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Reliability Demonstration for Long-Life Products Based on Hardened Testing Method and Gamma Process

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ABSTRACT Reliability demonstration testing is widely applied to the industry for the verification of products' certain reliability requirement. However, for long-life and high-reliability products, the sample size and testing time are both unacceptable. To shorten the testing time, the performance degradation data are used to predict whether one sample will fail by the end of the testing. To decrease the sample size or further shorten the testing time, the hardened testing method is considered. Thus, this paper proposes a hardened reliability demonstration testing method with the accelerated degradation model to demonstrate the structural reliability at a high confidence level. First, an accelerated gamma degradation model for the considered problem is formulated. Then, the transformation method of the reliability indexes under different stress levels is proposed. Finally, we develop the optimal testing plan to obtain the stress level, sample size, and average testing time by minimizing the total testing cost, and give the testing termination rules for one sample. A numerical example is given to demonstrate the availability of the proposed method on shortening the testing time and reducing the sample size.

INDEX TERMS Reliability demonstration testing, accelerated degradation, hardened stress level, gamma process, optimal testing plan.

NOMENCLATURE

ABBREVIATIONS

RDT	reliability demonstration testing
ZFT	zero-failure testing
SSI	stress-strength interference
ADT	accelerated degradation test
MTTF	mean time to failures
CDF	cumulative distribution function
PDF	probability density function

NOTATION

s	normal stress level
k	hardened coefficient
n	sample size
m	number of measurements on one sample
t_j	measurement time; $j = 0, 1, 2, \cdots, m$
τ	measurement interval
y_{ij}	the i^{th} sample's degradation measure at t_j
$f_{Y(t)}\left(y_{ij}\right)$	PDF of $Y(t)$ at t_j
$F_{Y(t)}\left(y_{ij}\right)$	CDF of $Y(t)$ at t_j
$\alpha_i(t)$	non-decreasing, continuous and real-valued shape parameter in a Gamma distribution for the i^{th} sample; $i = 1, 2, \cdots, n$
β_i	shape parameter in a Gamma distribution for the i^{th} sample
λ_i, c_i	parameters in $\alpha_i(t)$
D_f	pre-specified threshold
T_i	the i^{th} sample's lifetime
$F_{T_i}(t)$	the i^{th} sample's lifetime distribution
$\Phi(\cdot)$	standard normal distribution
$\Gamma(\cdot), \Gamma(\cdot,\cdot)$	Gamma function and incomplete Gamma function
A_F	accelerated factor
$h(ks)$	acceleration model; $\lambda = \lambda_i = h(ks)$
a, b	parameters in $h(ks)$
$L_i(\boldsymbol{\Theta})$	likelihood function for the i^{th} sample; $\boldsymbol{\Theta} = (a, b, k, \beta_i, c_i)$
$F_{T_i(ks)}(t(ks))$	failure probability under the hardened stress level ks
δ	specified type II error (consumer's risk)

δ_c	additional type II error incurred by hardened stress level and early termination
ρ	specified type I error (producer's risk)
R_s	specified reliability index to be demonstrated
t_s	time at which R_s is specified
R_s^{H}	pre-specified reliability index to be demonstrated under the hardened stress level ks
$F_{T_i(s)U}\left(t_s\left(s\right)\right)$	the upper bound of the type II error probability
TC	total cost for an acceleration degradation
t_a	average test time of all samples
n_a	the number of samples allowable for the test critical hardened coefficient
k_c	critical hardened coefficient
w_1	the cost of one sample
w_2	test cost per unit time incurred by manpower, electric power, etc.
w_3	cost of one measurement
w_4	risk cost caused by the hardened stress level

I. INTRODUCTION

Reliability demonstration testing (RDT) is required before the design typification and volume production, which is the examination link of the reliability design. On the one hand, RDT demonstrates that the design achieves the reliability specified in the product planning phase; on the other hand, RDT proves that the production process meets the requirements of manufacturing products specified in the product planning phase. In general, RDT methods can be divided into two categories: failure-based methods [1]–[5], and degradation-based methods [6], [7]. More specifically, failure-based methods can further be divided into two categories: (1) testing designs based on the number of failures [1], [2]; (2) testing designs based on failure times of each sample [3], [4].

Zero-failure testing (ZFT) is widely applied to the reliability demonstration at a high confidence level in industry because of its implementary convenience and simplicity without monitoring failures or measuring the performance degradation during the testing [8], [9]. In the application, Sun *et al.* [10] provided a zero-failure RDT plan based on the accelerated degradation testing and Weibull distribution by minimizing the expected testing time constrained by Type II error, sample size, and total cost. Guo *et al.* [11] proposed a flexible and practical method to estimate the system reliability and its confidence bounds when few or no failures occur during the subsystem testing. However, for long-life and high-reliability products, ZFT requires a large sample size and a long testing time, which makes it inefficient and high-cost. Therefore, how to reduce the sample size or shorten the testing time has become one crucial engineering problem for the reliability demonstration.

Much effort has been made to solve the problem about reducing the sample size, and there are two kinds of most widely used strategies: 1) the use of prior information [12]–[14]; 2) the hardened testing method [15], [16]. A. J. Fernndez [12] proposed some generalized beta prior models on fraction defective in the reliability testing plan, which significantly reduces the required sample size. Coolen *et al.* [13] proposed a zero-failure Bayesian reliability demonstration method for multi-task systems by minimizing a linear cost model considering the loss due to failure in the testing. George and Thomas [14] described a Bayesian approach for RDT and compared the Bayesian method with the classical statistical method. X. Beurtey [15] explored the relevant hardened testing method for the initiating explosive structure. Rong [16] proposed a hardened testing method based on the stress-strength interference (SSI) model to reduce the sample size.

In addition, the accelerated test methods and degradation data are often used to shorten the testing time for long-life products [17]–[22]. Baussaron [18] proposed a reliability demonstration method based on ADT using Wiener process model. Liu *et al.* [19] gave a Bayesian sequential procedure using estimated performance reliability life based on degradation test data. Guang and David [20] proposed a reliability demonstration method for long-life products based on the degradation testing and a wiener process Model. Zhang *et al.* [21] presented a reliability demonstration methodology for products with Gamma process by optimal accelerated degradation testing. Specifically, Yang [22] proposed a reliability demonstration testing method based on the performance degradation for long-life products, where the main idea is predicting whether or not a testing unit will fail by the end of the testing using the degradation measurements. As soon as there are sufficient data to make such a conclusion at a high confidence level, the unit can be censored to shorten the testing time. However, the drawback of this method is the increase of the sample size, which cannot meet the testing condition when the sample size is limited for the product. The hardened testing method is a feasible way to achieve shortening the testing time and reducing the sample size at the same time. On the one hand, it is reasonable to believe that the reliability index can be lowered under the hardened stress level. Thus, how to transform the reliability index under the normal stress level to that under the hardened stress level is a problem we will solve in this paper. On the other hand, the observed degradation data under the hardened stress level can be utilized to draw a conclusion about the failure state, which naturally requires the analysis method of the accelerated degradation testing (ADT). Moreover, the hardened testing method can also be used to further shorten the testing time.

