IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received June 5, 2017, accepted June 28, 2017, date of publication July 4, 2017, date of current version July 24, 2017.

Digital Object Identifier 10.1109/ACCESS.2017.2723462

Fixed-Time Group Tracking Control With Unknown Inherent Nonlinear Dynamics

YILUN SHANG AND YAMEI YE

School of Mathematical Sciences, Tongji University, Shanghai 200092, China

Corresponding author: Yilun Shang (shylmath@hotmail.com)

This work was supported in part by the National Natural Science Foundation of China under Grant 11505127, in part by the Shanghai Pujiang Program under Grant 15PJ1408300 and in part by the Program for Young Excellent Talents in Tongji University under Grant 2014KJ036.

ABSTRACT In this paper, the fixed-time group tracking problem for multi-agent systems with unknown inherent nonlinear dynamics is studied. A distributed tracking control protocol is introduced to ensure that the follower agents in each subgroup can track their respective leaders in a prescribed time regardless of the initial conditions. Compared with the existing works on group (tracking) consensus, we do not require the inter-group balance condition, and the leaders are allowed to interact with follower agents in different subgroups. Some conditions have been derived to choose appropriate control gains to achieve the fixed-time group tracking. Finally, numerical simulations are presented to illustrate the availability of our results.

INDEX TERMS Fixed-time consensus, group consensus, multi-agent system, inherent nonlinear dynamics, leader-follower.

I. INTRODUCTION

Over the past two decades, distributed cooperative control of multi-agent systems has attracted considerable attention [1], [2] because coordination control reduces system costs, enhances resilience against possible agent fault, breaches the size constraints, and increases flexibility in performance as compared to traditional monolithic ones. Coordination control of multi-agent systems has found a wide range of applications in areas including distributed computation, coordination of distributed sensor networks, cooperation of unmanned aerial vehicles, and formation of multi-robots etc.; see, e.g., [3]–[5]. Many control tasks in multi-agent systems can be boiled down to consensus problems [1], which aim to design distributed protocols and algorithms based only on the local relative information such that the states of all agents reach an agreement, i.e., converge to a consistent value. Various types of control protocols, such as average consensus protocols, leader-following consensus protocols, and event-based control protocols, have been proposed to deal with different agent dynamics and communication constraints; see the updated survey papers [2], [5] and references therein.

While most of the existing works are concerned with global consensus, namely, all the agents reach a common state, in many practical applications, there may be multiple consistent states as agents are often divided into some subgroups to carry out different cooperative tasks. Examples include team hunting of predators, obstacle avoidance of flocks and herds, coordinated military actions, and cooperative searching of autonomous vehicles for multiple objects. As an extension to global consensus protocols, group (or cluster) consensus protocols [6], [7] have been proposed to solve these issues, where the states of of multiple agents in each subnetwork converge to an individual consistent state asymptotically when information exchanges exist not only among agents within the same subnetwork but among those in different subnetworks. Group consensus problems have been studied intensively in recent years for both continuous-time (e.g., [6]–[10]) and discrete-time systems (e.g., [11]–[13]), to name just a few. However, most of the existing group consensus protocols have been designed to achieve group consensus when there is no leader or the ultimate consistent states are not explicitly provided. On the other hand, the leader-follower consensus problem (a.k.a., consensus tracking [14]) has been firstly motivated in [15], where a group of mobile autonomous agents (followers) asymptotically track the leader by exchanging their own state information with their neighbors. Consensus tracking protocols have many applications and have been further developed recently to solve group consensus tracking problems with multiple leaders for a second-order multi-agent system in [9] and to achieve event-triggered group consensus in [16], both in the manner of asymptotic convergence. The main goal of the current paper is to move a further step along this line of research by focusing on the convergence rate.

In the consensus problems, convergence rate is a significant performance indicator of the control strategies. Compared to the usual asymptotic algorithms, finite-time controller enjoys some attractive properties such as faster convergence rate, better disturbance rejection, and more robustness to uncertainties [17]. Finite-time consensus problems have been tackled for first-order [18], second-order [19], and inherent nonlinear/uncertain dynamics [20], [21]. It is worth noting that the settling time of the above finite-time control laws depends on the initial conditions of agents, which cannot guarantee a prescribed convergence time since the knowledge of initial conditions is usually not available in advance in distributed systems. To overcome this weakness, some new results based on the notion of fixed-time stability [22] have been reported recently, which allow an upper-bounded settling time independent of the initial conditions of the agents. In the leaderless scenario, fixed-time consensus protocols are proposed for multi-agent systems with integrator dynamics [23]–[25] under undirected communication topologies. The results are generalized in [26] and [27] to accommodate directed topologies. In [28], the fixed-time leader-follower consensus problem is treated for first-order multi-agent systems with unknown nonlinear inherent dynamics under undirected topologies. Two fixed-time tracking control protocols for second-order integrator systems with bounded input uncertainties are proposed in [29]. Very recently, fixed-time group consensus/synchronization has been addressed in [30] in the leaderless scenario.

Motivated by the above works, we in this paper consider the fixed-time group tracking problem for multi-agent systems with unknown inherent nonlinear dynamics. The contribution of this paper is highlighted as follows. First, compared with the existing results [9], [16], [28], [29], we generalize the leader-follower consensus problems by splitting the network into different subgroups and assigning a virtual leader to each subgroup of the system. We not only present the settling time regardless of the initial conditions, but address the unknown inherent nonlinear dynamics. Second, the proposed controllers enable group tracking without requiring the *inter-group balance condition* (c.f. Assumption 2), which is literally imposed on all the above mentioned works concerning group consensus problems, restricting the communication topology to a rigescent grouping. Third, we introduce a competition and cooperation mechanism for different groups, namely, the coupling strength between agents in different groups is allowed to be negative. Finally, our framework is less restrictive than most of the existing works dealing with group tracking in the sense that information exchange between leaders and followers in different subgroups is taken into consideration (c.f. Remark 1). We emphasize that, inspired by the controller design and convergence analysis in the recent work [28], the novelty of the current work lies in further dealing with group tracking scheme with multiple leaders in the fixed-time consensus framework and weakening some common assumptions in group consensus problems.

The rest of the paper is organized as follows. Section II gives some preliminaries and formulate the group tracking problem. Section III is devoted to the study of the fixed-time group tracking protocols. In Section IV, some examples are provided to illustrate the availability of the theoretical results. The paper is concluded in Section V.

II. NOTATIONS AND PRELIMINARIES

We begin with some notations that will be used throughout the paper. The size of a set S is denoted by $|S|$. Let $\mathbb{R}_+$ represent the set of non-negative real numbers. Let M^{T} be the transpose of a matrix M. For a symmetric matrix $M \in \mathbb{R}^{N\times N}$, $M > 0$ indicates that M is positive definite. The maximum and minimum eigenvalues are denoted by $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$, respectively. $1_N \in \mathbb{R}^N$ is a vector with all the entries being 1, and $\mathrm{diag}(a_1, \cdots, a_N) \in \mathbb{R}^{N\times N}$ is a diagonal matrix with diagonal entries $a_1, \cdots, a_N$. For a vector $x = (x_1, \cdots, x_N)^{\mathrm{T}} \in \mathbb{R}^N$ and $a \geq 0$, we define $\lfloor x\rceil^a = (\mathrm{sgn}(x_1)|x_1|^a, \cdots, \mathrm{sgn}(x_N)|x_N|^a)^{\mathrm{T}}$, where $\mathrm{sgn}(\cdot)$ is the signum function. For $p > 0$, the p-norm $\|\cdot\|_p$ is defined as $\|x\|_p = (\sum_{i=1}^N |x_i|^p)^{1/p}$ for a vector $x \in \mathbb{R}^N$. The following lemma connecting different norms is very instrumental in dealing with the fixed-time consensus problems, a proof of which can be found in [31].

Lemma 1: Let $x \in \mathbb{R}^N$ and $p > q > 0$. Then

$$\|x\|_p \leq \|x\|_q \leq N^{\frac{1}{q}-\frac{1}{p}}\|x\|_p.$$

In view of Lemma 1, we will simply denote $\|\cdot\|$ for some norm in a finite-dimensional linear space when the precise definition is not essential.

A. GRAPH THEORY

The communication topology of a multi-agent system can often be described by a graph [32]. Let $G = (V, E)$ be an undirected graph, where the node set $V = \{1, 2, \cdots, N\}$ represents N follower agents and the edge set $E \subseteq V \times V$ describes the information exchange among the followers. Define $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ as an associated weighted adjacency matrix of the graph, where $a_{ij} = a_{ji} \neq 0$ if $(i, j) \in E$ and $a_{ij} = 0$ otherwise. We will only consider undirected graphs in this work, and A satisfies $A^{\mathrm{T}} = A$.

