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Analog Network-Coded Modulation With Maximum Euclidean Distance: Mapping Criterion and Constellation Design

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ABSTRACT Physical-layer network coding holds the great potential of improving the power efficiency and the spectral efficiency for the two-stage transmission scheme. The first stage is the multiple access stage, where two source nodes (SN_1 and SN_2) simultaneously transmit to the relay node (RN). The second stage is the Broadcast stage, where the RN broadcasts to the two destination nodes (DN_1 and DN_2), after a denoising-and-mapping operation. In this paper, we investigate the joint network-coded modulation design of the two stages. A universal modulation framework is built, referred to as analog network-coded modulation strategy, which is more general than the former modulation design mechanism. More explicitly, we propose a joint design criterion to guarantee the forwarding reliability at the RN. The criterion ensures that the neighboring constellation points superposed at the RN are mapped to an identical constellation point for broadcasting if their Euclidean distance (ED) is less than a given threshold. This yields a non-convex polynomial optimization problem by minimizing the average transmission power and constraining the ED among the constellation points. By solving the problem, we propose two joint modulation design algorithms, termed as the Enhanced Semidefinite Relaxation Algorithm and the Fast-Relaxation Algorithm, respectively. The two algorithms can achieve the tradeoff between the communication performance and the computation resources. As for the Fast-Relaxation Algorithm, the theoretical performance boundary is derived in detail. Simulation results demonstrate the effectiveness of both the proposed algorithms by comparing symbol error rate performance with the existing modulation design methods.

INDEX TERMS Two-way relaying, network coding, modulation coding, physical layer, constellation design, optimization methods, relaxation methods, semidefinite programming, Gaussian randomization.

I. INTRODUCTION

Network coding has emerged as a promising and powerful solution to future communication networks for its huge potential of improving the power efficiency and the spectral efficiency. The underlying core idea behind network coding is the linear combination and compression of various traffic flows with the aid of network coding mapping operation such as XOR at *RN*. At their intended destination nodes, these network coded flows can be decoded by using some prior information.

In the past decades, the promise of huge gains of network coding has attracted extensive investigations [1]–[6]. Of particular practical interest is the application in the two-way relay-aided sub-network topology. Various network coding schemes for the two-way relay-aided topology have been proposed, as discussed in [7]–[14]. One typical scheme is the three stage Decode-and-Forward two-way relaying, where *RN* can decode each symbol transmitted in different stages and then combine the symbols by applying a network coding operation such as XOR for broadcasting in the third stage. Besides, the appearance of the two-stage two-way relaying further promotes the research on the network coding. One two-stage scheme is Amplify-and-Forward, where two source nodes simultaneously transmits to *RN*, and subsequently, *RN* broadcasts the received superposed signal after direct amplification, without denoising operation [6], [12]. Another two-stage scheme is Joint Decode-and-Forward, where *RN* is able to jointly decode both symbols simultaneously transmitted from the two source nodes. This scheme requires a limitation on the data rates of the two source nodes [11]. Moreover, a two-stage scheme equipped with PNC represents a radical departure from the conventional wireless design [13], [14] for its particular denoising-and-mapping operation at *RN*. The two-way relaying equipped with PNC scheme consists of two stages, namely MA stage and BC stage. At the MA stage, SN_1 and SN_2 simultaneously transmit signals to *RN*. The simultaneously arrived signals superpose at *RN* and *RN* can map them to its constellation by performing a denoising-and-mapping operation, rather than jointly decoding. Then at the BC stage, *RN* broadcasts the network-coded signals to both two destination nodes. The destination nodes can decode the messages by using their local prior information because they also act as source nodes. By allowing SN_1 and SN_2 to transmit simultaneously to *RN* and not treating the simultaneously arrived signal as interference, this scheme can boost the system throughput by 100% [14].

As for the PNC scheme, an underlying core problem that remains to be addressed is the constellation and network coding mapping design. Traditionally, the design of the network coding mapping is independent of the constellation design. In [15], the network coding mapping method based on closest-neighbor clustering is proposed, where the conventional quadrature phase-shift keying (QPSK) modulation is used at the MA stage. After the network coding design, the constellation is designed for the BC stage by using sphere packing, matched with the designed network coding mapping. In [16], the method of using Latin Squares is introduced to design the network coding mapping, where the same modulation order is assumed at the two source nodes. As for the constellation design, the M-ary phase-shift keying (M-PSK) modulation is used at both the MA stage and the BC stage. In [17], the network coding mapping is investigated for the special linear PNC scheme. The mapping method is designed by optimizing the coefficients of the linear combinations of two uplink messages, where the pulse amplitude modulation (PAM) is used. In [18], two constellation design methods are investigated for multiple-input multiple-output systems by respectively minimizing the upper boundary on the average bit error probability and maximizing the cutoff rate under an average transmit power constraint. In [19], a constellation design method is investigated for the relay channel model with orthogonal receive components. The method is presented by maximizing the mutual information between the transmitted signals and the received signals at the destination nodes.

In contrast to the existing works which use the conventional constellations, we jointly design the network coding mapping and the constellations at all the three nodes. We build a unified modulation framework for PNC, referred to as Analog Network-coded Modulation Strategy, which is more general than the mechanism in [15]–[17]. A simpler version of the framework is introduced in [20]. In the proposed framework, an analog mapping criterion is presented. According to the mapping criterion, *RN* is stipulated to map the neighboring received superposed constellation points, among which the ED is less than a given threshold, to an identical constellation point. This unifies the up-link and down-link modulation and simultaneously guarantees the mapping reliability at *RN*. The built framework holds with no restriction on the constellations of both the MA stage and the BC stage such that the potentially huge gains of PNC are ensured. Based on this framework, the network coding mapping and constellations of all the three nodes can be jointly designed by formulating an optimization problem. The problem is formulated by setting the constellations of SN_1, SN_2, and *RN* as variables and minimizing the total average transmission power while guaranteeing a target SER and transmission rate. Solving the optimization problem can directly provide the jointly designed constellations and network coding mapping for PNC. The problem is shown to be NP-hard. Two suboptimal algorithms with polynomial-complexity, respectively referred to as Enhanced-SDR Algorithm and Fast-relaxation Algorithm, are proposed. The Enhanced-SDR algorithm can achieve better approximate performance while the Fast-relaxation Algorithm has lower complexity. Hence the two algorithms achieve the tradeoff between communication performance and computation requirements. The theoretical boundaries are provided for both the algorithms. Especially, for the Fast-relaxation Algorithm, the theoretical boundary is derived in detail in this paper. The Monte-Carlo simulation results demonstrate the effectiveness of both the two proposed algorithms. It is shown that both the algorithms can obtain a significant SER gain over a traditional one-to-one mapping scheme as well as can outperform the modulation design method in [15].

The notations used in this paper are summarized in Table 1. The remainder of this paper is organized as follows. Section II describes the system model. In Section III, the unified modulation framework for PNC and the problem formulation are proposed. The Enhanced-SDR algorithm is described in Section IV. In Section V, the Fast-relaxation Algorithm and its approximation boundary analysis are investigated in detail. In Section VI, we discuss some practical issues for the proposed modulation framework. At last, simulation results and conclusions are presented in Section VII and Section VIII, respectively.

II. SYSTEM DESCRIPTION

Consider a bi-directional relaying, where two nodes communicate with each other through a relay node (*RN*) such that the two nodes act as both source nodes and destination nodes, as shown in FIGURE 1. We denote the two nodes by $SN_1(DN_1)$ and $SN_2(DN_2)$ respectively. PNC is considered here for its huge throughput gains. The two-way relaying equipped with PNC consists of two stages. The first stage is the MA stage, where SN_1 and SN_2 simultaneously transmit

TABLE 1. Main notations.

Symbol	Notation
M	the order of constellation at SN_1
N	the order of constellation at SN_2
$\mathcal{A}$	the constellation used at SN_1
$\mathcal{B}$	the constellation used at SN_2
$\mathcal{S}$	the constellation used at RN
$\mathcal{A} \times \mathcal{B}$	Cartesian product of $\mathcal{A}$ and $\mathcal{B}$
ε_A, ε_B	squared ED, and respectively represent decoding accuracy requirement at DN_1, DN_2
ε_R	squared ED, and represents the accuracy of successful forwarding at RN
a_i	constellation point in $\mathcal{A}$
b_j	constellation point in $\mathcal{B}$
(a'_i, b'_j)	minimum distance criterion estimates at RN
$\mathcal{C}$	network coding mapper/mapping method
s_{ij}	constellation point in $\mathcal{S}$ and $s_{ij} = \mathcal{C}(a_i, b_j)$
h_1, h_2	channel coefficients
n_1, n_2, n_R	noises at DN_1, DN_2 and RN respectively
σ^2	the variance of Additive White Gaussian Noise (AWGN)
$\mathbb{C}$	complex number
$\mathbb{R}$	real number
C_M^2	combinatorial number
$\mathbb{E}(\cdot)$	expectation operator
$\mathbf{tr}(\cdot)$	trace of corresponding matrix
$\mathbf{rank}(\cdot)$	rank of corresponding matrix
$x(i)$	the ith element of x
$x(i:j)$	a sub-vector of x by extracting the ith through jth element
$[x\ y]$	a new transversal vector constructed by vectors x, y
$\lvert \cdot \rvert$	the magnitude of complex number
$x \circ y$	the Hadamard product for vectors x and y
$\mathcal{M}$	specific matrix coefficient set in the Randomization approximation procedures.
$\mathbf{H}$	general coefficient matrix
$\mathcal{N}_c(\cdot)$	complex-valued circular normal distribution
$\mathcal{N}(\cdot)$	real-valued normal distribution
randn$(M, 1)$	$M \times 1$ normal randomization vector
$\mathbf{\Lambda}$	diagonal matrix
m_1, m_2, m_3	quantity of corresponding constraints
$\cdot \succeq 0$	semidefinite matrix
N_{rand}	number of randomizations in randomization approximation algorithm

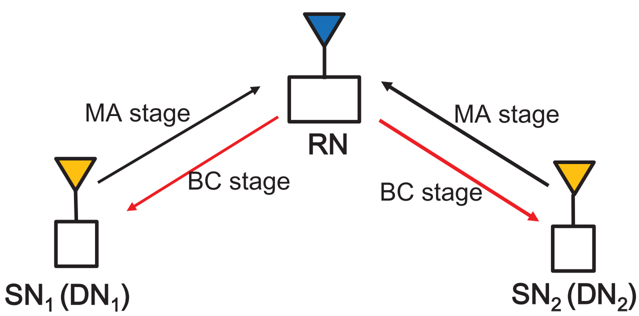

FIGURE 1. System Model. ***RN*** performs a denosing-and-mapping operation before broadcasting to ***DN***$_1$ and ***DN***$_2$.

signals to RN. The simultaneous arriving signals have additive nature at RN. Instead of decoding every arriving signal from SN_1 and SN_2, RN directly maps the superposed signal into a new constellation for broadcasting at the second stage. The second stage is the BC stage, where RN broadcasts the mapped signal to both DN_1 and DN_2. Both destination nodes can decode the intended messages by using their local signal as prior knowledge.