There are two classes of models for describing ADT data: general path model [23] and stochastic process model. The latter is more widely applied, such as the Wiener process and Gamma process [24]–[30]. The Wiener process with the attractive properties of the normal distribution is commonly used for some specific datasets, but is not proper for the monotonic degradation modeling [24], [25]. Thus, the Gamma process is considered for overcoming this deficiency because of its characteristic of the independent non-negative increment [30]. Abdel [26] was the first to propose the Gamma process as a proper model for the deterioration occurring randomly in time. Ling *et al.* [27] studied the degradation of light intensity of LEDs based on the accelerated degradation analysis assuming a Gamma degradation process and time-scale transformation. Tseng *et al.* [28] dealt with the optimal step-stress ADT plan with a Gamma degradation process by miniThus, recall thatmizing the approximate variance of the estimated mean time to failures (MTTF) constrained by the total experimental cost. Ye *et al.* [29] investigated the semiparametric inference of the simple Gamma process model and a random-effects variant.

Based on the good properties of Gamma process, and for the specific product whose performance degradation can be measured, this paper presents a RDT method based on the hardened testing method and the use of degradation measurements to both shorten the testing time and reduce the sample size in ZFT. This paper is organized as follows. In section II, an accelerated Gamma degradation model for the considered problem is formulated, and the relevant inferences are presented. In section III, the sample size model under the hardened stress level is given. In section IV, the optimal testing plan and the decision rules for terminating one sample's testing are developed. In section V, an numerical example is presented to illustrate the availability of the proposed method, and the robustness analysis against parameters affecting the optimal testing plan is given. In section VI, some concluding remarks are provided.

II. MODEL DESCRIPTION AND RELEVANT INFERENCES FOR GAMMA PROCESS

In this paper, we combine the use of degradation information with the hardened testing method to shorten the testing time and reduce the sample size at the same time, so the accelerated degradation model is required naturally.

A. ACCELERATED DEGRADATION MODEL BASED ON GAMMA PROCESS AND INFERENCES ON LIFETIME CHARACTERISTIC

Gamma process is a stochastic process with independent non-negative increments, which has a Gamma distribution with identical scale parameter. Gamma process is able to model the gradual damage monotonically accumulating over time, such as fatigue, erosion, crack growth, etc [30]. In mathematical terms, Gamma process is defined as follows. Suppose that n testing samples are placed at a constant hardened stress level of $ks(k \geq 1)$, and inspected at times $0 = t_0 < t_1 < t_2 < \cdots < t_m$, where k is the hardened coefficient, s is the normal stress level. During testing, the i^{th} sample's degradation is measured as y_{ij} at time t_j. The inspection is nondestructive, which can keep the same dispersion characteristic and time-varying characteristic. Thus, recall that Y has a Gamma distribution with shape parameter $\alpha_i > 0$ and scale parameter $\beta_i > 0$ if its probability density function (PDF) is given by:

$$f_Y\left(y_{ij}\right) = \mathrm{Ga}\left(y_{ij}\,|\alpha_i, \beta_i\right) = \frac{\beta_i^{\alpha_i}}{\Gamma\left(\alpha_i\right)} y_{ij}^{\alpha_i - 1} \exp\left(-\beta_i y_{ij}\right) \quad (1)$$

In addition, let $\alpha_i(t)$ be a non-decreasing, continuous and real-valued function for $t \geq 0$ with $\alpha_i(0) \equiv 0$. Thus, Y can be amended as $Y(t)$ denoting the degradation at time t. In conformity with the definition of the Gamma process $Y(t)$, the pdf of $Y(t)$ can be given by:

$$f_{Y(t)}\left(y_{ij}\right) = \mathrm{Ga}\left(y_{ij}\,|\alpha_i(t), \beta_i\right) \quad (2)$$

with

$$E\left(Y(t)\right) = \frac{\alpha_i(t)}{\beta_i}, \quad \mathrm{Var}\left(Y(t)\right) = \frac{\alpha_i(t)}{\beta_i^2} \quad (3)$$

$$F_{Y(t)}\left(y_{ij}\right) = \int_0^{y_{ij}} \frac{\beta_i^{\alpha_i(t)}}{\Gamma\left(\alpha_i(t)\right)} x^{\alpha_i(t)-1} \exp\left(-\beta_i x\right) \mathrm{d}x \quad (4)$$

and the following properties:

(1) $P\left(Y(0) = 0\right) = 1$;

(2) $Y(\tau) - Y(t) \sim \mathrm{Ga}\left(\alpha_i(\tau) - \alpha_i(t), \beta_i\right)$, for $\tau > t \geq 0$;

(3) The Gamma process has independent increments.

Empirical studies [30] show that the expected degradation in (3) at time t generally follows a power law, which is a key input in modeling the temporal variability in the degradation:

$$E\left(Y(t)\right) = \frac{\lambda_i t^{c_i}}{\beta_i}, \quad \left(\lambda_i, c_i > 0\right) \quad (5)$$

Thus, we let $\alpha_i(t) = \lambda_i t^{c_i}$ in this paper.

Suppose T is the time-to-failure of a product at which the degradation measure crosses a pre-specified threshold D_f. Under the settings of the above, the distribution of the i^{th} sample's lifetime T_i can then be written as:

$$\begin{aligned} F_{T_i}(t) &= \Pr\left\{T_i \leq t\right\} = \Pr\left\{Y(t) \geq D_f\right\} \\ &= \int_{D_f}^{+\infty} f_{Y(t)}\left(y_{ij}\right) \mathrm{d}y_{ij} = \frac{\Gamma\left(\lambda_i t^{c_i}, \beta_i D_f\right)}{\Gamma\left(\lambda_i t^{c_i}\right)} \end{aligned} \quad (6)$$

where $\Gamma\left(\lambda_i t^{c_i}, \beta_i D_f\right) = \int_{x=\beta_i D_f}^{+\infty} x^{\lambda_i t^{c_i}-1} \exp(-x)\mathrm{d}x$ is the incomplete Gamma function. Then, the i^{th} sample's reliability at time t is given by:

$$R_{T_i}(t) = 1 - F_{T_i}(t) = 1 - \frac{\Gamma\left(\lambda_i t^{c_i}, \beta_i D_f\right)}{\Gamma\left(\lambda_i t^{c_i}\right)} \quad (7)$$

In the actual application, Park and Padgett [31] proposed a two-parameter Birnbaum Saunders distribution to approximate the lifetime distribution. In our case, we have:

$$F_{T_i}(t) \approx 1 - \Phi\left(\frac{D_f \beta_i - \lambda_i t^{c_i}}{\sqrt{\lambda_i t^{c_i}}}\right) = \Phi\left(\frac{1}{p}\left(\sqrt{\frac{t^{c_i}}{q}} - \sqrt{\frac{q}{t^{c_i}}}\right)\right) \quad (8)$$

where $\Phi(\cdot)$ is the standard normal distribution, and $p = \sqrt{1/\left(\beta_i D_f\right)}$, $q = D_f \beta_i / \lambda_i$.

