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Iterative Learning Control for MIMO Singular Distributed Parameter Systems

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ABSTRACT This paper deals with the iterative learning control issue for multi-input multi-output singular distributed parameter systems (SDPSs) with parabolic and hyperbolic type, which described by coupled partial differential equations with singular matrix coefficients. Initially, applying the singular value decomposition theory to SDPSs, an equivalent dynamic decomposition form is derived. Then, the estimation of the relationship between the learning system substates and output tracking error are constructed in the light of P-type update learning scheme under some assumptions. Moreover, two sufficient conditions are presented to ensure that the tracking error is convergent in the sense of $\mathbf{L}_2$ norm by employing the contracting mapping principle as well as some basic differential inequalities. Finally, two numerical examples are shown to demonstrate the validity of the developed theoretical results.

INDEX TERMS Distributed parameter systems, learning systems, intelligent control, singular value decomposition, convergence.

I. INTRODUCTION

Singular Distributed Parameter Systems (SDPSs), which can be governed by partial differential equations with singular matrices arise in front of partial derivatives, deriving from the research of physical phenomena and industrial processes, such as nanoelectronics, transmission lines in signal propagation as well as atmospheric physics [1], [2]. The real world has provided the profound actual background for the research of SDPSs, involved in the fields of mathematics, material engineering, chemical biology, economics and so on [3]–[5]. SDPSs is also referred to as partial differential-algebraic equations. This system with an infinite dimensional state, distinguishing from the generalized state space system, but also distinct from the general distributed parameter system [6]. SDPSs with two administrative levels, one layer for the objective dynamic characteristics which described by partial differential equations, another layer is management characteristics of static properties which described by algebraic equations, whereas the normal system without static attribute. Not only will SDPSs unstable, but the system structure has changed also dramatically under the interference of unknown factors, such as causing the pulse behavior, this also makes the controller hard to implement [7].

Iterative Learning Control (ILC) is an intelligent control scheme that is suitable for the repetitive controlled system achieving perfect tracking over a finite time interval. Its basic idea is to utilize the former once or previous control information to amend and update the control inputs of current times, so repeatedly learning that the objective output will gradually achieve the complete tracking of reference trajectory. Although the convergent conditions of ILC are established with the help of rigorous mathematical analysis, it does not require a precise mathematical model [8]–[10]. Since the concept of ILC is coined by Arimoto in 1984, it has become a hotspot in the field of intelligent control. Not only ILC achieves fruitful results in practical applications, such as limb recovery robot [11], rapid thermal processing [12], semibatch chemical reactors [13], urban traffic systems [14], multi-agent systems [15], [16], but also employees into the theoretical analysis of various systems which contain switching systems, stochastic systems, pulse and distributed parameter systems (DPSs). However, there is no theoretical analysis of ILC for SDPSs at present.

In the past few years, the theoretical research of SDPSs have attracted increasing attention to academics, and some accomplishments have been presented in the literatures. Now, the study of SDPSs mainly concentrate in two aspects. On the one hand, the expression and characteristics of the solutions are considered. For example, the paper [17] exhibits the solution of coupled hyperbolic PDEs with singular matrix coefficients in view of the Fourier approach. In [18] and [19], the operator decomposition method and Empathy theory are respectively introduced to discuss the solvability problem of homogeneous constant SDPSs in Banach space. A boundary value problem for linear SDPSs is considered in terms of the separation of variables method and matrix pencil theory in [4]. On the other hand is the study of its control problem, such as, the robust exponential stability for uncertain SDPSs in the light of linear operator inequality is investigated in [20]. Based on the generalized operator semigroup theory and functional analysis method incorporated average dwell time approach, the control synthesis of SDPSs which including feedback stability and the well-posedness problem are concerned in Hilbert space, and some sufficient conditions are derived in [21]–[23]. The literature [24] studies sliding mode control scheme for SDPSs with perturbation using inherent function method. In [25], state feedback control approach is proposed for SDPSs with parabolic-elliptic type and the equivalent decomposition form is shown based on the spectrum analysis. In a word, the control theory of SDPSs combines singular systems theory with distributed parameter systems theory [26]–[28]. The researches of ILC for singular systems and DPSs are limited and only a little related results are reported. Reference [29] designs P-type ILC updating law in the frequency domain for linear inhomogeneous DPSs with the help of Laplace transform. Eigenspectrum-Based ILC scheme is considered for semi-linear DPSs and applying Galerkinąŕs approach with the eigenspectrum theoretics to reduce model in [30]. The papers [31], [32] provide the convergence conditions for uncertain linear DPSs with closed-loop and opened-loop P-type algorithm by applying contraction mapping principle respectively. Reference [33] utilizes frobenius norm to address ILC tracking problem for the fast subsystem of singular system canonical form with impulse behavior and the requirement of impulse controllable constraint. PD-type ILC law is presented for singular discrete systems with the aid of singular value decomposition transformation in [34]. However, to the best of the authors' knowledge, there is no report about the ILC of MIMO SDPSs with parabolic and hyperbolic type.

In this paper, we are concerned with the problem of ILC algorithm for two classes of MIMO singular parameter distributed systems which are parabolic and hyperbolic type, reformulated into its equivalent dynamic decomposition form by means of singular value decomposition theory. According to the substate variables of equal decomposition, we estimate the relationship between them and the output tracking error, thereby two Lemmas are simultaneously given. Using a typical P-type learning law, the two convergence conditions for SDPSs with parabolic and hyperbolic type are provided in terms of some integral inequalities and contraction mapping approach. Finally, two numerical simulations are performed to illustrate the effectiveness of the proposed controller.

The paper has the following structure. In Section II, problem formulation and system description are first given under some assumptions. We present the details for analysing the convergence conditions of output tracking error for the repetitive MIMO SDPSs in Section III. In consequence, based on the derivation in Section III, we focus on two numerical simulations in Section IV. At last, Section V concludes the paper and further discussions are shown.

Notations: The superscript 'T′ denotes the matrix transposition; $A > 0$ (respectively, $A < 0$) denotes a symmetric positive (respectively, negative) definite matrix. Define $V = (v_1, v_2, \cdots, v_n)$ is vector, then the Euclidean norm of its is $\|V\| = \sqrt{\sum_{i=1}^{n} v_i^2}$. If define V is matrix, $\|V\| = \sqrt{\lambda_{\max}(V^T V)}$ is its matrix norm, where $\lambda_{\max}(\cdot)$ is the maximum eigenvalue of V. If $Q_i \in \mathbf{L}^2(\Omega)$ $(i = 1, 2, \cdots, n)$, we define $Q = (Q_1, Q_2, \cdots, Q_n) \in \mathbb{R}^n \cap \mathbf{L}^2(\Omega)$, then $\|Q\|_{\mathbf{L}^2} = \{\int_\Omega Q^T(x)Q(x)dx\}^{\frac{1}{2}}$. For $f(x, t) : \Omega\times[0, T] \to \mathbb{R}^n, f(\cdot, t) \in \mathbb{R}^n\cap\mathbf{L}^2(\Omega), \forall t \in [0, T]$, its $(\mathbf{L}^2, \lambda)$ norm is defined as $\|f\|^2_{(\mathbf{L}^2,\lambda)} = \sup_{0\leqslant t\leqslant T} \{\|f(\cdot, t)\|^2_{\mathbf{L}^2} e^{-\lambda t}\}$ and $\nabla \triangleq \partial/\partial x$ denotes a gradient operator.

II. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

Consider the following MIMO singular distributed parameter systems

$$\begin{cases} E\dfrac{\partial^\alpha Z(\xi, t)}{\partial t^\alpha} = A\dfrac{\partial^2 Z(\xi, t)}{\partial \xi^2} + B(t)U(\xi, t), \\ Y(\xi, t) = H(t)Z(\xi, t) + L(t)U(\xi, t), \end{cases} \tag{1}$$

where $(\xi, t) \in \Omega\times[0, T], \Omega = [0, 1], \xi, t$ describe time coordinates and space coordinates respectively. $E \in \mathbb{R}^{n\times n}$ is singular constant matrix with $rank(E) = r < n, Z(\xi, t) \in \mathbb{R}^n$, $U(\xi, t) \in \mathbb{R}^m, Y(\xi, t) \in \mathbb{R}^s$ denote system state, control input and the output of system respectively, $A \in \mathbb{R}^{n\times n}$ is diagonal positive definite constant matrix, $B(t) \in \mathbb{R}^{n\times m}, H(t) \in \mathbb{R}^{s\times n}$, $L(t) \in \mathbb{R}^{s\times m}$ are time-varying bounded matrices. System type index $\alpha = 1$ or 2, the learning system (1) is turn into SDPSs with parabolic type under index value α taking one and when index value α is equal to two, the system (1) become hyperbolic SDPSs. The initial and boundary conditions of (1) are given as,

$$Z(\xi, t) = 0, \frac{\partial Z(\xi, t)}{\partial \upsilon} = 0, \quad (\xi, t) \in \partial\Omega \times [0, T], \tag{2}$$

$$Z(\xi, 0) = \varphi(\xi), \quad \frac{\partial Z(\xi, t)}{\partial t}|_{t=0} = \phi(\xi), \quad \xi \in \Omega. \tag{3}$$

where υ is the unit outward vector at the boundary $\partial\Omega$.

Remark 1: The SDPSs (1) is a hyperbolic or parabolic PDE-based system with singular matrix, which is used to describe a wide family of problems in natural science including the temperature distribution of composites and

and non-loss transient response of coupled transmission lines, etc. [4], [5].

Given a desired tracking target $Y_d(\xi, t)$, the control goal is to find a desired control input $U_d(\xi, t)$ such that when $k \to \infty$, the output of learning systems $Y_k(\xi, t)$ can track the reference trajectory $Y_d(\xi, t)$ as follows,

$$Y_d(\xi, t) = H(t)Z_d(\xi, t) + L(t)U_d(\xi, t).$$

where the repetitive iteration process satisfied

$$\begin{cases} E\dfrac{\partial^\alpha Z_k(\xi, t)}{\partial t^\alpha} = A\dfrac{\partial^2 Z_k(\xi, t)}{\partial \xi^2} + B(t)U_k(\xi, t), \\ Y_k(\xi, t) = H(t)Z_k(\xi, t) + L(t)U_k(\xi, t), \end{cases} \tag{4}$$

where k denotes the iteration number.

Assumption 2: Singular distributed parameter systems described by (1) are regular, impulse-free and direct transmission matrix $L(t)$ is row full rank.

Assumption 3: For a desired output $Y_d(\xi, t)$, there exists an unique $U_d(\xi, t)$ to meet the equations in the learning systems described by (4).

Assumption 4: In an iterative process (4), we assume following boundary and initial condition

$$Z_k(\xi, t) = 0, \quad \frac{\partial Z_k(\xi, t)}{\partial \upsilon} = 0, \ (\xi, t) \in \partial\Omega \times [0, T], \tag{5}$$

$$Z_k(\xi, 0) = \varphi(\xi) = Z_d(\xi, 0), \quad \frac{\partial Z_k(\xi, t)}{\partial t}|_{t=0} = \phi(\xi), \ \xi \in \Omega. \tag{6}$$

Remark 5: These assumptions of the SDPSs are acceptable and reasonable from the perspectives of theoretical analysis and practical industry process. Since under the Assumption 2 that the SDPSs is regular, this require exist a complex number s_0 to meet $\det(s_0E - A) \neq 0$, then the SDPSs described by (4) can be transformed into the Kronecker-Weierstrass equivalent form with nonsingular transformation matrices M and N [2], [20]. This requirement is also the general assumption in control theory of singular system and SDPSs [22]–[25]. Due to the SDPSs is without impulse, which indicates that the nilpotent matrix is zero [2], [22], [25]. The above two requirements are important condition for the stability of SDPSs. Owing to the desired control input $U_d(\xi, t)$ exists uniquely in Assumption 3, the uniform convergence of the control sequence $U_k(\xi, t)$ to $U_d(\xi, t)$ indicates that the output tracking errors will vanish. From Assumption 4, it is well posed initial-boundary value conditions for partial differential systems. Identical initial condition is necessary for ILC scheme on account of most industrial process often start at the same position.

Remark 6: The expression forms of the classical solution about second order hyperbolic SDPSs described by (1) under $\alpha = 2$ are presented based on the generalized inverse of bounded linear operators in Hilbert space [1], [7]. The constructive expression of the solution for parabolic SDPSs are discussed by means of the separation principle and generalized evolution operator in the literature [17], [22]. Those has provided the powerful support to the control synthesis of SDPSs with parabolic and hyperbolic type.

In this paper, the control objective is to utilize ILC controller to track the desired goal $Y_d(\xi, t)$ on the basic of measurable output $Y_k(\xi, t)$, such that the output error converges to zero when the iteration times tends to infinity. Thus, we will design controller and analyse convergence in the next section.

III. CONVERGENCE ANALYSIS

In this section, some Lemmas and two sufficient conditions for the output error of MIMO SDPSs with parabolic and hyperbolic type to be convergent under the sense of $\mathbf{L}_2$ norm are presented respectively.

The following Lemma 7 is derived from [35], which will be useful part for the proof of the following lemmas and theorems.

Lemma 7 [35]: Suppose $Z(\xi) \in \mathbf{C}^1[0, 1]$ be a vector function and $Z(0) = Z(1) = 0$, then the following inequality holds,

$$\int_0^1 Z^{\mathrm{T}}(s)Z(s)\mathrm{d}s \leqslant \frac{1}{6}\int_0^1 (\frac{\mathrm{d}Z(s)}{\mathrm{d}s})^{\mathrm{T}}\frac{\mathrm{d}Z(s)}{\mathrm{d}s}\mathrm{d}s. \tag{7}$$

In view of singular value matrix theory and Assumption 2 [2], there are existence with two nonsingular matrices $M \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{n\times n}$, such that

$$MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \ MAN = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}.$$

The learning systems (4) can be transformed into the equivalent decomposition form and expressed as follow,

$$\begin{cases} \dfrac{\partial^\alpha Z_{1k}(\xi, t)}{\partial t^\alpha} = A_1\dfrac{\partial^2 Z_{1k}(\xi, t)}{\partial \xi^2} + B_1(t)U_k(\xi, t), \\ 0 = \dfrac{\partial^2 Z_{2k}(\xi, t)}{\partial \xi^2} + B_2(t)U_k(\xi, t), \\ Y_k(\xi, t) = H_1(t)Z_{1k}(\xi, t) + H_2(t)Z_{2k}(\xi, t) \\ \qquad + L(t)U_k(\xi, t), \end{cases} \tag{8}$$

where $N^{-1}Z_k(\xi, t) = \begin{bmatrix} Z_{1k}(\xi, t) \\ Z_{2k}(\xi, t) \end{bmatrix}$, $MB(t) = \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix}$, $H(t)N = \begin{bmatrix} H_1(t) \ H_2(t) \end{bmatrix}$, $B_1(t) \in \mathbb{R}^{r\times m}$, $B_2(t) \in \mathbb{R}^{(n-r)\times m}$, $H_1(t) \in \mathbb{R}^{s\times r}$, $H_2(t) \in \mathbb{R}^{s\times n-r}$ are time-varying bounded matrices. $A_1 \in \mathbb{R}^{r\times r}$ is diagonal positive definite constant matrix. $Z_{1k}(\xi, t) \in \mathbb{R}^r$ and $Z_{2k}(\xi, t) \in \mathbb{R}^{(n-r)}$ denote the two substates of learning systems (4).

In this paper, the following P-type update learning controller is employed,

$$U_{k+1}(\xi, t) = U_k(\xi, t) + \Gamma(t)e_k(\xi, t), \tag{9}$$

where $\Gamma(t)$ is the learning gain matrix to be determined.

In order to facilitate better lead to the convergence condition of tracking error, we first estimate the relationship between output error and learning system substate variables.