Let us denote the constellations used at SN_1 and SN_2 to be $\mathcal{A} = \{a_1, \ldots, a_M\}$ and $\mathcal{B} = \{b_1, \ldots, b_N\}$ respectively, where $a_i, b_j \in \mathbb{C}$, $i = 1, \ldots, M$, $j = 1, \ldots, N$, and M, N denote the orders of respective constellations. Then at the MA stage, the signal received at RN is superposed and written as

$$y_R = h_1 x_1 + h_2 x_2 + n_R, \quad x_1 \in \mathcal{A},\ x_2 \in \mathcal{B}, \tag{1}$$

where h_1 and h_2 are the corresponding channel coefficients from SN_1 and SN_2 respectively and n_R is the Additive White Gaussian Noise (AWGN) with zero mean and a variance of σ^2. We use x_1 and x_2 to denote the signals respectively transmitted from SN_1 and SN_2. Their values are respectively from the sets $\mathcal{A}$ and $\mathcal{B}$. After receiving the superposed signal y_R, RN will perform a network coding mapper $\mathcal{C}$ to map it to the constellation used at RN for broadcasting at the BC stage.

We denote the constellation used at RN to be $\mathcal{S} = \{s_{ij} | i = 1, \ldots, M,\ j = 1, \ldots, N\}$. When SN_1 and SN_2 transmit a_i and b_j respectively, RN will broadcast s_{ij} correspondingly, *i.e.*, $s_{ij} = \mathcal{C}(a_i, b_j)$. In the mapping process, only denoising detection is performed, rather than joint decoding. The signals received at DN_1 and DN_2 can be respectively written as

$$y_1 = h_1 x_R + n_1, \quad x_R \in \mathcal{S} \tag{2}$$

$$y_2 = h_2 x_R + n_2, \quad x_R \in \mathcal{S}, \tag{3}$$

where x_R denotes the signal transmitted from RN, n_1 and n_2 are AWGN with zero mean and a variance of σ^2. By utilizing their own prior information, DN_1 and DN_2 can decode the intended signal respectively from y_1 and y_2. Given the above system model, we shall present a novel Analog Network-coded Modulation Strategy. This modulation strategy can guarantee the mapping reliability at RN and achieve the joint design of the network coding mapping and the constellations $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{S}$.

III. ANALOG NETWORK-CODED MODULATION STRATEGY

In this section, we firstly build a unified modulation framework, namely the Analog Network-coded Modulation Strategy, to unify the modulation design of both the up-link and down-link transmissions. Then based on this strategy, an optimization problem is formulated to achieve the joint design of the mapping method and the constellations for all three nodes.

In the Analog Network-coded Modulation Strategy, an analog mapping criterion is presented, which guarantees the mapping reliability at RN. As shown in FIGURE 2, at RN, the neighboring received constellation points, among which ED is less than a given threshold, are stipulated to be mapped to an identical constellation for broadcasting. We set a squared ED constraint ε_R at RN to quantize the "neighborhood." That is, if ED between two arbitrary received

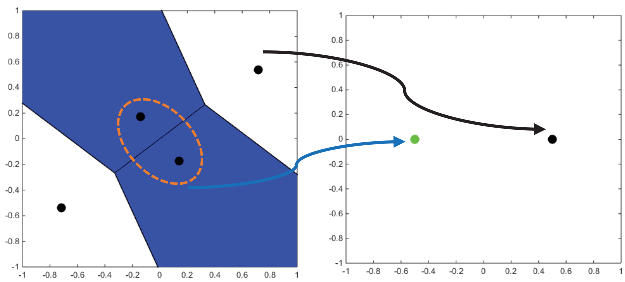

FIGURE 2. Analog Mapping Criterion at *RN*. The neighboring received constellation points, among which the ED is less than a given threshold, are stipulated to be mapped to an identical constellation point for broadcasting as depicted by Eq. (4)

superposed constellation points $h_1 a_i + h_2 b_j$ and $h_1 a_p + h_2 b_q$ is less than $\sqrt{\varepsilon_R}$, RN will forward the two points by an identical constellation point (i.e., $|s_{ij} - s_{pq}| = 0$). Conversely, if two received points are mapped to different broadcasted constellation points, their ED is required to be larger than $\sqrt{\varepsilon_R}$ to ensure the mapping reliability. These relations can be exactly described by the following formulation:

$$\left[\left|(h_1 a_i + h_2 b_j) - (h_1 a_p + h_2 b_q)\right|^2 - \varepsilon_R\right] \left|s_{ij} - s_{pq}\right|^2 \geq 0. \tag{4}$$

From Eq. (4), we can observe that the analog mapping criterion unifies the constellation design of both the up-link and down-link transmission. Meanwhile, by adjusting ε_R, we can limit the probability of failing forwarding at RN.

Besides the proposed analog mapping criterion, the necessary condition for DN_1 and DN_2 to successfully decode the intended signal is that:

$$\mathcal{C}(a_i, b_j) \neq \mathcal{C}(a_i, b_k), \quad \forall b_j \neq b_k \in \mathcal{B},\ a_i \in \mathcal{A} \tag{5}$$

$$\mathcal{C}(a_j, b_i) \neq \mathcal{C}(a_k, b_i), \quad \forall a_j \neq a_k \in \mathcal{A},\ b_i \in \mathcal{B}, \tag{6}$$

which is referred to as *Exclusive law* in this paper. We set two ED constraints ε_A and ε_B on the constellation $\mathcal{S}$ to limit the decoding reliability at DN_1 and DN_2, respectively. Then by using ε_A and ε_B, the necessary *Exclusive law* can be transformed into the following formulations:

$$\left|s_{ij} - s_{ik}\right|^2 \geq \varepsilon_A, \quad s_{ij} \neq s_{ik} \in \mathcal{S} \tag{7}$$

$$\left|s_{ji} - s_{ki}\right|^2 \geq \varepsilon_B, \quad s_{ji} \neq s_{ki} \in \mathcal{S}. \tag{8}$$

At the MA stage, to detect the received superposed signal y_R at RN, the minimum distance criterion [21] is adopted as follows:

$$(a_i', b_j') = \underset{(a_i, b_j) \in \mathcal{A} \times \mathcal{B}}{\arg\min} \left|y_R - (h_1 a_i + h_2 b_j)\right|, \tag{9}$$

where (a_i', b_j') denotes the obtained estimate of y_R. Note that the minimum distance criterion detection at RN is only used for quantize the received superposed signal rather than joint decoding. After the denoising detection, the designed network coding mapper $\mathcal{C}$ is performed to map the minimum distance estimate to constellation $\mathcal{S}$ for broadcasting. According to the proposed analog mapping criterion, the detection errors among the neighboring received points can be avoided so that the mapping reliability is guaranteed.

At the BC stage, by using its own prior knowledge a_i, DN_1 can successfully detect the desired signal through the the minimum distance criterion detection as follows:

$$\hat{b}_j' = \underset{b_j \in \mathcal{B}}{\arg\min} \left|y_1 - h_1 \mathcal{C}(a_i, b_j)\right|, \tag{10}$$

assuming the successful mapping at RN, *i.e.*, $\mathcal{C}(a_i', b_j') = \mathcal{C}(a_i, b_j)$. From Eq. (10), we can observe that by using the prior knowledge a_i, DN_1 only needs to operate the searching of N alternative constellation points in each communication round, no matter what order the constellation broadcasted by RN is. In the same way, DN_2 can detect the desired signal by using its own information b_j as follows:

$$\hat{a}_j' = \underset{a_j \in \mathcal{A}}{\arg\min} \left|y_2 - h_2 \mathcal{C}(a_i, b_j)\right|. \tag{11}$$

DN_2 only needs to search M alternative constellation points when performing the signal detection. This completes one round of communication between the two source nodes SN_1 and SN_2.

Based on the above modulation strategy, we formulate a unified optimization problem to jointly design the constellations and network coding mapper, aiming at achieving the optimal modulation design for PNC. The constraints in Eq. (4) and Eqs. (7-8) give the sufficient conditions for the modulation design to complete the overall communication between SN_1 and SN_2. Meanwhile, the ED constraints $\{\varepsilon_A,\ \varepsilon_B,\ \varepsilon_R\}$ can limit the error rate of signal detection at DN_1 and DN_2, as well as RN. It is well known that for a relay system, less average transmission power implies better constellation design when the modulation orders and the minimum ED of the constellations are both given. Following this rule, we are led to define the merit function $f : \mathcal{A}, \mathcal{B}, \mathcal{S} \rightarrow \mathbb{R}$ by the average transmission power:

$$f(A, B, S) = \frac{1}{M} A^H A + \frac{1}{N} B^H B + \frac{1}{MN} \sum_i \sum_j \left|s_{ij}\right|^2, \tag{12}$$

where we define $A \in \mathbb{C}^M = [\ldots, a_i, \ldots]_{a_i \in \mathcal{A}}$, $B \in \mathbb{C}^N = [\ldots, b_j, \ldots]_{b_j \in \mathcal{B}}$, and $S \in \mathbb{C}^{M \times N} = [s_{ij}]_{s_{ij} \in \mathcal{S}}$. Then the problem to design the optimal constellations $\{\mathcal{A},\ \mathcal{B},\ \mathcal{S}\}$ is turned into solving the novel optimization problem

$$(A^*, B^*, S^*) = \underset{\substack{A \in \mathbb{C}^M,\ B \in \mathbb{C}^N, \\ S \in \mathbb{C}^{M \times N}}}{\arg\min} f(A, B, S), \tag{13}$$

under constraints in Eq. (4) and Eqs. (7-8). Solving the problem can provide the optimal constellation design for the two-way relaying equipped with PNC, which minimizes transmission power whereas achieving the target SER and transmission rate.

Next, we focus on solving the formulated optimization problem. Observe that parts of the fourth order constraints in Eq. (4) are redundant when we simultaneously consider the constraints in Eqs. (7-8), where the exclusive law indicates that $s_{ij} \neq s_{ik}$ for the same i, $i = 1, \ldots, M$ and $s_{ki} \neq s_{ji}$ for the same $i = 1, \ldots, N$. Based on this observation, the redundant fourth order constraints can be transformed into equivalent quadratic constraints. By this equivalent transformation, the original optimization problem is rewritten as

$$\begin{aligned}
\min_{A,B,S}\ & \frac{1}{M}A^H A + \frac{1}{N}B^H B + \frac{1}{MN}\sum_i\sum_j |s_{ij}|^2 \\
\text{s.t. } & |s_{ij} - s_{ik}|^2 \geq \varepsilon_A, \quad \forall s_{ij}, s_{ik} \in \mathcal{S} \\
& |s_{ji} - s_{ki}|^2 \geq \varepsilon_B, \quad \forall s_{ji}, s_{ki} \in \mathcal{S} \\
& |h_1(a_j - a_k)|^2 \geq \varepsilon_R, \quad \forall a_j, a_k \in \mathcal{A} \\
& |h_2(b_j - b_k)|^2 \geq \varepsilon_R, \quad \forall b_j, b_k \in \mathcal{B} \\
& \left[|(h_1a_i + h_2b_j) - (h_1a_p + h_2b_q)|^2 - \varepsilon_R\right] |s_{ij} - s_{pq}|^2 \geq 0 \\
& a_i, a_p \in \mathcal{A}, \quad b_j, b_q \in \mathcal{B}, \ s_{ij}, s_{pq} \in \mathcal{S}, \ i \neq p, \ j \neq q.
\end{aligned} \tag{14}$$

It can be observed from problem (14) that the first two groups of constraints only limit constellations $\mathcal{S}$, likely, the third and the fourth groups only limit $\mathcal{A}$ and $\mathcal{B}$ respectively, and the last group jointly constrains $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{S}$. Furthermore, we can observe from the last group of constraints that only such points s_{ij} and s_{pq} that simultaneously satisfy $i \neq p$ and $j \neq q$ can be merged as one point (i.e., $s_{ij} = s_{pq}$). Otherwise, huge failing decoding will happen at DN_1 and DN_2 because of violating the Exclusive law. Those will be helpful in developing the related Gaussian randomization approximation method for the proposed algorithms later.