B. PARAMETER VARIATION RULE AND LIKELIHOOD FUNCTION

The stress affects the degradation of testing samples, and the failure mechanism cannot be changed under the hardened stress level, whose sufficient and necessary condition is the constant acceleration factor A_F proposed in [32]. One definition of the acceleration factor is given based on the accelerated failure time model :

$$A_{F(1,2)} = t_2/t_1 \tag{9}$$

where testing times t_1 and t_2 satisfy the following equation:

$$F_1(t_1) = F_2(t_2) \tag{10}$$

where $F_1(t_1)$ and $F_2(t_2)$ respectively represent the product's cumulative failure probability under different stress levels s_1 and s_2.Combining (9) with (10), we have:

$$F_1(t_1) = F_2\left(A_{F(1,2)}t_1\right) \tag{11}$$

Substituting (6) into (11), we can obtain:

$$\frac{\Gamma\left(\lambda_{i1}t_1^{c_{i1}}, \beta_{i1}D_f\right)}{\Gamma\left(\lambda_{i1}t_1^{c_{i1}}\right)} = \frac{\Gamma\left(\lambda_{i2}\left(A_{F(1,2)}t_1\right)^{c_{i2}}, \beta_{i2}D_f\right)}{\Gamma\left(\lambda_{i2}\left(A_{F(1,2)}t_1\right)^{c_{i2}}\right)} \tag{12}$$

To make (12) stand up for the arbitrary t_1, the following equation should be satisfied:

$$\begin{cases}\lambda_{i1}t_1^{c_{i1}} = \lambda_{i2}\left(A_{F(1,2)}t_1\right)^{c_{i2}} \\ \beta_{i1}D_f = \beta_{i2}D_f\end{cases}$$
$$\Rightarrow \begin{cases}A_{F(1,2)} = \left(\lambda_{i1}/\left(\lambda_{i2}t_1^{c_{i1}-c_{i2}}\right)\right)^{1/c_{i2}} = \left(\lambda_{i1}/\lambda_{i2}\right)^{1/c_{i2}} \\ \beta_{i1} = \beta_{i2}, c_{i1} = c_{i2}\end{cases} \tag{13}$$

Equation (13) shows us that λ_i should change with the stress level, and β_i, c_i should keep constant with the stress level. Thus, we let $\lambda_i = h(ks)$, and β_i, c_i have no relation with the stress, where $h(ks)$ is a link function reflecting the effect of the stress on the degradation process. For simplicity, and without loss of generality, two assumptions are made as follows:

(1) The measurement interval under the hardened stress level is pre-determined and the same for all samples.

(2) All samples share the same link function, namely, the expression of $h(ks)$ is suitable and the same for each sample, i.e., $\lambda = \lambda_i = h(ks)$, $(i = 1, 2, \cdots, n)$. The link function follows one of the following acceleration models, where $(a, b > 0)$:

a) Inverse power law model: $h(ks) = a \cdot (ks)^b$. Then, we have:

$$A_{F(ks,s)} = \left(\lambda_{i(ks)}/\lambda_{i(s)}\right)^{1/c_i} = k^{b/c_i} \tag{14}$$

b) Arrhenius model: $h(ks) = a \cdot \exp(-b/(ks))$. Then, we have:

$$A_{F(ks,s)} = \left(\lambda_{i(ks)}/\lambda_{i(s)}\right)^{1/c_i} = \exp\left(\frac{(k-1)b}{ksc_i}\right) \tag{15}$$

c) Exponential model: $h(ks) = a \cdot \exp(b \cdot ks)$. Then, we have:

$$A_{F(ks,s)} = \left(\lambda_{i(ks)}/\lambda_{i(s)}\right)^{1/c_i} = \exp\left(\frac{bs(k-1)}{c_i}\right) \tag{16}$$

Specifically, when $k = 1$, $A_{F(ks,s)} \equiv 1$ for all models. Thus, we can amend (6-8) as follows:

$$F_{T_i(ks)}(t(ks)) = \frac{\Gamma\left(h(ks)t^{c_i}, \beta_i D_f\right)}{\Gamma\left(h(ks)t^{c_i}\right)} \approx 1 - \Phi\left(\frac{D_f\beta_i - h(ks)t^{c_i}}{\sqrt{h(ks)t^{c_i}}}\right) \tag{17}$$

$$R_{T_i(ks)}(t(ks)) = 1 - \frac{\Gamma\left(h(ks)t^{c_i}, \beta_i D_f\right)}{\Gamma\left(h(ks)t^{c_i}\right)} \approx \Phi\left(\frac{D_f\beta_i - h(ks)t^{c_i}}{\sqrt{h(ks)t^{c_i}}}\right) \tag{18}$$

where 'ks' in parentheses indicates the stress level.

Under the settings of the above, we have the likelihood function for the i^{th} sample under the hardened stress level:

$$L_i(\mathbf{\Theta}) = \prod_{j=1}^{m} \frac{\beta_i^{\Delta\alpha_i(t_j)}\left(\Delta y_{ij}\right)^{\Delta\alpha_i(t_j)-1}}{\Gamma\left(\Delta\alpha_i\left(t_j\right)\right)} \exp\left(-\beta_i \cdot \Delta y_{ij}\right) \tag{19}$$

where $\mathbf{\Theta} = (a, b, k, \beta_i, c_i)$, $\Delta y_{ij} = y_{ij} - y_{i(j-1)}$, $\Delta\alpha_i\left(t_j\right) = h(ks) \cdot \left(t_j^{c_i} - t_{j-1}^{c_i}\right)$, and m equals that t_m divided by the measurement interval τ. Specifically, the values of parameters (a, b) can be determined by the grope testing, the value of k is determined during the testing design process.

III. SAMPLE SIZE UNDER HARDENED STRESS LEVEL

In general, when the product's life distribution is unknown, a binomial distribution can be used to plan the testing, the minimum sample size that ZFT requires can be calculated by:

$$n = \frac{\ln\delta}{\ln R_s} \tag{20}$$

where δ represents the specified type II error (consumer's risk), and R_s represents the reliability index that needs to be verified at a $100(1-\delta)\%$ confidence level.

The conventional ZFT considers at a $100(1-\delta)\%$ confidence level that the product's reliability is not lower than R_s when no samples fail at the specified time t_s under the normal stress level, and the type II error will not exceed δ caused only by the sampling error.

However, ZFT based on the accelerated degradation information incurs an additional type II error δ_c if the testing is terminated at t_m, where $t_m < t_s$. Thus, the probability of the total type II error is $\delta = \delta_s + \delta_c$, namely, $\delta_s = \delta - \delta_c$. If the testing is terminated until t_s under the normal stress level, then $\delta_c = 0$. As discussed above, we can amend (20) as:

$$n = \frac{\ln(\delta - \delta_c)}{\ln R_s} \tag{21}$$

As shown in (21), the sample size increase when δ_c is larger, but the increase is not significant if δ_c is a small fraction of δ, as shown in Fig. 1.

