, $|PN_{V'}(v_1)| + |PN_{V'}(v_2)| + |PN_{V'}(v_3)| \geqslant 3n-7-(k-3)-1 = 3n-5-k$.

Case 2.1.2: v_2 and v_3 share one common neighbor.

Then $|PN_S(v_1)| + |PN_S(v_2)| + |PN_S(v_3)| = 3n-7$. After a similar argument to Case 2.1.1, we get that V' has at most one node, say u, satisfying $|(N(u) \cup \{u\}) \cap (PN_S(v_1) \cup PN_S(v_2) \cup PN_S(v_3))| = 2$. Therefore, $|PN_{V'}(v_1)| + |PN_{V'}(v_2)| + |PN_{V'}(v_3)| \geqslant 3n-7-(k-3)-1 = 3n-5-k$.

Case 2.1.3: There are two nodes x, y between v_2 and v_3 (see Figure 5).

then $|PN_S(v_1)| + |PN_S(v_2)| + |PN_S(v_3)| = 3n-5$. If v_2, x, y, v_3 belong to a 6-node cycle, then V' has at most one

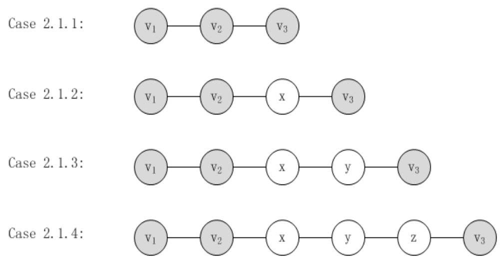

FIGURE 5. Case 2.1 of Lemma 14.

node, say u, satisfying $|(N(u) \cup \{u\}) \cap [PN_S(v_2) \cup PN_S(v_3)]| = 2$. And if v_1, v_2, x, y, v_3 belong to an 8-node cycle, according to Lemma 12, there exist at most two nodes in V', say w, such that $|N(w) \cap (PN_S(v_1) \cup PN_S(v_3))| = 2$. Therefore, $|PN_{V'}(v_1)| + |PN_{V'}(v_2)| + |PN_{V'}(v_3)| \geqslant 3n - 5 - (k-3) - 4 = 3n - 6 - k$.

Case 2.1.4: There are three nodes x, y, z between v_2 and v_3 (see Figure 5).

Then $|PN_S(v_1)| + |PN_S(v_2)| + |PN_S(v_3)| = 3n - 5$. If v_2, x, y, z, v_3 belong to an 8-node cycle, then according to Lemma 12, there exist at most three nodes in V', say u, such that $|N(u) \cap [PN_S(v_2) \cup PN_S(v_3)]| = 2$. Therefore, $|PN_{V'}(v_1)| + |PN_{V'}(v_2)| + |PN_{V'}(v_3)| \geqslant 3n - 5 - (k-3) - 3 = 3n - 5 - k$.

If there are at least four nodes between v_2 and v_3, the claim is obvious.

Case 2.2: Any two nodes of v_1, v_2, v_3 is not adjacent.

If v_1, v_2, v_3 belong to a 6-node cycle, then according to Lemma 13, the claim is true. Hence, we need only to discuss the situation that v_1, v_2, v_3 do not belong to a 6-node cycle. Consider the following cases :

Case 2.2.1: There exist two nodes in S, say v_1, v_2, such that they share one common neighbor.

If v_1, v_2, v_3 share one common neighbor, then according to condition i) of Lemma 13, the claim holds.

If v_1, v_2 share one common neighbor, say x, and v_2, v_3 share another common neighbor, say y, then according to condition iii) of Lemma 13, the claim holds.

If v_1, v_2 share one common neighbor, say x, and v_1, v_3(and v_2, v_3) share no common neighbors, then $|PN_S(v_1)| + |PN_S(v_2)| + |PN_S(v_3)| = 3n - 5$. If V' has some node u satisfying $|N(u) \cap [PN_S(v_1) \cup PN_S(v_2)]| = 2$, according to condition ii) of Lemma 13, the claim is true. Similarly, if there are two (three) nodes $x, y(x, y, z)$ between v_2 and v_3 and $v_2, x, y, v_3(v_2, x, y, z, v_3)$ can form one 6-node (8-node) cycle with two other nodes (three other nodes), a similar argument to the Case 2.1.3, 2.1.4 can be used.