Problem (14) is non-convex since all the constraints are nonconvex. In the subsequent two sections, for achieving the tradeoff between communication performance and computation resources, two different approximation algorithms are developed, referred to as Enhanced-SDR Algorithm and Fast-relaxation Algorithm respectively. Both two algorithms are with polynomial complexity. The Enhanced-SDR Algorithm can achieve better approximation performance while the Fast-relaxation Algorithm has lower complexity. The detailed comparison of the performance will be shown in Section VII.

IV. JOINT CONSTELLATION DESIGN BY THE ENHANCED-SDR ALGORITHM

In this section, the Enhanced-SDR Algorithm is developed to jointly design the mapping method and constellations used at all three nodes. This algorithm firstly relaxes the original problem into a semidefinite programming problem. Then a Gaussian randomization procedure is developed to extract approximate solutions to the original problem from the solutions of the relaxed semidefinite programming problem. At last, the constellations and network coding mapping method can be directly derived from the approximate solution.

For the sake of clarity, by defining

$$\begin{aligned}
z_1^T &= [\frac{1}{\sqrt{M}}A^T \ \frac{1}{\sqrt{N}}B^T] \in \mathbb{C}^{M+N} \\
z_2^T &= \frac{1}{\sqrt{MN}}[s_{11}\ s_{12}\ \ldots\ s_{MN}] \in \mathbb{C}^{1\times MN},
\end{aligned} \tag{15}$$

problem (14) can be rewritten as

$$\begin{aligned}
\min_{z_1,z_2}\ & z_1^H z_1 + z_2^H z_2 \\
\text{s.t. } & z_1^H \mathbf{H}_{1i} z_1 \geq 1, \quad i = 1, 2, \ldots, m_1 \\
& z_2^H \mathbf{H}_{2i} z_2 \geq 1, \quad i = 1, 2, \ldots, m_2 \\
& (z_1^H \mathbf{D}_i z_1 - 1)(z_2^H \mathbf{E}_i z_2) \geq 0, \quad i = 1, 2, \ldots, m_3,
\end{aligned} \tag{16}$$

where m_1, m_2, and m_3 denote the respective number of constraints, $\{\mathbf{H}_{1i}\}_{i=1}^{m_1}$, $\{\mathbf{H}_{2i}\}_{i=1}^{m_2}$, $\{\mathbf{D}_i\}_{i=1}^{m_3}$, and $\{\mathbf{E}_i\}_{i=1}^{m_3}$ denote the corresponding coefficient matrices. It can be verified that all the coefficient matrices are rank-one positive semidefinite, and

$$\begin{aligned}
m_1 &= C_M^2 + C_N^2 \\
m_2 &= MC_N^2 + NC_M^2 \\
m_3 &= 2C_M^2 C_N^2.
\end{aligned} \tag{17}$$

A. ENHANCED-SDR RELAXATION

In this subsection, Enhanced-SDR relaxation is developed to relax problem (16) into a solvable semidefinite programming problem. Solving the semidefinite programming problem can give us a lower boundray on the average transmission power, and also a group of matrix solutions, from which the constellations and network coding mapping method can be extracted.

Proposition 1: Problem (16) can be relaxed into the following semidefinite programming problem:

$$\begin{aligned}
\min_{Z_1,Z_2,Z_3,z_3}\ & \mathbf{tr}(Z_1) + \mathbf{tr}(Z_2) \\
\text{s.t. } & \mathbf{tr}(\mathbf{H}_{1i} Z_1) \geq 1, \quad i = 1, 2, \ldots, m_1 \\
& \mathbf{tr}(\mathbf{H}_{2i} Z_2) \geq 1, \quad i = 1, 2, \ldots, m_2 \\
& \mathbf{tr}(\mathbf{D}_j Z_1) - c_j z_3 = 1, \quad \mathbf{tr}(\mathbf{E}_j Z_2) - d_j z_3 = 0 \\
& \mathbf{tr}(\mathbf{F}_j Z_3) \geq 0, \quad j = 1, 2, \ldots, m_3 \\
& Z_1 \succeq 0, \quad Z_2 \succeq 0 \\
& \begin{bmatrix} 1 & z_3^H \\ z_3 & Z_3 \end{bmatrix} \succeq 0.
\end{aligned} \tag{18}$$

Proof: The main idea of the proof is to transform problem (16) into a quadratically constrained quadratic programming problem by introducing new variables and then apply the basic idea of SDR.

Firstly, by defining new variables

$$p_j = z_1^H \mathbf{D}_j z_1 - 1 \tag{19}$$
$$q_j = z_2^H \mathbf{E}_j z_2, \tag{20}$$

problem (16) can be equivalently rewritten as

$$\begin{aligned}
\min_{z_1,z_2,z_3} \quad & z_1^H z_1 + z_2^H z_2 \\
\text{s.t.} \quad & z_1^H \mathbf{H}_{1i} z_1 \geq 1, \quad i = 1, 2, \ldots, m_1 \\
& z_2^H \mathbf{H}_{2i} z_2 \geq 1, \quad i = 1, 2, \ldots, m_2 \\
& z_1^H \mathbf{D}_j z_1 - c_j^T z_3 = 1, \quad z_2^H \mathbf{E}_j z_2 - d_j^T z_3 = 0 \\
& z_3{}^T \mathbf{F}_j z_3 \geq 0, \quad j = 1, 2, \ldots, m_3,
\end{aligned} \tag{21}$$

where we define $z_3 \in \mathbb{R}^{2m_3} = [p_1 \ \ldots \ p_{m_3} \ q_1 \ \ldots q_{m_3}]$. By the introduction of series of new variables, the original problem (16) is transformed equivalently into a quadratically constrained quadratic programming problem so that semidefinite relaxation (SDR) can be applied. Problem (21) is nonconvex and NP-hard [22]. Hence we can conclude that the equivalent problem (16) is also NP-hard.

Observe that for arbitrary vector x and coefficient matrix $\mathbf{H}$, we have

$$x^H \mathbf{H} x = \mathbf{tr}(x^H \mathbf{H} x) = \mathbf{tr}(\mathbf{H} x x^H). \tag{22}$$

Note that xx^H is equivalent to a rank-one hermitian positive semidefinite matrix. Thus, by introducing new variables $\{Z_i = z_i z_i^H\}_{i=1}^3$, problem (21) can be transformed into the following equivalent formulation:

$$\begin{aligned}
\min_{Z_1,Z_2,Z_3,z_3} \quad & \mathbf{tr}(Z_1) + \mathbf{tr}(Z_2) \\
\text{s.t.} \quad & \mathbf{tr}(\mathbf{H}_{1i} Z_1) \geq 1, \quad i = 1, 2, \ldots, m_1 \\
& \mathbf{tr}(\mathbf{H}_{2i} Z_2) \geq 1, \quad i = 1, 2, \ldots, m_2 \\
& \mathbf{tr}(\mathbf{D}_j Z_1) - c_j z_3 = 1, \quad \mathbf{tr}(\mathbf{E}_j Z_2) - d_j z_3 = 0 \\
& \mathbf{tr}(\mathbf{F}_j Z_3) \geq 0, \quad j = 1, 2, \ldots, m_3 \\
& Z_1 \succeq 0, \quad Z_2 \succeq 0 \\
& Z_3 = z_3 z_3^H \\
& \mathbf{rank}(Z_1) = 1, \quad \mathbf{rank}(Z_2) = 1.
\end{aligned} \tag{23}$$

By applying SDR, we relax the nonconvex rank constraints $\mathbf{rank}(Z_1) = 1$ and $\mathbf{rank}(Z_2) = 1$, and relax $Z_3 = z_3 z_3^H$ into a convex positive semidefinite constraint $Z_3 \succeq z_3 z_3^H$. Noting that $Z_3 \succeq z_3 z_3^H$ is equivalent (Schur complement) to

$$\begin{bmatrix} 1 & z_3^H \\ z_3 & Z_3 \end{bmatrix} \succeq 0, \tag{24}$$

we obtain the relaxed version of problem (16) as problem (18). This completes the proof. ■

Problem (18) consists of one linear objective function, $m_1 + m_2 + 3m_3$ linear inequality constraints, and three semidefinite positive constraints. Hence it is a standard semidefinite programming problem. Semidefinite programming problem can now be solved efficiently and reliably to any arbitrary accuracy with polynomial complexity [22]. The max-size of the three matrix variables in the semidefinite programming problem is $2m_3 \times 2m_3$, and the number of the linear inequality constraints is $m_1 + m_2 + 3m_3$. As discussed in [23], interior point methods will take $O(\sqrt{m_3}\log(1/\epsilon))$ iterations, each requiring at most $O({m_3}^6)$ arithmetic operations. In contrast, the complexity of exhaustive search is $O(2^{2(M+N+MN)})$. The proposed algorithm reduces the computational complexity significantly.

B. RANDOMIZATION APPROXIMATION

This subsection develops a Gaussian randomization method for the Enhanced-SDR Algorithm to achieve the joint design of the constellations and network coding mapping method by utilizing the three matrix solutions obtained from *Proposition 1*. The Gaussian randomization method is described in detail in Algorithm 1.

When we obtain the solutions $\tilde{z}_1$, $\tilde{z}_2$ from Algorithm 1, the optimized constellations can be derived by

$$\mathcal{A} = \{a_i | a_i = \sqrt{M}\tilde{z}_1(i)\}_{i=1}^M \tag{27}$$

$$\mathcal{B} = \{b_j | b_j = \sqrt{N}\tilde{z}_1(j)\}_{j=M+1}^{M+N} \tag{28}$$

$$\mathcal{S} = \{s_{ij} | s_{ij} = \sqrt{MN}\tilde{z}_2((i-1)N + j)\}_{i=1,j=1}^{M,N}. \tag{29}$$

Correspondingly, the network coding mapper can be extracted from the constellations as

$$s_{ij} = \mathcal{C}(a_i, b_j). \tag{30}$$

That is when SN_1 and SN_2 transmit a_i, b_j respectively, RN will broadcast s_{ij}.

As shown in [24], a complex-valued SDR can achieve an approximation boundary of $8m$, where m denotes the number of quadratic constraints. By a similar procedure, a bound of $\tilde{\gamma} = 4(M^2N^2 + M^2 + N^2 - MN - M - N)$ can be obtained for the Enhanced-SDR algorithm, where we plug the values of m_1, m_2, and m_3 in Eq. (17).

Taking the computation requirements and the delay of the communication into consideration, we shall propose another novel algorithm, which has much lower computation complexity, to achieve the tradeoff between the communication performance and computation resource in the next section. Detailed performance boundary analysis will be presented for the algorithm.

