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Synchronization and Robust Synchronization for Fractional-Order Coupled Neural Networks

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ABSTRACT Synchronization and robust synchronization of fractional-order coupled neural networks (FCNNs) are considered in this paper. Different with the most published works on synchronization based on a special solution of an isolate node of the networks, we remove this restriction and introduce a more widely accepted definition of synchronization. Meanwhile, because of parametric uncertainties of network models, robust synchronization for FCNNs is investigated. In addition, by utilizing pinning control strategies, several sufficient conditions are derived to make sure that the considered networks can realize pinning synchronization and robust pinning synchronization. Finally, the correctness of the obtained results is substantiated by two given numerical examples.

INDEX TERMS Synchronization, robust synchronization, pinning control, fractional-order coupled neural networks.

I. INTRODUCTION

Over the past few decades, the growing attentions of researchers have been paid to neural networks (NNs) as a result of their widespread application in various fields, such as combinatorial optimization problems, signal processing and so on [1]–[5]. In the study of NNs, the discussion of dynamical behaviors is always a hot topic, such as synchronization [6], [7], stability [8]–[10] and so forth [11]–[16]. In [6], the authors focused on robust exponential synchronization for chaotic delayed NNs. Zhang *et al.* [8] paid their attentions to stability of discrete-time NNs with a time-varying delay. Asynchronous state estimation of Markov jump NNs taking into account jumping fading channels was considered by Tao *et al.* [12].

In order to describe physical phenomena more accurately, fractional-order NNs were built to replacing the classical integer-order NNs by researchers. Owing to the advantage of infinity memory compared with classical integer-order NNs, fractional-order NNs can model numerous phenomena more effective in various fields [17]–[19]. Thus, many researchers take much interest in the study of dynamical behaviors for fractional-order NNs. So far, lots of results of dynamical behaviors for fractional-order NNs sprung up very rapidly [20]–[33]. In [24], synchronization and Mittag-Leffler stability for memristor-based fractional-order NNs were investigated by introducing Lyapunov method. Gu *et al.* [27] considered the problem of synchronization-based parameter estimation for fractional-order NNs combined with adaptive control under parameter update law. In terms of linear feedback controller and fractional-order differential inequalities, some criteria were proposed to make ensure that fractional-order memristive BAM NNs realize synchronization in finite time [28].

Nowadays, the investigation of classical integer-order coupled neural networks (CNNs) have developed maturely due to its extensive applications in various areas [34]–[37]. Based on the cellular neural network, the authors presented a secure communication system [36]. However, because of the complexity, a few results of dynamical behaviors for FCNNs have been reported [38], [39], especially for synchronization [39]. Bao *et al.* [39] established several sufficient criteria to make sure that the considered network is synchronized by means of the fractional-order Lyapunov stability theorem. It should be pointed out that, because of the environmental noises and model errors, the exact values of FCNNs parameters usually may not be obtained. Thus, it is necessary to consider parametric uncertainties while we study the synchronization for FCNNs. To our knowledge, the results of synchronization for FCNNs have not been obtained.

Because network cannot synchronize by themselves in some circumstances, many researchers have made their effort to investigate pinning synchronization. Until now, lots of interesting works have been reported. However, most of researchers are devoted entirely to investigate the pinning synchronization for classical integer-order CNNs [40]–[43]. In [40], the authors made the CNNs achieve synchronization by using impulse pinning strategy, some inequality techniques and matrix decomposition methods. Hu *et al.* [41] focused on pinning synchronization for diffusively and linearly coupled inertial delayed NNs. The authors proposed some novel synchronization criteria for neural networks with hybrid coupling by utilizing some free weighting matrices and the appropriate Lyapunov-Krasovskii functional [42]. On the other hand, some researches have investigated the synchronization for fractional-order complex networks with pinning control and many interesting results have been got [44]–[46]. Nevertheless, very few works on pinning synchronization for FCNNs have been obtained. On account of the virtue of pinning control strategy and the phenomenon of parametric uncertainties, it is essential to pay our attention to studying the robust synchronization of FCNNs via pinning control here.

In this paper, we focus on the synchronization and robust synchronization issues for FCNNs. The main novelties lie in three aspects. First, the FCNNs models with and without parametric uncertainties are proposed. Second, several conditions are derived to let the considered network to realize synchronization and pinning synchronization. Third, considering the existences of environmental noises and model errors, robust synchronization and robust pinning synchronization for the FCNNs are investigated.

