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Cutset Bounds on the Capacity of MIMO Relay Channels

YOUNGMIN JEONG, (Member, IEEE), DUNG PHUONG TRINH, (Student Member, IEEE), AND HYUNDONG SHIN, (Senior Member, IEEE)
Department of Electronic Engineering, Kyung Hee University, Yongin-si 17104, South Korea

Corresponding author: Hyundong Shin (hshin@khu.ac.kr)

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ABSTRACT We analyze the ergodic capacity of multiple-input multiple-output (MIMO) Rayleigh-fading relay channels. We first derive the probability density function of a sum of independent complex central Wishart matrices—called the *central hyper-Wishart matrix*—and its joint eigenvalue density. We then derive a trace representation for the max-flow min-cut upper bound on the ergodic capacity of general full-duplex MIMO relay channels where each communicating node is equipped with N antennas and has access only to respective receive channel state information. We also establish the Schur monotonicity theorem for this cutset bound as a functional of the signal-to-noise ratios (SNRs) of three communication links. We further characterize the exact ergodic capacity in the regularity SNR regime where the upper and lower bounds coincide.

INDEX TERMS Cooperative relaying, cutset bound, ergodic capacity, hyper-Wishart matrix, multiple-input multiple-output (MIMO), Rayleigh fading.

I. INTRODUCTION

The problem of transmitting information over three-terminal communication channels was first introduced by van der Meulen in the pioneering work [1]. Following this first treatment for discrete memoryless relay channels, the seminal work [2] established the capacity theorems for degraded and reversely degraded memoryless relay channels as well as the upper and lower bounds on the capacity of general memoryless relay channels. The upper bound was developed using the max-flow min-cut theorem while the achievable lower bound involved block-Markov superposition coding. It was further shown that these two bounds—often called the *cutset* upper bound and *partial decode-and-forward* (DF) lower bound—coincide for a class of relay channels with orthogonal components and hence, the capacity for such relay channels is equal to the cutset bound [3].

Cooperative communication in wireless networks can be formulated as a relay channel where one or more relays help a pair of nodes to communicate [4], [5]. Motivated by this, a large wave of work has been recently spawned on the capacity analysis and relaying operations for wireless relay networks from last decades with the multiple-input multiple-output (MIMO) technology [6]–[14]. In particular, the upper and lower bounds on the ergodic capacity were developed in [11, Th. 4.1 and 4.2] for general full-duplex MIMO relay channels in the presence of Rayleigh fading and receive channel state information (CSI) only. These bounds resorted again to standard achievability and converse bounding techniques in [2] along with the fact that independent Gaussian codebooks at the source and relay are optimal coding strategies due to the channel uncertainty at these nodes—i.e., no transmit CSI (see also [6, Th. 8]).[1] Furthermore, the sufficient regularity conditions, under which the upper and lower bounds coincide and hence the ergodic capacity can be exactly characterized, were investigated for the high signal-to-noise ratio (SNR) case where all nodes have the same number of antennas. However, since the exact expressions for the bounds were presented in forms of expectations with respect to source-to-destination (S $\rightarrow$ D), source-to-relay (S $\rightarrow$ R), and relay-to-destination (R $\rightarrow$ D) Rayleigh-fading channel matrices, there is no analytic expression for the capacity in MIMO relay channels. Hence, the results were verified by the simulation study [11] or by the asymptotic analysis [12]–[14]. The key to the successful analysis for the capacity of MIMO

[1] With full CSI, the solutions to transmit covariance optimization problems at the source and relay for the cutset upper bound and partial DF lower bound were formulated in [15] as standard convex problems, which are tighter than [11, Th. 3.1 and 3.2].

relay channel relies on the marginal eigenvalue density of the sum of Wishart matrices. However, even in the case of two Wishart matrices with unequal covariance matrices, the eigenvalue distribution has an intractable form [16]–[18].[2] Hence, we are motivated to find its tractable form expression with application to MIMO relay channels.

In this paper, we analyze the upper and lower bounds on the ergodic capacity of a general full-duplex MIMO Rayleigh-fading relay channel where each communicating node is equipped with N antennas and has access only to respective receive CSI, which is referred to as a "N-MIMO relay channel". Specifically, the main results of the paper, which resort to the finite random matrix methodology developed in [19]–[25], can be summarized as follows.

- *Hyper-Wishart matrix:* We first derive the probability density function (PDF) of a sum of two independent complex central Wishart matrices (Theorem 1) and its joint eigenvalue density (Theorem 2 and Corollary 1). We refer this random matrix as the "central hyper-Wishart matrix" (Definition 1).
- *Shannon transform and its Schur monotonicity:* We derive the Shannon transform of the unordered marginal eigenvalue density of the hyper-Wishart matrix with scaled identity covariances as the closed-form trace representation (Theorem 3).[3] We show that this Shannon transform as a functional of scale variables is *monotonically decreasing in a sense of Schur (MDS)* or *Schur-concave* (Property 2).[4] We further assess the Shannon transform of its asymptotic spectrum as the antenna numbers tend to infinity (Theorem 4).
- *Ergodic Capacity:* We derive the closed-form trace expressions for the upper and lower bounds [11, Th. 4.1 and 4.2] on the ergodic capacity of general N-MIMO Rayleigh-fading relay channels (Theorem 5 and Remark 4). We establish the Schur monotonicity for the cutset bound as a functional of the SNRs of three communication links (Remark 3). We further characterize the exact ergodic capacity in the regularity SNR regime where the upper and lower bounds coincide and hence the ergodic capacity coincides with the cutset bound (Remark 5).

The rest of this paper is organized as follows. In Section II, we introduce a hyper-Wishart matrix and derive some distributional properties of its eigenvalues. Section III analyzes the cutset upper bound and partial DF lower bound on the ergodic capacity of N-MIMO Rayleigh-fading relay channels using the statistical results related to the hyper-Wishart matrix. In Section IV, we present some numerical results and finally, Section V concludes the paper. Throughout the paper, we shall adopt the notation: i) random variables are displayed in sans serif, upright fonts; their realizations in serif, italic fonts; and ii) vectors and matrices are denoted by bold lowercase and uppercase letters, respectively. We relegate the glossary of notation and symbols used in the paper to Appendix A.

II. MATHEMATICAL FRAMEWORK

We begin by introducing a new class of random matrices involving Wishart matrices and deriving some distributional properties of its eigenvalues, which will be extensively invoked in the capacity analysis.[5]

A. HYPER-WISHART MATRIX

Definition 1 (Hyper-WishartMatrix): Let $\mathsf{W}_i \sim \tilde{\mathcal{W}}_m(n_i, \Sigma_i)$, $i = 1, 2, \ldots, L$, be statistically independent complex central Wishart matrices where $n_i \geqslant m$.[6] Then, the Lth-order central hyper-Wishart matrix $\mathsf{W} \in \mathbb{C}^{m\times m} > \mathbf{0}$ with parameters m, $\boldsymbol{n} = (n_1, n_2, \ldots, n_L)$, and $\{\boldsymbol{\Sigma}_i\}_{i=1}^L$, denoted by $\mathsf{W} \sim \mathcal{H}\tilde{\mathcal{W}}_m^{(L)}\left(\boldsymbol{n}, \{\boldsymbol{\Sigma}_i\}_{i=1}^L\right)$, is defined as

$$\mathsf{W} = \sum_{i=1}^{L} \mathsf{W}_i. \tag{1}$$

Remark 1: If $\boldsymbol{\Sigma}_i = \boldsymbol{\Sigma}$ or $n_i = n$ in Definition 1, then the hyper-Wishart matrix W reduces to the complex Wishart matrix, that is, $\mathsf{W} \sim \tilde{\mathcal{W}}_m\left(\sum_{i=1}^L n_i, \boldsymbol{\Sigma}\right)$ or $\mathsf{W} \sim \tilde{\mathcal{W}}_m\left(n, \sum_{i=1}^L \boldsymbol{\Sigma}_i\right)$.