V. JOINT CONSTELLATION DESIGN BY THE FAST-RELAXATION ALGORITHM

In this section, the Fast-relaxation Algorithm is proposed, which has much lower complexity and can also achieve the joint design of the constellations for the physical-layer network-coded two-way relaying. This algorithm has the solving process similar to the Enhanced-SDR Algorithm. The difference is that the Fast-relaxation Algorithm directly relaxes the original problem, unlike the Enhanced-SDR introducing some new variables before relaxing. Meanwhile, the Fast-relaxation Algorithm takes advantage of the relations among the coefficient matrices to eliminate the redundant constraints in some certain subset of the problem domain. Due to those differences, the Fast-relaxation Algorithm has a significantly lower computation complexity than the Enhanced-SDR algorithm. Theoretical approximation ratio of the Fast-relaxation Algorithm is also presented in this section.

Algorithm 1 Gaussian Randomization Procedure of the Enhanced-SDR Algorithm

Input: Enhanced-SDR optimal solutions Z_1^*, Z_2^*, the number of randomizations N_{rand}.
for $l = 1, 2, \ldots, N_{rand}$ **do**
  Generate $\xi_2 \sim \mathcal{N}_c(0, Z_2^*)$.
  Construct

$$z_2^l = \frac{\xi_2}{\min_{1\le i\le m_2}\sqrt{\xi_2^H \mathbf{H}_{2i}\xi_2}}. \tag{25}$$

  for any $i, j = 1, \ldots, MN, i \neq j$ **do**
    Compute the squared distance

$$d_1^2 = |\sqrt{MN}(z_2^l(i) - z_2^l(j))|^2.$$

    if $d_1^2 < \min\{\varepsilon_A, \varepsilon_B\}$ **then**
      Construct $z_2^l(i) = z_2^l(j)$.[1]
      if the exclusive law is not satisfied **then**
        Eliminate above construction.
      end if
    end if
  end for
  for any $i, j = 1, \ldots, MN, i \neq j$ **do**
    Construct $z_2^l(i) = z_2^l(j)$.
    if the exclusive law is not satisfied **then**
      Eliminate above construction.
    end if
  end for
  Generate $\xi_1 \in \mathbb{C}^{M+N} \sim \mathcal{N}_c(0, Z_1^*)$.
  Construct

$$z_1^l = \frac{\xi_1}{\min\{\xi_1{}^H \mathcal{M}\xi_1\}}.^2 \tag{26}$$

end for
Determine $l^* = \underset{l=1,\ldots,N_{rand}}{\arg\min}\ (z_1^l)^H z_1^l + (z_2^l)^H z_2^l$.
for $k = 1, 2, \ldots, K$ **do**
  Compute the descent direction:

$$\Delta z_1 = -2z_1^{l^*}, \quad \Delta z_2 = -2z_2^{l^*}.$$

  Choose the step size:

$$t_1 = \delta_1|\text{randn}(M+N, 1)|, \quad t_2 = \delta_2.$$

  Generate:

$$z_1^k = z_1^{l^*} + t_1 \circ (\Delta z_1), \quad z_2^k = z_2^{l^*} + t_2\Delta z_2$$

  if z_1^k, z_2^k are feasible **then**
    if $(z_1^k)^H z_1^k + (z_2^k)^H z_2^k < (z_1^{l^*})^H z_1^{l^*} + (z_2^l)^H z_2^{l^*}$ **then**
      Update $z_1^{l^*} = z_1^k, z_2^{l^*} = z_2^k$.
    end
  end
end
Output The approximate solutions $\tilde{z}_1 = z_1^{l^*}, \tilde{z}_2 = z_2^{l^*}$ to (16).

[1]Note that this construction means there is a certain i that makes $(z_2^l)^H \mathbf{E}_i z_2^l = 0$ in problem (16).

[2]$\mathcal{M}$ contains all the coefficient matrices $\mathbf{H}_{1i}$, $i = 1, \ldots, m_1$ and such part of coefficient matrices $\mathbf{D}_i$ that corresponds to $(z_2^l)^H \mathbf{E}_i z_2^l \neq 0$ in problem (16).

In order to describe the algorithm conveniently, let f_i, g_i, h_i, and d_k denote the corresponding coefficient vectors in problem (14), and take that

$$\begin{aligned} z_1 &= \frac{1}{\sqrt{M}}A \in \mathbb{C}^M \\ z_2 &= \frac{1}{\sqrt{N}}B \in \mathbb{C}^N \\ z_3^T &= \frac{1}{\sqrt{MN}}[s_{11}\ s_{12}\ \ldots\ s_{MN}] \in \mathbb{C}^{1\times MN}. \end{aligned} \tag{31}$$

Then by defining $\{F_i := f_i f_i^H\}_{i=1}^{m_1}$, $\{G_j := g_j g_j^H\}_{j=1}^{m_2}$, $\{H_i := h_i h_i^H\}_{i=1}^{m_3}$, and $\{D_k := d_k d_k^H\}_{k=1}^{2m_1m_2}$, the original problem (14) can be equivalently rewritten as

$$\begin{aligned} \min_{z_1,z_2,z_3}\ & z_1^H z_1 + z_2^H z_2 + z_3^H z_3 \\ \text{s.t. } & z_1^H F_i z_1 \ge 1, \quad i = 1, 2, \ldots, m_1 \\ & z_2^H G_j z_2 \ge 1, \quad j = 1, 2, \ldots, m_2 \\ & z_3^H H_i z_3 \ge 1, \quad i = 1, 2, \ldots, m_3 \\ & \left(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}^H \begin{bmatrix} F_i & \pm f_i g_j^H \\ \pm g_j f_i^H & G_j \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} - 1\right)(z_3^H D_k z_3) \ge 0 \\ & i = 1, \ldots, m_1,\ j=1, \ldots, m_2,\ k=1, \ldots, 2m_1m_2. \end{aligned} \tag{32}$$

where we still use m_1, m_2 and m_3 to denote the respective number of constraints. It can be checked that in this section,

$$\begin{aligned} m_1 &= C_M^2 \\ m_2 &= C_N^2 \\ m_3 &= MC_N^2 + NC_M^2. \end{aligned} \tag{33}$$

Note that in this section, the values of m_1, m_2, and m_3 are different from those in Section IV. All the coefficient matrices $\{F_i\}_{i=1}^{m_1}$, $\{G_j\}_{j=1}^{m_2}$, $\{H_i\}_{i=1}^{m_3}$, and $\{D_k\}_{k=1}^{2m_1m_2}$ are still positive semidefinite and rank-one.

A. FAST-RELAXATION AND RANDOMIZATION APPROXIMATION

This subsection relaxes problem (32) directly into a new solvable semidefinite programming problem by taking good advantage of the coefficient relations. Then based on the matrix solutions of the obtained relaxed problem, a heuristic randomization approximation procedure is developed to jointly design the constellations and network coding mapping method.

Proposition 2: Problem (32) can be relaxed into the the following semidefinite programming problem:

$$\begin{aligned} \min\ & \mathbf{tr}(Z_1) + \mathbf{tr}(Z_2) + \mathbf{tr}(Z_3) \\ \text{s.t. } & \mathbf{tr}(F_i Z_1) \ge 1, \quad i = 1, 2, \ldots, m_1 \\ & \mathbf{tr}(G_j Z_2) \ge 1, \quad j = 1, 2, \ldots, m_2 \\ & \mathbf{tr}(H_i Z_3) \ge 1, \quad i = 1, 2, \ldots, m_3 \\ & Z_1 \succeq 0, \quad Z_2 \succeq 0,\ Z_3 \succeq 0. \end{aligned} \tag{34}$$

Proof: The proof begins with changing the optimization variables in problem (32) equivalently to $\{Z_i := z_i z_i^H\}_{i=1}^3$ and $Z_4 := z_2 z_1^H$ as Eq. (22). Then problem (32) can be transformed into the following equivalent formulation:

$$\begin{aligned}
\min\ & \mathbf{tr}(Z_1) + \mathbf{tr}(Z_2) + \mathbf{tr}(Z_3) \\
\text{s.t.}\ & \mathbf{tr}(F_i Z_1) \geq 1, \quad i = 1, 2, \ldots, m_1 \\
& \mathbf{tr}(G_j Z_2) \geq 1, \quad j = 1, 2, \ldots, m_2 \\
& \mathbf{tr}(H_i Z_3) \geq 1, \quad i = 1, 2, \ldots, m_3 \\
& \left[\mathbf{tr}(F_i Z_1) \pm 2\mathbf{tr}(f_i g_j^H Z_4) + \mathbf{tr}(G_j Z_2) - 1\right] \\
& \cdot \mathbf{tr}(D_k Z_3) \geq 0 \\
& i = 1, 2, \ldots, m_1, \quad j = 1, 2, \ldots, m_2, \\
& k = 1, 2, \ldots, 2m_1 m_2 \\
& \mathbf{rank}(Z_1) = 1, \quad \mathbf{rank}(Z_2) = 1, \ \mathbf{rank}(Z_3) = 1 \\
& Z_1 \succeq 0, \quad Z_2 \succeq 0, \ Z_3 \succeq 0 \\
& Z_4 = z_2 z_1^H. \qquad (35)
\end{aligned}$$

Similar to the discussion in subsection of Enhanced-SDR relaxation, by directly relaxing the rank constraints and $Z_4 = z_2 z_1^H$, we can relax problem (32) into the following Fast-relaxation problem:

$$\begin{aligned}
\min\ & \mathbf{tr}(Z_1) + \mathbf{tr}(Z_2) + \mathbf{tr}(Z_3) \\
\text{s.t.}\ & \mathbf{tr}(F_i Z_1) \geq 1, \quad i = 1, 2, \ldots, m_1 \\
& \mathbf{tr}(G_j Z_2) \geq 1, \quad j = 1, 2, \ldots, m_2 \\
& \mathbf{tr}(H_i Z_3) \geq 1, \quad i = 1, 2, \ldots, m_3 \\
& \left[\mathbf{tr}(F_i Z_1) \pm 2\mathbf{tr}(f_i g_j^H Z_4) + \mathbf{tr}(G_j Z_2) - 1\right] \\
& \cdot \mathbf{tr}(D_k Z_3) \geq 0 \\
& i = 1, 2, \ldots, m_1, \quad j = 1, 2, \ldots, m_2, \\
& k = 1, 2, \ldots, 2m_1 m_2 \\
& Z_1 \succeq 0, \quad Z_2 \succeq 0, \ Z_3 \succeq 0. \qquad (36)
\end{aligned}$$

This relaxation method is termed as Fast-relaxation because problem (36) has much lower computation complexity than the Enhanced-SDR relaxation problem (18). Although problem (36) is nonconvex due to the existence of the $2m_1m_2$ nonconvex quadratic constraints, we can quickly obtain the optimal solutions by taking full advantage of the relations among the coefficient matrices. These relations show that the nonconvex quadratic constraints in problem (36) are redundant in some feasible regions, which exactly be proved to contain the optimal solutions.