Case 2.2.2: Any two nodes of S do not share one common neighbor.

Then $|PN_S(v_1)| + |PN_S(v_2)| + |PN_S(v_3)| = 3n - 3$. If V' has some node u satisfying $|(N(u) \cup \{u\}) \cap [PN_S(v_1) \cup PN_S(v_2) \cup PN_S(v_3)]| \geqslant 2$, then a similar argument to Case 2.1 or Lemma 13 can be used for $\{u, v_1, v_2\}$. Otherwise, $|PN_{V'}(w_1)| + |PN_{V'}(w_2)| + |PN_{V'}(w_3)| \geqslant 3n - 3 - (k-3) = 3n - k \geqslant 3n - 6 - k$. ■

Lemma 15: In $S_n(n \geqslant 5)$, let V' be a k-node subset of $V(S_n)$, where $(1 \leqslant k \leqslant 3n-6)$, then $|N(V')| \geqslant kn - \frac{(k+2)^2}{3} + 1$.

Proof: We show this result by using induction on k. When $k = 1$, since each node of S_n has exactly $(n-1)$ neighbors, the result is true. when $k = 2$, since $|N(V')| \geqslant 2n - 4 \geqslant 2n - \frac{(2+2)^2}{3} + 1$, the result is true. When $k = 3$, since $|N(V')| \geqslant 3n - 7 \geqslant 3n - \frac{(3+2)^2}{3} + 1$, the result is true. And for $k = 4$, according to Lemma 10, $|N(V')| \geqslant 4n - 10 \geqslant 4n - \frac{(4+2)^2}{3} + 1$.

Now, suppose that the result holds for k $(k \geqslant 4)$. Next, we will discuss the case of $k+1$. By contrary, suppose that V has a subset V' with $|V'| = k+1$ satisfying $|N(V')| < (k+1)n - \frac{(k+3)^2}{3} + 1$. By Lemma 14, we have that V' has a node v satisfying $|PN_{V'}(v)| \geqslant n - 2 - \lfloor \frac{k+1}{3} \rfloor$. Let $V'' = V' - \{v\}$, the following cases will be discussed:

Case 1: v has a neighbor in V'.

$N(V'') = (N(V') - PN_{V'}(v)) \cup \{v\}$, then $|N(V'')| = |N(V')| - |PN_{V'}(v)| + 1 < (k+1)n - \frac{(k+3)^2}{3} + 1 - [n - 2 - \lfloor \frac{k+1}{3} \rfloor] + 1 = kn - \frac{k^2+6k-3}{3} + \lfloor \frac{k+1}{3} \rfloor$. Since $kn - \frac{(k+2)^2}{3} + 1 - [kn - \frac{k^2+6k-3}{3} + \lfloor \frac{k+1}{3} \rfloor] = \frac{2k-4}{3} - \lfloor \frac{k+1}{3} \rfloor = \frac{k-5}{3} + (\frac{k+1}{3} - \lfloor \frac{k+1}{3} \rfloor) \geqslant 0 (k \geqslant 5)$, $|N(V'')| < kn - \frac{(k+2)^2}{3} + 1$, a contradiction.

Case 2: v has no neighbor in V'.

Then $N(V'') = (N(V) - PN_{V'}(v))$, then $|N(V'')| = |N(V')| - |PN_{V'}(v)| < (k+1)n - \frac{(k+3)^2}{3} - [n - 2 - \lfloor \frac{k+1}{3} \rfloor] + 1 = kn - \frac{k^2+6k}{3} + \lfloor \frac{k+1}{3} \rfloor \leqslant kn - \frac{k^2+6k}{3} + \frac{k+1}{3} = kn - \frac{k^2+5k-1}{3}$. Since $kn - \frac{(k+2)^2}{3} + 1 - [kn - \frac{k^2+5k-1}{3}] = \frac{k-2}{3} > 0 (k \geqslant 5)$, $|N(V'')| < kn - \frac{(k+2)^2}{3} + 1$, a contradiction. ■