Theorem 1: Let $\mathsf{W} \sim \mathcal{H}\tilde{\mathcal{W}}_m^{(2)}(\boldsymbol{n}, \{\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2\})$ be the second-order central hyper-Wishart matrix. Then, the density of W for $\boldsymbol{W} > \mathbf{0}$ is given by

$$\begin{aligned} p_{\mathsf{W}}(\boldsymbol{W}) &= \frac{1}{\tilde{\Gamma}_m(n_1+n_2)} \det(\boldsymbol{\Sigma}_1)^{-n_1} \det(\boldsymbol{\Sigma}_2)^{-n_2} \det(\boldsymbol{W})^{n_1+n_2-m} \\ &\quad \times \operatorname{etr}\left(-\boldsymbol{\Sigma}_1^{-1}\boldsymbol{W}\right) {}_1\tilde{F}_1\left(n_2; n_1+n_2; \left(\boldsymbol{\Sigma}_1^{-1} - \boldsymbol{\Sigma}_2^{-1}\right)\boldsymbol{W}\right) \end{aligned} \tag{2}$$

Proof: See Appendix B. □

Theorem 2 (Joint Eigenvalue Density): Let $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_m > 0$ be the ordered eigenvalues of $\mathsf{W} \sim \mathcal{H}\tilde{\mathcal{W}}_m^{(2)}(\boldsymbol{n}, \{\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2\})$. Then, the joint density of $\lambda_1, \lambda_2, \ldots, \lambda_m$ is given in (3), as shown at the top of the next page, where $\boldsymbol{D} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$.

Proof: It follows readily from using the PDF $p_{\mathsf{W}}(\boldsymbol{W})$ in Theorem 1 and the unitary transformation $\boldsymbol{W} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^\dagger$ with Jacobian $J(\boldsymbol{W} \to \boldsymbol{U}, \boldsymbol{D}) = \prod_{i<j}^m \left(\lambda_j - \lambda_i\right)^2$. □

[2] The distribution of the sum of two complex Wishart matrices has been derived using the Harish-Chandra-Itzykson-Zuber unitary group integral when one of the covariance matrices is proportional to the identity matrix while the second is arbitrary [18].

[3] The Shannon transform is originally motivated by applications and its wireless engineering and random matrix theory applications have been developed in [23].

[4] Although the term "Schur-concave" is well entrenched in the literature, this terminology is misleading and the term "MDS" or "Schur-decreasing" is more pertinent [26]. See [21, Appendix I] and reference therein for basic notions of majorization and Schur monotonicity theory.

[5] There has been much attention to the distribution theory of random matrices (see [17], [27], [28], and references therein). Specifically, the Wishart distributions are great interest in multivariate statistical analysis [17], [27]. It was named in honor of John Wishart, who formulated a general distribution using a geometrical argument [29], from the one obtained by Fisher [30] in the bivariate case.

[6] When $n_i < m$, the complex Wishart matrix W_i be a singular Wishart matrix [28].

$$p_{\lambda_1,\lambda_2,\ldots,\lambda_m}(\lambda_1,\lambda_2,\ldots,\lambda_m) = \frac{1}{\tilde{\Gamma}_m(n_1+n_2)} \det(\boldsymbol{\Sigma}_1)^{-n_1}\det(\boldsymbol{\Sigma}_2)^{-n_2}\left[\prod_{k=1}^{m}\lambda_k^{n_1+n_2-m}\right]\left[\prod_{i<j}^{m}(\lambda_j-\lambda_i)^2\right] \times \int_{\boldsymbol{U}\in\mathcal{U}(m)} \operatorname{etr}\left(-\boldsymbol{\Sigma}_1^{-1}\boldsymbol{UDU}^\dagger\right){}_1\tilde{F}_1\left(n_2;n_1+n_2;\left(\boldsymbol{\Sigma}_1^{-1}-\boldsymbol{\Sigma}_2^{-1}\right)\boldsymbol{UDU}^\dagger\right)[d\boldsymbol{U}] \quad (3)$$

Corollary 1 (Scaled Identity Covariances): If $\boldsymbol{\Sigma}_1 = \alpha\boldsymbol{I}_m$ and $\boldsymbol{\Sigma}_2 = \beta\boldsymbol{I}_m$, $\alpha,\beta > 0$, in Theorem 2, then we have

$$p_{\lambda_1,\lambda_2,\ldots,\lambda_m}(\lambda_1,\lambda_2,\ldots,\lambda_m) = K_{m,n_1,n_2,\alpha,\beta} \times\left[\prod_{k=1}^{m}\lambda_k^{n_1+n_2-m}e^{-\lambda_k/\alpha}\right]\left[\prod_{i<j}^{m}(\lambda_j-\lambda_i)\right]\det(\boldsymbol{\Xi}) \quad (4)$$

where

$$K_{m,n_1,n_2,\alpha,\beta} = \left[\prod_{i=1}^{m}\alpha^{n_1}\beta^{n_2}(n_1+n_2-i)!\,(i-1)!\right]^{-1} \quad (5)$$

and $\boldsymbol{\Xi}$ is the $m\times m$ matrix whose (i,j)th entry Ξ_{ij} is given by

$$\Xi_{ij} = \lambda_i^{j-1}\,{}_1F_1\left(n_2-m+j;\,n_1+n_2-m+j;\,\frac{\beta-\alpha}{\alpha\beta}\lambda_i\right). \quad (6)$$

Proof: It follows immediately from Theorem 2 and [21, Corollory 3]. □

B. Shannon TRANSFORM

The Shannon transform of a nonnegative random variable x is defined as [23, Definition 2.12]

$$\mathcal{V}_{\mathsf{x}}(\gamma) \triangleq \mathbb{E}\{\ln(1+\gamma\mathsf{x})\} \quad (7)$$

where $\gamma \geqslant 0$. Again motivated by applications, for an $m\times m$ positive semidefinite random matrix $\mathbf{A}$, its Shannon transform is defined as [23]

$$\mathcal{V}_{\mathbf{A}}(\gamma) = \lim_{m\to\infty}\frac{1}{m}\mathbb{E}\{\ln\det(\boldsymbol{I}_m+\gamma\mathbf{A})\} = \int_0^\infty \ln(1+\gamma x)\,d\mathsf{F}^{\mathbf{A}}(x) \quad (8)$$

where $\mathsf{F}^{\mathbf{A}}(x)$ is the asymptotic empirical eigenvalue distribution (or spectrum) of $\mathbf{A}$. In particular, if the joint density of ordered eigenvalues of $\mathbf{A}$ is symmetric in all eigenvalues, then the Shannon transform of the unordered marginal eigenvalue $\lambda(\mathbf{A})$ of $\mathbf{A}$ is equal to

$$\mathcal{V}_{\lambda(\mathbf{A})}(\gamma) = \frac{1}{m}\mathbb{E}\{\ln\det(\boldsymbol{I}_m+\gamma\mathbf{A})\} \quad (9)$$

leading to asymptotical equivalence between the unordered marginal eigenvalue and empirical spectrum of $\mathbf{A}$ such that $\mathcal{V}_{\mathbf{A}}(\gamma) = \lim_{m\to\infty}\mathcal{V}_{\lambda(\mathbf{A})}(\gamma)$. For completeness, we first review these two key Shannon transforms involving the standard Wishart matrix, which have been well studied in the context of multiple-antenna communications.

Let $\mathbf{W}^\star(m,n) \sim \tilde{\mathcal{W}}_m\left(n,\frac{1}{m}\boldsymbol{I}_m\right)$ be the standard Wishart matrix and $\lambda^\star(m,n)$ be its unordered marginal eigenvalue. Then, $\lambda^\star(m,n)$ has the density function of the form in [19, eq. (42)] and its Shannon transform is given by [19, Th. II.1]

$$\mathcal{V}_{\lambda^\star(m,n)}(\gamma) = \frac{e^{m/\gamma}}{m}\sum_{i=0}^{m-1}\sum_{j=0}^{i}\sum_{k=0}^{2j}\sum_{\ell=0}^{n-m+k}\left\{\frac{(-1)^k(2j)!\,(n-m+k)!}{2^{2i-k}j!k!\,(n-m+j)!} \times\binom{2i-2j}{i-j}\binom{2j+2n-2m}{2j-k}E_{\ell+1}\left(\frac{m}{\gamma}\right)\right\} \quad (10)$$

which is the ergodic per-antenna capacity of MIMO Rayleigh-fading channels.[7] As $m,n\to\infty$ with $n/m\to\kappa$, the spectrum of the standard Wishart $\mathbf{W}^\star(m,n)$ converges almost surely to the Marčenko-Pastur density [23, eq. (1.12)] and its Shannon transform is given by [22, eq. (11)], [23, eq. (1.14)]

$$\mathcal{V}_{\mathbf{W}^\star(\kappa)}(\gamma) = \kappa\ln\left(1+\gamma-\frac{1}{4}\mathcal{F}(\gamma,\kappa)\right) + \ln\left(1+\gamma\kappa-\frac{1}{4}\mathcal{F}(\gamma,\kappa)\right)-\frac{\mathcal{F}(\gamma,\kappa)}{4\gamma} \quad (11)$$

where

$$\mathcal{F}(\gamma,\kappa) = \left(\sqrt{\gamma\left(1+\sqrt{\kappa}\right)^2+1}-\sqrt{\gamma\left(1-\sqrt{\kappa}\right)^2+1}\right)^2. \quad (12)$$

We now extend the Shannon transforms (10) and (11) to the case of a conical combination of two independent standard Wishart matrices, which includes hyper-Wishart matrices with scaled identity covariances as nontrivial cases.