Based on the observation above, we can prove that problem (36) has the same optimal value with the semidefinite programming problem (34). Let v_r and $\hat{v}_r$ respectively denote the optimal value of problems (36) and (34). Noting that problem (34) is the further relaxation of problem (36), we thus have

$$\hat{v}_r \leq v_r. \qquad (37)$$

We denote the optimal solutions to problem (34) by Z_1^*, Z_2^*, and Z_3^*. Then we have

$$\begin{aligned}
& \mathbf{tr}(F_i Z_1^*) \geq 1 \\
& \mathbf{tr}(G_j Z_2^*) \geq 1 \\
& \hat{v}_r = \mathbf{tr}(Z_1^*) + \mathbf{tr}(Z_2^*) + \mathbf{tr}(Z_3^*). \qquad (38)
\end{aligned}$$

We may as well suppose that $Z_4^* = \mathbf{0}$. Then it can be easily checked that in problem (36), we always have

$$\mathbf{tr}(F_i Z_1^*) \pm 2\mathbf{tr}(f_i g_j^H Z_4^*) + \mathbf{tr}(G_j Z_2^*) \geq 1. \qquad (39)$$

Meanwhile, since $Z_3^* \succeq 0$ and $\{D_k = d_k d_k^H\}_{k=1}^{2m_1m_2}$, it follows that

$$\mathbf{tr}(D_k Z_3^*) = \mathbf{tr}(d_k^H Z_3^* d_k) \geq 0. \qquad (40)$$

Hence the quadratic constraints always hold in problem (36). Considering that Z_1^*, Z_2^*, Z_3^* simultaneously satisfy all the other constraints in problem (36), we can conclude that Z_1^*, Z_2^*, Z_3^*, and $Z_4^* = \mathbf{0}$ are exactly a group of feasible solutions to problem (36). Hence, we have

$$v_r \leq \mathbf{tr}(Z_1^*) + \mathbf{tr}(Z_2^*) + \mathbf{tr}(Z_3^*). \qquad (41)$$

Finally, combining Eqs. (37) and (41), we can conclude that

$$v_r = \hat{v}_r. \qquad (42)$$

The proof is completed. ■

Problem (34) consists of a linear objective function, $m_1+m_2+m_3$ linear inequality constraints, and three positive-semidefinite constraints. Hence it belongs to a standard semidefinite programming problem. Since semidefinite programming problem (34) has much less number of constraints than semidefinite programming problem (18) obtained by Enhanced-SDR relaxation, problem (34) has much lower computation complexity as discussed before. TABLE 2 shows the computation complexity that the algorithms take by using interior point methods, where we respectively plug the values of m_1, m_2, and m_3 in Eqs. (17) and (33).

TABLE 2. Complexity comparison of the algorithms.

Algorithms	Exhaustive Search	Enhanced-SDR Algorithm	Fast-Relaxation Algorithm
Computation Complexity	exponential complexity $O(2^{2(M+N+MN)})$	a worst complexity of $O(MN\log(1/\epsilon))$ iterations, with each iteration taking at most $O(M^{12}N^{12})$ arithmetic operations	a worst complexity of $O(\sqrt{MN}\log(1/\epsilon))$ iterations, with each iteration taking at most $O(M^6N^6)$ arithmetic operations.

Similar to the Enhanced-SDR Algorithm, a heuristic randomization procedure is developed to design the optimized constellations and network coding mapping method for PNC by using the obtained optimal solutions Z_1^*, Z_2^*, Z_3^*, Z_4^*. The details of the randomization procedure to extract approximate solutions to problem (32) are depicted in Algorithm 2.

Algorithm 2 Gaussian Randomization Procedure of the Fast-Relaxation Algorithm

Input: Fast-relaxation optimal solutions Z_1^*, Z_2^*, Z_3^*, the number of randomizations N_{rand}, the number of iterations K, the step sizes of the iterations δ_1, δ_2.

for $l = 1, 2, \ldots, N_{rand}$ **do**

Generate $\xi_3 \sim \mathcal{N}_c(0, Z_3^*)$.

Construct

$$z_3^l = \frac{\xi_3}{\min_{1\le i\le m_3}\sqrt{\xi_3^H H_i \xi_3}}. \quad (43)$$

for any $i, j = 1, \ldots, MN, \ i \neq j$ **do**

Compute the squared distance

$$d_1^2 = |\sqrt{MN}(z_3^l(i) - z_3^l(j))|^2.$$

if $d_1^2 < \min\{\varepsilon_A, \ \varepsilon_B\}$ **then**

Construct $z_3^l(i) = z_3^l(j)$.

if the exclusive law is not satisfied **then**

Eliminate above construction.

end if

end if

end for

Generate $\hat{\xi} \sim \mathcal{N}_c(0, \widehat{Z})$, where $\widehat{Z} = \begin{bmatrix} Z_1^* & \mathbf{0} \\ \mathbf{0} & Z_2^* \end{bmatrix}$.

Construct

$$\hat{z}^l = \frac{\hat{\xi}}{\min\{\hat{\xi}^H \mathcal{M} \hat{\xi}\}}.^{3} \quad (44)$$

end for

Determine $l^* = \arg\min_{l=1,\ldots,N_{rand}} (\hat{z}^l)^H \hat{z}^l + (z_3^l)^H z_3^l$.

for $k = 1, 2, \ldots, K$ **do**

Compute the descent direction:

$\Delta\hat{z} = -2\hat{z}^{l^*}, \Delta z_3 = -2z_3^{l^*}$.

Choose the step size:

$t_1 = \delta_1|\text{randn}(M+N, 1)|, t_2 = \delta_2$.

Generate:

$\hat{z}^k = \hat{z}^{l^*} + t_1 \circ (\Delta\hat{z}), z_3^k = z_3^{l^*} + t_2\Delta z_3$

if $\hat{z}^k, \ z_3^k$ are feasible **then**

if $(\hat{z}^k)^H\hat{z}^k + (z_3^k)^H z_3^k < (\hat{z}^{l^*})^H\hat{z}^{l^*} + (z_3^{l^*})^H z_3^{l^*}$ **then**

Update $\hat{z}^{l^*} = \hat{z}^k, z_3^{l^*} = z_3^k$.

end

end

end

Output The approximate solutions $\tilde{z}_1 = \hat{z}^{l^*}, \tilde{z}_2 = z_3^{l^*}$ to problem (32).

Upon obtaining the solutions $\tilde{z}_1, \ \tilde{z}_2$, we can also derive the optimized constellations and mapping method by an operation similar to the formulations in Eqs. (27-30).

[3] $\mathcal{M}$ has the meaning similar to that in footnote.[2]

B. PERFORMANCE ANALYSIS

In order to evaluate the performance of the novel Fast-relaxation Algorithm, this subsection develops the detailed theoretical approximation ratio analysis by using the constructed solutions $\tilde{z}_1, \ \tilde{z}_2$ as above. We firstly develop two Lemmas. Then based on the two Lemmas, the theoretical bound is developed. The following *Lemma 1* will estimate the left-tail of the distribution of a convex quadratic form of a complex-valued circular normal random vector.

Lemma 1: Let $H \in \mathbb{C}^{n\times n}$, $X \in \mathbb{C}^{n\times n}$ be two arbitrary Hermitian positive semidefinite matrices (i.e., $H \succeq 0, X \succeq 0$) and $\mathbf{rank}(H) = 1$. Suppose $\xi \in \mathbb{C}^n$ is a random vector generated from the complex-valued normal distribution $\mathcal{N}_c(0, X)$. Then, for any $\gamma > 0$,

$$\Pr\{\xi^H H\xi < \gamma\mathbb{E}(\xi^H H\xi)\} \le \gamma. \quad (45)$$

Proof: See Appendix A. ■

Based on *Lemma 1*, we develop the following *Lemma 2* to estimate the right-tail of the distribution of $\tilde{z}_1, \ \tilde{z}_2$ constructed by Eqs. (43) and (44).

Lemma 2: Let $\Omega_i, \ X$ be arbitrary Hermitian positive semidefinite matrices that satisfy $\mathbf{tr}(\Omega_i X) \ \ge \ 1$ and $\mathbf{rank}(\Omega_i) = 1$ for any $i = 1, 2, \ldots, n$. Then for any $\gamma > 0$ and $\mu > 0$, if $\xi \sim \mathcal{N}_c(0, X)$, the following inequality holds:

$$\Pr\{\min_{1\le i\le n} \xi^H \Omega_i \xi \ge \gamma, \ ||\xi||^2 \le \mu\mathbf{tr}(X)\} \ge 1 - n\gamma - \frac{1}{\mu}. \quad (46)$$

Proof: See Appendix B. ■

Then by applying *Lemma 2*, we can now bound the performance of the Fast-relaxation Algorithm.

Theorem 1: For problem (32), suboptimal solutions can be obtained by the Fast-relaxation Algorithm with the boundary $\frac{3\pi}{2} \cdot \max\{m_1 + m_2 + 2m_1m_2, m_3\}$.

Proof: The proof begins with the knowledge that

$$\mathbf{tr}(F_i Z_1^*) \ge 1, \quad i = 1, \ldots, m_1 \quad (47)$$

$$\mathbf{tr}(G_j Z_2^*) \ge 1, \quad j = 1, \ldots, m_2, \quad (48)$$

where $Z_1^*, \ Z_2^*$ are the optimal solutions to problem (36). Then it can be verified that

$$\mathbf{tr}(\begin{bmatrix} F_i & \pm f_i g_j^H \\ \pm g_j f_i^H & G_j \end{bmatrix}\begin{bmatrix} Z_1^* & \mathbf{0} \\ \mathbf{0} & Z_2^* \end{bmatrix}) \ge 1, \quad \forall i, j. \quad (49)$$

According to *Lemma 2*, if letting $\hat{\xi} \sim \mathcal{N}_c(0, \widehat{Z})$, where we define

$$\widehat{Z} = \begin{bmatrix} Z_1^* & \mathbf{0} \\ \mathbf{0} & Z_2^* \end{bmatrix}, \quad (50)$$

we can get that

$$\Pr\{\min \hat{\xi}^H \mathcal{M}\hat{\xi} \ge \gamma, ||\hat{\xi}||^2 \le \mu\mathbf{tr}(\widehat{Z})\} \ge 1 - n\gamma - \frac{1}{\mu}. \quad (51)$$

In Eq. (51), n denotes the number of the coefficient matrices in $\mathcal{M}$. It can be checked that

$$n \leq m_1 + m_2 + 2m_1m_2. \quad (52)$$

We choose that

$$u = 3, \quad \gamma = \frac{3}{\pi n}(1 - \frac{1}{3}) = \frac{2}{\pi n}.$$

Plugging these choices of γ and μ into Eq. (46), we see that there is a positive probability (independent of problem size) of at least

$$1 - \gamma n - \frac{1}{\mu} = 1 - \frac{2}{\pi} - \frac{1}{3} = 0.03\ldots$$

that $\hat{\xi}$ generated by $\hat{\xi} \sim \mathcal{N}_c(0, \widehat{Z})$ satisfies

$$\min \hat{\xi}^H \mathcal{M}\hat{\xi} \geq \frac{2}{\pi n} \text{ and } ||\hat{\xi}||^2 \leq 3\mathbf{tr}(\widehat{Z}). \quad (53)$$

Likewise, we can conclude that there is also a positive probability that ξ_3 generated by $\xi_3 \sim \mathcal{N}_c(0, Z_3^*)$ satisfies

$$\min_{1\leq i\leq m_3} \xi_3^H H_i \xi_3 \geq \frac{2}{\pi m_3} \text{ and } ||\xi_3||^2 \leq 3\mathbf{tr}(Z_3^*). \quad (54)$$