Lemma 16: In $S_n(n \geqslant 4)$, let F be a subset $V(S_n)$ with $|F| \leqslant ln - \frac{(l+2)^2}{3}$ $(1 \leqslant l \leqslant n-2)$, $G_{induced}(V(S_n) - F)$ the induced subgraph of $V(S_n) - F$. And let $C_{sub}(G_{induced}(V(S_n) - F)) = \{C_1, C_2, \cdots, C_m\}$. Then the following conditions hold:

i) $\Sigma_{i=0}^{l-1} i |Card_i(C_{sub}(G_{induced}(V(S_n) - F)))| \leqslant l - 1$.

ii) There exists exactly one subset $C_i \in C_{sub}(G_{induced}(V(S_n) - F))$ with $|V(C_i)| \geqslant l$.

Proof: By induction on n. For $n = 4$, then $1 \leqslant l \leqslant 2$. For $l = 1$, we have $|F| \leqslant 1$, then the results are true. When $l = 2$, we have $|F| \leqslant 3 - \frac{1}{3}$. Since the connectivity of a 4-dimensional star graph is 3, the results hold when $n = 4$. Assume that the results hold for some $n - 1 \geqslant 4$. Now we consider the situation of $n > 5$. For convenience, we divide S_n into n $S_{n-1}s$, denoted by $H_1, H_2, \cdots, H_n$.

By induction on l. For $l = 1$, we have $|F| \leqslant n - 3$. The fact the connectivity of S_n is $n-1$ implies that the results are true. Assume that the results are true for some $l-1$ where $l - 1 \geqslant 1$. We shall prove that the results hold also for l. Suppose that, to the contrary, for some l, at least one of the condition i) and ii) is false. Next, we will derive a contradiction.

Case 1: $\Sigma_{i=0}^{l-1} i |Card_i(C_{sub}(G_{induced}(V(S_n) - F)))| \geqslant l$.

Let $T = \{Card_i(C_{sub}(G_{induced}(V(S_n) - F)))$ *where* $1 \leqslant i \leqslant l - 1\}$ and we can always find a subset $T' \subseteq T$ with $T' = \{C_1, C_2, \cdots, C_r\}$such that $| \cup_{i=1}^{r-1} C_i| \leqslant l - 1$ and

$|\cup_{i=1}^{r} C_i| \geqslant l$. Let $T_a = \cup_{i=1}^{r-1} C_i$ and $T_b = C_r$, $|T_a \cup T_b| = \alpha$. Note that $|T_a|, |T_b| \leqslant l-1$, then $\alpha \leqslant 2l-2$. Then $|F| \geqslant \alpha n - \frac{(\alpha+2)^2}{3} + 1$. Let $f(\alpha) = \alpha n - \frac{(\alpha+2)^2}{3} + 1 - [ln - \frac{(l+2)^2}{3}]$. After simplifying, $f(\alpha) = \frac{-\alpha^2+(3n-4)\alpha+l^2+4l-3ln+3}{3}$. Note that $\alpha \in [l, 2l-2]$, we have that $f(\alpha) \geqslant min\{f(l), f(2l-2)\}$. Furthermore, $f(l) = 1$ and $f(2l-2) = \frac{-3l^2+(3n+4)l+(7-6n)}{3} > 0$ $(l \geqslant 2)$, which implies that $f(\alpha) > 0$. On the other hand, from the assumption, we have that $|F| \leqslant ln - \frac{(l+2)^2}{3}$, a contradiction.

Case 2: There exist at least two $C_i \in C_{sub}(G_{induced}(V(S_n)-F))$ with $|V(C_i)| \geqslant l$.