Theorem 3: Let $\mathbf{W}_1 \sim \tilde{\mathcal{W}}_m\left(n_1,\frac{1}{m}\boldsymbol{I}_m\right)$ and $\mathbf{W}_2 \sim \tilde{\mathcal{W}}_m\left(n_2,\frac{1}{m}\boldsymbol{I}_m\right)$ be statistically independent and

$$\mathbf{W} = \alpha\mathbf{W}_1+\beta\mathbf{W}_2,\quad \alpha,\beta\geqslant 0. \quad (13)$$

Then, the Shannon transform of the unordered marginal eigenvalue $\lambda(\mathbf{W})$ of $\mathbf{W}$ is given by

$$\mathcal{V}_{\lambda(\mathbf{W})}(\gamma) = \begin{cases} \frac{1}{m}\operatorname{tr}\left(\boldsymbol{G}^{-1}\dot{\boldsymbol{G}}\right), & \text{if } \alpha\neq\beta \text{ and } \alpha,\beta>0 \\ \mathcal{V}_{\lambda^\star(m,n_1+n_2)}(\alpha\gamma), & \text{if } \alpha=\beta\neq 0 \\ \mathcal{V}_{\lambda^\star(m,n_2)}(\beta\gamma), & \text{if } \alpha=0 \\ \mathcal{V}_{\lambda^\star(m,n_1)}(\alpha\gamma), & \text{if } \beta=0 \end{cases} \quad (14)$$

where $\boldsymbol{G}$ and $\dot{\boldsymbol{G}}$ are $m\times m$ matrices whose (i,j)th entries are given in (15) and (16), as shown at the bottom of the next page, respectively. In particular, we denote $m\mathcal{V}_{\lambda(\mathbf{W})}(\gamma)$

[7] We can also express the Shannon transform $\mathcal{V}_{\lambda^\star(m,n)}(\gamma)$ as the trace representation using [20, Th. 2].

when $n_1 = n_2 = m$ at $\gamma = 1$ by $\psi_m(\alpha, \beta)$ as a functional of the conical scaling variables $(\alpha, \beta) \in \mathbb{R}_+^2$.

Proof: See Appendix C. □

Theorem 4: Let W be a conical combination of two $m \times m$ standard Wishart matrices as defined in Theorem 3. Then, as $m, n_1, n_2 \to \infty$ with limiting ratios $n_1/m \to \kappa_1$ and $n_2/m \to \kappa_2$, the Shannon transform of its asymptotic spectrum is given by

$$\mathcal{V}_{\mathsf{W}}(\gamma) = \begin{cases} \ln\left[\dfrac{(1+\alpha\gamma\eta)^{\kappa_1}(1+\beta\gamma\eta)^{\kappa_2}}{\eta}\right]+\eta-1, & \text{if } \alpha \neq \beta \text{ and } \\ & \alpha, \beta > 0 \\ \mathcal{V}_{\mathsf{W}^\star(\kappa_1+\kappa_2)}(\alpha\gamma), & \text{if } \alpha = \beta \neq 0 \\ \mathcal{V}_{\mathsf{W}^\star(\kappa_2)}(\beta\gamma), & \text{if } \alpha = 0 \\ \mathcal{V}_{\mathsf{W}^\star(\kappa_1)}(\alpha\gamma), & \text{if } \beta = 0 \end{cases} \tag{17}$$

where η is the unique positive solution of

$$\kappa_1 + \kappa_2 - \left(\frac{\kappa_1}{1+\alpha\gamma\eta} + \frac{\kappa_2}{1+\beta\gamma\eta}\right) = 1 - \eta. \tag{18}$$

Again, we denote $\mathcal{V}_{\mathsf{W}}(\gamma)$ when $n_1 = n_2 = m$ at $\gamma = 1$ by $\psi_\infty(\alpha, \beta)$ as a functional of $(\alpha, \beta) \in \mathbb{R}_+^2$.

Proof: See Appendix D. □

Remark 2: As κ_1 and κ_2 tend to infinity, W_1 and W_2 converge almost surely to $\boldsymbol{I}_m$ by the law of large numbers. Hence, we have

$$\lim_{\kappa_1,\kappa_2\to\infty} \mathcal{V}_{\mathsf{W}}(\gamma) \xrightarrow{\text{a.s.}} \ln\left[1 + \gamma(\alpha+\beta)\right]. \tag{19}$$

Property 1: By definition,

$$\frac{\partial}{\partial\alpha}\psi_m(\alpha, \beta) = \mathbb{E}\left\{\mathrm{tr}\left[(\boldsymbol{I}_n + \alpha\mathsf{W}_1 + \beta\mathsf{W}_2)^{-1}\mathsf{W}_1\right]\right\} \geqslant 0 \tag{20}$$

where the last inequality follows from positive definiteness of W_1 and W_2 along with the fact that the eigenvalues of $\boldsymbol{AB}$ are all positive for any $\boldsymbol{A}, \boldsymbol{B} > \boldsymbol{0}$. Therefore, we can see from (13) and (20) that the Shannon transform $\psi_m(\alpha, \beta)$ as a functional of the scaling variables $(\alpha, \beta) \in \mathbb{R}_+^2$ is symmetric in α and β; and increasing in each argument (α or β).

Property 2 (Schur Monotonicity): Since

$$\begin{aligned}\frac{\partial^2}{\partial\alpha^2}\psi_m(\alpha, t-\alpha) &= -\mathbb{E}\left\{\mathrm{tr}\left(\left[(\boldsymbol{I}_n+\alpha\mathsf{W}_1+(t-\alpha)\mathsf{W}_2)^{-1}(\mathsf{W}_1-\mathsf{W}_2)\right]^2\right)\right\} \\ &\leqslant 0,\end{aligned} \tag{21}$$

$\psi_m(\alpha, t-\alpha)$ is concave in α for each fixed $t \geqslant \alpha$. Therefore, it follows from [26, Proposition C.2.b] that the Shannon transform $\psi_m(\alpha, \beta)$ as a functional of $(\alpha, \beta) \in \mathbb{R}_+^2$ is MDS:

$$\psi_m(\alpha_1, \beta_1) \geqslant \psi_m(\alpha_2, \beta_2) \tag{22}$$

whenever $(\alpha_1, \beta_1) \preccurlyeq (\alpha_2, \beta_2)$ on $\mathbb{R}_+^2$. Let $\mathcal{L}_+^{(\gamma)}$ be a straight line on the nonnegative quadrant $\mathbb{R}_+^2$ such that

$$\mathcal{L}_+^{(\gamma)} = \left\{(\alpha, \beta) \in \mathbb{R}_+^2 : \alpha + \beta = \gamma\right\}. \tag{23}$$

Since $(\gamma/2, \gamma/2) \preccurlyeq (\alpha, \beta) \preccurlyeq (\gamma, 0)$ for all $(\alpha, \beta) \in \mathcal{L}_+^{(\gamma)}$ (see blue solid and red dashed lines in Fig. 1), the MDS property (22) reveals that

$$\psi_m(\gamma, 0) \leqslant \psi_m(\alpha, \beta) \leqslant \psi_m\left(\frac{\gamma}{2}, \frac{\gamma}{2}\right), \quad \forall(\alpha, \beta) \in \mathcal{L}_+^{(\gamma)}. \tag{24}$$

Moreover, since $\psi_m(\alpha, \beta)$ is (argument-wise) increasing and MDS, it follows from [26, Th. A.8] that

$$\psi_m(\alpha_1, \beta_1) \geqslant \psi_m(\alpha_2, \beta_2) \tag{25}$$

whenever $(\alpha_1, \beta_1) \preccurlyeq^{\mathrm{w}} (\alpha_2, \beta_2)$ on $\mathbb{R}_+^2$ (see blue and red shaded regions in Fig. 1). The geometrical illustration for the Schur monotonicity property of the Shannon transform $\psi_m(\alpha, \beta)$ on $\mathbb{R}_+^2$ is depicted in Fig. 1.