Let $\hat{\xi}$, ξ_3 be any vector satisfying Eq. (53) and Eq. (54). Note that $\hat{\xi}$, ξ_3 are independent with each other. Meanwhile, according to the randomization approximation procedure in Algorithm 2, a group of feasible solutions $\hat{z}^l$, z_3^l to problem (16) can be constructed with probability 1. Hence we have

$$\begin{aligned} v_m &\leq ||\hat{z}^l||^2 + ||z_3^l||^2 \\ &\leq \frac{||\hat{\xi}||^2}{min\, \hat{\xi}^H \mathcal{M}\hat{\xi}} + \frac{||\xi_3||^2}{\underset{1\leq i\leq m_3}{min}\ \xi_3^H H_i \xi_3} \\ &\leq \frac{3\mathbf{tr}(\widehat{Z})}{2/(\pi n)} + \frac{3\mathbf{tr}(Z_3^*)}{2/(\pi m_3)} \\ &\leq \frac{3\pi}{2} \cdot \max\{n, m_3\} \cdot (\mathbf{tr}(\widehat{Z}) + \mathbf{tr}(Z_3^*)) \\ &\leq \frac{3\pi}{2} \cdot \max\{m_1 + m_2 + 2m_1m_2, m_3\} \cdot (\mathbf{tr}(\widehat{Z}) + \mathbf{tr}(Z_3^*)) \\ &= \frac{3\pi}{2} \cdot \max\{m_1 + m_2 + 2m_1m_2, m_3\} \cdot v_r, \end{aligned} \quad (55)$$

where the last equality uses

$$v_r = \mathbf{tr}(Z_1^*) + \mathbf{tr}(Z_2^*) + \mathbf{tr}(Z_3^*) = \mathbf{tr}(\widehat{Z}) + \mathbf{tr}(Z_3^*). \quad (56)$$

Since the followed iteration process in the algorithm guarantees that each accepted step should have the lower transmission power than the last step, Eq. (55) still holds after the iterations. This completes the proof. ∎

VI. DISCUSSIONS ON THE PRACTICAL ISSUES

In this section, we discuss some practical issues of the proposed Analog Network-coded Modulation Strategy.

A. THE IMPLEMENTATION OF THE PROPOSED STRATEGY

In this subsection, in order to make the proposed strategy more practical, we propose an implementation method of designing the constellations and mappings offline as well as searching modulation books online.

We firstly adopt the precoding technique to achieve the channel difference to be 0 and then split the magnitude region of h_2/h_1, *i.e.* $|h_2/h_1|$, into small sections. Practically, the maximum transmission power at the source nodes is limited. As a result, for the regions where γ is sufficiently large or small, from the view of *RN*, the constellation from the worse channel will be much smaller than that from the other channel. That is, at the receiver of *RN*, a constellation is superposed by another constellation that is much smaller. This makes for this case, the conventional XOR mapping can effectively cluster the superposed constellation points. Similar discussion is also demonstrated in [15]. Based on this consideration, we focus on the joint mapping and modulation design within a bounded region with boundaries c_1 and c_2. The upper boundary c_2 can be selected sufficiently larger and the lower boundary c_1 can be selected sufficiently smaller. Then we split the bounded region into finite number of small sections. We can firstly provide the design result for each small section by using its mean value, and then store them as the modulation books in the nodes.

When taking the strategy into practice, we store the boundaries c_1, c_2, and the boundaries of all the small sections at the nodes. The decision regarding which mapping and constellation to use can be made by finding the small section in which $|h_2/h_1|$ falls. For the case where $|h_2/h_1|$ is outside the bounded region, we can use the conventional XOR mapping and modulation. For the case where $|h_2/h_1|$ is inside the bounded region, we use the mapping and modulation jointly designed by the proposed algorithms.

B. BIT-MAPPING DESIGN FOR THE OPTIMIZED CONSTELLATIONS

In this subsection, we provide a bit-mapping design method for the newly designed constellations based on the trellis coded modulation.

According to the proposed modulation framework, the denoising-and-mapping operation at *RN* is performed on symbol level, rather than bit level. Hence it is unnecessary to investigate the bit-mapping operation for the constellation $\mathcal{S}$ used by *RN*. Meanwhile, the bit-mapping operation for the constellations $\mathcal{A}$ and $\mathcal{B}$ cannot affect the denoising-and-mapping reliability at *RN*. This ensures that the bit-mapping operation cannot cause the ineffectiveness of the designed constellations and mapping.

For the bit-mapping design, a common principle is that only one bit differs between two neighboring constellation points, similar to the Gray code. However, as for the bit-mapping design for constellations $\mathcal{A}$ and $\mathcal{B}$ in PNC, what is different is that the one bit variation should happen between such two constellation points that are mapped to the two

Algorithm 3 Bit-Mapping Operation for the Newly Designed Constellation Based on Trellis Coded Modulation

Input: The newly designed constellation $\mathcal{A}$ (or $\mathcal{B}$).

Initial Set the superior-tree to be constellations $\mathcal{A}$.

Step 1: Compute and sort the ED between arbitrary two points in the superior-tree.

Step 2[4]**:** Find out the the two points corresponding to the minimum ED, and partition them into two different limbs. Label 0 and 1 on each branch respectively.

Step 3: Find out the two points that have the minimum ED in the rest points of the superior-tree.

Step 4[5]**:** Partition the two points above into the two limbs generated in step 2 according to the principle of maximizing the minimum ED.

Step 5: Repeat Step 3-Step 4 until the points in the superior-tree are all partitioned into the two limbs.

Step 6: Update the superior-tree by each limb obtained above.

Step 7: Repeat Step 2-Step 6 until the superior-tree contains only two points.

Step 8: Label the two points by 0 and 1 respectively.

Output: The bit-mapping result corresponds to each constellation point.

adjacent constellations points in $\mathcal{S}$. This is because the destination nodes decode the intended information by detecting the signal from RN, rather than detecting the signal directly from the corresponding source nodes. As a result, we should firstly find out the two points in $\mathcal{S}$ with minimum ED, and then find out the corresponding two points in $\mathcal{A}$ (or $\mathcal{B}$) and code them with only one bit variation. However, for different local prior information, the corresponding pairwise constellation points in $\mathcal{A}$ (or $\mathcal{B}$) are different. This makes it hard to achieve the bit-mapping that guarantees the one-bit variation.

Here we adopt the gist of the trellis coded modulation, which is based on the set partitioning methodology. Bit-mapping by set partitions firstly groups the symbols in a tree-like structure, separating them into two limbs of equal size. For each sub-tree, the symbols are further partitioned until the minimum ED satisfies the requirement. The binary bits 1 and 0 can be labeled on each branch to perform the bit-mapping. **Algorithm 3** provides the implementation process of the bit-mapping operation. The design criterion of the proposed bit-mapping method is that in each sub-tree, the minimum ED is larger than that in the superior-tree.

VII. SIMULATION RESULTS

In this section, simulation results are presented to demonstrate the potential of the joint constellation and network coding mapping design algorithms. We firstly show the joint design results of the constellation and the network coding mapping under different parameter configurations. Then the SER curves are demonstrated by Monte-Carlo simulation.

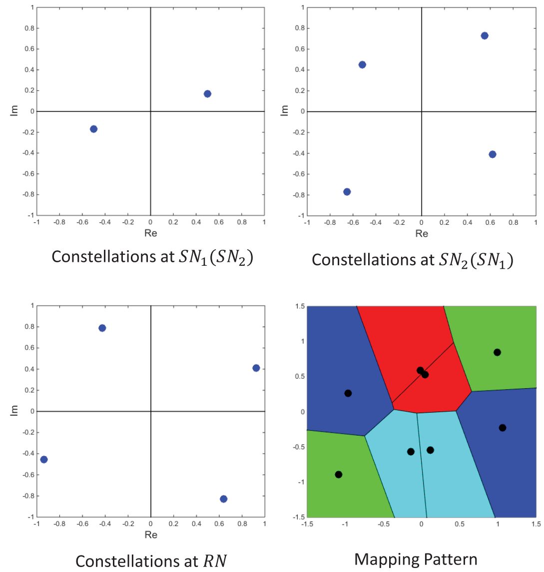

FIGURE 3. Illustration of the jointly optimized constellations and network coding mapping designed by Enhanced-SDR Algorithm. The parameters are configured as: $M = 2$, $N = 4$, $h_1 = 1$, $h_2 = 1$.

A. JOINTLY DESIGNED CONSTELLATIONS AND MAPPING PATTERNS

This subsection shows the design results of the constellations and the network coding mappings under different parameter configurations.

We firstly configure the simulation parameters as follows: $M = 2$, $N = 4$, $\varepsilon_A = 1$, $\varepsilon_B = 1$, $\varepsilon_R = 1$, $h_1 = 1$, $h_2 = 1$. The Enhanced-SDR Algorithm is firstly applied to solve the optimization problem, where $N_{rand} = 10^7$ randomizations are simulated because the average transmission power gain becomes very smaller when $N_{rand} > 10^7$. The other parameters in the algorithm are configured as: $K = 10^5$, $\delta_1 = 0.1$, $\delta_2 = 0.005$. The designed constellations and mapping pattern are shown in FIGURE 3. By assuming AWGN with zero mean, the network coding mapping at RN can be depicted by the Voronoi diagram as shown in the mapping pattern. When the signals received at RN are located at the regions within the same color, RN will broadcast them by an identical constellation point. Total four colors are contained in the mapping pattern, so correspondingly the constellation used at RN only contains four points as shown in FIGURE 3. For comparison, we run the Fast-Relaxation Algorithm under the same parameter configuration. The designed constellations and mapping pattern are shown in FIGURE 4.

Furthermore, the design results are demonstrated for the higher-order case where the parameters are configured as follows: $M = 4$, $N = 4$, $\varepsilon_A = 1$, $\varepsilon_B = 1$, $\varepsilon_R = 1$, $h_1 = 1$, $h_2 = (1+j)/2$. We choose $h_1 = 1$, $h_2 = (1+j)/2$ as the channel state because it belongs to a kind of particular channel conditions, referred to as *singular points* in [15], for which

[4]If there are different pairs of points corresponding to the minimum ED, we select an arbitrary pair.

[5]This step has totally two choices. The principle of maximizing the minimum ED means that we choose the one that can provide the larger minimum ED.