Let $C_1, C_2 \in C_{sub}(G_{induced}(V(S_n) - F))$ with $|V(C_1)|, V(C_2)| \geqslant l$. If $|V(C_1)| = \alpha \leqslant 2n-2$, then $|F| \geqslant |V(C_1)| \geqslant \alpha n - \frac{(\alpha+2)^2}{3} + 1$. Let $g(\alpha) = \alpha n - \frac{(\alpha+2)^2}{3} + 1 - [ln - \frac{(l+2)^2}{3}]$ where $l \leqslant \alpha \leqslant 2n-2$. Since $g(\alpha)$ is a quadratic function, $g(\alpha) \geqslant min\{g(l), g(2l-2)\}$. Note that $g(l) = 1$ and $g(2n-2) = \frac{2n^2-6n-3ln+(l+2)^2+3}{3} \geqslant 1$, which implies that $g(\alpha) \geqslant 1$, therefore, $|F| > ln - \frac{(l+2)^2}{3}$. This is a contradiction to the hypothesis. Therefore, $|V(C_1)| > 2n-2$. Similarly, $|V(C_2)| > 2n-2$. Let $V(C_1) \cap H_i = A_i$, $V(C_2) \cap H_i = B_i$ and $F \cap H_i = T_i$. We discuss the cases described as follows:

Case 2.1: $|T_i| > (l-1)(n-1) - \frac{(l+1)^2}{3}$ for some i.

$\Sigma_{j \neq i}^{n} |T_j| = |F| - |T_i| < ln - \frac{(l+2)^2}{3} - [(l-1)(n-1) - \frac{(l+1)^2}{3}] = (n-2) + \frac{l}{3} \leqslant 2n-6 (n \geqslant 5)$. Therefore, $\Sigma_{j \neq i}^{n} |T_j| < 2n-6$.

Since the connectivity of H_j is $n-2$, there is at most one $H_j (j \neq i)$ satisfying that $H_j - T_j$ is disconnected. If for each $j \in \{1, 2, \cdots, n\} (j \neq i)$, $H_j - T_j$ is connected. Then $L = V(S_n) - (F \cup H_i)$ is connected. Since each node of H_i has exactly one neighbor outside H_i and $\Sigma_{j \neq i}^{n} |T_j| \leqslant 2n-7$, there exist at most $2n-7$ nodes in $H_i - T_i$ which are not adjacent to L. Therefore, there exists only one component of $V(S_n) - F$ whose size is larger than $2n-2$, which is a contradiction. If there exists some $H_j (j \neq i)$ satisfying that $H_j - T_j$ is disconnected, we shall show the result is also true. Let L_a be the largest component of $H_a - T_a (1 \leqslant a \leqslant n)$. Note that $|T_j| \leqslant 2n-7 \leqslant 3n-11-\frac{1}{3} = 3(n-1) - \frac{(3+2)^2}{3} (n \geqslant 5)$, according to the induction hypothesis, $|H_j - T_j - L_j| \leqslant 2$. Since the number of edges between any two different H_j's, where each one of them is different from H_i, is $(n-2)!$. When $n \geqslant 5$, $(2n-7+2) < (n-2)!$, $A = \cup_{j=1, j \neq i}^{n} L_j$ is connected. Since each node of H_i has exactly one neighbor outside H_i and $\Sigma_{j \neq i}^{n} |T_j| \leqslant 2n-7$, there exist at most $(2n-7+2)$ nodes in $H_i - T_i$ which are not adjacent to A, which implies that there exists only one component of $V(S_n) - F$ whose size is larger than $2n-2$, this is a contradiction.

Case 2.2: $|T_i| \leqslant (l-1)(n-1) - \frac{(l+1)^2}{3}$ for all i.