III. CAPACITY OF *N*-MIMO RELAY CHANNELS

In this section, we analyze the cutset upper bound and partial DF lower bound on the ergodic capacity of MIMO Rayleigh-fading relay channels using the statistical results on the hyper-Wishart matrix obtained in Section II.

$$\begin{aligned}G_{ij} = {} & (n_1+n_2-m+i+j-2)!\left(\frac{\alpha\gamma}{m}\right)^{n_1+n_2-m+i+j-1} \\ & \times {}_2F_1\left(n_2-m+j, n_1+n_2-m+i+j-1; n_1+n_2-m+j; 1-\frac{\alpha}{\beta}\right)\end{aligned} \tag{15}$$

$$\begin{aligned}\dot{G}_{ij} = {} & \frac{(n_1+n_2-m+j-2)!\,(1-n_1-n_2+m-j)_{n_2-m+j}}{(n_2-m+j-1)!}\left(\frac{\beta-\alpha}{\alpha\beta\gamma/m}\right)^{1-n_1-n_2+m-j} \\ & \times\left[\sum_{k=0}^{n_1-1}\sum_{\ell=0}^{k+i-1}\frac{(1-n_1)_k\left(\frac{\beta-\alpha}{\alpha\beta\gamma/m}\right)^k}{k!\,(2-n_1-n_2+m-j)_k}\left(\frac{\alpha\gamma}{m}\right)^{k+i}(k+i-1)!\,e^{m/(\alpha\gamma)}E_{\ell+1}\left(\frac{m}{\alpha\gamma}\right)\right. \\ & \left. -\sum_{k=0}^{n_2-m+j-1}\sum_{\ell=0}^{k+i-1}\frac{(1-n_2+m-j)_k\left(\frac{\alpha-\beta}{\alpha\beta\gamma/m}\right)^k}{k!\,(2-n_1-n_2+m-j)_k}\left(\frac{\beta\gamma}{m}\right)^{k+i}(k+i-1)!\,e^{m/(\beta\gamma)}E_{\ell+1}\left(\frac{m}{\beta\gamma}\right)\right]\end{aligned} \tag{16}$$

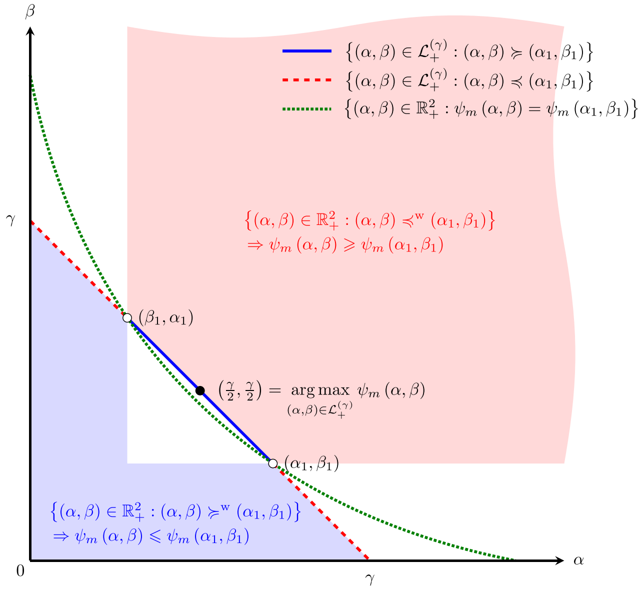

FIGURE 1. Geometrical illustration of Schur monotonicity of the Shannon transform $\psi_m(\alpha,\beta)$ on $\mathbb{R}_+^2$.

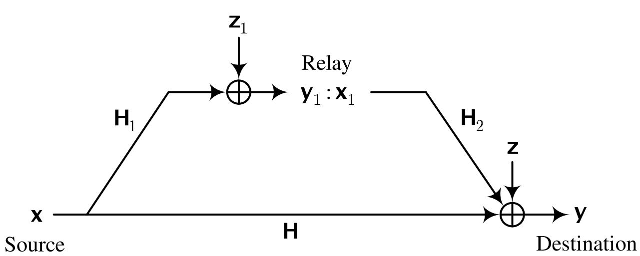

FIGURE 2. A general full-duplex N-MIMO relay channel.

A. CHANNEL MODEL

We consider a general full-duplex N-MIMO Rayleigh-fading relay channel with N-antenna source, relay, and destination nodes, as depicted in Fig 2. The received signals at the relay and destination can be written respectively as

$$\mathbf{y}_1 = \mathbf{H}_1\mathbf{x} + \mathbf{z}_1 \tag{26}$$

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{H}_2\mathbf{x}_1 + \mathbf{z} \tag{27}$$

where $\mathbf{x}$ and $\mathbf{x}_1$ are $N \times 1$ transmitted signals from the source and relay nodes with power constraints $\mathbb{E}\left\{\|\mathbf{x}\|^2\right\} \leqslant P$ and $\mathbb{E}\left\{\|\mathbf{x}_1\|^2\right\} \leqslant P_1$, respectively; $\mathbf{H}$, $\mathbf{H}_1$, and $\mathbf{H}_2$ are $N \times N$ Rayleigh-fading channel gain matrices for the S $\to$ D, S $\to$ R, and R $\to$ D communication links, respectively; and $\mathbf{z}$ and $\mathbf{z}_1$ are $N \times 1$ zero-mean complex additive white Gaussian noise (AWGN) vectors at the relay and destination, respectively. The entries of the channel matrices $\mathbf{H}$, $\mathbf{H}_1$, and $\mathbf{H}_2$ are independent and identically distributed (i.i.d.) circularly symmetric complex Gaussian $\mathcal{CN}(0, \Omega/N)$, $\mathcal{CN}(0, \Omega_1/N)$, and $\mathcal{CN}(0, \Omega_2/N)$, respectively, while those of the AWGN vectors $\mathbf{z}$ and $\mathbf{z}_1$ are i.i.d. $\mathcal{CN}\left(0, \sigma^2\right)$ and $\mathcal{CN}\left(0, \sigma_1^2\right)$, respectively. The transmitted signals $\mathbf{x}$ and $\mathbf{x}_1$ are statistical independent of the AWGN $\mathbf{z}$ and $\mathbf{z}_1$. Furthermore, all the random quantities $\mathbf{H}$, $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{z}$, and $\mathbf{z}_1$ are also mutually independent. The variances Ω and Ω_i, $i = 1, 2$, capture the variations of distance-related large-scale path loss between different transmit-receive node pairs, i.e., $\Omega = r^{-\nu}$ and $\Omega_i = r_i^{-\nu}$, where r and r_i denote the distances between respective communicating node pairs and ν is the power pass-loss exponent. Hence, we can parameterize the average received SNRs per antenna for the S $\to$ D, S $\to$ R, and R $\to$ D links as

$$\mathrm{SNR} = \frac{P}{r^{\nu}\sigma^2}, \quad \mathrm{SNR}_1 = \frac{P}{r_1^{\nu}\sigma_1^2}, \; \mathrm{SNR}_2 = \frac{P_1}{r_2^{\nu}\sigma^2} \tag{28}$$

respectively. We further assume that each node has access to respective receive CSI only, i.e., the source has no CSI, the relay has access to $\mathbf{H}_1$ only, and the destination has access to $\mathbf{H}$ and $\mathbf{H}_2$.[8]