For convenience, we introduce the following new marks,

$$\bar{U}_k(\xi, t) \triangleq U_{k+1}(\xi, t) - U_k(\xi, t) \tag{10}$$

$$\bar{Z}_{ik}(\xi, t) \triangleq Z_{ik+1}(\xi, t) - Z_{ik}(\xi, t), \quad i = 1, 2. \tag{11}$$

Lemma 8: Consider the SDPSs learning process described by (4) under Assumptions 2∼4. and P-type learning law, then the estimation of $Z_{2k}(\xi, t)$ holds as follows,

$$\|\bar{Z}_{2k}(\cdot, t)\|^2_{\mathbf{L}^2} \leqslant \|B_2(t)\Gamma(t)\|^2\|e_k(\cdot, t)\|^2_{\mathbf{L}^2}.$$

Proof: According to the second formula of systems (8) and subtracting the k times iterative process by the $k+1$ times process, then the following gives

$$\frac{\partial^2(Z_{2k+1}(\xi, t)-Z_{2k}(\xi, t))}{\partial \xi^2} = -B_2(t)(U_{k+1}(\xi, t) - U_k(\xi, t)).$$

By (10) and (11), we get

$$\frac{\partial^2 \bar{Z}_{2k}(\xi, t)}{\partial \xi^2} = -B_2(t)\bar{U}_k(\xi, t). \tag{12}$$

Two sides of (12) left multiply by $(Z_{2k+1}(\xi, t) - Z_{2k}(\xi, t))^{\mathrm{T}}$, and integrating about ξ on Ω, we can obtain

$$\int_\Omega \bar{Z}^T_{2k}(\xi, t)\frac{\partial^2 \bar{Z}_{2k}(\xi, t)}{\partial \xi^2}\,\mathrm{d}\xi = -\int_\Omega \bar{Z}^T_{2k}(\xi, t)B_2(t)\bar{U}_k(\xi, t)\mathrm{d}\xi. \tag{13}$$

The left hand side of equation (13), we can use part integral with respect to ξ to address it as follows,

$$\begin{aligned}\int_\Omega \bar{Z}^{\mathrm{T}}_{2k}(\xi, t)\frac{\partial^2 \bar{Z}_{2k}(\xi, t)}{\partial \xi^2}\,\mathrm{d}\xi &= \bar{Z}^{\mathrm{T}}_{2k}(\xi, t)\frac{\partial \bar{Z}_{2k}(\xi, t)}{\partial \xi}\Big|_{\partial\Omega} \\ &\quad - \int_\Omega (\frac{\partial \bar{Z}_{2k}(\xi, t)}{\partial \xi})^{\mathrm{T}}\frac{\partial \bar{Z}_{2k}(\xi, t)}{\partial \xi}\mathrm{d}\xi.\end{aligned} \tag{14}$$

In view of initial boundary conditions (5), we deduce that

$$\bar{Z}_{2k}(\xi, t) = Z_{2k+1}(\xi, t) - Z_{2k}(\xi, t) = 0, (\xi, t) \in \partial\Omega \times [0, T],$$

so, replacing it into formula (14), we have

$$\int_\Omega \bar{Z}^{\mathrm{T}}_{2k}(\xi, t)\frac{\partial^2 \bar{Z}_{2k}(\xi, t)}{\partial \xi^2}\,\mathrm{d}\xi = -\|\frac{\partial \bar{Z}_{2k}(\cdot, t)}{\partial \xi}\|^2_{\mathbf{L}^2}. \tag{15}$$

Substituting (15) into formula (13), we obtain

$$\|\frac{\partial \bar{Z}_{2k}(\cdot, t)}{\partial \xi}\|^2_{\mathbf{L}^2} = \int_\Omega \bar{Z}^{\mathrm{T}}_{2k}(\xi, t)B_2(t)\bar{U}_k(\xi, t)\mathrm{d}\xi. \tag{16}$$

Due to space variable ξ meet boundary conditions (6) and we use Lemma 7 and by (14), the following gives,

$$\begin{aligned}\|\bar{Z}_{2k}(\cdot, t)\|^2_{\mathbf{L}^2} &\leqslant \frac{1}{6}\|\frac{\partial \bar{Z}_{2k}(\cdot, t)}{\partial \xi}\|^2_{\mathbf{L}^2} \\ &\leqslant \frac{1}{6}\int_\Omega \bar{Z}^{\mathrm{T}}_{2k}(\xi, t)B_2(t)\bar{U}_k(\xi, t)\mathrm{d}\xi.\end{aligned} \tag{17}$$

Applying Hölder inequality and P-type learning law (9) into the right hand side of (17), we have

$$\begin{aligned}\|\bar{Z}_{2k}(\cdot, t)\|^2_{\mathbf{L}^2} \leqslant& \frac{1}{12}\|B_2(t)\Gamma(t)\|^2\|e_k(\cdot, s)\|^2_{\mathbf{L}^2} \\ &+ \frac{1}{12}\|\bar{Z}_{2k}(\cdot, t)\|^2_{\mathbf{L}^2}.\end{aligned} \tag{18}$$

Rearranging the inequality (18), we can obtain that,

$$\begin{aligned}\|\bar{Z}_{2k}(\cdot, t)\|^2_{\mathbf{L}^2} &\leqslant \frac{1}{11}\|B_2(t)\Gamma(t)\|^2\|e_k(\cdot, s)\|^2_{\mathbf{L}^2} \\ &\leqslant \|B_2(t)\Gamma(t)\|^2\|e_k(\cdot, s)\|^2_{\mathbf{L}^2}.\end{aligned} \tag{19}$$

The proof of Lemma 8 is end.

Through Lemma 8, we can discover that substate component in SDPSs (4) all have certain restraint relations with the tracking error which has provided the convenience for the next theorem proposed. Next, we will carry on the thorough analysis to convergence conditions for MIMO SDPSs with parabolic and hyperbolic type.

Theorem 9: Consider the P-type learning law (9) employed to the repetitive MIMO SDPSs (1) with parabolic type under $\alpha = 1$, and meeting Assumptions 2∼4. If for all $t \in [0, T]$, the gain matrix $\Gamma(t)$ satisfies

$$\|I - L(t)\Gamma(t)\|^2 + \|H(t)\|^2\|B_2(t)\Gamma(t)\|^2 \leqslant \rho_1, \quad 2\rho_1 \in [0, 1),$$

then the $\mathbf{L}_2$ norm of output error converge to zero for all $t \in [0, T]$ as $k \to \infty$, i.e.,

$$\lim_{k\to\infty} \|e_k(\cdot, t)\|_{\mathbf{L}^2} = 0, \quad \forall\, t \in [0, T].$$

Proof: According to the P-type learning law (9) and the output equation of systems (4), then the following gives

$$\begin{aligned}e_{k+1}(\xi, t) &= e_k(\xi, t) - Y_{k+1}(\xi, t) + Y_k(\xi, t) \\ &= e_k(\xi, t) - L(t)\big(U_{k+1}(\xi, t) - U_k(\xi, t)\big) \\ &\quad - H(t)(Z_{k+1}(\xi, t) - Z_k(\xi, t)) \\ &= (I - L(t)\Gamma(t))e_k(\xi, t) - H(t)\bar{Z}_k(\xi, t).\end{aligned} \tag{20}$$

Introducing the following new marks

$$\tilde{e}_k(\xi, t) \triangleq (I - L(t)\Gamma(t))e_k(\xi, t),$$
$$\tilde{H}_k(\xi, t) \triangleq -H(t)\bar{Z}_k(\xi, t).$$

Then, left multiply (20) by $e^{\mathrm{T}}_{k+1}(\xi, t)$, we have

$$\begin{aligned}&e^{\mathrm{T}}_{k+1}(\xi, t)e_{k+1}(\xi, t) \\ &\quad = (\tilde{e}_k(\xi, t) + \tilde{H}_k(\xi, t))^{\mathrm{T}}(\tilde{e}_k(\xi, t) + \tilde{H}_k(\xi, t)) \\ &\quad \leqslant 2(\tilde{e}^{\mathrm{T}}_k(\xi, t)\tilde{e}_k(\xi, t) + \tilde{H}_k^{\mathrm{T}}(\xi, t)\tilde{H}_k(\xi, t)) \\ &\quad \leqslant 2(\|\tilde{e}_k(\xi, t)\|^2 + \|\tilde{H}_k(\xi, t)\|^2) \\ &\quad \leqslant 2\rho\|e_k(\xi, t)\|^2 + 2h(\|\bar{Z}_{1k}(\xi, t)\|^2 + \|\bar{Z}_{2k}(\xi, t)\|^2),\end{aligned} \tag{21}$$

where introducing the following marks

$$\rho = \max_{0\leqslant t\leqslant T}\{\|I - L(t)\Gamma(t)\|^2\}, \quad h = \max_{0\leqslant t\leqslant T}\{\|H(t)\|^2\}.$$