Let L_i be the largest component of $H_i - T_i$. According to the induction hypothesis, we have that $|L_i| \geqslant l-1$ and $|H_i - T_i - L_i| \leqslant l-2$, which implies that $|H_i - L_i| \leqslant (l-2) + (l-1)(n-1) - \frac{(l+1)^2}{3}$. Since there are $(n-2)!$ edges between H_i and H_j, and each node of H_i is adjacent to at most one node in H_j, and $(n-2)! - [2(l-2) + 2((l-1)(n-1) - \frac{(l+1)^2}{3})] \geqslant (n-2)! - \frac{4n^2-16n-2}{3} > 0$ $(n \geqslant 6)$, there exists at least one edge from L_i to L_j, which implies that L_i, L_j are connected$(n \geqslant 6)$. And for $n = 5$, $|T_i|, |T_j| \leqslant \frac{-l^2+10l-13}{3} \leqslant 3 - \frac{1}{3}$, which implies that $H_i - T_i$ $(H_j - T_j)$ is connected. Therefore, L_i, L_j are connected for $n = 5$. Furthermore, $\cup_{i=1}^{n} L_i$ is the largest component of $G_{induced}(V(S_n) - F)$, then either $V(C_1) \subseteq \cup_{i=1}^{n} V(H_i - T_i - L_i)$ or $V(C_2) \subseteq \cup_{i=1}^{n} V(H_i - T_i - L_i)$. Without loss of generality, let $V(C_1) \subseteq \cup_{i=1}^{n} V(H_i - T_i - L_i)$, then $A_i \subseteq V(H_i - T_i - L_i)$. For each pair i, j $(i \neq j)$, $\alpha = |A_i| \leqslant l-2$ and $\beta = |A_j| \leqslant l-2$, $|N(A_i) \cap V(H_i)| + |N(A_j) \cap V(H_j)| \geqslant \alpha(n-1) - \frac{(\alpha+2)^2}{3} + 1 + \beta(n-1) - \frac{(\beta+2)^2}{3} + 1 \geqslant (\alpha+\beta)(n-1) - \frac{(\alpha+\beta+2)^2}{3} + 1$ $(*)$. For $A_1, A_2, \cdots, A_n$, we can always find a positive integer $1 \leqslant r \leqslant n$ such that $|\cup_{i=1}^{r-1} A_i| \leqslant l-2$, $|\cup_{i=1}^{r} A_i| \geqslant l-1$. Since $|\cup_{i=1}^{r} A_i| \leqslant 2(l-2) \leqslant 2n-8$ and $|V(C_1)| > 2n-2$, there exist three nodes $u, v, w \in \cup_{j=r+1}^{n} A_j$. According to Lemma 15 and $(*)$, $|N(\cup_{j=r+1}^{n} A_j) \cap (\cup_{j=r+1}^{n} H_j)| \geqslant |N(\{u, v, w\}) \cap (\cup_{j=r+1}^{n} H_j)| \geqslant 3(n-1) - \frac{5^2}{3} + 1 = 3n - 10 - \frac{1}{3}$. Let $\gamma = \Sigma_{i=1}^{r} |A_i|$. Since $\Sigma_{i=1}^{r} |N(A_i) \cap H_i| \geqslant \gamma(n-1) - \frac{(\gamma+2)^2}{3} + 1$, $|F| \geqslant \Sigma_{i=1}^{r} |N(A_i) \cap H_i| + |N(\cup_{j=r+1}^{n} A_j) \cap (\cup_{j=r+1}^{n} H_j)| \geqslant \gamma(n-1) - \frac{(\gamma+2)^2}{3} + 1 + 3n - 10 - \frac{1}{3}$, where $l-1 \leqslant \gamma \leqslant 2(l-2)$. According to the properties of the quadratic function, $\gamma(n-1) - \frac{(\gamma+2)^2}{3} + 1 \geqslant (l-1)(n-1) - \frac{(l+1)^2}{3} + 1$, where $l-1 \leqslant \gamma \leqslant 2(l-2)$. Therefore, $|F| \geqslant (l-1)(n-1) - \frac{(l+1)^2}{3} + 1 + 3n - 10 - \frac{1}{3}$. Since $(l-1)(n-1) - \frac{(l+1)^2}{3} + 1 + 3n - 10 - \frac{1}{3} - [ln - \frac{(l+2)^2}{3}] = 2n - 7 - \frac{l+1}{3} > 0$ $(n \geqslant 5)$, we conclude that $|F| > ln - \frac{(l+2)^2}{3}$, a contradiction.

Combining the above cases, our proof is completed. ■

IV. THE T/S-DIAGNOSABILITY OF THE STAR GRAPH

Definition 17: Let S be a system, t and s are two positive integers $(t \leqslant s)$. S is called to be t/s-diagnosable if a set of units L with $|L| \leqslant s$, which contains all faulty units of S, can be located provided the number of faulty units in the system S is no more than t.

Lemma 18: Let $G = (V, E)$ be a system, G' a connected subgraph of G with $|V(G')| \geqslant t+1$. Assume that G has at most t faulty vertices. If the test results in G' are all 0s, then G' doesn't contain any faulty vertices.