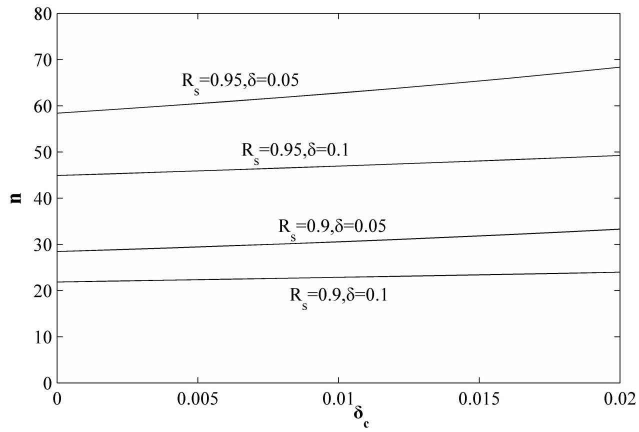

FIGURE 1. Sample sizes at various values of δ_c.

Obviously, the reliability index R_s pre-determined by the engineering experience should be lowered under the hardened stress level. This paper presents a method to transform the reliability index under the normal stress level to that under the hardened stress level. First, we can obtain the reliability at time t_s based on (18):

$$R_{T_i(ks)}\left(t_s\left(ks\right)\right) \approx \Phi\left(\frac{D_f \beta_i - h\left(ks\right) t_s{}^{c_i}}{\sqrt{h\left(ks\right) t_s{}^{c_i}}}\right) \tag{22}$$

The values of D_f, t_s is pre-determined, and when $h(ks)$, c_i is determined, $R_{T_i(ks)}\left(t_s\left(ks\right)\right)$ can be regarded as the monotonic function of β_i. As mentioned in section II, the values of parameters (a, b) can be determined by the grope testing, so λ_i is only related to k as shown in (14-16). As for c_i, some classical values are given for the degradation of concrete by Ellingwood and Mori [33]: corrosion of reinforcement (linear: $c_i = 1$), sulfate attack (parabolic: $c_i = 2$), and diffusion-controlled aging (square root: $c_i = 0.5$). There is often engineering knowledge available about the shape of the expected deterioration, so the parameter c_i can be assumed constant for different samples and can also be determined by the grope testing [23]. Thus, if one sample can achieve the reliability R_s at t_s, then we have (23) under the normal stress level ($k = 1$) according to (22):

$$\Phi\left(\frac{D_f \beta_{is} - h\left(s\right) t_s{}^{c}}{\sqrt{h\left(s\right) t_s{}^{c}}}\right) = R_s$$
$$\Rightarrow \beta_{is} = \frac{\left(\Phi^{-1}\left(R_s\right).\sqrt{h(s) t_s{}^{c}} + h(s) t_s{}^{c}\right)}{D_f} \tag{23}$$

Combining (22) with (23), then we can easily obtain the reliability index (24) that the sample should achieve under the hardened stress level ($k > 1$), and the sample size under the hardened stress level can be expressed as (25):

$$R_s^{\mathrm{H}} = \Phi\left(\frac{\left(\Phi^{-1}\left(R_s\right).\sqrt{h(s) t_s{}^{c}} + h(s) t_s{}^{c}\right) - h(ks) t_s{}^{c}}{\sqrt{h(ks) t_s{}^{c}}}\right) \tag{24}$$

$$n = \frac{\ln\left(\delta - \delta_c\right)}{\ln R_s^{\mathrm{H}}} \tag{25}$$

To make the reliability index R_s^{H} under the hardened stress level more accurate, the grope testing should be conducted for multiple samples to assure that the deviation of the parameter value is as small as possible.

Just to show the effect of the hardened testing method on the decrease of the sample size, we assume $R_s = 0.9$, $s = 10^4$KPa, $t_s = 10000$d, $D_f = 60$, $h\left(ks\right) = 0.02 \cdot (ks)^{0.5}$, $c = 0.5$. Thus, we can obtain the sample sizes at various values of k and δ_c , as shown in Fig. 2.

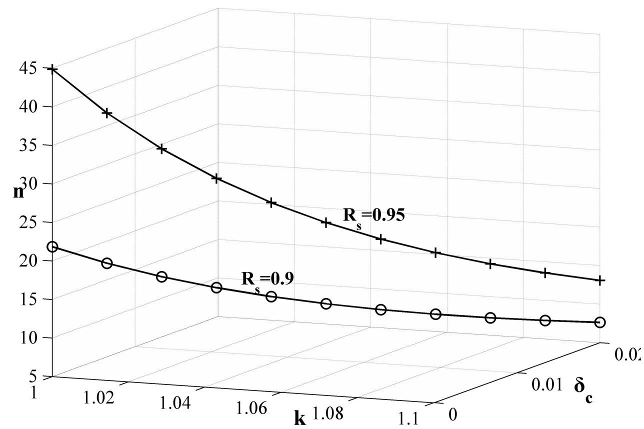

FIGURE 2. Sample sizes at various values of k and δ_c.

Fig. 2 shows us that although the value of δ_c increases, the general trend of the sample size goes down when the hardened coefficient just increases a little, which is very beneficial for ZFT of expensive products.

IV. OPTIMAL PLANS AND TESTING TERMINATION RULES FOR ACCELERATED DEGRADATION ZFT

Based on the failure probability and the sample size under the hardened stress level, we give the optimal plans and testing termination rules for the accelerated degradation ZFT by minimizing the cost function.

A. TWO TYPES OF ERROR PROBABILITIES

On the one hand, $F_{T_i(ks)}\left(t\left(ks\right)\right)$ is the conditional failure probability, which is implicitly conditional on t_m and $\lambda = h(ks)$ according to (17). On the other hand, two types of error probabilities are mainly related to the estimated errors of model parameters. As we can see, although the principle of the constant acceleration factor stipulates that parameter β_i should keep constant under the hardened stress level, there will always be a certain error when we directly apply $\hat{\beta}_i$ estimated under the hardened stress level to infer that whether the sample will fail at t_s under the normal stress level.

From the analysis above, we assume that the true value of parameter β_i is uniformly distributed in a certain interval $\left[\hat{\beta}_i - \zeta, \hat{\beta}_i + \zeta\right]$, where $\hat{\beta}_i$ is the estimated value of parameter β_i for the i^{th} sample, ζ is a known constant.

If one sample is tested until t_m under the hardened stress level of ks $(k \geq 1)$, and for this sample that has $y_{im}\left(t_s\left(ks\right)\right) < D_f$, then the failure probability $F_{T_i(s)}\left(t_s\left(s\right)\right)$ at time t_s under the normal stress level represents the type II error (consumer's risk) incurred by the hardened stress level and the early termination of the testing at t_m. In addition, considering $\beta_i \in \left[\hat{\beta}_i - \zeta, \hat{\beta}_i + \zeta\right]$, we can obtain the upper bound of the type II error probability $F_{T_i(s)U}\left(t_s\left(s\right)\right)$ as:

$$F_{T_i(s)U}(t_s(s)) \approx 1-\Phi\left(\frac{D_f\left(\hat{\beta}_i-\zeta\right)-h(s){t_s}^c}{\sqrt{h(s){t_s}^c}}\right), \quad (k=1) \tag{26}$$

where $\hat{\beta}_i = h(ks)\cdot {t_m}^c/E\left(Y\left(t_m\left(ks\right)\right)\right)$ according to (5), and $E\left(Y\left(t_m\left(ks\right)\right)\right) \approx \sum_{i=1}^{n} y_{im}\left(ks\right)/n$, n is the sample size.