To investigate the group consensus, a *grouping* $\mathcal{G} = \{\mathcal{G}_1, \cdots, \mathcal{G}_K\}$ of the graph G is defined by dividing its node set into disjoint groups $\{\mathcal{G}_k\}_{k=1}^K$. In other words, $\mathcal{G}$ satisfies $\cup_{k=1}^K \mathcal{G}_k = V$ and $\mathcal{G}_k \cap \mathcal{G}_{k'} = \emptyset$ for $k \neq k'$. To fix the notation, we write $\mathcal{G}_1 = \{1, \cdots, r_1\}$, $\mathcal{G}_2 = \{r_1 + 1, \cdots, r_2\}$, $\cdots$, $\mathcal{G}_K = \{r_{K-1} + 1, \cdots, N\}$. Let $r_0 = 0$. We assume that $a_{ij} \geq 0$ if $i, j \in \mathcal{G}_k$ for some k. Namely, the interactions between agents in the same group are cooperative. Naturally, $\mathcal{G}_k$ $(1 \leq k \leq K)$ inherit the structure of G in the sense of induced subgraph [32]. For each $1 \leq k \leq K$, the Laplacian matrix of $\mathcal{G}_k$ is defined as $L_k = (l_{ij}) \in \mathbb{R}^{|\mathcal{G}_k|\times|\mathcal{G}_k|}$ with $l_{ii} = \sum_{j\in\mathcal{G}_k, j\neq i} a_{ij}$ and $l_{ij} = -a_{ij}$ for $i \neq j$. It is well-known that L_k is positive semi-definite and zero is an eigenvalue of L_k with the eigenvector $1_{|\mathcal{G}_k|}$. Furthermore, we define $L \in \mathbb{R}^{N\times N}$ as a block matrix, where the K diagonal blocks are $L_1, \cdots, L_K$, and all other entries equal the corresponding entries (i.e., with the same positions) in the matrix $-A$.

Let $\Theta = \{\theta_1, \theta_2, \cdots, \theta_K\}$ be the set of K virtual leader agents. The topology of the leader-follower multi-agent system can be characterized by the weighted matrix

$$H = L + \sum_{k=1}^{K} Q_k := \begin{pmatrix} H_1 & H_{12} & \cdots & H_{1K} \\ H_{21} & H_2 & \cdots & H_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ H_{K1} & H_{K2} & \cdots & H_K \end{pmatrix} \in \mathbb{R}^{N\times N}, \quad (1)$$

where $H_k \in \mathbb{R}^{|\mathcal{G}_k|\times|\mathcal{G}_k|}$ and $Q_k = \mathrm{diag}(a_{1\theta_k}, \cdots, a_{N\theta_k}) \in \mathbb{R}^{N\times N}$ represents the information exchange between the N followers and the k-th leader for $1 \le k \le K$. The weights $\{a_{i\theta_k}\}$ can be either non-negative (i.e., cooperative) or non-positive (i.e., competitive). To delineate the overall communications between the followers and the virtual leaders, we define a new (undirected) graph $\bar{G}$ on $V \cup \{0\}$ by attaching a new node 0 to G and adding an edge between $i \in V$ and 0 whenever $\sum_{k=1}^{K} a_{i\theta_k} > 0$. The node 0 can be viewed as a super leader.

The following lemma characterizes the property of the diagonal blocks $\{H_k\}_{k=1}^{K}$ in (1), which will play a key role in the convergence analysis of tracking error system.

Lemma 2: Fix $k \in \{1, \cdots, K\}$. H_k is positive definite if and only if the following conditions hold.

(a) *Each follower in $\mathcal{G}_k$ has a path to the super leader in the graph $\bar{G}$;*

(b) *For all $i \in \mathcal{G}_k$, $\sum_{k'=1}^{K} a_{i\theta_{k'}} \ge 0$, meaning that the overall relationship between the K leaders and each follower i is cooperative.*

Proof (Sufficiency): Suppose that $S_1, \cdots, S_\rho$ with $\rho \ge 1$ are the connected components of $\mathcal{G}_k$ (viewed as an induced subgraph of G). Let $V(S_l)$ be the node set of S_l (with nodes arranged according to $S_1, \cdots, S_\rho$ without loss of generality) and write $|V(S_l)| = s_l$ for $l = 1, \cdots, \rho$. Recall that L_k is the Laplacian matrix of $\mathcal{G}_k$, which is of the form of a block diagonal matrix $L_k = \mathrm{diag}(L_{k,1}, \cdots, L_{k,\rho})$, where $L_{k,l} \in \mathbb{R}^{s_l\times s_l}$ is the Laplacian matrix of S_l. For each $1 \le k' \le K$, we write $Q_{k'}|_{\mathcal{G}_k} \in \mathbb{R}^{|\mathcal{G}_k|\times|\mathcal{G}_k|}$ for the k-th diagonal block if $Q_{k'}$ is partitioned as $Q_{k'} = \mathrm{diag}(Q_{k'}|_{\mathcal{G}_1}, \cdots, Q_{k'}|_{\mathcal{G}_K})$. We further partition $Q_{k'}|_{\mathcal{G}_k}$ according to the pattern of L_k as $Q_{k'}|_{\mathcal{G}_k} = \mathrm{diag}(Q_{k'}|_{\mathcal{G}_{k,1}}, \cdots, Q_{k'}|_{\mathcal{G}_{k,\rho}})$.

It follows from (1) that $H_k = \mathrm{diag}(L_{k,1} + \sum_{k'=1}^{K} Q_{k'}|_{\mathcal{G}_{k,1}}, \cdots, L_{k,\rho} + \sum_{k'=1}^{K} Q_{k'}|_{\mathcal{G}_{k,\rho}})$. Following the comments above Lemma 2, the conditions (a) and (b) indicate that $\sum_{k'=1}^{K} Q_{k'}|_{\mathcal{G}_{k,l}}$ is a positive definite diagonal matrix for each $l = 1, \cdots, \rho$. Thanks to the structure of L_k which has zero row sum, we are led to the conclusion that H_k is strictly diagonally dominant, which in turn implies that H_k is invertible employing the Levy-Desplanques theorem [33]. Since L_k is positive semi-definite and $\sum_{k'=1}^{K} a_{i\theta_{k'}} \ge 0$ holds for all $i \in \mathcal{G}_k$, H_k is also positive semi-definite. Therefore, H_k must be positive definite.

(Necessity): If (a) is not true, then there exists an integer $l \in \{1, \cdots, \rho\}$ such that there is no path connecting the component S_l to the super leader in $\bar{G}$. Hence, the diagonal elements in $\sum_{k'=1}^{K} Q_{k'}|_{\mathcal{G}_{k,l}}$ are non-positive. Since zero is an eigenvalue of $L_{k,l}$, we have $\lambda_{\min}(L_{k,l} + \sum_{k'=1}^{K} Q_{k'}|_{\mathcal{G}_{k,l}}) \le \lambda_{\min}(L_{k,l}) + \lambda_{\max}(\sum_{k'=1}^{K} Q_{k'}|_{\mathcal{G}_{k,l}}) \le 0$ by Weyl's theorem [33]. This implies that H_k has a non-positive eigenvalue, which contradicts the condition that H_k is positive definite.

If (b) does not hold, then there exists some $i \in \mathcal{G}_k$ such that $\sigma := \sum_{k'=1}^{K} a_{i\theta_{k'}} < 0$. The entries of L_k can be chosen small (in modules) enough so that $\lambda_{\max}(L_k) < -\sigma$ by using the Geršgorin disk theorem [33]. Another application of Weyl's theorem yields $\lambda_{\min}(H_k) \le \lambda_{\max}(L_k) + \lambda_{\min}(\sum_{k'=1}^{K} Q_{k'}|_{\mathcal{G}_k}) < 0$, which contradicts the condition that H_k is positive definite. □

Remark 1: In the construction of communication topology $\bar{G}$, we consider the information exchange between followers in each subgroup $\mathcal{G}_k$ and its own leader θ_k as well as leaders for other groups. It is worth noting that this is more general than the recent works [9], [16], [27] on group tracking control, where followers can only have access to its own leader in each subgroup. As we will see below, this flexible framework would enable the agents in $\mathcal{G}_k$ to track actually any leader in Θ provided an appropriate protocol is in use.