II. PRELIMINARIES

A. NOTATIONS

Let $\mathbb{R} = (-\infty, +\infty)$. $0 \leqslant P \in \mathbb{R}^{n\times n}$ $(0 > P \in \mathbb{R}^{n\times n}, 0 < P \in \mathbb{R}^{n\times n}, 0 \geqslant P \in \mathbb{R}^{n\times n})$ denotes that matrix P is symmetric and semi-positive (negative, positive, semi-negative) definite. $\otimes$ stands for the Kronecker product of two matrices. $\lambda_m(\cdot)$ $(\lambda_M(\cdot))$ means the minimum (maximum) eigenvalue of the corresponding matrix. For any vector $e(t) = (e_1(t), e_2(t), \ldots, e_n(t))^T \in \mathbb{R}^n$, we denote

$$\|e(t)\|_2 = \sqrt{\sum_{i=1}^{n} e_i^2(t)}.$$

B. DEFINITIONS

Definition 1 (See [47]): The Caputo fractional derivative can be given by

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} \dot{f}(\tau) d\tau,$$

where $t \geqslant t_0$, $0 < \alpha < 1$, $f \in C^n([t_0, +\infty), \mathbb{R})$ is an arbitrary integrable function, $\Gamma(\alpha)$ denotes the gamma function defined as $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$. In the following discussion, we always deem that $0 < \alpha < 1$.

Some properties of the fractional derivatives are given as follows:

Property 2:

$$D_t^\alpha(\varepsilon_1 y(t) + \varepsilon_2 z(t)) = \varepsilon_1 D_t^\alpha y(t) + \varepsilon_2 D_t^\alpha z(t),$$

where ε_1 and ε_2 are real constants, $y(t), z(t) \in \mathbb{R}^n$.

Property 3: The Caputo fractional derivative of a constant function is always zero.

Definition 4 (See [48]): The Mittag-Leffler function define as

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)},$$

where $z \in \mathbb{C}$ and $\alpha > 0$. And, the Mittag-Leffler function in two parameters can be presented as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)},$$

where $z \in \mathbb{C}, \alpha > 0$ and $\beta > 0$. For $\beta = 1$, one has $E_{\alpha,1} = E_\alpha, E_{1,1}(z) = e^z$.

For convenience, we introduce the Laplace transform of Caputo fractional derivative and Mittag-Leffler function in two parameters respectively described as

$$\mathcal{L}\{D_t^\alpha f(t)\} = s^\alpha F(s) - s^{\alpha-1} f(t_0), \quad \text{for } 0 < \alpha < 1,$$

and

$$\mathcal{L}\{t^{\beta-1} E_{\alpha,\beta}(-kt^\alpha)\} = \frac{s^{\alpha-\beta}}{s^\alpha + k}, \quad \mathcal{R}(s) > |k|^{\frac{1}{\alpha}},$$

where $t \geqslant 0, k \in \mathbb{R}$; s denotes the variable in Laplace domain; $\mathcal{R}(s)$ is the real part of s; $F(s)$ is the Laplace transform of $f(t)$; $\mathcal{L}\{\cdot\}$ is the Laplace transform operator.

C. LEMMAS

Lemma 5 (See [47]): For a vector $y(t) \in \mathbb{R}^n$ of derivable functions, we have the following inequality:

$$D_t^\alpha(y^T(t) Q y(t)) \leqslant 2y^T(t) Q D_t^\alpha y(t),$$

where $\alpha \in (0, 1], t \geqslant t_0$ and $0 < Q \in \mathbb{R}^{n\times n}$ is a constant matrix.

Lemma 6 (See [49]): For any matrix $0 < Q \in \mathbb{R}^{n\times n}$ and vectors $\alpha_1, \alpha_2 \in \mathbb{R}^n$, the following inequality holds:

$$2\alpha_1^T \alpha_2 \leqslant \alpha_1^T Q \alpha_1 + \alpha_2^T Q^{-1} \alpha_2.$$

Lemma 7: Let $c \in \mathbb{R}, A, B, C, D$ be matrices with suitable dimensions. Then the Kronecker product has the following properties:

(1) $(cA) \otimes B = A \otimes (cB)$;
(2) $(A \otimes B)^T = A^T \otimes B^T$;
(3) $(A + B) \otimes C = A \otimes C + B \otimes C$;
(4) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

III. SYNCHRONIZATION FOR FCNNs

In this section, we analyze synchronization issue for FCNNs. First, several criteria of synchronization are derived for FCNNs. Then, pinning control strategy is proposed to ensure that the considered network can realize synchronization.