B. CUTSET UPPER BOUND

The max-flow min-cut argument yields the upper bound on the ergodic capacity (nats/s/Hz) of the general N-MIMO Rayleigh-fading relay channel as follows [11, Th. 4.1]:

$$\begin{aligned}\langle \mathsf{c} \rangle \leqslant \min\Bigg\{&\mathbb{E}_{\mathbf{H},\mathbf{H}_1} \ln\det\left(\boldsymbol{I}_N + \frac{P}{\sigma^2 N}\mathbf{H}^{\dagger}\mathbf{H} + \frac{P}{\sigma_1^2 N}\mathbf{H}_1^{\dagger}\mathbf{H}_1\right),\\ &\mathbb{E}_{\mathbf{H},\mathbf{H}_2} \ln\det\left(\boldsymbol{I}_N + \frac{P}{\sigma^2 N}\mathbf{H}\mathbf{H}^{\dagger} + \frac{P_1}{\sigma^2 N}\mathbf{H}_2\mathbf{H}_2^{\dagger}\right)\Bigg\}.\end{aligned} \tag{29}$$

The key step for developing this bound is to show that optimal coding strategies use independent Gaussian codebooks for $\mathbf{x}$ and $\mathbf{x}_1$ at the source and relay by virtue of the fact that the source has no CSI and the relay has access only to receive CSI (i.e., on $\mathbf{H}_1$).[9]

Theorem 5 (Cutset Bound): Let $\mathrm{snr} = \mathrm{SNR}/N$ and $\mathrm{snr}_i = \mathrm{SNR}_i/N$, $i = 1, 2$, be the normalized SNR parameters. Then, the cutset bound on the ergodic capacity (nats/s/Hz) of the general N-MIMO Rayleigh-fading relay channel is given in closed form by

$$\langle \mathsf{c} \rangle \leqslant \psi_N\left(\mathrm{snr}, \min\left\{\mathrm{snr}_1, \mathrm{snr}_2\right\}\right). \tag{30}$$

Proof: It follows readily from (29), Theorem 3, and Remark 1. □

Remark 3: The cutset upper bound (30) is only a function of snr and $\min\{\mathrm{snr}_1, \mathrm{snr}_2\}$ due to the symmetric antenna configuration of the N-MIMO channel. Specifically, the cutset bound as a functional of snr and $\min\{\mathrm{snr}_1, \mathrm{snr}_2\}$ is increasing and MDS (see Property 2).

[8] We assume that the relay can cancel out the self-interference with full CSI at the relay [11]. The optimal power allocation problems under different operating conditions were investigated in [6], [31], and [32]. Recently, it was shown that the isotropic Grassmann input distribution achieves an upper bound on the cutset bound in MIMO relay channels when no CSI is available at any of the communicating nodes [33].

[9] In case of the fixed channel or assuming full CSI, the transmitted signals $\mathbf{x}$ and $\mathbf{x}_1$ from the source and relay are in contrast correlated for maximizing the achievable information rate in the multiple access part of cuts with complete cooperation between the source and relay nodes [2], [11], [15].

Remark 4 (Partial DF Lower Bound): Using block-Markov superposition coding and again independent codebooks for the source and relay, the partial DF lower bound on the ergodic capacity of general MIMO Rayleigh-fading relay channels has been also given in [11, Th. 4.2]. For the general N-MIMO relay channel, we obtain this lower bound in closed form, using the same line of the proof of Theorem 5, as follows:

$$\langle \mathsf{c} \rangle \geqslant \max\Big\{ \psi_N(\mathrm{snr}, 0), \min\{\psi_N(0, \mathrm{snr}_1), \psi_N(\mathrm{snr}, \mathrm{snr}_2)\}\Big\}. \tag{31}$$

If $\mathrm{snr}_1 \leqslant \mathrm{snr}_2$, the lower bound is particularized using (25) to

$$\langle \mathsf{c} \rangle \geqslant \psi_N(\max\{\mathrm{snr}, \mathrm{snr}_1\}, 0). \tag{32}$$

Remark 5 (Regularity for Capacity Achieving): When the channel is physically degenerate such that i) $\mathrm{snr} = 0$ (no S $\to$ D link) and ii) $\mathrm{snr}_1 = 0$ or $\mathrm{snr}_2 = 0$ (relay off), the upper and lower bounds in (30) and (31) coincide and hence, the capacity for such N-MIMO relay channels can be characterized exactly as follows:

$$\langle \mathsf{c} \rangle = \begin{cases} \psi_N(0, \min\{\mathrm{snr}_1, \mathrm{snr}_2\}), & \text{if } \mathrm{snr} = 0 \\ \psi_N(\mathrm{snr}, 0), & \text{if } \mathrm{snr}_1 = 0 \text{ or } \mathrm{snr}_2 = 0 \end{cases} \tag{33}$$

where the first case corresponds to general full-duplex dual-hop MIMO relaying in which the minimum information flow capability between the S $\to$ R and R $\to$ D links behaves as a bottleneck for the achievable rate; and the second case boils down to the MIMO capacity in the absence of relaying [19, Th. III.1]. In addition to these two physically degenerate cases, we can also characterize the capacity in a regular SNR regime $\mathcal{R}$ as

$$\langle \mathsf{c} \rangle_{\mathcal{R}} = \psi_N(\mathrm{snr}, \mathrm{snr}_2) \tag{34}$$

where[10]

$$\mathcal{R} = \Big\{(\mathrm{snr}, \mathrm{snr}_1, \mathrm{snr}_2) \in \mathbb{R}_+^3 : \psi_N(0, \mathrm{snr}_1) \geqslant \psi_N(\mathrm{snr}, \mathrm{snr}_2)\Big\}.$$

IV. NUMERICAL RESULTS

In this section, we provide some numerical results to illustrate our analysis.

A. VERIFICATION

To verify our analysis, we depicted the upper bound on ergodic capacity $\langle \mathsf{c} \rangle$ in nats/s/Hz and its scaled asymptote $N\langle \mathsf{c} \rangle^\star$ in nats/s/Hz as a function of $\min(\mathrm{snr}_1, \mathrm{snr}_2)$ in Fig. 3 when $\mathrm{snr} = 25$ dB and $N = 2, 3, 4$. The analysis result using (30) in Theorem 5 agrees exactly with the simulation result obtained using [11, Th. 4.1]. We can also see that

[10]The sufficient conditions for regularity were characterized in [11] for the high-SNR regime and the scalar case.

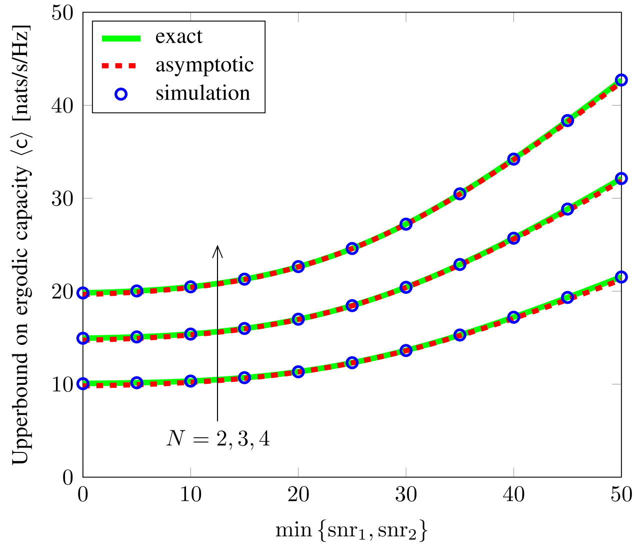

FIGURE 3. Upper bound on ergodic capacity $\langle \mathsf{c} \rangle$ in nats/s/Hz and its scaled asymptote as a function of $\min\{\mathrm{snr}_1, \mathrm{snr}_2\}$ when $\mathrm{snr} = 25$ dB, and $N = 2, 3, 4$.