Integrating both side of (21) about ξ on Ω, it meets

$$\begin{aligned}\|e_{k+1}(\cdot, t)\|^2_{\mathbf{L}^2} \leqslant& 2\rho\|e_k(\cdot, t)\|^2_{\mathbf{L}^2} + 2h\|\bar{Z}_{1k}(\cdot, t)\|^2_{\mathbf{L}^2} \\ &+ 2h\|\bar{Z}_{2k}(\cdot, t)\|^2_{\mathbf{L}^2}.\end{aligned} \tag{22}$$

From the first equation of systems (8) by $k+1$ times and k times learning process, we can obtain

$$\frac{\partial\big(Z_{1k+1}(\xi,t)-Z_{1k}(\xi,t)\big)}{\partial t}=A_1\frac{\partial^2\big(Z_{1k+1}(\xi,t)-Z_{1k}(\xi,t)\big)}{\partial\xi^2}+B_1(t)(U_{k+1}(\xi,t)-U_k(\xi,t)).$$

By brief marks, we have

$$\frac{\partial(\bar{Z}_{1k}(\xi,t))}{\partial t}=A_1\frac{\partial^2\bar{Z}_{1k}(\xi,t)}{\partial\xi^2}+B_1(t)\bar{U}_k(\xi,t). \tag{23}$$

Two sides of (23) multiplied $(Z_{1k+1}(\xi,t)-Z_{1k}(\xi,t))^{\mathrm{T}}$, we can get that

$$\frac{1}{2}\frac{\partial[\bar{Z}_{1k}^{\mathrm{T}}(\xi,t)\bar{Z}_{1k}(\xi,t)]}{\partial t}=\bar{Z}_{1k}^{\mathrm{T}}(\xi,t)A_1\frac{\partial^2\bar{Z}_{1k}(\xi,t)}{\partial\xi^2}+\bar{Z}_{1k}^{\mathrm{T}}(\xi,t)B_1(t)\bar{U}_k(\xi,t). \tag{24}$$

Integrating the both sides of (24) with respect to ξ over Ω, it satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\bar{Z}_{1k}(\cdot,t)\|_{\mathbf{L}^2}^2\big)&=2\int_\Omega\bar{Z}_{1k}^{\mathrm{T}}(\xi,t)A_1\frac{\partial^2\bar{Z}_{1k}(\xi,t)}{\partial\xi^2}\mathrm{d}\xi\\&\quad+2\int_\Omega\bar{Z}_{1k}^{\mathrm{T}}(\xi,t)B_1(t)\bar{U}_k(\xi,t)\mathrm{d}\xi\\&\triangleq I_1+I_2.\end{aligned} \tag{25}$$

We will address $I_i(i=1,2)$ in the following. Tackling I_1 in the light of Green formula, then following meets that

$$\begin{aligned}I_1&=2\int_\Omega\bar{Z}_{1k}^{\mathrm{T}}(\xi,t)A_1\frac{\partial^2\bar{Z}_{1k}(\xi,t)}{\partial\xi^2}\mathrm{d}\xi\\&=2\sum_{i=1}^{r}\sum_{j=1}^{r}\int_\Omega\bar{Z}_{1ki}^{\mathrm{T}}(\xi,t)A_{1ij}\frac{\partial^2\bar{Z}_{1kj}(\xi,t)}{\partial\xi^2}\mathrm{d}\xi\\&=2\sum_{i=1}^{r}\sum_{j=1}^{r}\int_{\partial\Omega}\bar{Z}_{1kj}^{\mathrm{T}}(\xi,t)A_{1ij}\frac{\partial\bar{Z}_{1ki}(\xi,t)}{\partial\nu}\mathrm{d}S\\&\quad-2\sum_{i=1}^{r}\sum_{j=1}^{r}\int_\Omega\nabla\bar{Z}_{1kj}^{\mathrm{T}}(\xi,t)A_{1ij}\nabla\bar{Z}_{1kj}(\xi,t)\mathrm{d}\xi\\&\triangleq I_{11}+I_{12}.\end{aligned} \tag{26}$$

For I_{11} with the help of boundary condition (6), we can obtain

$$\frac{\partial\bar{Z}_{1k}(\xi,t)}{\partial\nu}=\frac{\partial Z_{1k+1}(\xi,t)}{\partial\nu}-\frac{\partial Z_{1k}(\xi,t)}{\partial\nu}=0.$$

So it deduces $I_{11}=0$. Then we deal with I_{12},

$$\begin{aligned}I_{12}&=-2\sum_{i=1}^{r}\sum_{j=1}^{r}\int_\Omega\nabla\bar{Z}_{1kj}^{\mathrm{T}}(\xi,t)A_{1ij}\nabla\bar{Z}_{1kj}(\xi,t)\mathrm{d}\xi\\&=-2\sum_{i=1}^{r}\int_\Omega\nabla\bar{Z}_{1ki}^{\mathrm{T}}(\xi,t)A_{1ii}\nabla\bar{Z}_{1ki}(\xi,t)\mathrm{d}\xi.\\&\leqslant-2\lambda_{min}(A_1)\sum_{i=1}^{r}\int_\Omega\nabla\bar{Z}_{1k}^{\mathrm{T}}(\xi,t)\nabla\bar{Z}_{1k}(\xi,t)\mathrm{d}\xi\\&\leqslant-2\lambda_{min}(A_1)\|\nabla\bar{Z}_{1k}(\cdot,t)\|_{\mathbf{L}^2}^2.\end{aligned}$$

Because $A_1>0$, then $I_{12}\leqslant 0$ by combining (26), thus we have

$$I_1=I_{11}+I_{12}\leqslant 0. \tag{27}$$

For I_2, using Hölder inequality, we can find

$$\begin{aligned}I_2&=2\int_\Omega\bar{Z}_{1k}^{\mathrm{T}}(\xi,t)B_1(t)\bar{U}_k(\xi,t)\,\mathrm{d}\xi\\&\leqslant\lambda_{\max\limits_{0\leqslant t\leqslant T}}(B_1^{\mathrm{T}}(t)B_1(t))\|\bar{U}_k(\cdot,t)\|_{\mathbf{L}^2}^2+\|\bar{Z}_{1k}(\cdot,t)\|_{\mathbf{L}^2}^2.\end{aligned} \tag{28}$$

Thus, according to the result of formulas (25)∼(28), the following gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\bar{Z}_{1k}(\cdot,t)\|_{\mathbf{L}^2}^2\big)&\leqslant\lambda_{\max\limits_{0\leqslant t\leqslant T}}(B_1^{\mathrm{T}}(t)B_1(t))\|\bar{U}_k(\cdot,t)\|_{\mathbf{L}^2}^2\\&\quad+\|\bar{Z}_{1k}(\cdot,t)\|_{\mathbf{L}^2}^2\\&\leqslant\|\bar{Z}_{1k}(\cdot,t)\|_{\mathbf{L}^2}^2+g\|\bar{U}_k(\cdot,t)\|_{\mathbf{L}^2}^2,\end{aligned} \tag{29}$$

where $g=\lambda_{\max\limits_{0\leqslant t\leqslant T}}(B_1^{\mathrm{T}}(t)B_1(t))$.

For inequality (29) integrating about t, and using Bellman-Gronwall inequality, we have

$$\begin{aligned}\|\bar{Z}_{1k}(\cdot,t)\|_{\mathbf{L}^2}^2&\leqslant g\int_0^t e^{(t-s)}\|\bar{U}_k(\cdot,s)\|_{\mathbf{L}^2}^2\mathrm{d}s\\&\quad+e^t\|\bar{Z}_{1k}(\cdot,0)\|_{\mathbf{L}^2}^2.\end{aligned} \tag{30}$$

On the other hand, according to the iterative learning control law (8), we can obtain

$$\|U_{k+1}(\cdot,t)-U_k(\cdot,t)\|_{\mathbf{L}^2}^2\leqslant c\|e_k(\cdot,t)\|_{\mathbf{L}^2}^2, \tag{31}$$

where $c=\lambda_{\max\limits_{0\leqslant t\leqslant T}}(\Gamma(t)^{\mathrm{T}}\Gamma(t))$.