The following corollary is immediate from Lemma 2.

Corollary 1: Fix $k \in \{1, \cdots, K\}$. Suppose that $\mathcal{G}_k$ (viewed as an induced subgraph of G) is connected. If $\sum_{k'=1}^{K} a_{i\theta_{k'}} \ge 0$ holds for all $i \in \mathcal{G}_k$, with at least one of these inequalities being strict, then H_k is positive definite.

B. FIXED-TIME STABILITY

Consider the general differential inclusion

$$\dot{x}(t) \in F(t, x(t)), \quad x(0) = x_0, \quad (2)$$

where $x \in \mathbb{R}^n$ and $F : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an upper semi-continuous convex-valued mapping such that the set $F(t, x)$ is non-empty for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ and $F(t, 0) = 0$ for $t > 0$. The solutions of (2) are understood in the sense of Filippov [34].

Definition 1 [17]: The origin of system (2) is globally finite-time stable if there is a function $T : \mathbb{R}^n \to \mathbb{R}_+$, called settling time function, such that for all $x_0 \in \mathbb{R}^n$, the solution $x(t, x_0)$ of system (2) is defined and $x(t, x_0) \in \mathbb{R}^n$ for $t \in [0, T(x_0))$ and $\lim_{t\to T(x_0)} x(t, x_0) = 0$.

Definition 2 [22]: The origin of system (2) is a globally fixed-time equilibrium if it is globally finite-time stable and the settling-time function $T(x_0)$ is bounded; namely, there is some $T_{\max} > 0$ satisfying $T(x_0) \le T_{\max}$ for any $x_0 \in \mathbb{R}^n$.

For example, the origin of the simple scalar system $\dot{x} = -x^{1/3}$ is globally finite-time stable with $T(x_0) = \frac{3}{2}\sqrt[3]{|x_0|^2}$. The origin of $\dot{x} = -\lfloor x\rceil^{1/3} - \lfloor x\rceil^2$ is globally fixed-time stable because $T(x_0) \le \pi$ for any $x_0 \in \mathbb{R}$.

Lemma 3 [23]: If there is a continuously differentiable positive definite and radially unbounded function

$\mathsf{V}:\mathbb{R}^n\to\mathbb{R}_+$ *such that*

$$\sup_{t>0,y\in F(t,x)} y\frac{\partial \mathsf{V}(x)}{\partial x}\le -a\mathsf{V}^p(x)-b\mathsf{V}^q(x)\quad \textit{for } x\neq 0,$$

with $a,b>0$*,* $p=1-(1/\mu)$*,* $q=1+(1/\mu)$*, and* $\mu>1$*. Then the origin of the system (2) is globally fixed-time stable and the following estimate of the settling time holds:*

$$T(x_0)\le T_{\max}=\frac{\pi\mu}{2\sqrt{ab}},\qquad \forall x_0\in\mathbb{R}^n.$$

This lemma provides a good estimate of the settling time independent of the initial conditions, which will be used to analyze the fixed-time convergence of group tracking protocols.

C. PROBLEM FORMULATION

Now we are in the position to formulate our fixed-time group consensus tracking problem. Consider the following multi-agent system with N follower agents and K virtual leaders governed by

$$\begin{aligned}\dot{x}_i&=f(t,x_i)+u_i,\quad i\in V,\\ \dot{x}_{\theta_k}&=f(t,x_{\theta_k})+u_{\theta_k},\quad k\in\{1,\cdots,K\},\end{aligned}\tag{3}$$

where $x_i\in\mathbb{R}^m$ (resp. $x_{\theta_k}\in\mathbb{R}^m$) is the state of agent i (resp. leader θ_k) and $u_i\in\mathbb{R}^m$ (resp. $u_{\theta_k}\in\mathbb{R}^m$) is the control input of agent i (resp. leader θ_k). The input u_{θ_k} is assumed to be bounded by a known constant ω, i.e., $\|u_{\theta_k}\|\le\omega$ for all $1\le k\le K$. The function $f:\mathbb{R}_+\times D\longrightarrow\mathbb{R}^m$ represents the uncertain dynamics of an agent, which is continuous in t and $D\subseteq\mathbb{R}^m$ is a domain containing the origin. Since f in general is a nonlinear function, we assume that there exist positive constants ℓ_1, ℓ_2 and ℓ_3 such that

$$\begin{aligned}\|f(t,x_1)-f(t,x_2)\|^2&\le \ell_1+\ell_2\|x_1-x_2\|^2\\ &\quad+\ell_3\|x_1-x_2\|^4\end{aligned}\tag{4}$$

holds for all $x_1,x_2\in D$ and $t\ge 0$.

Remark 2: The condition (4) is also adopted in [28], which is more general than most of the previous works concerning nonlinear inherent dynamics [16], [20], [29]. In fact, (4) encompasses the usual Lipschitz condition ($\ell_1=\ell_3=0$), quasi-Lipschitz condition ($\ell_3=0$) and some essentially polynomial models in mechanical sciences, including the polynomial friction models [35] and the polynomial magneto-rheological damper dynamics [36].

The communication topology is assumed to satisfy the following assumption.

Assumption 1: Each follower in $\mathcal{G}_k$ ($1\le k\le K$) has a path to the super leader 0 in the graph $\bar{G}$; and $\sum_{k'=1}^K a_{i\theta_{k'}}\ge 0$ for all $i\in V$.

It follows from Lemma 2 that H_k is positive definite for $1\le k\le K$. The goal of this paper is to design suitable distributed protocols u_i such that followers in each subgroup track the corresponding virtual leader in a pre-defined time $T_{\max}$, i.e., $x_i(t)=x_{\theta_k}(t)$ for all $i\in\mathcal{G}_k$ ($1\le k\le K$) and $t\ge T_{\max}$. For simplicity, $m=1$ is assumed in the following. However, the analysis is valid for $m>1$ exploiting the properties of the Kronecker product.

III. FIXED-TIME GROUP TRACKING PROTOCOLS

In this section, for ease of presentation, we first design the fixed-time group tracking control law under the inter-group balance condition, and see how it can be treated in general scenarios.

A. CONSENSUS UNDER INTER-GROUP BALANCE CONDITION

Assumption 2: In the weighted adjacency matrix A of G, we assume that $\sum_{j\in\mathcal{G}_{k'}}a_{ij}=0$ for all $i\in\mathcal{G}_k$ and $k\neq k'$.

Remark 3: Recall that the interaction between different groups is allowed to be cooperative or competitive. The above condition indicates a balance of influence between an agent in a subgroup and all agents in any other subgroup. This condition (and a relaxed variant replacing 0 by a constant $c_{kk'}$ depending only on k and k') is essential in most of the literature concerning group consensus; see, e.g., [6]–[10], [12], [30]. We will see in the next subsection on how to revise the main result if it is violated.

For the leader-follower multi-agent system (3), we introduce the control protocol for $i\in\mathcal{G}_k$, $k=1,\cdots,K$ as follows:

$$\begin{aligned}u_i&=\alpha\left\lfloor\sum_{j\in\mathcal{G}_k\cup\theta_k}a_{ij}(x_j-x_i)\right\rceil^2+\left\lfloor\sum_{k'\neq k}\sum_{j\in\mathcal{G}_{k'}}a_{ij}x_j\right\rceil^2\\ &\quad+\beta\left(\sum_{j\in\mathcal{G}_k\cup\theta_k}a_{ij}(x_j-x_i)\right)+\left(\sum_{k'\neq k}\sum_{j\in\mathcal{G}_{k'}}a_{ij}x_j\right)\\ &\quad+\gamma\,\mathrm{sgn}\left(\sum_{j\in\mathcal{G}_k\cup\theta_k}a_{ij}(x_j-x_i)\right)\\ &\quad+\mathrm{sgn}\left(\sum_{k'\neq k}\sum_{j\in\mathcal{G}_{k'}}a_{ij}x_j\right),\end{aligned}\tag{5}$$

where $\alpha,\beta,\gamma>0$ are positive control gains. Note that only local information between neighboring agents is needed. Recall that the follower i is cooperative (competitive) with the leader θ_k if $a_{i\theta_k}>0$ (<0). Here, the negative weights $\{a_{i\theta_k}\}$ are allowed but not required.