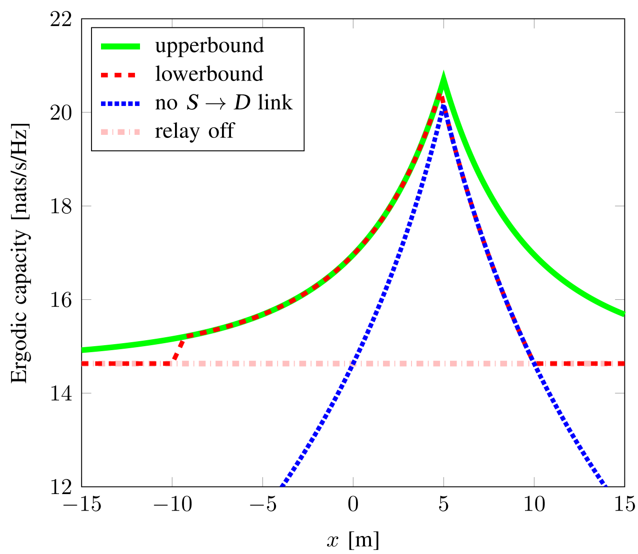

FIGURE 4. Ergodic capacity $\langle \mathsf{c} \rangle$ in nats/s/Hz as a function of the relay location at the position x when $P = P_1 = -23$ dBm, $\sigma^2 = \sigma_1^2 = -101.11$ dBm, $N = 2$, and $\nu = 4$. The source and destination are located on a line at $x = 0$ and $x = 10$, respectively.

the scaled asymptotic capacity, which can be obtained by replacing the exact expression $\psi_N(\mathrm{snr}, \min\{\mathrm{snr}_1, \mathrm{snr}_2\})$ with $N\psi_\infty(\mathrm{snr}, \min\{\mathrm{snr}_1, \mathrm{snr}_2\})$ in Theorem 5, accurately predicts the exact results, specially when $\mathrm{snr} = \min(\mathrm{snr}_1, \mathrm{snr}_2)$.

B. CAPACITY ACHIEVABILITY

In this subsection, we consider an N-MIMO relay channel with a small cell size to illustrate capacity achievability [34], [35]. We set the transmission powers and noise powers to $P = P_1 = -23$ dBm and $\sigma^2 = \sigma_1^2 = -101.11$ dBm, leading to $P/\sigma^2 = P/\sigma_1^2 = P_1/\sigma_1^2 = 78.11$ dB, which are the average receive SNRs at

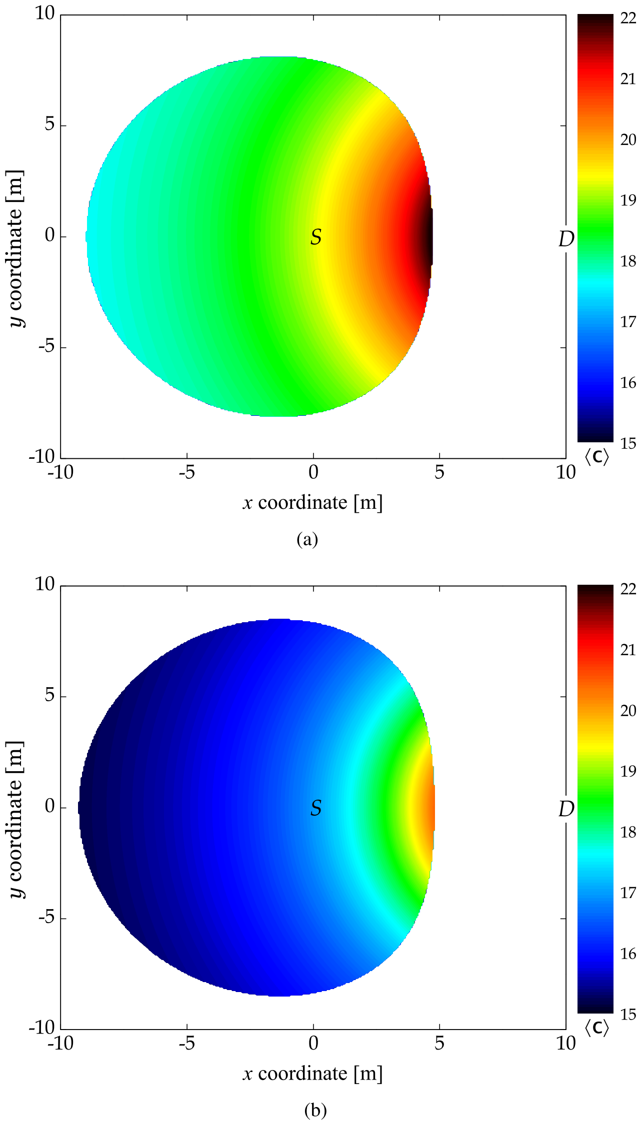

FIGURE 5. **Capacity achieving region and their ergodic capacity ⟨c⟩ in nats/s/Hz for a N-MIMO Rayleigh-fading relay channel for (a) $\nu = 3.5$ and (b) $\nu = 4$ when $N = 2$. The relay located at (x, y) in the 20×20 grid with the source and destination located at the coordinates $(0, 0)$ and $(10, 0)$, respectively.**

a reference distance of 1 meter away from the source and relay.

Fig. 4 shows the upper and lower bounds on the ergodic capacity ⟨c⟩ in nats/s/Hz as a function of the relay location at the position x when the source and destination are located on a line at $x = 0$ and $x = 10$, respectively. We set the power pass-loss exponent $\nu = 4$ for all links and $N = 2$. For a comparison, we also plot the physically degenerated channels (no S → D link and relay off channels) as shown in (33). We can observe that the upper and lower bounds on the ergodic capacity are coincided when the relay is located at $x \in (-9.3, 4.8)$. In this case, the normalized SNR parameters are in the regular SNR regime $\mathcal{R}$ as shown in Remark 5 and the capacity ⟨c⟩ is equal to $\psi_2(\mathsf{snr}, \mathsf{snr}_2)$.

To further characterize the sufficient condition for capacity achieving, we plot the ergodic capacity ⟨c⟩ in nats/s/Hz for a N-MIMO Rayleigh-fading relay channel in Fig. 5 where the relay is located at (x, y) in the 20×20 grid, and the source and destination are located at the coordinates $(0, 0)$ and $(10, 0)$, respectively, for (a) $\nu = 3.5$ and (b) $\nu = 4$ when $N = 2$. The color in contour plots represents the ergodic capacity in nats/s/Hz for the corresponding relay position at (x, y) when the upper and lower bounds coincide. We can observe that the regular SNR regime $\mathcal{R}$ increases with the pass-loss exponent ν while decreasing the achievable capacity. This sufficient regularity condition for capacity achievability depends on the number of antennas, fading, and path-loss exponent, as expected.

V. CONCLUSIONS

In this paper, we developed a framework to characterize ergodic capacity of MIMO Rayleigh-fading relay channels by introducing a cental hyper-Wishart matrix. We derived the PDF of the sum of two independent complex central Wishart matrices and its joint eigenvalue density which enable us to derive the closed-form formula for the upper and lower bounds on ergodic capacity of MIMO Rayleigh-fading relay channels and its asymptote with Shannon transform of the unordered marginal eigenvalue density of hyper-Wishart matrix. It has been shown that i) the Shannon transform as a functional of scale variables has a MDS property; and ii) the regularity SNR regime where the upper and lower bounds coincide is characterized by the number of antennas, fading, and path-loss exponent.