In the same way, we can obtain the upper bound of the type I error probability (producer's risk) as:

$$1-F_{T_i(s)U}(t_s(s)) \approx \Phi\left(\frac{D_f\left(\hat{\beta}_i+\zeta\right)-h(s){t_s}^c}{\sqrt{h(s){t_s}^c}}\right), \quad (k=1) \tag{27}$$

B. OPTIMIZATION MODEL BASED ON COST FUNCTION

Obviously, the cost of the accelerated degradation ZFT comes from the sample size n and the testing time t_m. However, if the hardened stress level cannot be controlled very well, the product may be damaged while it will not be damaged under the normal stress level. Thus, we cannot ignore the risk cost caused by the hardened stress level. In addition, t_m varies from sample to sample, so we use an average testing time t_a of all samples for the cost modeling, from which we can obtain the average measurement times $m = t_a/\tau$. Thus, the total cost $\mathrm{TC}\left(n, t_a, k\right)$ can be expressed as:

$$\mathrm{TC}\left(n, t_a, k\right) = w_1 n + w_2 t_a + w_3 nm + w_4\left(k-1\right) \tag{28}$$

where $w_1 n$ is the sample cost; $w_2 t_a$ is the cost of conducting an accelerated degradation ZFT; $w_3 nm$ is the cost of measurements; $w_4\left(k-1\right)$ is the risk cost.

Combining (25) with (28), we can obtain the more concrete expression as follows:

$$\mathrm{TC}\left(\delta_c, \eta, k\right) = \frac{\ln\left(\delta-\delta_c\right)}{\ln R_s^{\mathrm{H}}}\left(w_1 + w_3\frac{\eta t_s}{\tau}\right) + w_2\eta t_s + w_4\left(k-1\right) \tag{29}$$

where $\eta = t_m/t_s \approx t_a/t_s$.

Actually, there are two ways to connect the characteristic between the normal testing and the hardened testing: the first one is what we discuss above, i.e., the testing time t_s keep the same, but the reliability index R_s is lowered under the hardened stress level; the second one is that the reliability index keep the same, but the testing time should be shortened to achieve R_s under the hardened stress level, where the testing time under the hardened stress level is t_s/A_F. The second one cannot decrease the sample size, but can further shorten the testing time, and the total cost can be expressed as:

$$\mathrm{TC}\left(\delta_c, \eta, k\right) = \frac{\ln\left(\delta-\delta_c\right)}{\ln R_s}\left(w_1 + w_3\frac{\eta t_s}{A_F\tau}\right) + w_2\frac{\eta t_s}{A_F} + w_4(k-1) \tag{30}$$

Equation (30) is generally applied to the situation where the testing cost per unit time is very expensive, whereas (29) is generally applied to the situation where each sample's cost is very expensive or the sample size is very limited.

To obtain the optimal values of (δ_c, η, k), we need to minimize the total cost, as shown in (29) and (30). However, $\mathrm{TC}\left(\delta_c, \eta, k\right)$ should satisfy some constraints given by the engineering experience: 1) the probability of the type II error caused by the hardened stress level and the early termination of the testing for any sample cannot exceed δ_c/n [22]; 2) the sample size cannot be larger than the number n_a that is available for the testing; 3) the hardened stress level ks must not exceed the critical stress level $k_c s$, i.e., $1 \leq k < k_c$, or a mass of products may be damaged. Based on these optimization criteria above, and assuming that the sample size for the grope testing is x, then the optimization model combining with the cost function (29) can be written as:

$$\min_{(\delta_c,\eta,k)} \left\{\frac{\ln\left(\delta-\delta_c\right)}{\ln R_s^{\mathrm{H}}}\left(w_1+w_3\frac{\eta t_s}{\tau}\right)+w_2\eta t_s+w_4(k-1)\right\} \tag{31}$$

$$\text{subject to}\ \begin{cases} 1-\Phi\left(\dfrac{D_f\left(\frac{h(ks)\cdot(\eta t_s)^c}{E(Y(t_m(ks)))}-\zeta\right)-h(s){t_s}^c}{\sqrt{h(s){t_s}^c}}\right) \leq \dfrac{\delta_c \ln R_s^{\mathrm{H}}}{\ln\left(\delta-\delta_c\right)} \\ \dfrac{\ln\left(\delta-\delta_c\right)}{\ln R_s^{\mathrm{H}}} \leq n_a \Rightarrow 0 \leq \delta_c \leq \delta - \left(R_s^{\mathrm{H}}\right)^{n_a} \\ 1 \leq k \leq k_c \\ 0 < \eta \leq 1 \end{cases} \tag{32}$$

Because the hardened coefficient k is a variable that needs to be determined and cannot be known in advance, the degradation measure $y_{im}(ks)$ under the hardened stress level cannot be known, which makes $E\left(Y\left(t_m\left(ks\right)\right)\right) \approx \sum_{i=1}^{n} y_{im}\left(ks\right)/n$ incalculable. Considering the mean degradation measure $E\left(Y\left(t_m\left(ks\right)\right)\right)$ at t_m under the hardened stress level can be approximated as $E\left(Y\left(t_s\left(s\right)\right)\right)$ at t_s under the normal stress level, we have:

$$\begin{cases} 1-\Phi\left(\dfrac{D_f\left(\frac{h(ks)\cdot(\eta t_s)^c}{E(Y(t_s(s)))}-\zeta\right)-h(s){t_s}^c}{\sqrt{h(s){t_s}^c}}\right) \\ \quad\le \dfrac{\delta_c \ln R_s^{\mathrm{H}}}{\ln(\delta-\delta_c)} \\ \dfrac{\ln(\delta-\delta_c)}{\ln R_s^{\mathrm{H}}} \le n_a \Rightarrow 0\le\delta_c\le\delta-\left(R_s^{\mathrm{H}}\right)^{n_a} \\ E\left(Y\left(t_s(s)\right)\right)=\sum_{i=1}^{x} y_{is}(s)/x \\ 1\le k\le k_c \\ 0<\eta\le 1 \end{cases} \tag{33}$$