Proof: To the contrary, suppose that G' contains some faulty vertex u. Since the test results in G' are all 0s, each vertex of $N_{G'}(u)$ is faulty. Furthermore, all vertices of $N_{G'}(N_{G'}(u))$ are faulty. Since G' is a connected subgraph of G, each vertex of $V(G')$ is faulty. Hence, the system has more than t faulty vertices, a contradiction. ■

Lemma 19 [35]: Let $G = (V, E)$ represent a system, $Y = \{v_i \in V | 1 \leqslant i \leqslant l, (v_j, v_k) \notin E \ (1 \leqslant j, k \leqslant l, j \neq k)(l \geqslant 3)\}$. Let β be the cardinality of $N(Y)$, then the system is not $(\beta+1)/[\beta+(l-1)]$-diagnosable.

Now, we begin to discuss the t/s-diagnosability of n-dimensional star graph.

Theorem 20: $S_n(n \geqslant 5)$ is not $[(n-2)l+2]/[(n-1)l]$-diagnosable $(3 \leqslant l \leqslant n-1)$.

Proof: According to Lemma 19 and Lemma 7, we conclude that the result is true. ■

Theorem 21: $S_n(n \geqslant 5)$ is $\lfloor ln - \frac{(l+2)^2}{3} \rfloor / (\lfloor ln - \frac{(l+2)^2}{3} \rfloor + l - 2)$-diagnosable, where $2 \leqslant l \leqslant n-2$.

Proof: We employ $G = (V, E)$ to represent the graph of S_n, R to represent the collection of all faulty units in S_n with $|R| \leqslant \lfloor ln - \frac{(l+2)^2}{3} \rfloor$. Let $C_{sub}(G_{induced}(V - R)) = \{C_1, C_2, \cdots, C_m\}$ represent the collection of all components in $G-R$. By Lemma 16, we have the two conditions described as follows:

i) $\Sigma_{i=0}^{l-1} i|Card_i(C_{sub}(G_{induced}(V-R)))| \leqslant l-1$.

ii) $C_{sub}(G_{induced}(V-R))$ has exactly one C_i such that $|V(C_i)| \geqslant l$.

Let $S = \cup_{i=0}^{l-1}(\cup Card_i(C_{sub}(G_{induced}(V-R))))$ and $\alpha = |S| \leqslant l-1$. We will discuss the cases described as follows:

Case 1: $\alpha < l-1$.

By above condition ii), we have $C_i = G_{induced}(V - R - S)$, which implies that $G_{induced}(V-R-S)$ is a component. On the other hand, we have that $|C_i| \geqslant n! - (l-2) - \lfloor ln - \frac{(l+2)^2}{3} \rfloor > \lfloor ln - \frac{(l+2)^2}{3} \rfloor$ $(n \geqslant 5)$. Since the test results in C_i are $0s$, we conclude that all nodes of C_i are fault-free by Lemma 18. Since $|V - V(C_i)| \leqslant \lfloor ln - \frac{(l+2)^2}{3} \rfloor + l - 2$ and $V - V(C_i)$ contains all faulty nodes in S_n, the result is true.

Case 2: $\alpha = l-1$.

We claim that R doesn't have any node v satisfying $N(v) \subseteq R \cup S$. By contrary, let $R' = R - \{v\}$, then we have that $|N(S \cup \{v\})| \leqslant |R'| \leqslant \lfloor ln - \frac{(l+2)^2}{3} \rfloor - 1$, a contradiction to Lemma 15. Therefore, $N(R) \subset V(C_i)$. On the other hand, the similar argument to the proof of Case 1 can be used to prove that all nodes in C_i are fault-free here. Hence, all nodes of R can be diagnosed to be faulty correctly. The result is true. ■

Theorem 22: $S_n(n \geqslant 5)$ is $\beta/(\beta + l - 2)$-diagnosable, where β and l are two positive integers satisfying the following conditions :$\beta \in (\lfloor (l-1)n - \frac{(l+1)^2}{3}, \lfloor ln - \frac{(l+2)^2}{3} \rfloor]$, $l \in [2, n-2]$.