APPENDIX A
GLOSSARY OF NOTATION AND SYMBOLS

$(\cdot)^\dagger$	Transpose conjugate
$\mathbb{R}_+^n$	Nonnegative orthant: $\mathbb{R}_+^n = \{(x_1, x_2, \ldots, x_n) : x_i \geqslant 0, \forall i\}$
$\jmath$	Imaginary unit: $\jmath = \sqrt{-1}$
$\mathrm{tr}\,(\boldsymbol{A})$	Trace of a matrix $\boldsymbol{A}$
$\mathrm{etr}\,(\boldsymbol{A})$	Exponential trace of a matrix $\boldsymbol{A}$: $\mathrm{etr}\,(\boldsymbol{A}) = \exp\,(\mathrm{tr}\,\boldsymbol{A})$
$\boldsymbol{I}_n$	$n \times n$ identity matrix
$\mathrm{diag}\,(\cdot)$	Diagonal matrix
$\boldsymbol{A} > \boldsymbol{B}$	Löwner partial ordering for Hermitian matrices $\boldsymbol{A}$ and $\boldsymbol{B}$: $\boldsymbol{A} > \boldsymbol{B}$ means that $\boldsymbol{A} - \boldsymbol{B}$ is positive definite
$\boldsymbol{a} \preccurlyeq \boldsymbol{b}$	Majorization ordering [21, Definition 3]: $\boldsymbol{a} \preccurlyeq \boldsymbol{b}$ means that $\boldsymbol{a}$ is majorized by $\boldsymbol{b}$
$\boldsymbol{a} \preccurlyeq^{\mathrm{w}} \boldsymbol{b}$	Weak majorization ordering [26, Definition A.2]: $\boldsymbol{a} \preccurlyeq^{\mathrm{w}} \boldsymbol{b}$ means that $\boldsymbol{a}$ is weakly supermajorized by $\boldsymbol{b}$
$\oplus$	Direct sum of matrices: $\boldsymbol{A} \oplus \boldsymbol{B} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{bmatrix}$
$(a)_n$	Pochhammer symbol: $(a)_n = a\,(a+1)\cdots(a+n-1)$, $(a)_0 = 1, a \neq 0$

${}_pF_q(\cdot)$	Generalized hypergeometric function of a scalar argument [36, eq. (9.14.1)]
$E_n(z)$	nth-order exponential integral function [19]: $E_n(z) = \int_1^\infty e^{-zx}x^{-n}dx$, $n = 0, 1, 2, \ldots, \Re\{z\} > 0$
$\tilde{\Gamma}_m(z)$	Complex matrix-variate gamma function [37, Definition 6.1]
${}_p\tilde{F}_q(\cdot)$	Hypergeometric function of a matrix argument [21, eq. (76)]
$\mathcal{U}(m)$	Unitary group of order m: $\mathcal{U}(m) = \left\{\boldsymbol{U} \in \mathbb{C}^{m\times m} : \boldsymbol{U}\boldsymbol{U}^\dagger = \boldsymbol{I}_m\right\}$
$[d\boldsymbol{U}]$	Unitary invariant Haar measure on the unitary group $\mathcal{U}(m)$ normalized to make the total volume unity
$(d\boldsymbol{X})$	Exterior product of the differential elements of $\boldsymbol{X}$
$\mathbb{E}\{\cdot\}$	Expectation operator
$p_{\mathsf{x}}(x)$	Probability density function of x
$\mathcal{CN}(\mu, \sigma^2)$	Circularly symmetric complex Gaussian distribution with mean μ and variance σ^2
$\tilde{\mathcal{N}}_{m,n}(\boldsymbol{M}, \boldsymbol{\Sigma}, \boldsymbol{\Psi})$	Complex matrix-variate Gaussian distribution [19, Definition II.1]
$\tilde{\mathcal{W}}_m(n, \boldsymbol{\Sigma})$	Complex central Wishart distribution [19, Definition II.2]

APPENDIX B
PROOF OF THEOREM 1

The joint density of $\mathbf{W}_1$ and $\mathbf{W}_2$ is given by

$$\begin{aligned} &p_{\mathbf{W}_1,\mathbf{W}_2}(\boldsymbol{W}_1, \boldsymbol{W}_2) \\ &= \frac{1}{\tilde{\Gamma}_m(n_1)\tilde{\Gamma}_m(n_2)} \det(\boldsymbol{\Sigma}_1)^{-n_1}\det(\boldsymbol{\Sigma}_2)^{-n_2}\det(\boldsymbol{W}_1)^{n_1-m} \\ &\quad\times \det(\boldsymbol{W}_2)^{n_2-m}\operatorname{etr}\left(-\boldsymbol{\Sigma}_1^{-1}\boldsymbol{W}_1 - \boldsymbol{\Sigma}_2^{-1}\boldsymbol{W}_2\right) \end{aligned} \tag{35}$$

for $\boldsymbol{W}_1 > \mathbf{0}$, $\boldsymbol{W}_2 > \mathbf{0}$. Making the transformations

$$\mathbf{W} = \mathbf{W}_1 + \mathbf{W}_2 \tag{36}$$

$$\mathbf{V} = \mathbf{W}^{-1/2}\mathbf{W}_2\mathbf{W}^{-1/2} \tag{37}$$

with Jacobian $J(\boldsymbol{W}_1, \boldsymbol{W}_2 \to \boldsymbol{W}, \boldsymbol{V}) = \det(\boldsymbol{W})^m$, we get the joint density of $\mathbf{W}$ and $\mathbf{V}$ as follows:

$$\begin{aligned} &p_{\mathbf{W},\mathbf{V}}(\boldsymbol{W}, \boldsymbol{V}) \\ &= \frac{1}{\tilde{\Gamma}_m(n_1)\tilde{\Gamma}_m(n_2)} \det(\boldsymbol{\Sigma}_1)^{-n_1}\det(\boldsymbol{\Sigma}_2)^{-n_2} \\ &\quad\times \det(\boldsymbol{W})^{n_1+n_2-m}\det(\boldsymbol{V})^{n_2-m}\det(\boldsymbol{I}_m - \boldsymbol{V})^{n_1-m} \\ &\quad\times \operatorname{etr}\left(-\boldsymbol{\Sigma}_1^{-1}\boldsymbol{W}\right)\operatorname{etr}\left\{\boldsymbol{W}^{1/2}\left(\boldsymbol{\Sigma}_1^{-1} - \boldsymbol{\Sigma}_2^{-1}\right)\boldsymbol{W}^{1/2}\boldsymbol{V}\right\}, \end{aligned} \tag{38}$$

for $\boldsymbol{W} > \mathbf{0}$, $\mathbf{0} < \boldsymbol{V} < \boldsymbol{I}_m$. We then obtain the density of $\mathbf{W}$ given in (39), as shown at the bottom of this page. Now integrating out with respect to $\boldsymbol{V}$ with the help of [37, Example 6.3], we arrive at the desired result (2).

$$\begin{aligned} p_{\mathbf{W}}(\boldsymbol{W}) = &\frac{1}{\tilde{\Gamma}_m(n_1)\tilde{\Gamma}_m(n_2)} \det(\boldsymbol{\Sigma}_1)^{-n_1}\det(\boldsymbol{\Sigma}_2)^{-n_2}\det(\boldsymbol{W})^{n_1+n_2-m}\operatorname{etr}\left(-\boldsymbol{\Sigma}_1^{-1}\boldsymbol{W}\right) \\ &\times \int_{\mathbf{0}<\boldsymbol{V}<\boldsymbol{I}_m} \det(\boldsymbol{V})^{n_2-m}\det(\boldsymbol{I}_m - \boldsymbol{V})^{n_1-m}\operatorname{etr}\left\{\boldsymbol{W}^{1/2}\left(\boldsymbol{\Sigma}_1^{-1} - \boldsymbol{\Sigma}_2^{-1}\right)\boldsymbol{W}^{1/2}\boldsymbol{V}\right\}(d\boldsymbol{V}) \end{aligned} \tag{39}$$

APPENDIX C
PROOF OF THEOREM 3

When $\alpha = \beta$, $\alpha = 0$ or $\beta = 0$, the problem boils down to the Wishart cases and hence, we can obtain the corresponding $\psi_m(\alpha, \beta)$ using (10) together with Remark 1.