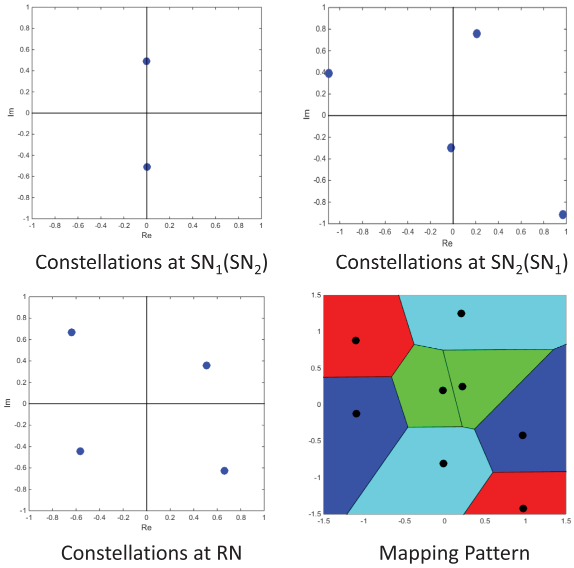

FIGURE 4. **Illustration of the jointly optimized constellations and network coding mapping designed by Fast-relaxation Algorithm. The parameters are configured as: $M = 2$, $N = 4$, $h_1 = 1$, $h_2 = 1$.**

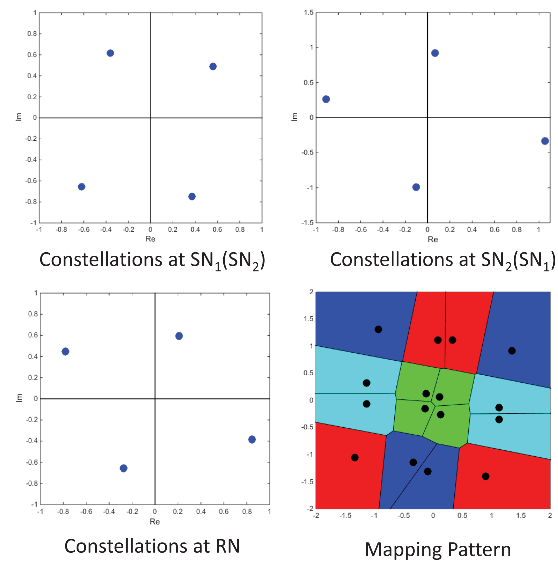

FIGURE 6. **Illustration of the jointly designed constellations and network coding mapping by Fast-relaxation Algorithm. The parameters are configured as: $M = 4$, $N = 4$, $h_1 = 1$, $h_2 = (1 + j)/2$.**

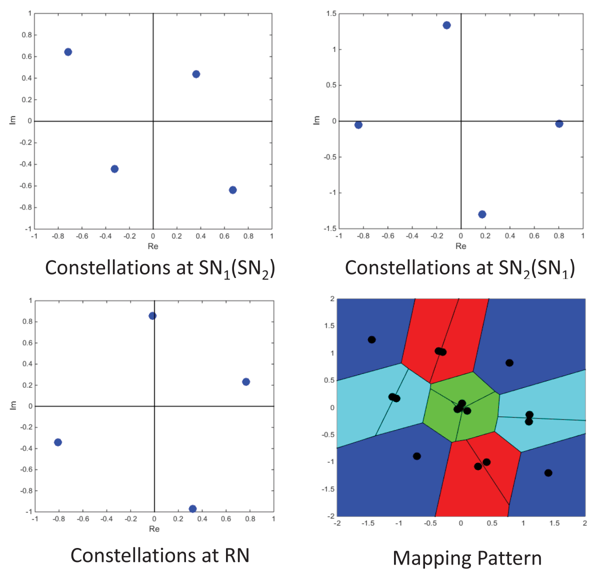

FIGURE 5. **Illustration of the jointly designed constellations and network coding mapping by Enhanced-SDR Algorithm. The parameters are configured as: $M = 4$, $N = 4$, $h_1 = 1$, $h_2 = (1 + j)/2$.**

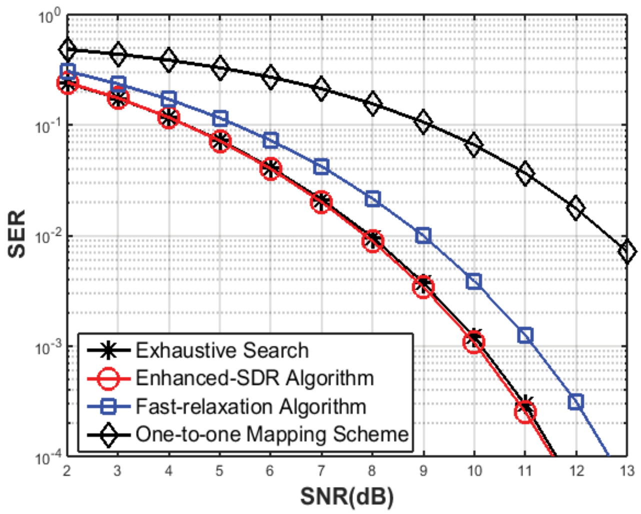

FIGURE 7. **SER performance comparison among the constellations respectively designed by Enhanced-SDR algorithm, Fast-relaxation algorithm, and exhaustive search. The parameters are configured as: $M = 2$, $N = 4$, $h_1 = 1$, $h_2 = 1$.**

the conventional 4-ary XOR network coding cannot cluster all the neighboring points effectively. FIGURE 5 shows the constellations designed by the Enhanced-SDR Algorithm. The simulation parameters are configured as: $N_{rand} = 10^7$, $K = 10^6$, $\delta_1 = 0.1$, $\delta_2 = 0.005$. FIGURE 6 shows the constellations designed by the Fast-relaxation Algorithm. The simulation parameters are configured as: $N_{rand} = 10^8$, $K = 10^6$, $\delta_1 = 0.1$, $\delta_2 = 0.005$.

In the next subsection, we will demonstrate and discuss the SER performance of the designed constellations and network coding mapping patterns.

B. PERFORMANCE COMPARISON OF THE ALGORITHMS

In this subsection, in order to directly evaluate the performance of the jointly designed constellations and mapping patterns, the end-to-end SER performances are demonstrated by Monte-Carlo simulation.

FIGURE 7 shows four different SER curves that correspond to four different constellation design methods. The black diamond curve is the benchmark one, which is given by an elaborate one-to-one mapping scheme. In this scheme, BPSK and QPSK are respectively used at SN_1 and SN_2, and correspondingly 8QAM is used at RN. We use this case that has limited optimization for comparison to observe the gains obtained through the proposed optimized

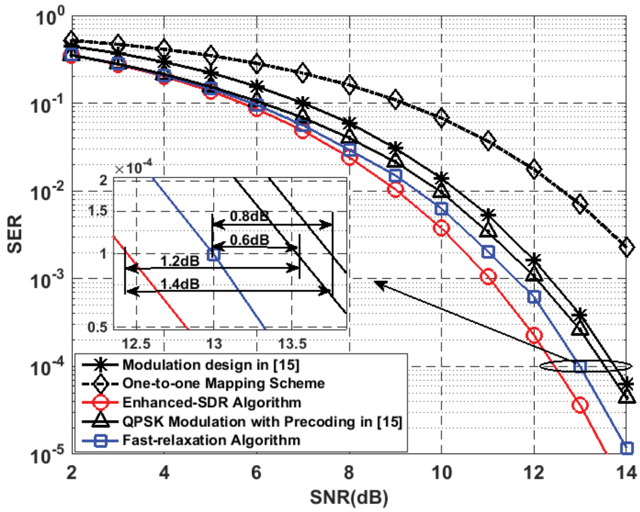

FIGURE 8. SER performance comparison among the constellations respectively designed by Enhanced-SDR algorithm, Fast-relaxation algorithm, and the algorithms in [15]. The parameters are configured as: $M = 4$, $N = 4$, $h_1 = 1$, $h_2 = (1 + j)/2$.

algorithms. The black asterisk curve in the figure represents the optimal performance given by the constellations designed by an elaborate exhaustive search, where BPSK and QPSK are used at SN_1 and SN_2, and QPSK is also used at RN for broadcasting. In this search, we firstly find out all possible network coding mappings. Then we search the optimal constellation for each kind of network coding mapping. The red circle cure is simulated from the constellations shown in FIGURE 3, designed by Enhanced-SDR Algorithm. The blue square curve is obtained from the constellations in FIGURE 4, designed by Fast-relaxation Algorithm. It is shown that both the algorithms can achieve a significant SER gain over the benchmark one.

It can also be observed that the Enhanced-SDR Algorithm outperforms the Fast-relaxation Algorithm in SER performance. However, as discussed in TABLE 2, by using interior point methods, the Enhanced-SDR Algorithm requires a worst complexity of $O(MN\log(1/\epsilon))$ iterations, with each iteration taking at most $O(M^{12}N^{12})$ arithmetic operations, while the Fast-relaxation Algorithm requires only $O(\sqrt{MN}\log(1/\epsilon))$ iterations, with each iteration taking only $O(M^6N^6)$ arithmetic operations. The Fast-relaxation Algorithm outperforms the Enhanced-SDR Algorithm in complexity. Hence the two proposed algorithms achieve the tradeoff between the communication performance and the computation resouces.

The SER curves of the constellations in FIGURE 5 and FIGURE 6 are shown in FIGURE 8. The black diamond curve is simulated by the one-to-one mapping strategy, where QPSK modulation is used at the MA stage and 16QAM is used at the BC stage. The proposed algorithms can still achieve a significant SER gain over the benchmark one. Besides, we compare the SER performance of the jointly designed constellations with that of the existing method in [15], in which the network coding mapping and the constellations are successively designed by the numerical methods. It is shown that the jointly designed modulations and mappings by the Enhanced-SDR algorithm and the Fast-relaxation algorithm outperform the design result in [15] by near 1.4 and 0.8 dB respectively at the SER of 10^{-4}. The black triangle curve is simulated by the case, where the precoding technique is adopted (to set the channel phase difference to be 0), and QPSK modulation is used at all the three nodes. This case is shown to provide the best performance for the modulation design mechanism in [15]. It is demonstrated that the joint design results under the proposed modulation framework can respectively outperform it by about 1.2 and 0.6 dB.

C. BIT ERROR RATE (BER) PERFORMANCE SIMULATION OF THE PROPOSED BIT-MAPPING METHOD

In this subsection, in order to demonstrate the effectiveness of the proposed bit-mapping method in the last section, we provide the BER performance simulations and the bit-mapping results.

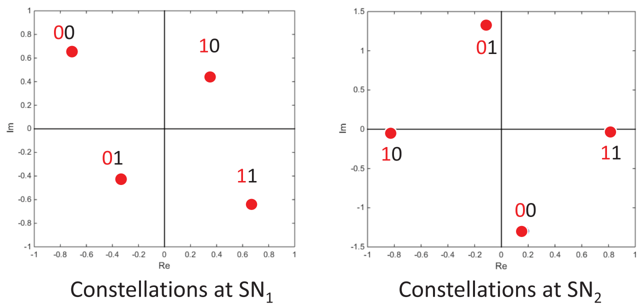

FIGURE 9. Bit-mapping methods designed by Algorithm 3 for the constellations at the source nodes shown in FIGURE 5.

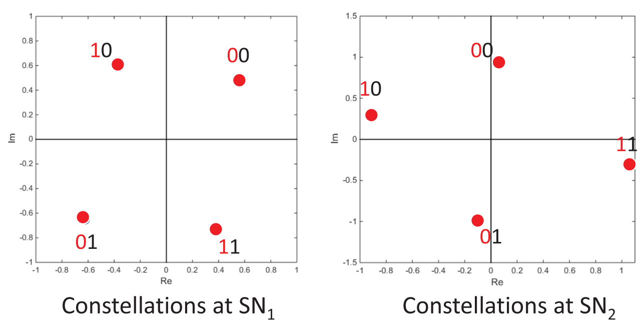

FIGURE 10. Bit-mapping methods designed by Algorithm 3 for the constellations at the source nodes shown in FIGURE 6.