Substituting (31) into (30), we have

$$\begin{aligned}\|\bar{Z}_{1k}(\cdot,t)\|_{\mathbf{L}^2}^2&\leqslant gc\int_0^t e^{(t-s)}\|e_k(\cdot,s)\|_{\mathbf{L}^2}^2\mathrm{d}s\\&\quad+e^t\|\bar{Z}_{1k}(\cdot,0)\|_{\mathbf{L}^2}^2.\end{aligned} \tag{32}$$

Because of the conditions of initial value (6), we can get

$$\|\bar{Z}_{1k}(\cdot,t)\|_{\mathbf{L}^2}^2\leqslant gc\int_0^t e^{(t-s)}\|e_k(\cdot,s)\|_{\mathbf{L}^2}^2\mathrm{d}s. \tag{33}$$

Now, we return to error convergence. Replacing (33) and Lemma 8 into (22), we can have that

$$\begin{aligned}&\|e_{k+1}(\cdot,t)\|_{\mathbf{L}^2}^2\\&\quad\leqslant 2\rho\|e_k(\cdot,t)\|_{\mathbf{L}^2}^2+2h\|\bar{Z}_{1k}(\cdot,t)\|_{\mathbf{L}^2}^2\\&\qquad+2h\|\bar{Z}_{2k}(\cdot,t)\|_{\mathbf{L}^2}^2,\\&\quad\leqslant 2\rho\|e_k(\cdot,t)\|^2+2hgc\int_0^t e^{(t-s)}\|e_k(\cdot,s)\|_{\mathbf{L}^2}^2\mathrm{d}s\\&\qquad+2hb_\gamma\|e_k(\cdot,t)\|_{\mathbf{L}^2}^2,\end{aligned} \tag{34}$$

where $b_\gamma=\max\limits_{0\leqslant t\leqslant T}\|B_2(t)\Gamma(t)\|^2$.

Taking both side of formula (34) multiplied by $e^{-\lambda t}$ where sufficiently large constant $\lambda > 1$, it satisfies that

$$\begin{aligned}&\|e_{k+1}(\cdot,t)\|^2_{\mathbf{L}^2}e^{-\lambda t}\\ &\quad\leqslant 2hgc\int_0^t e^{-(\lambda-1)(t-s)}e^{-\lambda s}\|e_k(\cdot,s)\|^2_{\mathbf{L}^2}\mathrm{d}s\\ &\qquad+2\rho\|e_k(\cdot,t)\|^2e^{-\lambda t}+2hb_\gamma\|e_k(\cdot,t)\|^2_{\mathbf{L}^2}e^{-\lambda t}. \quad (35)\end{aligned}$$

By the definition of $(\mathbf{L}^2,\lambda)$ norm, it becomes

$$\begin{aligned}\|e_{k+1}\|^2_{(\mathbf{L}^2,\lambda)} &\leqslant 2\rho\|e_k\|^2_{(\mathbf{L}^2,\lambda)}+2hb_\gamma\|e_k\|^2_{(\mathbf{L}^2,\lambda)}\\ &\quad+\frac{2hgc}{\lambda-1}\|e_k\|_{(\mathbf{L}^2,\lambda)}\\ &\leqslant (2\rho+2hb_\gamma+\frac{2hgc}{\lambda-1})\|e_k\|^2_{(\mathbf{L}^2,\lambda)}. \quad (36)\end{aligned}$$

In view of convergent conditions in Theorem 9, we have $2\rho+2hb_\gamma\leqslant 2\rho_1<1$, so we can find a λ which sufficiently large to meet the condition

$$2\rho+2hb_\gamma+\frac{2hgc}{\lambda-1}<1. \quad (37)$$

By the formula (36) and (37), it satisfies

$$\lim_{k\to\infty}\|e_k\|^2_{(\mathbf{L}^2,\lambda)}=0,\quad \forall t\in[0,T]. \quad (38)$$

Finally, in view of the following inequality

$$\|e_k(\cdot,t)\|^2_{\mathbf{L}^2}=(\|e_k(\cdot,t)\|^2_{\mathbf{L}^2}e^{-\lambda t})e^{\lambda t}\leqslant\|e_k\|^2_{(\mathbf{L}^2,\lambda)}e^{\lambda T}. \quad (39)$$

Thus, we obtain

$$\lim_{k\to\infty}\|e_k(\cdot,t)\|_{\mathbf{L}^2}=0,\quad \forall t\in[0,T]. \quad (40)$$

This is conclusion of the Theorem 9.

Remark 10: On the one hand, compare with the convergence condition of general distributed parameter systems in paper [30]–[32], we can observe that convergence condition in Theorem 9 are more than the second item. This is determined by the levels of the SDPSs which have two administrative levels. On the other hand, compare with the general singular system, for instance, the iterative convergence condition of literature [33], [34], we discover the result is consistent and all concern with transformed system matrix under the equivalent dynamic decomposition.

Theorem 11: Consider the P-type learning law (9) applied to the repetitive hyperbolic singular distributed parameter systems (1) under take index value $i = 2$ and meeting Assumptions 2~4. If for all $t\in[0,T]$, the gain matrix $\Gamma(t)$ satisfies

$$\|I-L(t)\Gamma(t)\|^2+\|H(t)\|^2\|B_2(t)\Gamma(t)\|^2\leqslant\rho_2,\qquad 2\rho_2\in[0,1),$$

then the $\mathbf{L}_2$ norm of output error converge to zero for all $t\in[0,T]$ as $k\to\infty$, i.e.,

$$\lim_{k\to\infty}\|e_k(\cdot,t)\|_{\mathbf{L}^2}=0,\forall\, t\in[0,T].$$

Proof: The same decomposition form as (8), it also can turn into the following

$$\begin{cases}\dfrac{\partial^2 Z_{1k}(\xi,t)}{\partial t^2}=A_1\dfrac{\partial^2 Z_{1k}(\xi,t)}{\partial\xi^2}+B_1(t)U_k(\xi,t),\\ 0=\dfrac{\partial^2 Z_{2k}(\xi,t)}{\partial\xi^2}+B_2(t)U_k(\xi,t),\\ Y_k(\xi,t)=H_1(t)Z_{1k}(\xi,t)+H_2(t)Z_{2k}(\xi,t)\\ \qquad+L(t)U_k(\xi,t),\end{cases} \quad (41)$$

According to the definition of $\bar{Z}_{1k}(\xi,t)$ by first equation in (41), we can obtain that

$$\frac{\partial^2\bar{Z}_{1k}(\xi,t)}{\partial t^2}=A_1\frac{\partial^2\bar{Z}_{1k}(\xi,t)}{\partial\xi^2}+B_1(t)\bar{U}_k(\xi,t). \quad (42)$$

Taking $(\frac{\partial\bar{Z}_{1k}(\xi,t)}{\partial t})^{\mathrm{T}}$ left multiply by (42), we can get

$$\begin{aligned}(\frac{\partial\bar{Z}_{1k}(\xi,t)}{\partial t})^{\mathrm{T}}\frac{\partial^2\bar{Z}_{1k}(\xi,t)}{\partial t^2}&=(\frac{\partial\bar{Z}_{1k}(\xi,t)}{\partial t})^{\mathrm{T}}A_1\frac{\partial^2\bar{Z}_{1k}(\xi,t)}{\partial\xi^2}\\ &\quad+(\frac{\partial\bar{Z}_{1k}(\xi,t)}{\partial t})^{\mathrm{T}}B_1(t)\bar{U}_k(\xi,t).\end{aligned} \quad (43)$$

In the two hand side of (43), we multiply constant two and integral above ξ on Ω, then the following gives

$$\begin{aligned}&2\int_0^1(\frac{\partial\bar{Z}_{1k}(\xi,t)}{\partial t})^{\mathrm{T}}\frac{\partial^2\bar{Z}_{1k}(\xi,t)}{\partial t^2}\mathrm{d}\xi\\ &\quad=2\int_0^1(\frac{\partial\bar{Z}_{1k}(\xi,t)}{\partial t})^{\mathrm{T}}A_1\frac{\partial^2\bar{Z}_{1k}(\xi,t)}{\partial\xi^2}\mathrm{d}\xi\\ &\qquad+2\int_0^1(\frac{\partial\bar{Z}_{1k}(\xi,t)}{\partial t})^{\mathrm{T}}B_1(t)\bar{U}_k(\xi,t)\mathrm{d}\xi.\\ &\quad\triangleq I_1+I_2.\end{aligned}$$