Let $\tilde{x}_i=x_i-x_{\theta_k}$ be the tracking error for $i\in\mathcal{G}_k$, $1\le k\le K$. Let $\bar{a}=\max_{i\in\mathcal{G}_k,j\in\mathcal{G}_{k'},k\neq k'}|a_{ij}|$. Denote by $W=\mathrm{diag}(H_1,\cdots,H_K)\in\mathbb{R}^{N\times N}$ the block diagonal matrix. We have $\lambda_{\max}(W)=\max_{1\le k\le K}\lambda_{\max}(H_k)$ and $\lambda_{\min}(W)=\min_{1\le k\le K}\lambda_{\min}(H_k)$.

Theorem 1: Under Assumptions 1 and 2, the multi-agent system (3) with protocol (5) and

$$\begin{aligned}\alpha&=\frac{(\bar{a}^2N^3+\sqrt{\ell_3N})\lambda_{\max}^{3/2}(W)}{\lambda_{\min}^{7/2}(W)}+\frac{\rho\sqrt{N}}{2\sqrt{2}\lambda_{\min}^{3/2}(W)},\\ \beta&=\frac{\bar{a}N+\sqrt{\ell_2}}{\lambda_{\min}(W)},\end{aligned}$$

$$\gamma = 1+\omega+\sqrt{\ell_1 \max_{1\le k\le K}|\mathcal{G}_k|}+\frac{\rho}{\sqrt{2\lambda_{\min}(W)}}$$

with $\rho > 0$, achieves the convergence of the tracking errors $\tilde{x}_i$, $i \in V$ to zero in a finite time, which is bounded by $T_{\max} = \pi/\rho$.

Proof: For $1 \le k \le K$, let $\delta_k = (\tilde{x}_{r_{k-1}+1}, \cdots, \tilde{x}_{r_k})^{\mathrm{T}} \in \mathbb{R}^{|\mathcal{G}_k|}$. Let $\tilde{x} = (\delta_1^{\mathrm{T}}, \cdots, \delta_K^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^N$ be the tracking error vector. Furthermore, let $\tilde{f}_k = (f(t, x_{r_{k-1}+1}) - f(t, x_{\theta_k}), \cdots, f(t, x_{r_k}) - f(t, x_{\theta_k}))^{\mathrm{T}} \in \mathbb{R}^{|\mathcal{G}_k|}$ and $U_k = u_{\theta_k}1_{|\mathcal{G}_k|} \in \mathbb{R}^{|\mathcal{G}_k|}$ for $1 \le k \le K$. Recall that we have set $m = 1$. By using (1) and Assumption 2, the dynamics of the tracking errors can be obtained from (3) and (5) as

$$\dot{\delta}_k = -\alpha\lfloor H_k\delta_k\rceil^2 + \left\lfloor \sum_{k'\neq k} H_{kk'}\delta_{k'}\right\rceil^2 - \beta H_k\delta_k + \sum_{k'\neq k} H_{kk'}\delta_{k'} - \gamma\,\mathrm{sgn}(H_k\delta_k) + \mathrm{sgn}\left(\sum_{k'\neq k} H_{kk'}\delta_{k'}\right) + \tilde{f}_k - U_k, \quad 1 \le k \le K. \tag{6}$$

Define the Lyapunov function as

$$\mathsf{V} = \frac{1}{2}\tilde{x}^{\mathrm{T}}W\tilde{x} = \frac{1}{2}\sum_{k=1}^{K}\delta_k^{\mathrm{T}}H_k\delta_k. \tag{7}$$

By Lemma 2, W is positive definite. Let $\mathsf{V}_k = \frac{1}{2}\delta_k^{\mathrm{T}}H_k\delta_k$, and hence $\mathsf{V} = \sum_{k=1}^{K}\mathsf{V}_k$. The time derivative of (7) along the solution of (6) is given by

$$\dot{\mathsf{V}} = \sum_{k=1}^{K}\delta_k^{\mathrm{T}}H_k\dot{\delta}_k = \sum_{k=1}^{K}\dot{\mathsf{V}}_k, \tag{8}$$

where

$$\dot{\mathsf{V}}_k = -\alpha\delta_k^{\mathrm{T}}H_k\lfloor H_k\delta_k\rceil^2 + \delta_k^{\mathrm{T}}H_k\left\lfloor \sum_{k'\neq k} H_{kk'}\delta_{k'}\right\rceil^2 - \beta\delta_k^{\mathrm{T}}H_k^2\delta_k + \delta_k^{\mathrm{T}}H_k\left(\sum_{k'\neq k} H_{kk'}\delta_{k'}\right) - \gamma\delta_k^{\mathrm{T}}H_k\mathrm{sgn}(H_k\delta_k) + \delta_k^{\mathrm{T}}H_k\mathrm{sgn}\left(\sum_{k'\neq k} H_{kk'}\delta_{k'}\right) + \delta_k^{\mathrm{T}}H_k\tilde{f}_k - \delta_k^{\mathrm{T}}H_kU_k. \tag{9}$$

In the sequel, we estimate the eight terms in (8) and (9) separately. It follows from the Courant-Fischer theorem [33] that $\|W\tilde{x}\|_2^2 \ge \lambda_{\min}(W)\tilde{x}^{\mathrm{T}}W\tilde{x} = 2\lambda_{\min}(W)\mathsf{V}$. Therefore, the first term can be estimated as

$$-\alpha\sum_{k=1}^{K}\delta_k^{\mathrm{T}}H_k\lfloor H_k\delta_k\rceil^2 = -\alpha\|W\tilde{x}\|_3^3 \le -\alpha N^{-1/2}(2\lambda_{\min}(W)\mathsf{V})^{3/2}$$

using Lemma 1. For the second term, we obtain

$$\begin{aligned}
&\delta_k^{\mathrm{T}}H_k\left\lfloor \sum_{k'\neq k} H_{kk'}\delta_{k'}\right\rceil^2\\
&\le \sum_{i\in\mathcal{G}_k}\left|\sum_{j\in\mathcal{G}_k\cup\theta_k} a_{ij}(x_j-x_i)\right|\cdot\left|\sum_{k'\neq k}\sum_{j\in\mathcal{G}_{k'}} a_{ij}x_j\right|^2\\
&\le \bar{a}^2\left(\sum_{i\in\mathcal{G}_k}\left|\sum_{j\in\mathcal{G}_k\cup\theta_k} a_{ij}(\tilde{x}_j-\tilde{x}_i)\right|\right)\cdot\left|\sum_{k'\neq k}\sum_{j\in\mathcal{G}_{k'}}\tilde{x}_j\right|^2\\
&\le \bar{a}^2\|H_k\delta_k\|_1\|\tilde{x}\|_1^2,
\end{aligned}$$

where we have used Assumption 2 in the second inequality. Hence, summing over k and by a repeated use of Lemma 1 and the Courant-Fischer theorem, we obtain

$$\begin{aligned}
&\sum_{k=1}^{K}\delta_k^{\mathrm{T}}H_k\left\lfloor \sum_{k'\neq k} H_{kk'}\delta_{k'}\right\rceil^2\\
&\le \bar{a}^2\|\tilde{x}\|_1^2\|W\tilde{x}\|_1 \le \bar{a}^2N\|\tilde{x}\|_2^2\|W\tilde{x}\|_1 \le \frac{\bar{a}^2N}{\lambda_{\min}^2(W)}\|W\tilde{x}\|_1^3\\
&\le \frac{\bar{a}^2N^{5/2}}{\lambda_{\min}^2(W)}\|W\tilde{x}\|_2^3 \le \frac{\bar{a}^2N^{5/2}}{\lambda_{\min}^2(W)}(2\lambda_{\max}(W)\mathsf{V})^{3/2}.
\end{aligned} \tag{10}$$

The third term is $-\beta\sum_{k=1}^{K}\delta_k^{\mathrm{T}}H_k^2\delta_k = -\beta\|W\tilde{x}\|_2^2$ and the fourth term can be bounded similarly as in (10) by

$$\begin{aligned}
&\sum_{k=1}^{K}\delta_k^{\mathrm{T}}H_k\left(\sum_{k'\neq k} H_{kk'}\delta_{k'}\right)\\
&\le \bar{a}\|\tilde{x}\|_1\|W\tilde{x}\|_1 \le \bar{a}\sqrt{N}\|\tilde{x}\|_2\|W\tilde{x}\|_1\\
&\le \frac{\bar{a}\sqrt{N}}{\lambda_{\min}(W)}\|W\tilde{x}\|_2\|W\tilde{x}\|_1 \le \frac{\bar{a}N}{\lambda_{\min}(W)}\|W\tilde{x}\|_2^2.
\end{aligned}$$