Likewise, we can obtain the optimization model based on the cost function (30):

$$\min_{(\delta_c,\eta,k)}\left\{\frac{\ln(\delta-\delta_c)}{\ln R_s}\left(w_1+\frac{w_3\eta t_s}{A_F\tau}\right)+\frac{w_2\eta t_s}{A_F}+w_4(k-1)\right\} \tag{34}$$

$$\text{subject to}\begin{cases} 1-\Phi\left(\dfrac{D_f\left(\frac{h(ks)\cdot(\eta t_s)^c}{E(Y(t_s(s)))}-\zeta\right)-h(s){t_s}^c}{\sqrt{h(s){t_s}^c}}\right) \\ \quad\le \dfrac{\delta_c \ln R_s}{\ln(\delta-\delta_c)} \\ \dfrac{\ln(\delta-\delta_c)}{\ln R_s} \le n_a \Rightarrow 0\le\delta_c\le\delta-\left(R_s^{H}\right)^{n_a} \\ A_{F(ks,s)}=\exp\left(\dfrac{(k-1)\,b}{ksc}\right) \\ E\left(Y\left(t_s(s)\right)\right)=\sum_{i=1}^{x} y_{is}(s)/x \\ 1\le k\le k_c \\ 0<\eta\le 1 \end{cases} \tag{35}$$

C. TEST TERMINATION RULES FOR ONE SAMPLE

The degradation measure for each sample is different, so the termination time of the testing may be different for each sample. Therefore, it is necessary to give the testing termination rules for each sample. In this paper, we quote the decision rules in [22] for terminating the testing of one sample.

As discussed above, we can calculate and update the upper bounds of two types of error probabilities $F_{T_i(s)U}\left(t_s(s)\right)$ and $1-F_{T_i(s)U}\left(t_s(s)\right)$ according to the Gamma process and the newest measurement data during testing. For one sample that has $y_{im}\left(t_s(ks)\right) < D_f$, then $F_{T_i(s)U}\left(t_s(s)\right)$ represents the upper bound of the type II error (consumer's risk) incurred by the hardened stress level and the early termination of the testing. When t_m is small, the type II error is too large to be acceptable. As t_m increases, the type II error decreases. When $F_{T_i(s)U}\left(t_s(s)\right)$ is smaller than the acceptable value, we can terminate the testing. Similarly, For one sample that has $y_{im}\left(t_s(ks)\right) > D_f$, then $1-F_{T_i(s)U}\left(t_s(s)\right)$ represents the upper bound of the type I error, which is too large at the beginning of the testing and decreases as t_m increases. When $1-F_{T_i(s)U}\left(t_s(s)\right)$ is smaller than the acceptable value, we can terminate the testing. However, there will be some situations where $y_{im}\left(t_s(ks)\right)$ is very close to D_f, then the testing should be continued until a large t_m or even t_s. Thus, the decision rules for terminating the testing of one sample are as follows:

(1) If $F_{T_i(s)U}\left(t_s(s)\right) \le \dfrac{\delta_c}{n}$, as shown in Fig. 3, terminate the testing after the m^{th} measurement. The sample will not fail at t_s under the normal stress level, and the testing time is reduced to t_m.

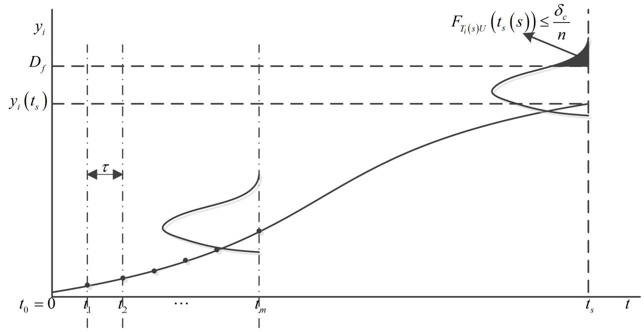

FIGURE 3. $F_{T_i(s)U}\left(t_s(s)\right)$ is small enough, leading to rule (1).

(2) If $1-F_{T_i(s)U}\left(t_s(s)\right) \le \dfrac{\rho}{n}$, as shown in Fig. 4, terminate the testing after the m^{th} measurement, where ρ represents the producer's risk . The sample will fail at t_s under the normal stress level, and the testing time is reduced to t_m.

(3) If $F_{T_i(s)U}\left(t_s(s)\right) > \dfrac{\delta_c}{n}$ and $1-F_{T_i(s)U}\left(t_s(s)\right) > \dfrac{\rho}{n}$, as shown in Fig. 5, continue the testing until (1) or (2) is satisfied or the testing time reaches t_s.

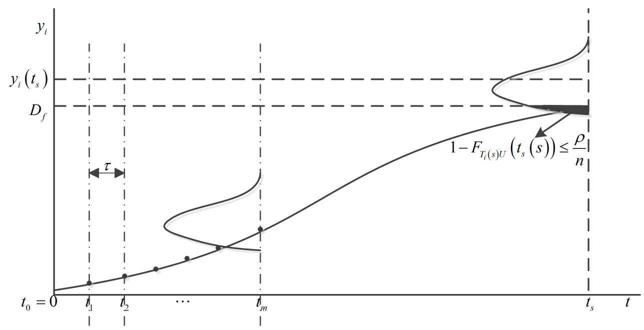

FIGURE 4. $F_{T_i(s)U}\left(t_s(s)\right)$ is large enough, leading to rule (2).

V. NUMERICAL EXAMPLE

The magnet ring is a crucial structure which assures to produce the steady space magnetic field. There are mainly two kinds of magnet rings: 1) Mn-Zn magnet ring which is generally used to make inductors, transformers, magnetic heads and aerial rods, etc; 2) Ni-Zn magnet which is generally used to make communication equipment lines, anti-jamming filter lines and data cables, etc. Considering its wide applications and important functions, the reliability must be assured even after a long-time use. In this paper, we use the

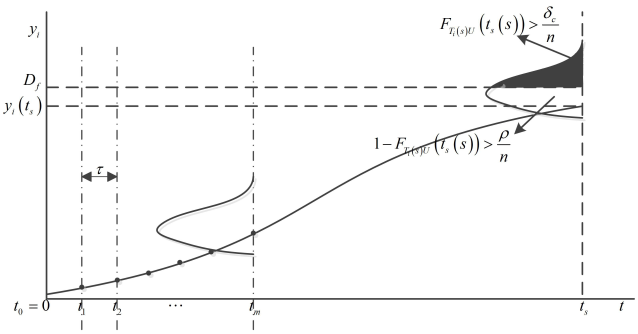

FIGURE 5. $F_{T_i(s)U}(t_s(s))$ **is neither too large nor too small.**

TABLE 1. **Opimization results.**

Variable values			Minimal total cost	Optimal testing scheme		
δ_c	η	k	$\mathrm{TC}(\delta_c, \eta, k)$	Sample size	Testing time	Stress level
0.056	0.53	1.84	\$19720	9	4609.7h	539.9K

magnetic flux to describe the performance characteristic, and the magnetic flux will gradually decrease because of complex environmental factors, such as temperature. We assume that if the degradation measure of magnetic flux exceeds 3%, then the magnet ring fails. To produce high-reliability and long-life magnet rings, the reliability index of RDT is required to achieve 0.98 at 1 year. We want to demonstrate this reliability at a 90% confidence level.