A. SYNCHRONIZATION FOR FCNNs

A FCNNs consisting of N identical nodes is described as

$$D_t^\alpha w_i(t) = -Cw_i(t) + Bf(w_i(t)) + J + d\sum_{j=1}^{N} A_{ij}\Gamma w_j(t), \tag{1}$$

where $i = 1, 2, \ldots, N$, $0 < \alpha < 1$; $w_i(t) = (w_{i1}(t), w_{i2}(t), \ldots, w_{in}(t))^T \in \mathbb{R}^n$ denotes the state for node i; N means the number of nodes; $f(w_i(t)) = (f_1(w_{i1}(t)), f_2(w_{i2}(t)), \ldots, f_n(w_{in}(t)))^T$, $f_j(\cdot)(j = 1, 2, \ldots, n)$ is the activation function of j-th neuron; $B = (b_{ij})_{n\times n}$ and $0 < C = \mathrm{diag}(c_1, c_2, \ldots, c_n)$ are the real matrices; $\Gamma = (\gamma_{ij})_{n\times n}$ corresponds to the inner coupling matrix; $J = (J_1, J_2, \ldots, J_n)^T$ is the external input vector; $0 < d \in \mathbb{R}$ denotes the overall coupling strength; $A = (A_{ij})_{N\times N}$ is the coupling configuration matrix of network which is defined as

$$\begin{cases} A_{ij} = A_{ji} > 0\ (i \neq j), & \text{if there is a connection between nodes } i \text{ and } j,\\ A_{ij} = 0\ (i \neq j), & \text{otherwise},\\ A_{ii} = -\sum\limits_{\substack{j=1\\ j\neq i}}^{N} A_{ij}, & i = 1, 2, \ldots, N. \end{cases}$$

In this paper, the following assumption and definition will be needed.

Assumption 8: The function $f_j(\cdot)$ is said to satisfies the Lipschitz condition if there exists positive constant ψ_j such that

$$|f_j(\rho_1) - f_j(\rho_2)| \leqslant \psi_j|\rho_1 - \rho_2|$$

for any $\rho_1, \rho_2 \in \mathbb{R}$, where $|\cdot|$ is the absolute value.

Definition 9 (See [50]): The FCNNs (1) realizes synchronization if

$$\lim_{t\to\infty}\left\| w_i(t) - \frac{1}{N}\sum_{j=1}^{N} w_j(t)\right\|_2 = 0, \quad i = 1, 2, \ldots, N.$$

Define $\bar{w}(t) = \frac{1}{N}\sum_{j=1}^{N} w_j(t)$. Then, we have

$$\begin{aligned} D_t^\alpha \bar{w}(t) &= \frac{1}{N}\sum_{j=1}^{N} D_t^\alpha w_j(t)\\ &= \frac{1}{N}\sum_{j=1}^{N}\Big[-Cw_j(t) + d\sum_{s=1}^{N} A_{js}\Gamma w_s(t)\\ &\quad + Bf(w_j(t)) + J\Big]\\ &= -\frac{C}{N}\sum_{j=1}^{N} w_j(t) + \frac{d}{N}\sum_{j=1}^{N}\sum_{s=1}^{N} A_{js}\Gamma w_s(t)\\ &\quad + \frac{1}{N}\sum_{j=1}^{N} Bf(w_j(t)) + J\\ &= -\frac{C}{N}\sum_{j=1}^{N} w_j(t) + \frac{1}{N}\sum_{j=1}^{N} Bf(w_j(t)) + J. \end{aligned}$$

It should be pointed out that $\frac{d}{N}\sum_{j=1}^{N}\sum_{s=1}^{N} A_{js}\Gamma w_s(t) = \frac{d}{N}\sum_{s=1}^{N}\sum_{j=1}^{N} A_{js}\Gamma w_s(t) = 0$ according to the definition of matrix A, that is $\sum_{j=1}^{N} A_{js} = 0$.