In FIGURE 9 and FIGURE 10, we provide the bit mapping results for the constellations shown in FIGURE 5 and FIGURE 6. Note that the denoising-and-mapping at RN is performed by per-symbol operation. Hence it is unnecessary to perform the bit-mapping operation for the constellation used at RN. FIGURE 11 demonstrates the effectiveness of

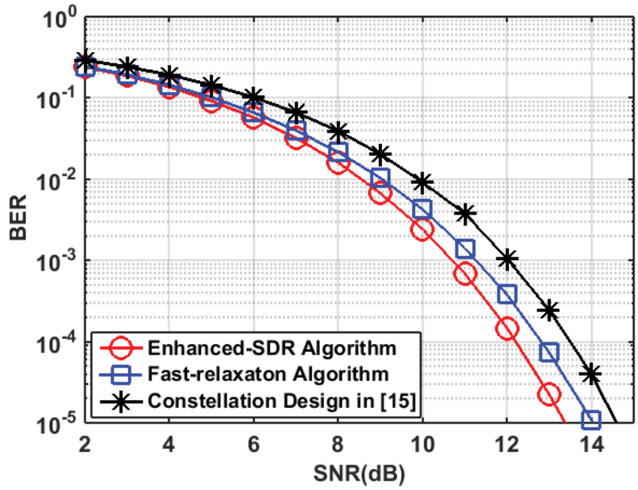

FIGURE 11. BER performance simulated respectively by the bit-mapping results in FIGURE 9 and FIGURE 10. In [15], Gray coding is applied for the QPSK Modulation at the source nodes.

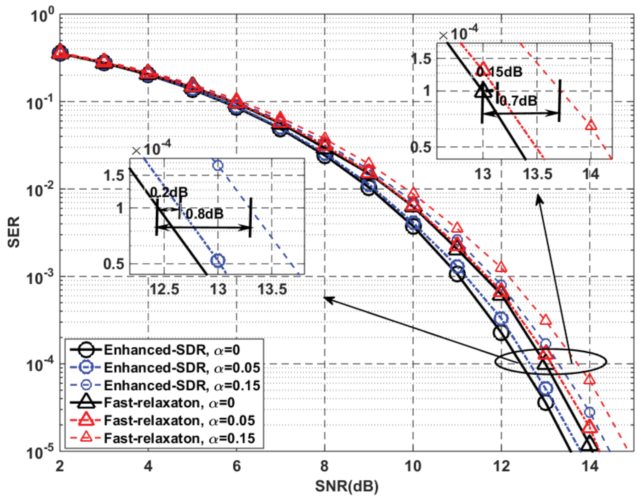

FIGURE 12. Illustration of the SER performance for different error radiuses of α.

the proposed bit-mapping method, where the BER simulations for the bit-mapping results shown in FIGURE 9 and FIGURE 10 are provided. We also provide the BER simulation result for the modulation mechanism in [15], where Gray coding mapping and QPSK modulation are used at the two source nodes. It is demonstrated that the BER performance by the proposed bit-mapping method can outperform that of the mechanism in [15].

D. PERFORMANCE INVESTIGATION FOR THE IMPERFECT CHANNEL STATE INFORMATION (CSI)

In this subsection, we provide the simulation results for the scenarios where the channel estimation errors are considered.

We adopt the additive model to depict the channel imperfection as

$$h = \hat{h} + e, \tag{57}$$

where we define $h = h_2/h_1$, $\hat{h}$ denotes the channel estimate and e denotes the estimation error. We assume that the error satisfies the following elliptic model:

$$\|e\| \leq \alpha, \tag{58}$$

where α denotes the radius of the error region. This model is motivated by considering the channel estimation as the main source of the channel uncertainty [25, Sec. 4.1]. Given the above imperfect CSI model, we simulate the worst-case SER performance under different error region radiuses α, as shown in FIGURE 12. It can be observed that the jointly designed constellations and mappings are not sensitive to the perturbation of the channel errors, although the performance degradation can also increase as the channel error becomes larger.

VIII. CONCLUSION

This paper investigated the joint optimization of the mapping method and constellation design for the two-way relaying networks quipped with PNC. A unified Analog Network-coded Modulation Strategy is build, where a novel analog mapping criterion is proposed. Based on this strategy, we formulated an optimization problem by minimizing the average transmitting power while guaranteeing the SER requirements and transmission rate. This problem is proved to be NP-hard and two polynomial-complexity approximation algorithms, referred to as Enhanced-SDR Algorithm and Fast-relaxation Algorithm, are proposed to solve it. Monte-Carlo simulation results show that the two proposed algorithms can achieve the tradeoff between the communication performance and computation resource. Besides, it is also shown that the proposed algorithms can outperform the existing design methods.

APPENDIX A
PROOF OF LEMMA 1

Since the matrix $X \succeq 0$ is hermitian matrix and has rank $r := \mathbf{rank}(X)$, we can diagonalize it by some $U \in \mathbb{C}^{n\times r}$ as

$$U^H X U = I_r. \tag{59}$$

Similarly, since $H \succeq 0$ is hermitian matrix, we perform an eigen-decomposition operation as

$$U^H H U = Q \Lambda Q^H, \tag{60}$$

where $Q^H \in \mathbb{C}^{r\times r}$ is the corresponding unitary matrix, and $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0$. Since $\mathbf{rank}(H) = 1$, $U^H H U$ has rank at most $\bar{r} = 1$. Thus, we have

$$\lambda_i = 0, \quad \forall i = 2, \ldots, r. \tag{61}$$

Let $\xi \sim \mathcal{N}_c(0, X)$ and $\bar{\xi} = Q^H U^H \xi$. It is readily checked that

$$\bar{\xi} \sim \mathcal{N}_c(0, I_r). \tag{62}$$

$$\begin{aligned}
&\Pr\{\min_{1\le i\le n} \xi^H\Omega_i\xi \ge \gamma,\ ||\xi||^2 \le \mu\mathbf{tr}(X)\}\\
&\quad = \Pr\{\xi^H\Omega_i\xi \ge \gamma,\ \forall i=1,\ldots,n,\ \text{and } ||\xi||^2 \le \mu\mathbf{tr}(X)\}\\
&\quad \ge \Pr\{\xi^H\Omega_i\xi \ge \gamma\mathbf{tr}(\Omega_i X),\ \forall i=1,\ldots,n,\ \text{and } ||\xi||^2 \le \mu\mathbf{tr}(X)\}\\
&\quad = \Pr\{\xi^H\Omega_i\xi \ge \gamma\mathbb{E}(\xi^H\Omega_i\xi),\ \forall i=1,\ldots,n,\ \text{and } ||\xi||^2 \le \mu\mathbb{E}(||\xi||^2)\}\\
&\quad = 1-\Pr\{\xi^H\Omega_i\xi < \gamma\mathbb{E}(\xi^H\Omega_i\xi),\ \text{for some } i,\ \text{or } ||\xi||^2 > \mu\mathbb{E}(||\xi||^2)\}\\
&\quad \ge 1-\sum_{i=1}^{n}\Pr\{\xi^H\Omega_i\xi < \gamma\mathbb{E}(\xi^H\Omega_i\xi)\} - \Pr\{||\xi||^2 > \mu\mathbb{E}(||\xi||^2)\}\\
&\quad \ge 1-n\gamma-\frac{1}{\mu},
\end{aligned} \tag{73}$$

Furthermore, ξ is statistically identical to $UQ\bar{\xi}$, so that $\xi^H H\xi$ is statistically identical to

$$\bar{\xi}^H Q^H U^H HUQ\bar{\xi} = \bar{\xi}^H\Lambda\bar{\xi} = \sum_{i=1}^{\bar{r}}\lambda_i|\bar{\xi}(i)|^2 = \lambda_1|\bar{\xi}(1)|^2. \tag{63}$$

Then we have

$$\begin{aligned}
&\Pr\{\xi^H H\xi < \gamma\mathbb{E}(\xi^H H\xi)\}\\
&\quad = \Pr\{\lambda_1|\bar{\xi}(1)|^2 < \gamma\mathbb{E}(\lambda_1|\bar{\xi}(1)|^2)\}\\
&\quad = \Pr\{\lambda_1|\bar{\xi}(1)|^2 < \lambda_1\gamma\mathbb{E}(|\bar{\xi}(1)|^2)\}.
\end{aligned} \tag{64}$$

If $\lambda_1 = 0$, then this probability is zero, which proves Eq. (45). If $\lambda_1 > 0$, then we have

$$\Pr\{\xi^H H\xi < \gamma E(\xi^H H\xi)\} = \Pr\{|\bar{\xi}(1)|^2 < \gamma\mathbb{E}(|\bar{\xi}(1)|^2)\}. \tag{65}$$

Recall that the probability density function of a complex-valued circular normal random variable $t \sim \mathcal{N}_c(0,\sigma^2)$ is

$$f(t) = \frac{1}{\pi\sigma^2}e^{-\frac{|t|^2}{\sigma^2}}, \quad \forall t \in \mathbb{C}, \tag{66}$$

where σ is the standard deviation. In polar coordinates, the density function can be written as:

$$f(\rho,\theta) = \frac{\rho}{\pi\sigma^2}e^{-\frac{\rho^2}{\sigma^2}}, \quad \forall\rho\in[0,+\infty),\ \theta\in[0,2\pi), \tag{67}$$

from which we can see that the argument θ of the complex-valued normal variable is uniformly distributed over $[0,2\pi)$ and the modulus ρ follows a Rayleigh distribution with density function:

$$f(\rho) = \begin{cases} \frac{2\rho}{\sigma^2}e^{-\frac{\rho^2}{\sigma^2}} & \text{if } \rho \ge 0;\\ 0 & \text{if } \rho < 0. \end{cases} \tag{68}$$

Then, it is readily concluded that the square of modulus $u = \rho^2$ follows a exponential distribution with density function:

$$f(u) = \begin{cases} \frac{1}{\sigma^2}e^{-\frac{u}{\sigma^2}} & \text{if } u \ge 0;\\ 0 & \text{if } u < 0. \end{cases} \tag{69}$$

Considering $\bar{\xi}(1) \sim \mathcal{N}_c(0,1)$, we thus have

$$\begin{aligned}
&\mathbb{E}(|\bar{\xi}(1)|^2) = 1\\
&\Pr\{|\bar{\xi}(1)|^2 \le \gamma\} = 1-e^{-\gamma}.
\end{aligned} \tag{70}$$

Substitute this into Eq. (65) and we get that

$$\begin{aligned}
&\Pr\{\xi^H H\xi < \gamma\mathbb{E}(\xi^H H\xi)\}\\
&\quad = \Pr\{|\bar{\xi}(1)|^2 < \gamma\}\\
&\quad \le 1-e^{-\gamma} \le \gamma,
\end{aligned} \tag{71}$$

which completes the proof.

APPENDIX B
PROOF OF LEMMA 2

Considering $\mathbb{E}(\xi\xi^H) = X$, we have

$$\mathbb{E}(\xi^H\Omega_i\xi) = \mathbf{tr}(\Omega_i X) \ge 1, \quad \forall i = 1,2,\ldots,n. \tag{72}$$

Thus, for any $\gamma > 0$ and $\mu > 0$, we have Eq. (73), as shown at the top of this page, where the last step uses *Lemma* 1 and Markov's inequality:

$$\Pr\{||\xi||^2 > \mu\mathbb{E}(||\xi||^2)\} \le \frac{1}{\mu}. \tag{74}$$

This completes the proof.

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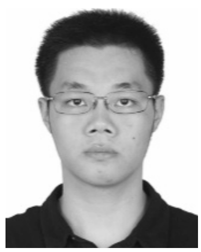

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