Introducing new mark $\dot{\bar{Z}}_{1k}(\xi,t)\triangleq\frac{\partial\bar{Z}_{1k}(\xi,t)}{\partial t}$, it meets

$$2\int_0^1(\frac{\partial\bar{Z}_{1k}(\xi,t)}{\partial t})^{\mathrm{T}}\frac{\partial^2\bar{Z}_{1k}(\xi,t)}{\partial t^2}\mathrm{d}\xi=\frac{\mathrm{d}(\|\dot{\bar{Z}}_{1k}(\cdot,t)\|^2_{\mathbf{L}^2})}{\mathrm{d}t}. \quad (44)$$

Dealing with I_1 by Green formula, we can have that

$$\begin{aligned}I_1&=2\int_\Omega(\frac{\partial\bar{Z}_{1k}(\xi,t)}{\partial t})^{\mathrm{T}}A_1\frac{\partial^2\bar{Z}_{1k}(\xi,t)}{\partial\xi^2}\mathrm{d}\xi\\ &=2\sum_{i=1}^r\sum_{j=1}^r\int_\Omega(\frac{\partial\bar{Z}_{1ki}(\xi,t)}{\partial t})^{\mathrm{T}}A_{1ij}(\frac{\partial^2\bar{Z}_{1kj}(\xi,t)}{\partial\xi^2})\mathrm{d}\xi\\ &=2\sum_{i=1}^r\sum_{j=1}^r\int_{\partial\Omega}(\frac{\partial\bar{Z}_{1ki}(\xi,t)}{\partial t})^{\mathrm{T}}A_{1ij}\frac{\partial\bar{Z}_{1kj}(\xi,t)}{\partial\nu}\mathrm{d}S\\ &\quad-2\sum_{i=1}^r\sum_{j=1}^r\int_\Omega\nabla\dot{\bar{Z}}^{\mathrm{T}}_{1ki}(\xi,t)A_{1ij}\nabla\bar{Z}_{1kj}(\xi,t)\mathrm{d}\xi.\end{aligned}$$

According to the boundary condition (6), the follows can be given

$$I_1 = -2\sum_{i=1}^{r}\int_\Omega \nabla\dot{\bar{Z}}_{1ki}^{\mathrm{T}}(\xi,t)A_{1ii}\nabla\bar{Z}_{1ki}(\xi,t)\mathrm{d}\xi \leqslant -\lambda_{\min}(A_1)\frac{\mathrm{d}(\|\nabla\bar{Z}_{1k}(\cdot,t)\|^2_{\mathbf{L}^2})}{\mathrm{d}t}. \quad (45)$$

For I_2 on the basis of learning law (9) and Hölder inequality, we can find

$$\begin{aligned} I_2 &= 2\int_\Omega (\frac{\partial\bar{Z}_{1k}(\xi,t)}{\partial t})^{\mathrm{T}}B_1(t)\bar{U}_k(\xi,t)\mathrm{d}\xi \\ &= 2\int_\Omega (\frac{\partial\bar{Z}_{1k}(\xi,t)}{\partial t})^{\mathrm{T}}B_1(t)\Gamma(t)e_k(\xi,t)\mathrm{d}\xi \\ &= b_\Gamma\|e_k(\cdot,t)\|^2_{\mathbf{L}^2} + \|\dot{\bar{Z}}_{1k}(\cdot,t)\|^2_{\mathbf{L}^2}. \quad (46)\end{aligned}$$

where $b_\Gamma = \max_{0\leqslant t\leqslant T}\|B_1(t)\Gamma(t)\|^2$.

So combining (43) and (45)~(46), then the following can induce

$$\frac{\mathrm{d}(\|\dot{\bar{Z}}_{1k}(\cdot,t)\|^2_{\mathbf{L}^2})}{\mathrm{d}t} \leqslant -\lambda_{\min}(A_1)\frac{\mathrm{d}\|\nabla\bar{Z}_{1k}(\cdot,t)\|^2_{\mathbf{L}^2}}{\mathrm{d}t} + \|\dot{\bar{Z}}_{1k}(\cdot,t)\|^2_{\mathbf{L}^2} + b_\Gamma\|e_k(\cdot,t)\|^2_{\mathbf{L}^2}. \quad (47)$$

It equals to the following,

$$\frac{\mathrm{d}(\|\dot{\bar{Z}}_{1k}(\cdot,t)\|^2_{\mathbf{L}^2} + \lambda_{\min}(A_1)\|\nabla\bar{Z}_{1k}(\cdot,t)\|^2_{\mathbf{L}^2})}{\mathrm{d}t} \leqslant \|\dot{\bar{Z}}_{1k}(\cdot,t)\|^2_{\mathbf{L}^2} + b_\Gamma\|e_k(\cdot,t)\|^2_{\mathbf{L}^2}.$$

Because $A_1 > 0$, for all $t \in [0,T]$, so

$$\begin{aligned}&\frac{\mathrm{d}(\|\dot{\bar{Z}}_{1k}(\cdot,t)\|^2_{\mathbf{L}^2} + \lambda_{\min}(A_1)\|\nabla\bar{Z}_{1k}(\cdot,t)\|^2_{\mathbf{L}^2})}{\mathrm{d}t} \\ &\leqslant \|\dot{\bar{Z}}_{1k}(\cdot,t)\|^2_{\mathbf{L}^2} + b_\Gamma\|e_k(\cdot,t)\|^2_{\mathbf{L}^2} \\ &\leqslant \|\dot{\bar{Z}}_{1k}(\cdot,t)\|^2_{\mathbf{L}^2} + \lambda_{\min}(A_1)\|\nabla\bar{Z}_{1k}(\cdot,t)\|^2_{\mathbf{L}^2} + b_\Gamma\|e_k(\cdot,t)\|^2_{\mathbf{L}^2}. \quad (48)\end{aligned}$$

In view of the Bellman-Gronwall inequality and initial condition (6), we have

$$\begin{aligned}&\|\dot{\bar{Z}}_{1k}(\cdot,t)\|^2_{\mathbf{L}^2} + \lambda_{\min}(A_1)\|\nabla\bar{Z}_{1k}(\cdot,t)\|^2_{\mathbf{L}^2} \\ &\leqslant (\|\dot{\bar{Z}}_{1k}(\cdot,0)\|^2_{\mathbf{L}^2} + \lambda_{\min}(A_1)\|\nabla\bar{Z}_{1k}(\cdot,0)\|^2_{\mathbf{L}^2})e^t + b_\Gamma\int_0^t e^{(t-s)}\|e_k(\cdot,s)\|^2_{\mathbf{L}^2}ds \\ &\leqslant b_\Gamma\int_0^t e^{(t-s)}\|e_k(\cdot,s)\|^2_{\mathbf{L}^2}ds. \quad (49)\end{aligned}$$

On the basis of Lemma 7, we get

$$\|\bar{Z}_{1k}(\cdot,t)\|^2_{\mathbf{L}^2} \leqslant \frac{1}{6}\|\nabla\bar{Z}_{1k}(\cdot,t)\|^2_{\mathbf{L}^2}. \quad (50)$$

Therefore, by combining (49) and (50), we easily obtain

$$\|\bar{Z}_{1k}(\cdot,t)\|^2_{\mathbf{L}^2} \leqslant \frac{b_\Gamma}{6\lambda_{\min}(A_1)}\int_0^t e^{(t-s)}\|e_k(\cdot,s)\|^2_{\mathbf{L}^2}ds. \quad (51)$$

Multiplying both sides of (51) by $e^{-\lambda t}$ where sufficiently large constant $\lambda > 1$, then the following meets

$$\begin{aligned}&\|\bar{Z}_{1k}(\cdot,t)\|^2_{\mathbf{L}^2}e^{-\lambda t} \\ &\leqslant \frac{b_\Gamma}{6\lambda_{\min}(A_1)}\int_0^t e^{-(\lambda-1)(t-s)}\|e_k(\cdot,s)\|^2_{\mathbf{L}^2}e^{-\lambda s}ds \\ &\leqslant \frac{b_\Gamma}{6\lambda_{\min}(A_1)}\|e_k\|^2_{(\mathbf{L}^2,\lambda)}\int_0^t e^{-(\lambda-1)(t-s)}ds \\ &\leqslant \frac{b_\Gamma}{6\lambda_{\min}(A_1)(\lambda-1)}\|e_k\|^2_{(\mathbf{L}^2,\lambda)}. \quad (52)\end{aligned}$$