Defining error vector $e_i(t) = w_i(t) - \bar{w}(t)$ which is described as follows:

$$\begin{aligned} D_t^\alpha e_i(t) &= -Ce_i(t) + Bf(w_i(t)) - \frac{1}{N}\sum_{j=1}^{N} Bf(w_j(t))\\ &\quad + d\sum_{j=1}^{N} A_{ij}\Gamma e_j(t), \quad i = 1, 2, \ldots, N. \end{aligned} \tag{2}$$

Through out this paper, we denote that

$$\Psi = \mathrm{diag}(\psi_1^2, \psi_2^2, \ldots, \psi_n^2),$$
$$e(t) = (e_1^T(t), e_2^T(t), \ldots, e_N^T(t))^T.$$

Theorem 10: The FCNNs (1) is synchronized, if there exists a matrix $0 < P \in \mathbb{R}^{n\times n}$ such that

$$I_N \otimes M + dA \otimes (P\Gamma + \Gamma^T P) < 0, \tag{3}$$

where $M = -PC - CP + PBB^TP + \Psi$.

Proof: For system (2), we construct the Lyapunov functional as

$$V_1(t) = \sum_{i=1}^{N} e_i^T(t)Pe_i(t). \tag{4}$$

Then, combining with Lemma 5, one has

$$\begin{aligned} D_t^\alpha V_1(t) &\leqslant 2\sum_{i=1}^{N} e_i^T(t)PD_t^\alpha e_i(t)\\ &= 2\sum_{i=1}^{N} e_i^T(t)P\Big[Bf(\bar{w}(t)) - Bf(\bar{w}(t))\\ &\quad - Ce_i(t) + Bf(w_i(t)) - \frac{1}{N}\sum_{j=1}^{N} Bf(w_j(t))\\ &\quad + d\sum_{j=1}^{N} A_{ij}\Gamma e_j(t)\Big]\\ &= -2\sum_{i=1}^{N} e_i^T(t)PCe_i(t)\\ &\quad + 2d\sum_{i=1}^{N}\sum_{j=1}^{N} e_i^T(t)A_{ij}P\Gamma e_j(t)\\ &\quad + 2\sum_{i=1}^{N} e_i^T(t)PB\big[f(w_i(t)) - f(\bar{w}(t))\big]\\ &\quad + 2\sum_{i=1}^{N} e_i^T(t)PB\Big[f(\bar{w}(t)) - \frac{1}{N}\sum_{j=1}^{N} f(w_j(t))\Big]. \end{aligned} \tag{5}$$

According to Assumption 8, one gets

$$\begin{aligned} &2\sum_{i=1}^{N} e_i^T(t)PB\big[f(w_i(t)) - f(\bar{w}(t))\big]\\ &\quad \leqslant \sum_{i=1}^{N} e_i^T(t)PBB^TPe_i(t) + \sum_{i=1}^{N} e_i^T(t)\Psi e_i(t). \end{aligned} \tag{6}$$

From $\sum_{i=1}^{N} e_i^T(t) = 0$, we know that

$$\sum_{i=1}^{N} e_i^T(t)PB\big[f(\bar{w}(t)) - \frac{1}{N}\sum_{j=1}^{N} f(w_j(t))\big] = 0. \tag{7}$$

By combining with (5)–(7), we know that

$$\begin{aligned} D_t^\alpha V_1(t) &\leqslant \sum_{i=1}^{N} e_i^T(t)(PBB^TP + \Psi)e_i(t) \\ &\quad + 2d\sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij}e_i^T(t)P\Gamma e_j(t) \\ &\quad - 2\sum_{i=1}^{N} e_i^T(t)PCe_i(t) \\ &= e^T(t)\big[I_N \otimes M + dA \otimes (P\Gamma + \Gamma^T P)\big]e(t) \\ &\leqslant \zeta_1 e^T(t)e(t), \end{aligned} \tag{8}$$

where $\zeta_1 = \lambda_M\big(I_N \otimes M + dA \otimes (P\Gamma + \Gamma^T P)\big) < 0$.

From (4), we get

$$\lambda_m(P)e^T(t)e(t) \leqslant V_1(t) \leqslant \lambda_M(P)e^T(t)e(t). \tag{9}$$

From (8) and (9), it is easily to know that

$$D_t^\alpha V_1(t) \leqslant l_1 V_1(t), \tag{10}$$

where $l_1 = \frac{\zeta_1}{\lambda_M(P)} < 0$. Thus, it is easily to get that

$$D_t^\alpha V_1(t) + q(t) = l_1 V_1(t), \tag{11}$$

where the function $q(t)$ is nonnegative.