The fifth term equals $-\gamma\sum_{k=1}^{K}\delta_k^{\mathrm{T}}H_k\mathrm{sgn}(H_k\delta_k) = -\gamma\|W\tilde{x}\|_1$. We estimate the sixth term as

$$\sum_{k=1}^{K}\delta_k^{\mathrm{T}}H_k\mathrm{sgn}\left(\sum_{k'\neq k} H_{kk'}\delta_{k'}\right) \le \sum_{k=1}^{K}\|H_k\delta_k\|_1 = \|W\tilde{x}\|_1.$$

In the light of (4) and the Cauchy-Schwarz inequality, the seventh term is upper-bounded by

$$\begin{aligned}
&\sum_{k=1}^{K}\delta_k^{\mathrm{T}}H_k\tilde{f}_k\\
&\le \sum_{k=1}^{K}\|H_k\delta_k\|_2\|\tilde{f}_k\|_2\\
&\le \sum_{k=1}^{K}\|H_k\delta_k\|_2\left(\sqrt{\ell_1|\mathcal{G}_k|}+\sqrt{\ell_2}\|\delta_k\|_2+\sqrt{\ell_3}\|\delta_k\|_4^2\right)\\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

Note that

$$I_1 \leq \sum_{k=1}^{K} \|H_k\delta_k\|_1 \sqrt{\ell_1|\mathcal{G}_k|} \leq \|W\tilde{x}\|_1 \sqrt{\ell_1 \max_{1\leq k\leq K} |\mathcal{G}_k|}.$$

By the Courant-Fischer theorem, we have $I_2 \leq \sum_{k=1}^{K} \frac{\sqrt{\ell_2}}{\lambda_{\min}(H_k)} \cdot \|H_k\delta_k\|_2^2 \leq \frac{\sqrt{\ell_2}}{\lambda_{\min}(W)}\|W\tilde{x}\|_2^2$. For I_3, we have

$$\begin{aligned} I_3 &= \sqrt{\ell_3}\sum_{k=1}^{K}\|H_k\delta_k\|_2^3 \frac{\|\delta_k\|_4^2}{\|H_k\delta_k\|_2^2} \\ &\leq \sqrt{\ell_3}\sum_{k=1}^{K}\|H_k\delta_k\|_2^3 \frac{\|\delta_k\|_2^2}{\|H_k\delta_k\|_2^2} \\ &\leq \frac{\sqrt{\ell_3}}{\lambda_{\min}^2(W)}\sum_{k=1}^{K}\|H_k\delta_k\|_2^3, \end{aligned} \tag{11}$$

where the first inequality is due to Lemma 1 and the second inequality is because $\frac{\|\delta_k\|_2}{\|H_k\delta_k\|_2} \leq \lambda_{\min}^{-1}(H_k) \leq \lambda_{\min}^{-1}(W)$. Another application of Lemma 1 shows that the right-hand side of (11) is upper-bounded by $\frac{\sqrt{\ell_3}}{\lambda_{\min}^2(W)}\|W\tilde{x}\|_2^3$, which together with the last inequality in (10) yields

$$I_3 \leq \frac{\sqrt{\ell_3}}{\lambda_{\min}^2(W)}(2\lambda_{\max}(W)\mathsf{V})^{3/2}.$$

Finally, recall the boundedness of u_{θ_k}, and the eighth term can be calculated as $-\sum_{k=1}^{K}\delta_k^{\mathrm{T}}H_kU_k \leq \omega\sum_{k=1}^{K}\|H_k\delta_k\|_1 = \omega\|W\tilde{x}\|_1$.

Now, we obtain from (8) that

$$\begin{aligned} \dot{\mathsf{V}} \leq &\Bigg(-\alpha N^{-1/2}(2\lambda_{\min}(W))^{3/2} + \frac{\bar{a}^2N^{5/2}}{\lambda_{\min}^2(W)}(2\lambda_{\max}(W))^{3/2} \\ &+ \frac{\sqrt{\ell_3}(2\lambda_{\max}(W))^{3/2}}{\lambda_{\min}^2(W)}\Bigg)\mathsf{V}^{3/2} \\ &- \left(\beta - \frac{\bar{a}N + \sqrt{\ell_2}}{\lambda_{\min}(W)}\right)\|W\tilde{x}\|_2^2 \\ &- \left(\gamma - 1 - \omega - \sqrt{\ell_1 \max_{1\leq k\leq K}|\mathcal{G}_k|}\right)\|W\tilde{x}\|_1. \end{aligned} \tag{12}$$

Taking α, β, γ as in the statement of Theorem 1, and noting that $\|W\tilde{x}\|_1 \geq \|W\tilde{x}\|_2 \geq (2\lambda_{\min}(W)\mathsf{V})^{1/2}$, we derive that $\dot{\mathsf{V}} \leq -\rho\mathsf{V}^p - \mathsf{V}^q$ with $\mu = 2$, $p = 1 - (1/\mu)$ and $q = 1 + (1/\mu)$. According to Lemma 3, one can conclude that the origin of the system (6) is globally fixed-time stable and the settling time is bounded by $T_{\max} = \pi/\rho$. The proof is completed. □

Remark 4: Notice that the convergence time upper-bound is independent of the initial conditions of the network and can be adjusted arbitrarily by tuning the controller parameters α and γ (through ρ). When $K = 1$, i.e., there exists only one leader, the problem reduces to fixed-time consensus tracking, which has been solved in [28]. Extension to group tracking control here invokes major changes in dealing with the weighted block matrix H, which encodes the interactions of agents between and within subnetworks (and their respective leaders).

B. CONSENSUS WITHOUT INTER-GROUP BALANCE CONDITION

It is clear that the protocol (5) may not drive the multi-agent system to fixed-time convergence if Assumption 2 is violated, because the tracking error system (6) is no longer valid.

For any $1 \leq k \leq K$, $i \in \mathcal{G}_k$, set $\eta_{ik'} = \sum_{j\in\mathcal{G}_{k'}} a_{ij}$. For each $i \in \mathcal{G}_k$, we partition the set $\{1, \cdots, K\}\backslash\{k\}$ into two subsets Φ_i^0 and Φ_i^1 by $\Phi_i^0 = \{k'|k' \neq k, \eta_{ik'} = 0\}$ and $\Phi_i^1 = \{k'|k' \neq k, \eta_{ik'} \neq 0\}$. Evidently, $\Phi_i^1 = \emptyset$ for all $i \in V$ if Assumption 2 holds. In general, we propose the following assumption for the information exchange between the followers and the virtual leaders.

Assumption 3: For each agent $i \in \mathcal{G}_k$, $1 \leq k \leq K$, we assume $a_{i\theta_{k'}} = -\eta_{ik'}$ for $k' \in \Phi_i^1$.

Remark 5: Assumption 3 means that the coupling strength between a follower agent i in subgroup $\mathcal{G}_k$ and the leader $\theta_{k'}$ for any other subgroup $\mathcal{G}_{k'}$ satisfying $\eta_{ik'} \neq 0$ is specified as $-\eta_{ik'}$. An interesting implication is that some virtual leaders may be designed in practical applications such as multi-robot systems to compensate the influence of agents in other subgroups so that the inter-group balance condition (Assumption 2) can be lifted. It is also worthy of noting that $\eta_{ik'}$ involves only local information within the neighborhood of agent i, and hence can be easily computed in a distributed manner.

For the leader-follower multi-agent system (3), we here introduce a modified distributed control law for $i \in \mathcal{G}_k$, $k = 1, \cdots, K$ as follows:

$$\begin{aligned} u_i = &\alpha\left\lfloor\sum_{j\in\mathcal{G}_k\cup\theta_k} a_{ij}(x_j - x_i)\right\rceil^2 + \left\lfloor\sum_{k'\in\Phi_i^0}\sum_{j\in\mathcal{G}_{k'}} a_{ij}x_j\right\rceil^2 \\ &+ \left\lfloor\sum_{k'\in\Phi_i^1}\sum_{j\in\mathcal{G}_{k'}} a_{ij}(x_j - x_{\theta_{k'}})\right\rceil^2 \\ &+ \beta\left(\sum_{j\in\mathcal{G}_k\cup\theta_k} a_{ij}(x_j - x_i)\right) + \left(\sum_{k'\in\Phi_i^0}\sum_{j\in\mathcal{G}_{k'}} a_{ij}x_j\right) \\ &+ \left(\sum_{k'\in\Phi_i^1}\sum_{j\in\mathcal{G}_{k'}} a_{ij}(x_j - x_{\theta_{k'}})\right) \\ &+ \gamma\,\mathrm{sgn}\left(\sum_{j\in\mathcal{G}_k\cup\theta_k} a_{ij}(x_j - x_i)\right) \\ &+ \mathrm{sgn}\left(\sum_{k'\in\Phi_i^0}\sum_{j\in\mathcal{G}_{k'}} a_{ij}x_j\right) \\ &+ \mathrm{sgn}\left(\sum_{k'\in\Phi_i^1}\sum_{j\in\mathcal{G}_{k'}} a_{ij}(x_j - x_{\theta_{k'}})\right), \end{aligned} \tag{13}$$

where $\alpha, \beta, \gamma > 0$ are positive control gains.