On the other hand, according to the estimation of error (22), we have

$$\begin{aligned}&\|e_{k+1}\|^2_{(\mathbf{L}^2,\lambda)} \\ &\leqslant 2\rho\|e_k\|^2_{(\mathbf{L}^2,\lambda)} + 2h(\|\bar{Z}_{1k}\|^2_{(\mathbf{L}^2,\lambda)} + \|\bar{Z}_{2k}\|^2_{(\mathbf{L}^2,\lambda)}), \\ &\leqslant (2\rho + 2hb_\gamma + \frac{hb_\Gamma}{3\lambda_{\min}(A_1)(\lambda-1)})\|e_k\|^2_{(\mathbf{L}^2,\lambda)}. \quad (53)\end{aligned}$$

Because $2\rho + 2hb_\gamma \leqslant 2\rho_2 < 1$, so we can find λ which is big enough to meet the condition

$$2\rho + 2hb_\gamma + \frac{hb_\Gamma}{3\lambda_{\min}(A_1)(\lambda-1)} < 1. \quad (54)$$

Combining (53) and (54), the following satisfies

$$\lim_{k\to\infty}\|e_k\|^2_{(\mathbf{L}^2,\lambda)} = 0. \quad (55)$$

Finally, the rest of proof is same as (39), we can obtain

$$\lim_{k\to\infty}\|e_k(\cdot,t)\|_{\mathbf{L}^2} = 0, \quad \forall t \in [0,T]. \quad (56)$$

This completes the proof of Theorem 11.

IV. NUMERICAL SIMULATIONS

In order to show the effectiveness of the proposed P-type learning scheme for MIMO SDPSs in this paper, two specific numerical examples are given as follows

$$\begin{cases} E\dfrac{\partial^\alpha Z(\xi,t)}{\partial t^\alpha} = A\dfrac{\partial^2 Z(\xi,t)}{\partial\xi^2} + B(t)U(\xi,t), \\ Y(\xi,t) = H(t)Z(\xi,t) + L(t)U(\xi,t), \end{cases}$$

where

$$E = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.5 \end{bmatrix},$$

$$B(t) = \begin{bmatrix} 0.3 & 0.2e^{-3t} \\ 0.2 & 0.1 \\ 0 & 0.3 \end{bmatrix}, \quad L(t) = \begin{bmatrix} 1.03e^{-0.9t} & 0 \\ 0 & 0.8 \end{bmatrix},$$

$$H(t) = \begin{bmatrix} 0.8t & 0.2 & 0.6 \\ 0.3 & 0.1e^{-1.5t} & 0.4 \end{bmatrix}.$$

In this example, $Z(\xi,t) \in \mathbb{R}^3$, $U(\xi,t) \in \mathbb{R}^2$, $Y(\xi,t) \in \mathbb{R}^2$ and we choose spatial variable $\xi \in [0,1]$ as well as time variable $t \in [0,1]$. The desired reference trajectory is selected as follows

$$Y_d(\xi,t) = \begin{bmatrix} Y_{1d}(\xi,t) \\ Y_{2d}(\xi,t) \end{bmatrix} = \begin{bmatrix} -3sin(6\xi)sin(2\pi t) \\ 2sin(3\pi\xi)(1-e^{-2t}) \end{bmatrix}.$$

The P-type ILC controller is setted as

$$U_{k+1}(\xi,t) = U_k(\xi,t) + \Gamma(t)e_k(\xi,t),$$

and we take gain matrix as $\Gamma(t) = \begin{bmatrix} 0.98 & 0.02e^{-0.6t} \\ 0 & 1.12 \end{bmatrix}$.

In this numerical simulation, the iterative initial value conditions are setted as

$$Z_{ik}(\xi,0) = 0, \quad i = 1,2,\ k = 1,2,\cdots,$$

and the boundary conditions are

$$Z_{ik}(0,t) = Z_{ik}(1,t) = 0, \quad i = 1,2,\ k = 1,2,\cdots.$$

There exists two nonsingular transform matrices

$$M = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},\ N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

such that

$$MEN = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad MAN = \begin{bmatrix} A_1 & 0 \\ 0 & 1 \end{bmatrix},$$

where $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.8 \end{bmatrix}$.

Then, the equivalent decomposition form are obtained as follows

$$\begin{cases} \dfrac{\partial^\alpha Z_{1k}(\xi,t)}{\partial t^\alpha} = A_1 \dfrac{\partial^2 Z_{1k}(\xi,t)}{\partial \xi^2} + B_1(t)U_k(\xi,t), \\ 0 = \dfrac{\partial^2 Z_{2k}(\xi,t)}{\partial \xi^2} + B_2(t)U_k(\xi,t), \\ Y_k(\xi,t) = H_1(t)Z_{1k}(\xi,t) + H_2(t)Z_{2k}(\xi,t) \\ \qquad\qquad + L(t)U_k(\xi,t), \end{cases} \tag{57}$$

where $B_1(t) = \begin{bmatrix} 1.5 & e^{-3t} \\ 0.4 & 0.2 \end{bmatrix}$, $B_2(t) = \begin{bmatrix} 0 & 0.6 \end{bmatrix}$,

$$H_1(t) = \begin{bmatrix} 0.8t & 0.2 \\ 0.3 & 0.1e^{-1.5t} \end{bmatrix}, \quad H_2(t) = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}.$$

Calculating the convergent condition by utilizing the above parameters, then the following indicates

$$\|I - L(t)\Gamma(t)\|^2 + \|H(t)\|^2\|B_2(t)\Gamma(t)\|^2 < 0.5,$$

which satisfies the convergent condition of Theorem 9 and Theorem 11.

Example 12 (The Simulation of Parabolic SDPSs ($\alpha = 1$): Initially, the initial control input is set to zero and we assign the initial-boundary value conditions at first iteration. Meanwhile, the discretization model of SDPSs through employing forward difference form is derived. Output state can be obtained by the control input and discrete system model. Then, using P-type learning law to calculate the next control input subsequently. Finally, repeating the previous steps until the tracking errors reach the setting precision. The result of parabolic type are shown in the Fig.1∼Fig.6 and Fig.9.

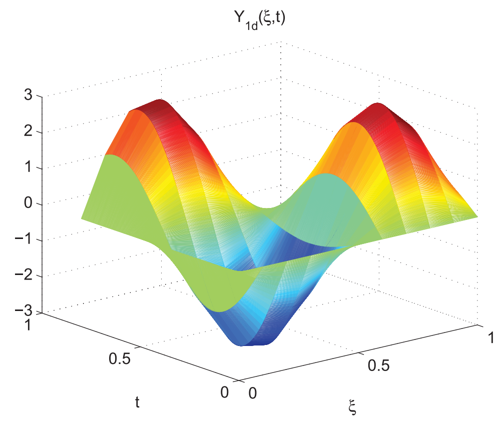

FIGURE 1. Desired surface $Y_{1d}(\xi,t)$ for SDPSs.

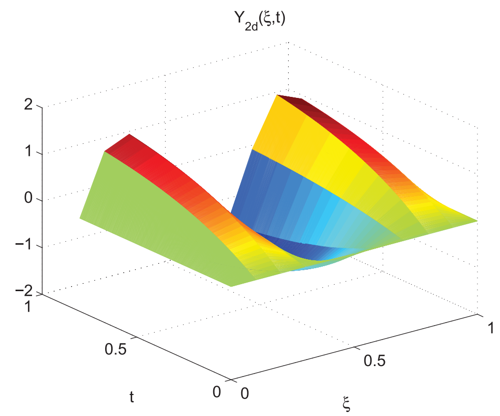

FIGURE 2. Desired surface $Y_{2d}(\xi,t)$ for SDPSs.