Taking $\mathcal{L}\{\cdot\}$ of (11), one gets

$$s^\alpha \mathcal{V}_1(s) - s^{\alpha-1}V_1(0) + \mathcal{Q}(s) = l_1\mathcal{V}_1(s), \tag{12}$$

where $\mathcal{V}_1(s) = \mathcal{L}\{V_1(t)\}$, $\mathcal{Q}(s) = \mathcal{L}\{q(t)\}$.

Hence, we obtain

$$\mathcal{V}_1(s) = \frac{s^{\alpha-1}}{s^\alpha - l_1}V_1(0) - \mathcal{Q}(s)\frac{1}{s^\alpha - l_1}. \tag{13}$$

Applying the Laplace inverse transform to (13), we have

$$V_1(t) = V_1(0)E_\alpha(l_1 t^\alpha) - q(t) * t^{\alpha-1}E_{\alpha,\alpha}(l_1 t^\alpha), \tag{14}$$

where $*$ represents the convolution operator.

Since $t^{\alpha-1}$ and $E_{\alpha,\alpha}(l_1 t^\alpha)$ are nonnegative functions, it follows from (9) and (14) that

$$\lambda_m(P)e^T(t)e(t) \leqslant V_1(t) \leqslant V_1(0)E_\alpha(l_1 t^\alpha), \tag{15}$$

which leads to that

$$0 \leqslant e^T(t)e(t) \leqslant \frac{V_1(0)}{\lambda_m(P)}E_\alpha(l_1 t^\alpha). \tag{16}$$

It should be pointed out that $E_\alpha(l_1 t^\alpha)$ is completely monotonic and decreases much faster than the function $e^{l_1 t}$, for $0 < \alpha < 1$ and $l_1 < 0$ (see [47]). Therefore, we have a conclusion that $\lim_{t\to+\infty} e^T(t)e(t) = 0$, that is $\lim_{t\to+\infty}\|e(t)\|_2 = 0$. Therefore, the FCNNs (1) realizes synchronization.

B. PINNING SYNCHRONIZATION FOR FCNNs

In the preceding discussion, we analyze the synchronization problem for FCNNs. However, the considered network cannot realize synchronization by themselves sometimes. Thus, pinning controllers are applied to first r nodes in the network for the purpose of ensuring that the considered network is synchronized in this subsection.

A FCNNs consisting of N identical nodes is described as

$$\begin{cases} D_t^\alpha w_i(t) = -Cw_i(t) + Bf(w_i(t)) + d\sum_{j=1}^{N} A_{ij}\Gamma w_j(t), \\ \qquad + J + u_i(t), \quad i = 1, 2, \ldots, r, \\ D_t^\alpha w_i(t) = -Cw_i(t) + Bf(w_i(t)) + d\sum_{j=1}^{N} A_{ij}\Gamma w_j(t), \\ \qquad + J, \quad i = r+1, r+2, \ldots, N, \\ u_i(t) = -k_i\left(w_i(t) - \frac{1}{N}\sum_{j=1}^{N} w_j(t)\right), \end{cases} \tag{17}$$

where $u_i(t)$ is the feedback controller; $d, \alpha, f(w_i(t)), w_i(t), B, \Gamma, A = (A_{ij})_{N\times N}, C$ and J have the same definitions as in (1); $k_i > 0$ is the control gain.

Let $\bar{w}(t) = \frac{1}{N}\sum_{j=1}^{N} w_j(t)$. Then, we have

$$\begin{aligned} D_t^\alpha \bar{w}(t) &= \frac{1}{N}\sum_{j=1}^{N} D_t^\alpha w_j(t) \\ &= -\frac{C}{N}\sum_{j=1}^{N} w_j(t) + \frac{d}{N}\sum_{j=1}^{N}\sum_{s=1}^{N} A_{js}\Gamma w_s(t) \\ &\quad + \frac{1}{N}\sum_{j=1}^{N} Bf(w_j(t)) + J - \frac{1}{N}\left(\sum_{j=1}^{r} u_j(t)\right) \\ &= -\frac{C}{N}\sum_{j=1}^{N} w_j(t) + \frac{1}{N}\sum_{j=1}^{N} Bf(w_j(t)) + J \\ &\quad - \frac{1}{N}\left(\sum_{j=1}^{r} u_j(t)\right). \end{aligned}$$

The error vector $e_i(t) = w_i(t) - \bar{w}(t)$ is govern by

$$\begin{aligned} D_t^\alpha e_i(t) &= -Ce_i(t) + Bf(w_i(t)) - \frac{1}{N}\sum_{j=1}^{N} Bf(w_j(t)) \\ &\quad + d\sum_{j=1}^{N} A_{ij}\Gamma e_j(t) - k_i e_i(t) + \frac{1}{N}\sum_{j=1}^{N} k_j e_j(t), \end{aligned} \tag{18}$$

where $i = 1, 2, \ldots, N$; $k_i > 0$ for $i = 1, 2, \ldots, r$ and $k_i = 0$ for $i = r+1, r+2, \ldots, N$.