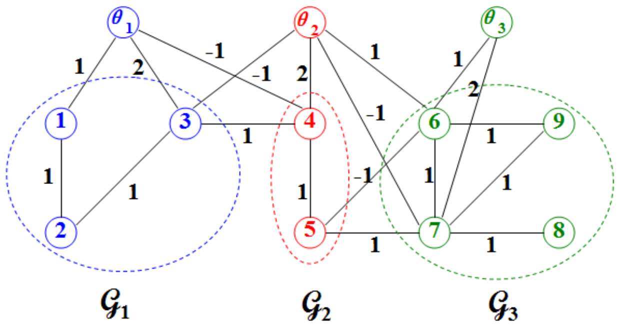

FIGURE 1. Communication topology for Example 1.

Theorem 2: Under Assumptions 1 and 3, the multi-agent system (3) with protocol (13) and

$$
\alpha = \frac{(\bar{a}^2 N^3 + \sqrt{\ell_3 N})\lambda_{\max}^{3/2}(W)}{\lambda_{\min}^{7/2}(W)} + \frac{\rho\sqrt{N}}{2\sqrt{2}\lambda_{\min}^{3/2}(W)},
$$

$$
\beta = \frac{\bar{a}N + \sqrt{\ell_2}}{\lambda_{\min}(W)},
$$

$$
\gamma = 1 + \omega + \sqrt{\ell_1 \max_{1\le k\le K} |\mathcal{G}_k| + \frac{\rho}{\sqrt{2\lambda_{\min}(W)}}}
$$

with $\rho > 0$, achieves the convergence of the tracking errors $\tilde{x}_i$, $i \in V$ to zero in a finite time, which is bounded by $T_{\max} = \pi/\rho$.

Note that from Assumption 3, the tracking error dynamics (6) can be reproduced, and hence Theorem 2 follows exactly from the proof of Theorem 1. Clearly, Theorem 1 is a special case of Theorem 2 with $\Phi_i^1 = \emptyset$ for all $i \in V$.

IV. NUMERICAL EXAMPLES

In this section, we present numerical examples to validate our theoretical results and illustrate the flexibility of our developed framework.

Example 1 (Three Leaders): In this example, we consider multi-agent system (3) with $K = 3$ leaders and $N = 9$ follower agents having $\mathcal{G}_1 = \{1, 2, 3\}$, $\mathcal{G}_2 = \{4, 5\}$, and $\mathcal{G}_3 = \{6, 7, 8, 9\}$. The network topology G together with its associated weights is shown in Fig. 1. Note that the leaders are by no means influenced by the followers as the weights shown between the leaders and the followers only appear in the controllers u_i's for the followers (see Eqs. (5) and (13)). It is easy to see that Assumptions 1 and 3 hold. The inherent nonlinear dynamics is chosen as $f(t, x_i) = 0.2\sin(x_i)$ for all $i \in \{1, \cdots, 9\} \cup \{\theta_1, \theta_2, \theta_3\}$. The condition (4) holds with $\ell_1 = \ell_3 = 0$ and $\ell_2 = 0.04$. The control inputs for the three leaders are taken as $u_{\theta_1} = -1$, $u_{\theta_2} = 1 + \cos(t)$, and $u_{\theta_3} = 2\cos(t)$; they are bounded by $\omega = 2$. By taking $\rho = 200$, we obtain from Theorem 2 an explicit estimation of the settling time $T_{\max} \approx 0.015$, which is independent of the initial conditions of the system. The group consensus tracking behaviors are shown with a small initial condition in Fig. 2(a) and a large initial condition in Fig. 2(b). One can see that the convergence time for both cases is less than 5×10^{-4}. The theoretical upper bound seems to be quite conservative for this scenario. Note that the initial conditions for the leaders are also different in these two situations for the sake of clear illustration. In view of the conservativeness of the estimation, a more practical settling time may be obtained by simulating the dynamical system for the followers with sufficiently large initial conditions. This is feasible because the fixed-time convergence is theoretically guaranteed and thus the convergence time will tend to a finite limit as the initial conditions increase. As such, we estimate the convergence time for the cases (a) and (b), respectively, as 4×10^{-4} and 10^{-5}.

Example 2 (Merging Two of the Leaders): Here, we merge the two leaders θ_1 and θ_2 in Example 1 so that they have the same dynamics and is denoted by a new θ_1. The corresponding network architecture is depicted in Fig. 3, which contains two subgroups $\mathcal{G}_1 = \{1, 2, 3, 4, 5\}$ and $\mathcal{G}_2 = \{6, 7, 8, 9\}$. One can see that Assumptions 1 and 3 still hold. For the

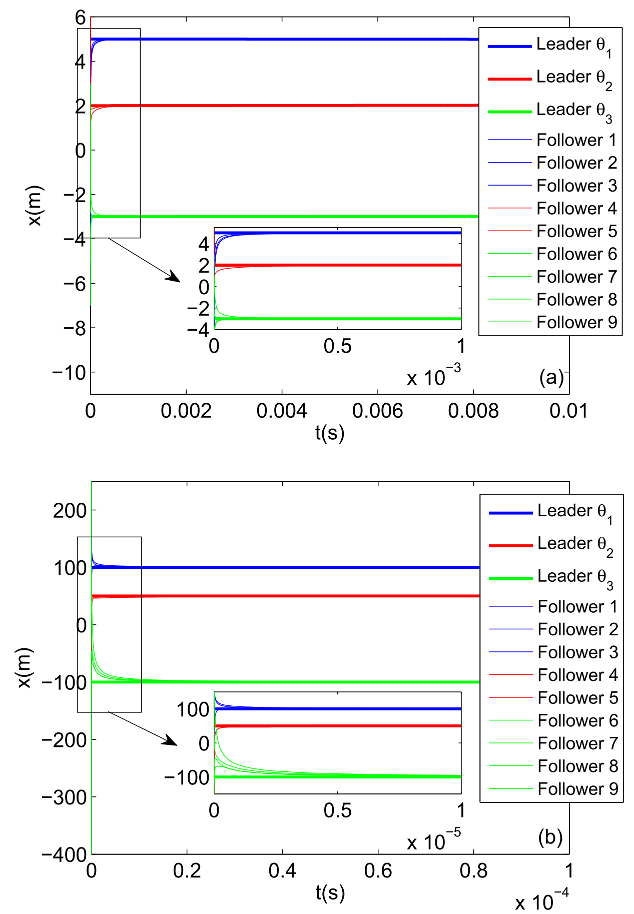

FIGURE 2. Fixed-time group tracking consensus for multi-agent systems (3), (13) and communication topology shown in Example 1. (a) is for $(x_{\theta_1}(0), x_{\theta_2}(0), x_{\theta_3}(0), x_1(0), x_2(0), x_3(0), x_4(0), x_5(0), x_6(0), x_7(0), x_8(0), x_9(0)) = (5, 2, -3, 2, -4, -3, 6, 0, -5, 3, 1, -7)$; and (b) is for $(x_{\theta_1}(0), x_{\theta_2}(0), x_{\theta_3}(0), x_1(0), x_2(0), x_3(0), x_4(0), x_5(0), x_6(0), x_7(0), x_8(0), x_9(0)) = (100, 50, -100, -50, 200, 150, -200, 75, -125, -400, 250, 100)$.

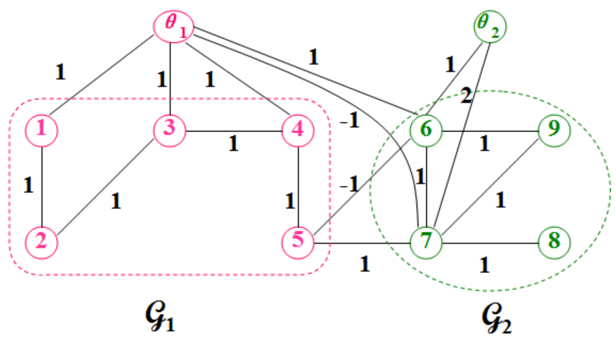

FIGURE 3. Communication topology for Example 2.