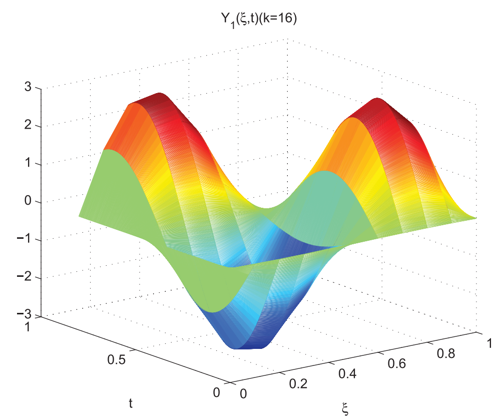

FIGURE 3. Actual output surface $Y_{1k}(\xi,t)$ for SDPSs.

Example 13 (The Simulation of Hyperbolic SDPSs ($\alpha = 2$): In this example, we consider hyperbolic SDPSs described by (57), then we should provide boundary conditions that both the initial state profiles are $\varphi_1(\xi) = 0.02\xi, \varphi_2(\xi) = 0.01\sin\xi, \varphi_3(\xi) = 0.03\sin\pi\xi$ and $\phi_1(\xi) = \phi_2(\xi) = \phi_3(\xi) = 0$, the input value of the controller at the beginning of learning process are set to be zero. The rest are the same as

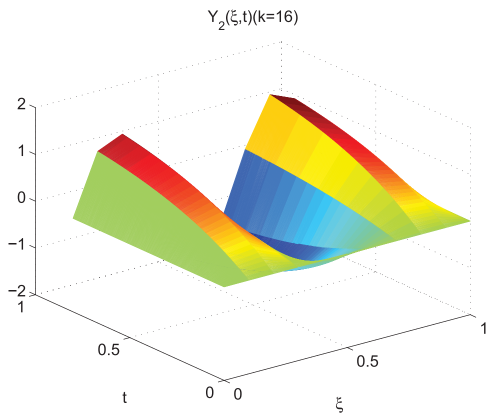

FIGURE 4. Actual output surface $Y_{2k}(\xi, t)$ for SDPSs.

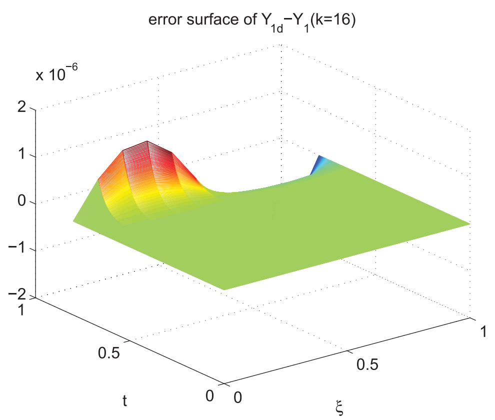

FIGURE 5. Error surface $e_{1k}(\xi, t)$ in parabolic SDPSs.

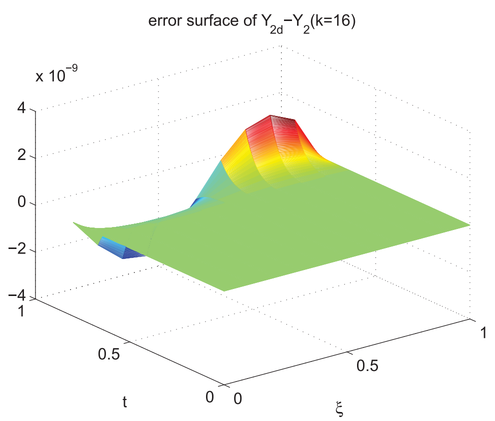

FIGURE 6. Error surface $e_{2k}(\xi, t)$ in parabolic SDPSs.

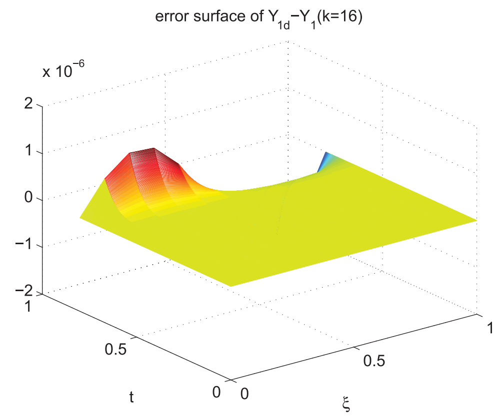

FIGURE 7. Error surface $e_{1k}(\xi, t)$ in hyperbolic SDPSs.

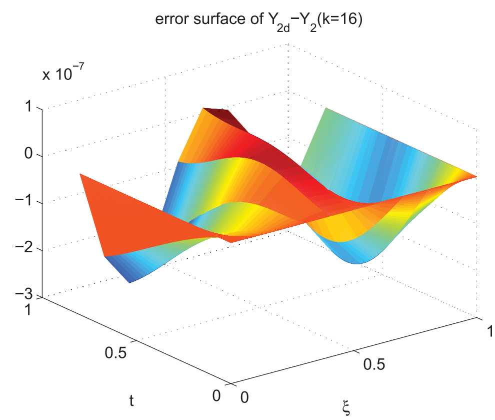

FIGURE 8. Error surface $e_{2k}(\xi, t)$ in hyperbolic SDPSs.

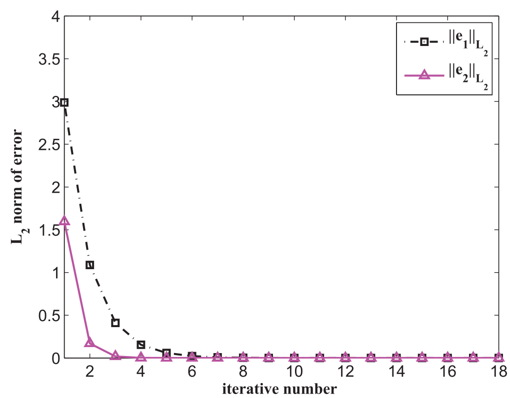

FIGURE 9. Max error-iterative number curve in parabolic SDPSs.

those in parabolic SDPSs. This simulation results have been presented in Fig.7∼Fig.8 and Fig.10.

Fig.1 and Fig.2 depict two given output target surface respectively. Fig.3 as well as Fig.4 show the actual tracking surface at the 16th iteration. Comparing Fig.1 with Fig.3, we can discover that objectives are utterly close to actual surface. The Fig.9 denotes that two tracking error value are almost approach to zero after nine iterations. As shown in Fig.5∼Fig.6, the maximum absolute error of $Y_1(\xi, t)$ and $Y_2(\xi, t)$ with high accuracy are 1.3345×10^{-6}, 2.3253×10^{-9} in sixteen iterations respectively. As shown in Fig.7∼Fig.8, the maximum absolute error of $Y_1(\xi, t)$ and $Y_2(\xi, t)$ with high accuracy are 1.5534×10^{-6}, 2.5982×10^{-7} in sixteen iterations respectively. This simulation results confirm the effectiveness of P-type ILC law on the basis of Fig.9 and Fig.10.

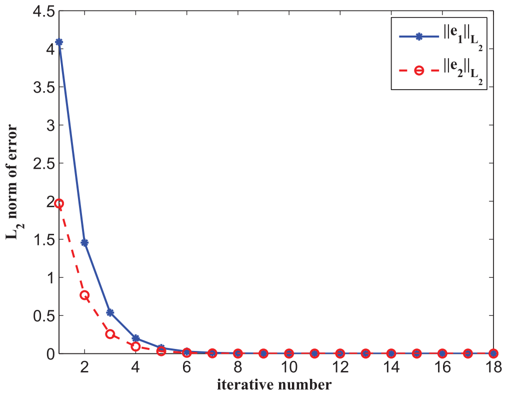

FIGURE 10. Max error-iterative number curve in hyperbolic SDPSs.

V. CONCLUSION

For two classes of MIMO SDPSs, whose dynamics are described by partial differential equations and executing in the repeatable environment, a P-type ILC controller is utilized to track the given desired goal. The control object is to ensure the convergence of output tracking error under the sense of $\mathbf{L}_2$ norm. Meanwhile, two lemmas are proposed to make the foreshadowing for the later theorem. The sufficient conditions and rigorous proof for MIMO SDPSs are given in terms of the equivalent dynamic decomposition form as well as some differential inequalities. At last, two numerical examples are given to show the effectiveness of theoretical results in the light of forward difference form. Research on SDPSs with parabolic and hyperbolic type also provide a reference for our further study on the ILC of SDPSs with external perturbation or the drifting initial value in learning process.

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