Theorem 11: The FCNNs (17) achieves synchronization if there exists a matrix $0 < P \in \mathbb{R}^{n\times n}$ such that

$$I_N \otimes M + dA \otimes (P\Gamma + \Gamma^T P) - 2K \otimes P < 0, \tag{19}$$

where $M = -PC - CP + PBB^TP + \Psi$ and $K = \mathrm{diag}(k_1, k_2, \ldots, k_N)$.

Proof: For error system (18), take $V_1(t)$ as in (4). Then, one has

$$\begin{aligned} D_t^\alpha V_1(t) &\leqslant 2\sum_{i=1}^{N} e_i^T(t)PD_t^\alpha e_i(t) \\ &= 2\sum_{i=1}^{N} e_i^T(t)P\Big[-Ce_i(t)+Bf(w_i(t))-k_ie_i(t) \\ &\quad +Bf(\bar{w}(t))-Bf(\bar{w}(t))-\frac{1}{N}\sum_{j=1}^{N}Bf(w_j(t)) \\ &\quad +d\sum_{j=1}^{N}A_{ij}\Gamma e_j(t)+\frac{1}{N}\sum_{j=1}^{N}k_je_j(t)\Big] \\ &= -2\sum_{i=1}^{N} e_i^T(t)PCe_i(t)-2\sum_{i=1}^{N}k_ie_i^T(t)Pe_i(t) \\ &\quad +2\sum_{i=1}^{N}e_i^T(t)PB\Big[f(\bar{w}(t))-\frac{1}{N}\sum_{j=1}^{N}f(w_j(t))\Big] \\ &\quad +2\sum_{i=1}^{N}e_i^T(t)PB\big[f(w_i(t))-f(\bar{w}(t))\big] \\ &\quad +2\sum_{i=1}^{N}e_i^T(t)P\left(\frac{1}{N}\sum_{j=1}^{N}k_je_j(t)\right) \\ &\quad +2d\sum_{i=1}^{N}\sum_{j=1}^{N}e_i^T(t)A_{ij}P\Gamma e_j(t). \end{aligned} \tag{20}$$

From $\sum_{i=1}^{N} e_i^T(t)=0$, we know that

$$2\sum_{i=1}^{N}e_i^T(t)P\left(\frac{1}{N}\sum_{j=1}^{N}k_je_j(t)\right)=0. \tag{21}$$

Therefore,

$$\begin{aligned} D_t^\alpha V_1(t) &\leqslant e^T(t)\big[I_N\otimes M+dA\otimes(P\Gamma+\Gamma^TP)-2K\otimes P\big]e(t) \\ &\leqslant \zeta_2e^T(t)e(t), \end{aligned} \tag{22}$$

where $\zeta_2=\lambda_M\big(I_N\otimes M+dA\otimes(P\Gamma+\Gamma^TP)-2K\otimes P\big)<0$.

The rest of the proof for $\lim_{t\to+\infty}\|e(t)\|_2=0$ run as in Theorem 10. Therefore, the network (17) realizes pinning synchronization.

IV. ROBUST SYNCHRONIZATION FOR FCNNs

We focus on the robust synchronization and robust pinning synchronization for FCNNs in this section. By using some inequality techniques and choosing suitable Lyapunov functional, several criteria of robust synchronization and robust pinning synchronization for FCNNs are proposed.

A. ROBUST SYNCHRONIZATION FOR FCNNs

In fact, the FCNNs may contain parameteric uncertainties due to the existence of environmental noises or model errors in many circumstances. Therefore, in this subsection, we consider a FCNNs with parameteric uncertainties consisting of N identical nodes described as

$$\begin{aligned} D_t^\alpha w_i(t) &= -Cw_i(t)+J+d\sum_{j=1}^{N}A_{ij}\Gamma w_j(t) \\ &\quad +Bf(w_i(t)),\quad i=1,2,\ldots,N, \end{aligned} \tag{23}$$

where α, $w_i(t), f(w_i(t))$ and J have the same definitions as in (1). The parameters d, C, B, Γ and A can be changed in some given precision described as follows:

$$\begin{cases} d^{(I)}:=\{0<d^-\leqslant d\leqslant d^+,\forall d\in d^{(I)}\}; \\ C^{(I)}:=\{C=\mathrm{diag}(c_i):C^-\leqslant C\leqslant C^+,0<c_i^-\leqslant c_i\leqslant c_i^+, \\ \qquad i=1,2,\ldots,n,\forall C\in C^{(I)}\}; \\ B^{(I)}:=\{B=(b_{ij})_{n\times n}:b_{ij}^-\leqslant b_{ij}\leqslant b_{ij}^+,\ i=1,2,\ldots,n, \\ \qquad j=1,2,\ldots,n,\forall B\in B^{(I)}\}; \\ \Gamma^{(I)}:=\{\Gamma=(\gamma_{ij})_{n\times n}:\gamma_{ij}^-\leqslant\gamma_{ij}\leqslant\gamma_{ij}^+,\ i=1,2,\ldots,n, \\ \qquad j=1,2,\ldots,n,\forall\Gamma\in\Gamma^{(I)}\}; \\ A^{(I)}:=\{A=(A_{ij})_{N\times N}:A_{ij}^-\leqslant A_{ij}(i\neq j)\leqslant A_{ij}^+, \\ \qquad i=1,2,\ldots,N,j=1,2,\ldots,N,\forall A\in A^{(I)}\}. \end{cases} \tag{24}$$

For convenience, we define

$$\begin{cases} \hat{b}_{ij}=\max\{|b_{ij}^-|,|b_{ij}^+|\},\quad i=1,2,\ldots,n,\ j=1,2,\ldots,n; \\ \hat{\gamma}_{ij}=\max\{|\gamma_{ij}^-|,|\gamma_{ij}^+|\},\quad i=1,2,\ldots,n,\ j=1,2,\ldots,n; \\ \hat{A}_{ij}(i\neq j)=A_{ij}^+,\quad \hat{A}_{ii}=\sum_{\substack{j=1\\ j\neq i}}^{N}A_{ij}^+,\ i=1,2,\ldots,N, \\ j=1,2,\ldots,N. \end{cases}$$

Let $\bar{w}(t)=\frac{1}{N}\sum_{j=1}^{N}w_j(t)$. Thereby, one gets

$$D_t^\alpha\bar{w}(t)=-\frac{C}{N}\sum_{j=1}^{N}w_j(t)+\frac{1}{N}\sum_{j=1}^{N}Bf(w_j(t))+J.$$

The error vector $e_i(t)=w_i(t)-\bar{w}(t)$ is presented as follows:

$$\begin{aligned} D_t^\alpha e_i(t) &= -Ce_i(t)+Bf(w_i(t))-\frac{1}{N}\sum_{j=1}^{N}Bf(w_j(t)) \\ &\quad +d\sum_{j=1}^{N}A_{ij}\Gamma e_j(t),\quad i=1,2,\ldots,N. \end{aligned} \tag{25}$$

Theorem 12: The FCNNs (23) with the parameter ranges defined by (24) is robustly synchronized if there exists a matrix $0<\hat{P}=\mathrm{diag}(\hat{p}_1,\hat{p}_2,\ldots,\hat{p}_n)$ such that

$$-2\hat{P}C^-+(\xi_B+d^+\xi_A)\hat{P}^2+\Psi+d^+\xi_\Gamma I_n<0, \tag{26}$$

where $\xi_B=\sum_{i=1}^{N}\sum_{j=1}^{N}\hat{b}_{ij}^2$, $\xi_A=\sum_{i=1}^{N}\sum_{j=1}^{N}\hat{A}_{ij}^2$ and $\xi_\Gamma=\sum_{i=1}^{N}\sum_{j=1}^{N}\hat{\gamma}_{ij}^2$.