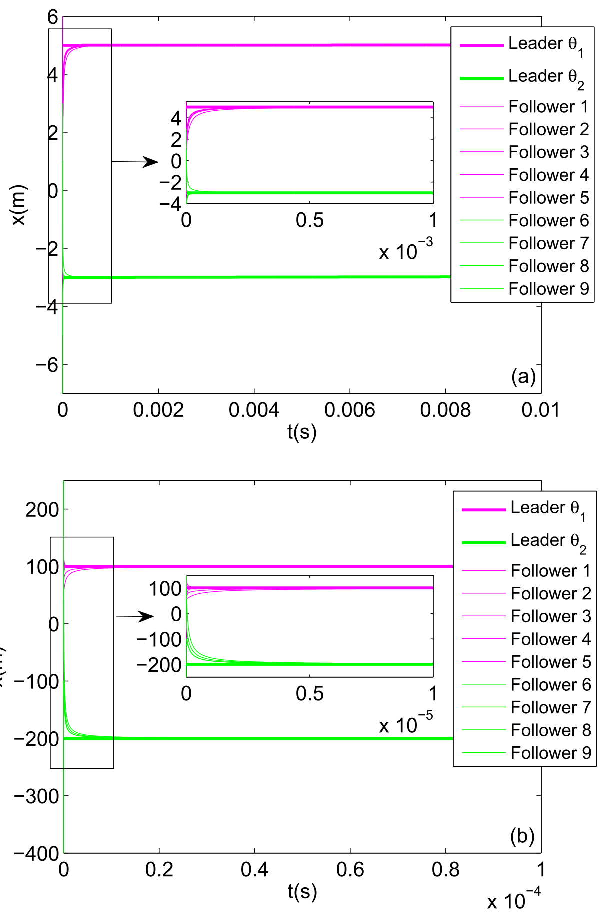

FIGURE 4. Fixed-time group tracking consensus for multi-agent systems (3), (13) and communication topology shown in Example 2. (a) is for $(x_{\theta_1}(0), x_{\theta_2}(0), x_1(0), x_2(0), x_3(0), x_4(0), x_5(0), x_6(0), x_7(0), x_8(0), x_9(0)) = (5, -3, 2, -4, -3, 6, 0, -5, 3, 1, -7)$; and (b) is for $(x_{\theta_1}(0), x_{\theta_2}(0), x_1(0), x_2(0), x_3(0), x_4(0), x_5(0), x_6(0), x_7(0), x_8(0), x_9(0)) = (100, -200, -50, 200, 150, -200, 75, -125, -400, 250, 100)$.

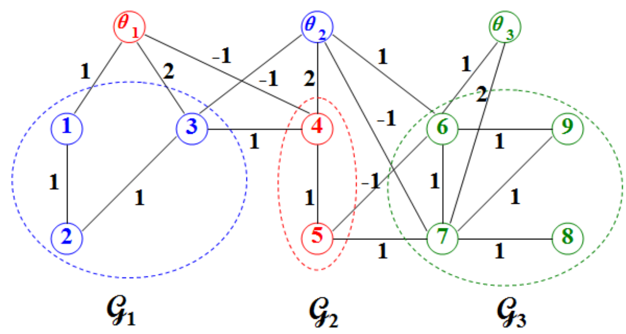

FIGURE 5. Communication topology for Example 3.

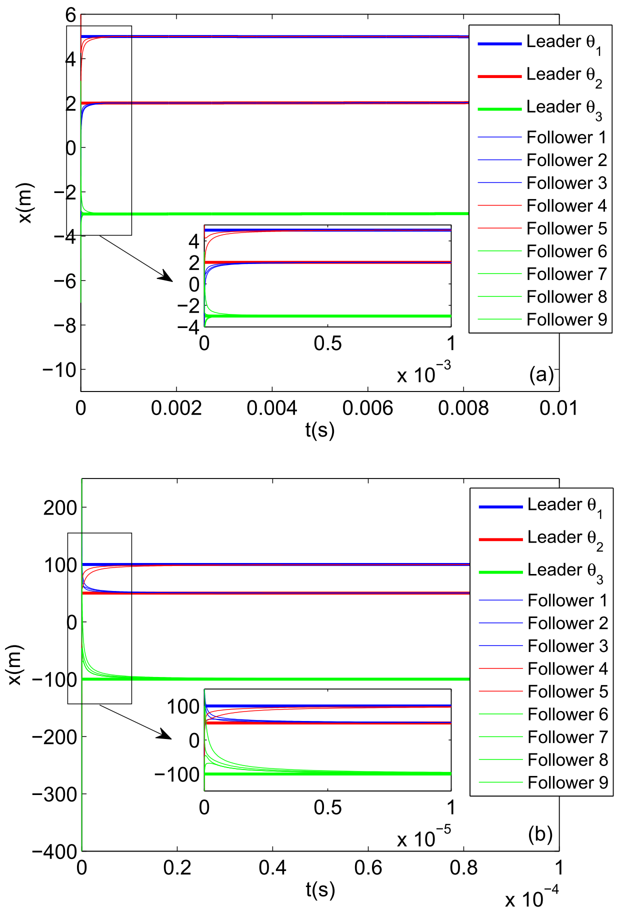

FIGURE 6. Fixed-time group tracking consensus for multi-agent systems (3), (13) and communication topology shown in Example 3. (a) is for $(x_{\theta_1}(0), x_{\theta_2}(0), x_{\theta_3}(0), x_1(0), x_2(0), x_3(0), x_4(0), x_5(0), x_6(0), x_7(0), x_8(0), x_9(0)) = (5, 2, -3, 2, -4, -3, 6, 0, -5, 3, 1, -7)$; and (b) is for $(x_{\theta_1}(0), x_{\theta_2}(0), x_{\theta_3}(0), x_1(0), x_2(0), x_3(0), x_4(0), x_5(0), x_6(0), x_7(0), x_8(0), x_9(0)) = (100, 50, -100, -50, 200, 150, -200, 75, -125, -400, 250, 100)$.

new leader θ_1, the control input is taken as $u_{\theta_1} = \cos(t)$. The control input for the new leader θ_2, the inherent nonlinear dynamics, and ρ are unchanged. Again, it follows from Theorem 2 that the estimated upper bound of settling time is $T_{\max} \approx 0.015$ regardless of the initial conditions. Analogously, the fixed-time group consensus tracking behavior is shown in Fig. 4 for two different initial conditions. We see that the follower agents 1-5 now track their common new leader θ_1. The convergence time for both cases in Fig. 4 is less than 5×10^{-4}. Similar practical convergence time as in Example 1 is applicable here.

Example 3 (Cross-Tracking): In this example, we get back to the scenario in Example 1 with three subgroups $\mathcal{G}_1 = \{1, 2, 3\}$, $\mathcal{G}_2 = \{4, 5\}$, and $\mathcal{G}_3 = \{6, 7, 8, 9\}$. But now we let the first group $\mathcal{G}_1$ track θ_2 and let the second group $\mathcal{G}_2$ track θ_1. The network topology is shown in Fig. 5, which is literally the same as Fig. 1. It is easy to see that Assumptions 1 and 3 hold, and the system parameters, including the control inputs for leaders, inherent nonlinear dynamics and ρ, are the same as in Example 1. Therefore, Theorem 2 implies an explicit estimation of the settling time $T_{\max} \approx 0.015$, which is independent of the initial conditions of the system. We show the group consensus tracking behaviors for two different initial conditions in Fig. 6. As one would expect, the cross-tracking is realized in fixed-time for both cases, which is less than 5×10^{-4}.

V. CONCLUSION

In this paper, we study the fixed-time group consensus tracking with unknown inherent nonlinear dynamics, while previous works mainly address fixed-time global consensus or finite-time group consensus problems. We present a general fixed-time tracking control protocol which accommodates uncertain nonlinear dynamics without assuming the inter-group balance condition. The leaders for each subgroup of the multi-agent system are allowed to interact with agents in other subgroups. Some conditions have been derived to choose appropriate gains to achieve the group tracking in a prescribed time independent of the initial conditions. Finally, some numerical simulations are provided to illustrate the availability of our obtained theoretical results. For future work, it would be interesting to consider the fixed-time group consensus tracking for directed networks, which is more general in the real world. Multi-agent systems with time-delays and hybrid dynamics [37] are challenging problems to be investigated.

ACKNOWLEDGMENTS

The authors would like to thank associate editor and anonymous reviewers for their insightful comments that have improved the presentation of the paper.