A. OPTIMAL RDT DESIGN

The traditional ZFT requires 114 samples to be tested for 1 year, which will be unrealistic if the sample size is limited to less than 114. To decrease the sample size and shorten the testing time, we use the proposed method to demonstrate the reliability 0.98 at a 90% confidence level.

To obtain the optimal scheme of the accelerated degradation ZFT, we select temperature to be the hardened stress and Arrhenius model to be the acceleration model. In addition, we assume that the normal temperature is 293K (20°C), and the highest temperature cannot exceed 673K (400°C), i.e., the critical hardened coefficient is about 2.29 ($1 \le k \le 2.29$). Assuming that the average degradation measure at t_s and the estimated values of parameters have been obtained by the grope testing (the degradation data come from 10 similar magnet rings that have been used in the engineering): $\sum_{i=1}^{10} y_{is}(s)/10 \approx 1.55$, $\hat{a} = 0.06$, $\hat{b} = 300.23$, $\hat{c} = 0.61$. The relevant testing costs are determined by the engineering experience: $w_1 = \$30$ one sample, $w_2 = \$50$ per day, $w_3 = \$5$ per measurement, $w_4 = \$10000$; the pre-specified $D_f = 3\%$, $\zeta = 0.05$; the allowable sample size $n_a = 15$; the type I error (producer's risk) $\rho = 0.1$, the type II error (customerâĂŹs risk) $\delta = 0.1$; $t_s = 8760\text{h}$, $R_s = 0.98$; the measurement interval $\tau = 120\text{h}$. Taking the above data into (31) and (33), then we have the optimization model,

TABLE 2. **Opimization results.**

Variable value			Minimal total cost	Optimal testing scheme		
δ_c	η	k	$\mathrm{TC}(\delta_c, \eta, k)$	Sample size	Testing time	Stress level
0.016	0.63	2.29	\$45206	123	2126.5h	673K

as shown in (36) and (37).

$$\min_{(\delta_c,\eta,k)} \left\{ \begin{array}{l} \frac{\ln(0.1-\delta_c)}{\ln R_s^{\mathrm{H}}}\left(30+5\times\frac{8760\eta}{120}\right) \\ +50\times 365\eta + 10000(k-1) \end{array} \right\} \tag{36}$$

subject to

$$\left\{ \begin{array}{l} 1-\Phi\left(\frac{3\left(\frac{0.06\exp(-300.23/293k)\cdot(8760\eta)^{0.61}}{E(Y(t_s(s)))} - 0.05\right) - 0.06\exp(-300.23/293)(8760)^{0.61}}{\sqrt{0.06\exp(-300.23/293)(8760)^{0.61}}} \right) \\ \le \frac{\delta_c \ln R_s^{\mathrm{H}}}{\ln(0.1-\delta_c)} \\ \frac{\ln(0.1-\delta_c)}{\ln R_s^{\mathrm{H}}} \le 15 \Rightarrow 0 \le \delta_c \le 0.1 - \left(R_s^H\right)^{15} \\ R_s^{\mathrm{H}} \\ = \Phi\left(\frac{\begin{array}{l} \Phi^{-1}(0.98).\sqrt{0.06\exp(-300.23/293)(8760)^{0.61}} \\ +0.06\exp(-300.23/293)(8760)^{0.61} \\ -0.06\exp(-300.23/(293k))(8760)^{0.61} \end{array}}{\sqrt{0.06\exp(-300.23/(293k))(8760)^{0.61}}} \right) \\ E(Y(t_s(s))) = \sum_{i=1}^{10} y_{is}(s)/10 \approx 1.55 \\ 1 \le k \le 2.29 \\ 0 < \eta \le 1 \end{array} \right. \tag{37}$$

Solving the optimization model, we can obtain the optimal values of (δ_c, η, k), the minimal total cost and the optimal testing plan, as shown in Table 1:

Table 1 tells us that we should conduct an accelerated degradation ZFT for 9 samples at the temperature of 539.9K (266.9°C). The average testing time is 4609.7h, and the average measurement times is $m = t_a/\tau \approx 38$.

As we can see, the average testing time is a little long. If we cannot accept the testing time, we can use the hardened testing method to further shorten the testing time, but the sample size will be larger than 114. Thus, let $n_a = 150$, we can have the optimization model based on (34) and (35) as:

$$\min_{(\delta_c,\eta,k)} \left\{ \begin{array}{l} \dfrac{\ln(0.1-\delta_c)}{\ln 0.98}\left(30+5\times\dfrac{8760\eta}{120\cdot A_{F(ks,s)}}\right) \\ +\dfrac{50\times 365\eta}{A_{F(ks,s)}}+10000(k-1) \end{array} \right\} \tag{38}$$

subject to

$$\begin{cases} 1-\Phi\left(\dfrac{\left\{ \begin{array}{l} 3\left(\begin{array}{l}\frac{0.06\exp(-300.23/293k)\cdot(8760\eta)^{0.61}}{E(Y(t_s(s)))} \\ -0.05\end{array}\right) \\ -0.06\exp(-300.23/293)(8760)^{0.61}\end{array}\right\}}{\sqrt{0.06\exp(-300.23/293)(8760)^{0.61}}}\right) \\ \quad \le \dfrac{\delta_c \ln 0.98}{\ln(0.1-\delta_c)} \\ \dfrac{\ln(0.1-\delta_c)}{\ln 0.98} \le 150 \Rightarrow 0 \le \delta_c \le 0.1-(0.98)^{150} \\ A_{F(ks,s)} = \exp\left(\dfrac{300.23\cdot(k-1)}{293\cdot 0.61\cdot k}\right) \\ E(Y(t_s(s))) = \sum_{i=1}^{10} y_{is}(s)/10 \approx 1.55 \\ 1 \le k \le 2.29 \\ 0 < \eta \le 1 \end{cases} \tag{39}$$

In the same way, we can obtain the optimal values of (δ_c, η, k), the minimal total cost and the optimal testing plan, as shown in Table 2:

As we can see from Table 2, the average testing time is shortened about 2500h, but the minimal total cost increases about \$25000 and the hardened coefficient is too large. The sample size is acceptable, but on the whole, using the hardened testing method to decrease the sample size is more acceptable and economic. Therefore, the following testing termination rules for one sample is given based on Table 1.

(1) If $F_{T_i(s)U}(t_s(s)) \le 6.24\times10^{-3}$, terminate the testing of the sample. The sample passes the testing.

(2) If $1-F_{T_i(s)U}(t_s(s)) \le 0.011$, terminate the testing of the sample. The sample fails to pass the testing.

(3) If $F_{T_i(s)U}(t_s(s)) > 6.24\times10^{-3}$ and $1-F_{T_i(s)U}(t_s(s)) > 0.011$, continue the testing until (1) or (2) is satisfied or the testing time reaches t_s.

B. DEGRADATION DATA ANALYSIS FOR ONE TESTING SAMPLE

Suppose that the 9 samples of magnet rings are tested at the hardened temperature of 539.9K (266.9°C). We use Monte-Carlo method to generate a set of degradation data, and a part of the degradation data and relevant analysis of one sample are shown in Table 3.