For $\alpha \neq \beta$ and $\alpha, \beta > 0$, we have $\mathbf{W} \sim \mathcal{H}\tilde{\mathcal{W}}_m^{(2)}\left(\{n_1, n_2\}, \left\{\frac{\alpha}{m}\boldsymbol{I}_m, \frac{\beta}{m}\boldsymbol{I}_m\right\}\right)$ by definition. Let

$$\Delta = \ln\det(\boldsymbol{I}_m + \gamma\mathbf{W}). \tag{40}$$

Then, the moment generating function of Δ can be written as

$$\begin{aligned} \phi_\Delta(s) &= \mathbb{E}_{\mathbf{Z}}\left\{e^{s\Delta}\right\} \\ &= \mathbb{E}_{\mathbf{W}}\left\{\det(\boldsymbol{I}_m + \gamma\mathbf{W})^s\right\}. \end{aligned} \tag{41}$$

Using Corollary 1 and the same steps leading to [20, Th. 1] or [21, Th. 10], we get

$$\phi(s) = K_{m,n_1,n_2,\alpha\gamma/m,\beta\gamma/m}\det\{\boldsymbol{G}(s)\} \tag{42}$$

where $\boldsymbol{G}(s)$ is the $m \times m$ matrix whose (i, j)th entry is given by

$$\begin{aligned} G_{ij}(s) = &\int_0^\infty (1+\lambda)^s \lambda^{n_1+n_2-m+i+j-2}e^{-\lambda m/(\alpha\gamma)} \\ &\times {}_1F_1\left(n_2-m+j; n_1+n_2-m+j; \frac{\beta-\alpha}{\alpha\beta\gamma/m}\lambda\right)d\lambda. \end{aligned} \tag{43}$$

Let $\boldsymbol{G} = \boldsymbol{G}(0)$ and $\dot{\boldsymbol{G}} = d\boldsymbol{G}(s)/ds\big|_{s=0}$ for notational simplicity, and let $\boldsymbol{G}_{[1]} = \boldsymbol{G}^{-1}\dot{\boldsymbol{G}}$ be the dimatrix of $\boldsymbol{G}(s)$ at $s = 0$ [20, Definition 2], where the (i, j)th entry of $\dot{\boldsymbol{G}}$ is given by

$$\begin{aligned} \dot{G}_{ij} = &\int_0^\infty \ln(1+\lambda)\lambda^{n_1+n_2-m+i+j-2}e^{-\lambda m/(\alpha\gamma)} \\ &\times {}_1F_1\left(n_2 - m + j; n_1 + n_2 - m + j; \frac{\beta - \alpha}{\alpha\beta\gamma/m}\lambda\right)d\lambda. \end{aligned} \tag{44}$$

Then, it follows from [20, eq. (10)] that

$$\begin{aligned} \mathbb{E}\{\Delta\} &= \frac{d}{ds}\ln\phi_\Delta(s)\bigg|_{s=0} \\ &= \frac{d}{ds}\ln\det\{\boldsymbol{G}(s)\}\bigg|_{s=0} \\ &= \operatorname{tr}\left(\boldsymbol{G}_{[1]}\right). \end{aligned} \tag{45}$$

Now, we obtain the (i, j)th entry of $\boldsymbol{G}$ as (15) with the help of the integral identity [36, eq. (7.522.9)]. To evaluate the (i, j)th entry of $\dot{\boldsymbol{G}}$ in (44), we use the identities [38, eq. (7.11.1.12)] and [19, eqs. (44) and (46)] successively, and then arrive at the result (16).

APPENDIX D
PROOF OF THEOREM 4

When $\alpha = \beta$, $\alpha = 0$ or $\beta = 0$, the problem boils down to the Wishart cases and hence, we can obtain the corresponding $\psi_\infty(\alpha, \beta)$ using (11) together with Remark 1.

For $\alpha \neq \beta$ and $\alpha, \beta > 0$, let us decompose a matrix $\mathsf{W} \sim \mathcal{H}\tilde{\mathcal{W}}_m^{(2)}\left(\{n_1, n_2\}, \left\{\frac{\alpha}{m}\boldsymbol{I}_m, \frac{\beta}{m}\boldsymbol{I}_m\right\}\right)$ into $\mathsf{W} = \bar{\mathsf{H}}\bar{\mathsf{T}}\bar{\mathsf{H}}^\dagger$, then $\bar{\mathsf{H}} \sim \tilde{\mathcal{N}}_{m,n_1+n_2}\left(\mathbf{0}, \frac{1}{\sqrt{m}}\boldsymbol{I}_m, \boldsymbol{I}_{n_1+n_2}\right)$ is the Grammian matrix and $\bar{\mathsf{T}} = \alpha\boldsymbol{I}_{n_1} \oplus \beta\boldsymbol{I}_{n_2}$ is the positive definite matrix. Following [23, Definitions 2.11 and 2.12], the transforms for the matrix $\bar{\mathsf{T}}$ are given by

$$\begin{aligned}\eta_{\bar{\mathsf{T}}}(\gamma) &= \lim_{n_1,n_2\to\infty}\frac{1}{n_1+n_2}\operatorname{tr}\left(\left(\boldsymbol{I}_{n_1+n_2}+\gamma\bar{\mathsf{T}}\right)^{-1}\right)\\ &= \frac{1}{\kappa_1+\kappa_2}\left(\frac{\kappa_1}{1+\alpha\gamma}+\frac{\kappa_2}{1+\beta\gamma}\right), \end{aligned} \quad (46)$$

$$\begin{aligned}\mathcal{V}_{\bar{\mathsf{T}}}(\gamma) &= \lim_{n_1,n_2\to\infty}\frac{1}{n_1+n_2}\ln\det\left\{\boldsymbol{I}_{n_1+n_2}+\gamma\bar{\mathsf{T}}\right\}\\ &= \frac{1}{\kappa_1+\kappa_2}\left(\kappa_1\ln(1+\alpha\gamma)+\kappa_2\ln(1+\beta\gamma)\right), \end{aligned} \quad (47)$$

respectively. Then, using [23, Th. 2.39], we have

$$\begin{aligned}\mathcal{V}_{\mathsf{W}}(\gamma) &= \lim_{m\to\infty}\frac{1}{m}\mathbb{E}\left\{\ln\det\left(\boldsymbol{I}_m+\gamma\mathsf{W}\right)\right\}\\ &= \mathcal{V}_{\bar{\mathsf{H}}\bar{\mathsf{T}}\bar{\mathsf{H}}^\dagger}(\gamma)\\ &= (\kappa_1+\kappa_2)\,\mathcal{V}_{\bar{\mathsf{T}}}(\gamma\eta) - \ln\eta + \eta - 1 \end{aligned} \quad (48)$$

where η is the solution to

$$\kappa_1+\kappa_2 = \frac{1-\eta}{1-\eta_{\bar{\mathsf{T}}}(\gamma\eta)} \quad (49)$$

which complete the proof.

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YOUNGMIN JEONG (S'09–M'15) received the B.E., M.E., and Ph.D. degrees in electronics and radio engineering from Kyung Hee University, Yongin-si, South Korea, in 2009, 2011, and 2015, respectively. He is currently a Post-Doctoral Associate jointly at Kyung Hee University and the Massachusetts Institute of Technology, Cambridge, MA, USA. His research interests include wireless communications and information theory with current emphasis on MIMO systems, network interference, vehicular communication networks, molecular communications, and quantum communication networks.

DUNG PHUONG TRINH (S'14) received the B.S. degree in electronics and telecommunications from the Hanoi University of Science and Technology, Vietnam, in 2012, and the M.E. degree in electronics and radio engineering from Kyung Hee University, South Korea, in 2014, where she is currently pursuing the Ph.D. degree in electronic engineering. She was a Visiting Scholar with the Massachusetts Institute of Technology, Cambridge, MA, USA, from 2016 to 2017. Her current research interests include wireless communications and information theory with current emphasis on MIMO systems, vehicular communication networks, and molecular communications.

HYUNDONG SHIN (S'01–M'04–SM'11) received the B.S. degree in electronics engineering from Kyung Hee University, South Korea, in 1999, and the M.S. and Ph.D. degrees in electrical engineering from Seoul National University, Seoul, South Korea, in 2001 and 2004, respectively. During his post-doctoral research at the Massachusetts Institute of Technology, Cambridge, MA, USA, from 2004 to 2006, he was with the Wireless Communication and Network Sciences Laboratory within the Laboratory for Information Decision Systems.

In 2006, he joined Kyung Hee University, where he is currently a Professor with the Department of Electronic Engineering. His research interests include wireless communications and information theory with current emphasis on MIMO communication systems, cooperative communication networks, cognitive radio and networks, network interference, vehicular communication networks, physical-layer security, location-aware radios and networks, molecular communications, and quantum information science.

Dr. Shin received the Knowledge Creation Award in the field of computer science from the Korean Ministry of Education, Science and Technology in 2010, the IEEE Communications Society's Guglielmo Marconi Prize Paper Award in 2008, and the William R. Bennett Prize Paper Award in 2012. He served as a Technical Program Co-Chair of the IEEE WCNC (PHY Track 2009) and the IEEE Globecom (the Communication Theory Symposium 2012 and the Cognitive Radio and Networks Symposium 2016). He was an Editor of the IEEE Transactions on Wireless Communications from 2007 to 2012 and the IEEE Communications Letters from 2013 to 2015.