REFERENCES

[1] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol. 95, no. 1, pp. 215–233, Jan. 2007.

[2] Y. Cao, W. Yu, W. Ren, and G. Chen, "An overview of recent progress in the study of distributed multi-agent coordination," *IEEE Trans. Ind. Informat.*, vol. 9, no. 1, pp. 427–438, Feb. 2013.

[3] J. A. Fax and R. M. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1465–1476, Sep. 2004.

[4] W. Ren, R. W. Beard, and E. M. Atkins, "Information consensus in multivehicle cooperative control," *IEEE Control Syst.*, vol. 27, no. 2, pp. 71–82, Apr. 2007.

[5] X. Ge, F. Yang, and Q.-L. Han, "Distributed networked control systems: A brief overview," *Inf. Sci.*, vol. 380, pp. 117–131, Feb. 2017.

[6] J. Yu and L. Wang, "Group consensus in multi-agent systems with switching topologies and communication delays," *Syst. Control Lett.*, vol. 59, pp. 340–348, Jun. 2010.

[7] J. Qin and C. Yu, "Cluster consensus control of generic linear multi-agent systems under directed topology with acyclic partition," *Automatica*, vol. 49, pp. 2898–2905, Sep. 2013.

[8] Y. Han, W. Lu, and T. Chen, "Achieving cluster consensus in continuous-time networks of multi-agents with inter-cluster non-identical inputs," *IEEE Trans. Autom. Control*, vol. 60, no. 3, pp. 793–798, Mar. 2015.

[9] Q. Cui, D. Xie, and F. Jiang, "Group consensus tracking control of second-order multi-agent systems with directed fixed topology," *Neurocomputing*, vol. 218, pp. 286–295, Dec. 2016.

[10] Y. Shang, "Couple-group consensus of continuous-time multi-agent systems under Markovian switching topologies," *J. Franklin Inst.*, vol. 352, no. 11, pp. 4826–4844, 2015.

[11] Y. Chen, J. Lü, F. Han, and X. Yu, "On the cluster consensus of discrete-time multi-agent systems," *Syst. Control Lett.*, vol. 60, no. 7, pp. 517–523, 2011.

[12] B. Hou, F. Sun, H. Li, Y. Chen, and G. Liu, "Scaled cluster consensus of discrete-time multi-agent systems with general directed topologies," *Int. J. Syst. Sci.*, vol. 47, no. 16, pp. 3839–3845, 2016.

[13] Y. Shang, "A combinatorial necessary and sufficient condition for cluster consensus," *Neurocomputing*, vol. 216, pp. 611–616, Dec. 2016.

[14] C. Sun, G. Hu, and L. Xie, "Robust consensus tracking for a class of high-order multi-agent systems," *Int. J. Robust Nonlinear Control*, vol. 26, pp. 578–598, Feb. 2016.

[15] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 988–1001, Jun. 2003.

[16] Z. Tu, D. Zhang, X. Xia, and H. Yu, "Event-triggered group consensus of leader-following multi-agent systems with nonlinear dynamics," in *Proc. 35th Chin. Control Conf.*, Chengdu, China, 2016, pp. 7885–7890.

[17] S. P. Bhat and D. S. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM J. Control Optim.*, vol. 38, no. 3, pp. 751–766, Jan. 2000.

[18] L. Wang and F. Xiao, "Finite-time consensus problems for networks of dynamic agents," *IEEE Trans. Autom. Control*, vol. 55, no. 4, pp. 950–955, Apr. 2010.

[19] Y. Zhao, Z. Duan, and G. Wen, "Finite-time consensus for second-order multi-agent systems with saturated control protocols," *IET Control Theory Appl.*, vol. 9, no. 3, pp. 312–319, 2015.

[20] Y. Cao and W. Ren, "Finite-time consensus for multi-agent networks with unknown inherent nonlinear dynamics," *Automatica*, vol. 50, no. 10, pp. 2648–2656, 2014.

[21] Y. Wang, Y. Song, M. Krstic, and C. Wen, "Fault-tolerant finite time consensus for multiple uncertain nonlinear mechanical systems under single-way directed communication interactions and actuation failures," *Automatica*, vol. 63, pp. 374–383, Jan. 2016.

[22] A. Polyakov, "Nonlinear feedback design for fixed-time stabilization of linear control systems," *IEEE Trans. Autom. Control*, vol. 57, no. 8, pp. 2106–2110, Aug. 2012.

[23] S. E. Parsegov, A. E. Polyakov, and P. S. Shcherbakov, "Fixed-time consensus algorithm for multi-agent systems with integrator dynamics," in *Proc. 4th IFAC Workshop Distrib. Estimation Control Netw. Syst.*, 2013, pp. 110–115.

[24] Y. Zuo and L. Tie, "A new class of finite-time nonlinear consensus protocols for multi-agent systems," *Int. J. Control*, vol. 87, no. 2, pp. 363–370, Feb. 2014.

[25] Z. Zuo and L. Tie, "Distributed robust finite-time nonlinear consensus protocols for multi-agent systems," *Int. J. Syst. Sci.*, vol. 47, no. 6, pp. 1366–1375, 2016.

[26] J. Fu and J.-Z. Wang, "Finite-time consensus for multi-agent systems with globally bounded convergence time under directed communication graphs," *Int. J. Control*, vol. 90, no. 9, pp. 1807–1817, 2016, doi: 10.1080/00207179.2016.1223348.

[27] Y. Shang and Y. Ye, "Leader-follower fixed-time group consensus control of multiagent systems under directed topology," *Complexity*, vol. 2017, Mar. 2017, Art. no. 3465076.

[28] M. Defoort, A. Polyakov, G. Demesure, M. Djemai, and K. Veluvolu, "Leader-follower fixed-time consensus for multi-agent systems with unknown non-linear inherent dynamics," *IET Control Theory Appl.*, vol. 9, no. 14, pp. 2165–2170, 2015.

[29] J. Fu and J. Wang, "Fixed-time coordinated tracking for second-order multi-agent systems with bounded input uncertainties," *Syst. Control Lett.*, vol. 93, pp. 1–12, Jul. 2016.

[30] X. Liu and T. Chen. (2015). "Fixed-time cluster synchronization for complex networks via pinning control." [Online]. Available: https://arxiv.org/abs/1509.03350
[31] G. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed. New York, NY, USA: Cambridge Univ. Press, 1952.
[32] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton, NJ, USA: Princeton Univ. Press, 2010.
[33] R. A. Horn and C. R. Johnson, *Matrix Analysis*. New York, NY, USA: Cambridge Univ. Press, 1985.
[34] A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides*. Dordrecht, The Netherlands: Kluwer, 1988.
[35] X. Yang and C. R. Liu, "A new stress-based model of friction behavior in machining and its significant impact on residual stresses computed by finite element method," *Int. J. Mech. Sci.*, vol. 44, no. 4, pp. 703–723, 2002.
[36] H. Du, K. Y. Sze, and J. Lam, "Semi-active $H\infty$ control of vehicle suspension with magneto-rheological dampers," *J. Sound Vibrat.*, vol. 283, pp. 981–996, May 2005.
[37] Y. Shang, "Consensus in averager-copier-voter networks of moving dynamical agents," *Chaos*, vol. 27, p. 023116, Feb. 2017.

YILUN SHANG received the B.S. and Ph.D. degrees in mathematics from Shanghai Jiao Tong University, China, in 2005 and 2010, respectively. He was a Post-Doctoral Fellow successively with the Department of Computer Science, The University of Texas at San Antonio, the SUTD-MIT International Design Centre, Singapore University of Technology and Design, and the Einstein Institute of Mathematics, Hebrew University of Jerusalem from 2010 to 2014. He is currently an Associate Professor with the School of Mathematical Sciences, Tongji University. He is also an International Visiting Fellow with the Department of Mathematical Sciences, University of Essex. He was a recipient of the 2016 Dimitrie Pompeiu Prize and serves as an Associate Editor of the IEEE Access.

His research interests include the structure and dynamics of complex networks, multi-agent systems, biomathematics, social dynamics, random graph theory, and probabilistic combinatorics.

YAMEI YE received the B.S. degree in mathematics from the Zhejiang University of Technology, China, in 2016. She is currently pursuing the M.S. degree with the School of Mathematical Sciences, Tongji University. Her research interests include agent-based modeling and simulation of multi-agent systems. She received the Honorable Mention of Mathematical Contest in Modeling in 2015 and the Second Class Prize of National Post-Graduate Mathematical Contest in Modeling in 2016.