As we can see from Table 3, when we have the fourteenth measurement at 1680h, the estimated values of $\hat{\alpha}_i(t_j)$ and $\hat{\beta}_i$ are 3.1846 and 2.8930 ($\hat{\alpha}_i(t_j)$ changes with time, $\hat{\beta}_i$ changes with the newest degradation data). The upper bound of the type II error probability is 0.0955, which leads to the termination rule (3). Therefore, the test should be continued.

TABLE 3. Degradation data analysis for one testing sample.

m	t_j	y_{ij}	$\hat{\alpha}_i(t_j)$	$\hat{\beta}_i$	$\hat{\beta}_i-\zeta$	Type I Error	Decision
5	600	0.6457	1.6994	2.6318	2.5818	0.1654	3
6	720	0.6762	1.8993	2.8087	2.7587	0.1152	3
7	840	0.6863	2.0865	3.0403	2.9903	0.0673	3
8	960	0.8656	2.2636	2.6151	2.5651	0.1708	3
9	1080	0.8949	2.4322	2.7179	2.6679	0.1395	3
10	1200	0.8990	2.5937	2.8851	2.8351	0.0973	3
11	1320	0.9140	2.7489	3.0076	2.9576	0.0729	3
12	1440	0.9196	2.8988	3.1522	3.1022	0.0505	3
13	1560	1.0418	3.0438	2.9217	2.8717	0.0894	3
14	1680	1.1008	3.1846	2.8930	2.8430	0.0955	3
:	:	:	:	:	:	:	:
25	3000	1.3293	4.5358	3.4122	3.3622	0.0242	3
26	3120	1.3348	4.6457	3.4804	3.4304	0.0197	3
27	3240	1.3431	4.7539	3.5395	3.4895	0.0163	3
28	3360	1.3793	4.8605	3.5239	3.4739	0.0171	3
29	3480	1.3802	4.9657	3.5978	3.5478	0.0135	3
30	3600	1.3843	5.0694	3.6621	3.6121	0.0109	3
31	3720	1.3979	5.1718	3.6997	3.6497	0.0096	3
32	3840	1.4074	5.2730	3.7466	3.6966	0.0081	3
33	3960	1.4172	5.3729	3.7912	3.7412	0.0070	3
34	4080	1.4322	5.4716	3.8204	3.7704	0.0063	3
35	4200	1.4381	5.5692	3.8726	3.8226	0.0052	1

Obviously, when we have the thirty-fifth measurement at 4200h, the estimated values of $\hat{\alpha}_i(t_j)$ and $\hat{\beta}_i$ are 5.5692 and 3.8726. The upper bound of the type II error probability is 0.0052 leading to the termination rule (1). Therefore, the testing should be terminated, and the sample passes the testing.

C. ROBUSTNESS ANALYSIS OF OPTIMAL TESTING PLAN

On the one hand, we use $E(Y(t_s(s)))$ pre-estimated from the grope degradation testing to approximate $E(Y(t_m(ks)))$, so the relative error on the optimal testing plan caused by this approximation should be discussed based on the robustness analysis against the true value of $E(Y(t_m(ks)))$. On the other hand, the certain error ζ for $\hat{\beta}_i$ is determined by the engineering experience, the relative error on the optimal testing plan caused by the engineering experience should also be discussed based on the robustness analysis against the true value of ζ.

To perform a numerical evaluation, suppose the true values of $E(Y(t_m(ks)))$ and ζ are the values given by the above example, i.e., $E(Y(t_m(ks))) = 1.55$, $\zeta = 0.05$, and that pre-estimated values deviate $\pm10\%$ from the true values. Then the optimal testing plan are respectively calculated based on different combinations of the deviations, the relative error compared with the optimal testing plan shown in Table 1 and Table 2 can be seen in Table 4 and 5. The relative error is defined as $(x_d - x_t)/x_t$, where x_d is the approximated value, and x_t is the true value. From Table 4 and 5, we can give the following three conclusions:

(1) The sample size and stress level have strong robustness against the pre-estimated values of $E(Y(t_m(ks)))$ and ζ,

TABLE 4. Relative errors against optimazation results (Table 1) caused by parameter errors.

Deviations (%)		Relative error (%)			
$E(Y(t_m(ks)))$	ζ	$\mathrm{TC}(\delta_c, \eta, k)$	Sample size	Testing time	Stress level
-10	-10	-8.2	0	-11.7	-3.83
-10	0	-8.1	0	-11.5	-3.79
-10	10	-8.0	0	-11.4	-3.75
0	-10	1.1	0	-0.16	-0.005
0	0	0	0	0	0
0	10	1.3	0	0.16	0.004
10	-10	10.5	0	11.5	3.67
10	0	10.6	0	11.72	3.72
10	10	10.7	0	11.9	3.77

TABLE 5. Relative errors against optimazation results (Table 2) caused by parameter errors.

Deviations (%)		Relative error (%)			
$E(Y(t_m(ks)))$	ζ	$\mathrm{TC}(\delta_c, \eta, k)$	Sample size	Testing time	Stress level
-10	-10	-10.1	0	-15.9	-0.3
-10	0	-10.0	0	-15.7	-0.3
-10	10	-10.0	0	-15.6	-0.3
0	-10	-0.11	0	-0.18	-0.3
0	0	0	0	0	0
0	10	0.11	0	0.18	0.3
10	-10	10.6	0	16.6	0.3
10	0	10.7	0	16.8	0.3
10	10	10.8	0	17.0	0.3

we may not be worried about the error on the sample size and stress level.

(2) The value of ζ has very small effect on the optimal testing plan.

(3) The value of $E(Y(t_m(ks)))$ has moderate effect on the total cost $\mathrm{TC}(\delta_c, \eta, k)$ and average testing time. Moreover, when the pre-estimated value is larger than the true value, both the total cost and average testing time increase, on the contrary, both the total cost and average testing time decrease.

To sum up, the robustness of the optimal testing plan against $E(Y(t_m(ks)))$ and ζ is great, i.e., we can believe the rationality of the optimal testing plan obtained by the proposed method.

VI. CONCLUSION

For the reliability demonstration of long-life and high-reliability products, we propose the accelerated degradation ZFT method under the hardened stress level to enrich the traditional ZFT with binomial distribution, which aims to shorten the testing time and reduce the sample size, and make an improvement on the degradation bogey test (DBT) method proposed by Yang [22]. The results of the given example show that the sample size can be dramatically reduced under the hardened condition when the testing time can still be shortened, and the robustness of the optimal testing plan is good.

The innovation points lie in two aspects. Firstly, we employ the Gamma process and accelerated degradation model to deduce the failure probability model, which is applied to predict whether or not one sample will fail by the end of the testing and shorten the testing time. Secondly, the hardened testing method is considered for reducing the sample size or further shortening the testing time.

The numerical example shows that the method we propose can efficiently shorten the testing time and decrease the sample size, and the robustness is great as well. The optimal testing plan obtained by the proposed method is reasonable and feasible, which has the significance for the reliability demonstration of long-life and high-reliability products in some way.

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