Proof: we use $G = (V, E)$ to denote the graph of S_n, and R to denote the set of all faulty nodes in S_n with $\lfloor (l-1)n - \frac{(l+1)^2}{3} \rfloor < |R| \leqslant \lfloor ln - \frac{(l+2)^2}{3} \rfloor$. Let $C_{sub}(G_{induced}(V-R)) = \{C_1, C_2, \cdots, C_m\}$. And a similar argument to the proof of Theorem 4.2 can be used to prove the result. ■

The results that S_n is $(n-1)$-diagnosable and $(2n-4)/(2n-4)$-diagnosable have been obtained in previous literatures. Our studies show that S_n is $\lfloor ln - \frac{(l+2)^2}{3} \rfloor / (\lfloor kn - \frac{(l+2)^2}{3} \rfloor + l - 2)$-diagnosable where $2 \leqslant l \leqslant n-2$, but not $[(n-2)l+2]/(n-1)l$-diagnosable $(3 \leqslant l \leqslant n-1)$. In other words, our results show that the t/s-diagnosability of S_n is about $(s-t+2)$ times as large as t-diagnosability of it. Figure 6 describes the relationship of several diagnosabilities of S_n.

For some integer $l \in [2, n-2]$ and some integer $\beta \in (\lfloor (l-1)n - \frac{(l+1)^2}{3}, \lfloor ln - \frac{(l+2)^2}{3} \rfloor]$, by Theorem 22 it is guaranteed that S_n is t/s-diagnosable $(t = \beta, s = t + l - 2)$. In the next section, for a t/s -diagnosable system S_n, in order to locate one set of nodes containing all faulty nodes with size of less than or equal to s, a t/s diagnosis algorithm will be presented.

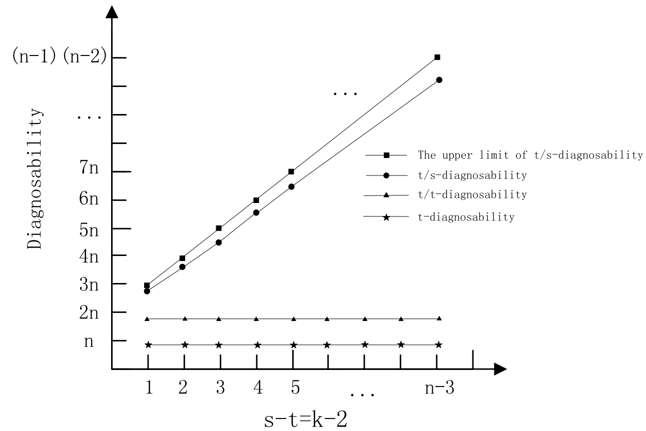

FIGURE 6. The comparison of several diagnosabilities of S_n.

V. A FAST T/S DIAGNOSIS ALGORITHM OF S_n

For the t/s-diagnosable system S_n, and a syndrome ω, our t/s-diagnosis algorithm needs to determine the largest component of fault-free nodes, for this reason, we present firstly an algorithm called Depth-First search (DFS) (see Algorithm 1).

Algorithm 1 DFS

Input:

Input S_n, a syndrome ω, let $L = S = \varnothing$ and a fault bound T.

Output:

Output The set S.

1: Choose a node $p \in V(S_n) - L$ and let $S = \{p\}$.

2: $DFS(p)$:
for each $q \in N(p)$
if $\omega(q, p) = \omega(p, q) = 0$,
$S = S \cup \{q\}$ and $DFS(q)$.

3: If $|S| \geqslant T + 1$, then goto step 4. If $|S| \leqslant T$, then let $L = L \cup S$ and goto step 1.

4: Output S.

It is worth noting that for given a syndrome ω and a bound $(T \leqslant \lfloor ln - \frac{(l+2)^2}{3} \rfloor, l \leqslant n-2, n \geqslant 5)$, the set of nodes S, which comprises all nodes in the largest component of fault-free nodes in S_n, can always be output by Algorithm 1. In fact, Lemma 16 guarantees that such a component with size of larger than or equal to $T+1$ is existing and unique. On the other hand, Lemma 18 guarantees that all nodes in such a component are fault-free.

Next, we will propose another algorithm, called Isolating-Fast Faults (in brief, IFF), to locate the set of nodes L with size of less than or equal to s for a t/s-diagnosable S_n $(t \leqslant \lfloor ln - \frac{(l+2)^2}{3} \rfloor, l \leqslant n-2, n \geqslant 5)$ (see Algorithm 2 for details).

Theorem 23: For S_n, let $N = n!$ denote the order number of S_n, then the algorithm IFF has time complexity $O(Nlog_2 N)$.

Algorithm 2 IFF

Input:

Input S_n, a syndrome ω, an integer l ($l \leqslant n-2$), a fault bound T ($T \leqslant \lfloor ln - \frac{(l+2)^2}{3} \rfloor$).

Output:

Output Three sets of units: P, Q, R, where the units of P are faulty, the units of R are fault-free, the units of U are unknown.

1: Let $P = R = U = \varnothing$. And call the algorithm DFS to output the set of nodes of the largest fault-free component S.

2: $R = R \cup S$ and $P = P \cup N(R)$.

3: If $V = R \cup P$, output the sets R and P, goto step 5.

4: If $|P| = T$, then $R = V - P$ and output the sets R and P. Otherwise, $U = V - R - P$ and output the sets P, R, U.

5: END.

TABLE 1. The results identified by the algorithm isolating-fast faults in the 6-dimensional star graph.

k	1	2	3	4
faulty nodes t	3	6	9	12
detected faulty nodes	3	6	9	12

TABLE 2. The results identified by the algorithm isolating-fast faults in the 8-dimensional star graph.

k	1	2	3	4	5	6
faulty nodes t	5	10	15	20	23	26
detected faulty nodes	5	10	15	20	23	26

Proof: To locate the set of vertices of the largest component S, step 1 costs $Nlog_2N$ time. In step 2, calculating $N(R)$ costs at most $O(N)$ time, calculating the set R and P costs $O(1)$ time, thus, step 2 costs $O(1+N+1) = O(N)$ time. Besides, step 3, step 4 and step 5 cost $O(1)$ time. Consequently, the time of running the algorithm IFF is $O(Nlog_2N + N + 1) = O(Nlog_2N)$. ■

Next, in order to evaluate the effectiveness of the algorithm Isolating-Fast faults, we design an experience of computer simulation as follows. We randomly deploy $t = \lfloor kn - \frac{(k+2)^2}{3} \rfloor$ faulty nodes in S_n. For the sake of convenience, we suppose that the probability, which the test result of each faulty vertex testing another vertex is 1, is 0.5. After running the algorithm for 100,000 times, we obtain the results of simulations on the 6(8)-dimensional star graph shown in the Table 1(2). And it is clear that the algorithm successfully identifies all faulty nodes in the system. It is worth mentioning that for the sake of guaranteeing the reliability of the outcomes of the simulation, we utilize the software Java and the advanced hardware including Intel Core i7 CPU 3.3 GHz, 16 GB DRAM, 64-bit Windows 7 OS to program and execute the algorithm.

VI. CONCLUSIONS

S_n is an useful topology, we proved the result that for S_n if $S \subset V(S_n)$ with $|S| \leqslant ln - \frac{(l+2)^2}{3}$ ($1 \leqslant l \leqslant n-2$) then $S_n - S$ has unique component with size more than or equal to l. By this component, we proposed a t/s diagnosis algorithm to locate a set of vertices containing all faulty vertices with size of no exceeding s. At the same time, we introduced a sufficient condition to judge that S_n is not t/s-diagnosable (namely $t = (n-2)l+2$, $s = (n-1)l$ and $3 \leqslant l \leqslant n-1$) and another sufficient condition to judge that S_n is $t_1/(t_1+l-2)$ -diagnosable ($t_1 \in (\lfloor (l-1)n - \frac{(l+1)^2}{3}, \lfloor ln - \frac{(l+2)^2}{3} \rfloor]$, $l \in [2, n-2]$). Our results are obtained based on the PMC. As is known to all that the comparison model, the generalization of the PMC model, is another classical model in system-level fault diagnosis. It is attractive to extend the outcomes of this paper from the PMC model to the comparison model.

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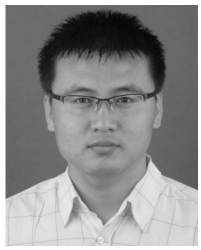

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