Proof: For error system (25), we take the Lyapunov functional as

$$V_2(t)=\sum_{i=1}^{N}e_i^T(t)\hat{P}e_i(t). \tag{27}$$

Then, one has

$$
\begin{aligned}
D_t^{\alpha}V_2(t) &\leqslant 2\sum_{i=1}^{N} e_i^T(t)\hat{P}D_t^{\alpha}e_i(t)\\
&= 2\sum_{i=1}^{N} e_i^T(t)\hat{P}\big[-Ce_i(t)+Bf(w_i(t))+Bf(\bar{w}(t))\\
&\quad -Bf(\bar{w}(t))-\frac{1}{N}\sum_{j=1}^{N}Bf(w_j(t))+d\sum_{j=1}^{N}A_{ij}\Gamma e_j(t)\big]\\
&= -2\sum_{i=1}^{N} e_i^T(t)\hat{P}Ce_i(t)\\
&\quad +2d\sum_{i=1}^{N}\sum_{j=1}^{N}A_{ij}e_i^T(t)\hat{P}\Gamma e_j(t)\\
&\quad +2\sum_{i=1}^{N} e_i^T(t)\hat{P}B\big[f(w_i(t))-f(\bar{w}(t))\big]\\
&\quad +2\sum_{i=1}^{N} e_i^T(t)\hat{P}B\big[f(\bar{w}(t))-\frac{1}{N}\sum_{j=1}^{N}f(w_j(t))\big]. \qquad (28)
\end{aligned}
$$

From $\sum_{i=1}^{N} e_i^T(t)=0$, we know that

$$
\sum_{i=1}^{N} e_i^T(t)\hat{P}B\big[f(\bar{w}(t))-\frac{1}{N}\sum_{j=1}^{N}f(w_j(t))\big]=0. \qquad (29)
$$

According to Assumption 8, one gets

$$
\begin{aligned}
&2\sum_{i=1}^{N} e_i^T(t)\hat{P}B\big[f(w_i(t))-f(\bar{w}(t))\big]\\
&\leqslant \sum_{i=1}^{N} e_i^T(t)\hat{P}BB^T\hat{P}e_i(t)+\sum_{i=1}^{N} e_i^T(t)\Psi e_i(t)\\
&\leqslant e^T(t)\big[I_N\otimes(\xi_B\hat{P}^2+\Psi)\big]e(t). \qquad (30)
\end{aligned}
$$

From Lemma 6, it is easily to get that

$$
\begin{aligned}
&2d\sum_{i=1}^{N}\sum_{j=1}^{N} e_i^T(t)A_{ij}\hat{P}\Gamma e_j(t)\\
&= 2de^T(t)\big[(A\otimes\hat{P})(I_N\otimes\Gamma)\big]e(t)\\
&\leqslant de^T(t)\big[A^2\otimes\hat{P}^2\big]e(t)+de^T(t)\big[I_N\otimes(\Gamma^T\Gamma)\big]e(t)\\
&\leqslant d\xi_A e^T(t)\big(I_N\otimes\hat{P}^2\big)e(t)+d\xi_\Gamma e^T(t)e(t)\\
&\leqslant d^+\xi_A e^T(t)\big(I_N\otimes\hat{P}^2\big)e(t)+d^+\xi_\Gamma e^T(t)e(t). \qquad (31)
\end{aligned}
$$

Combined with (28), (29), (30) and (31), we know that

$$
\begin{aligned}
D_t^{\alpha}V_2(t) &\leqslant -2e^T(t)(I_N\otimes(\hat{P}C^-)e(t)\\
&\quad +e^T(t)\big[I_N\otimes(\Psi+\xi_B\hat{P}^2)\big]e(t)\\
&\quad +d^+\xi_\Gamma e^T(t)e(t)+d^+\xi_A e^T(t)\big(I_N\otimes\hat{P}^2\big)e(t)\\
&= e^T(t)\big[I_N\otimes\big(-2\hat{P}C^-+(\xi_B+d^+\xi_A)\hat{P}^2\\
&\quad +\Psi+d^+\xi_\Gamma I_n\big)\big]e(t)\\
&\leqslant \zeta_3 e^T(t)e(t), \qquad (32)
\end{aligned}
$$

where $\zeta_3=\lambda_M\big(-2\hat{P}C^-+(\xi_B+d^+\xi_A)\hat{P}^2+\Psi+d^+\xi_\Gamma I_n\big)<0$.

From (27), we get

$$
\lambda_m(\hat{P})e^T(t)e(t)\leqslant V_2(t)\leqslant \lambda_M(\hat{P})e^T(t)e(t). \qquad (33)
$$

Combined with (32) and (33), it is easily to know that

$$
D_t^{\alpha}V_2(t)\leqslant l_2V_2(t), \qquad (34)
$$

where $l_2=\frac{\zeta_3}{\lambda_M(\hat{P})}<0$. Thereby, we have

$$
D_t^{\alpha}V_2(t)+m(t)=l_2V_2(t), \qquad (35)
$$

where the function $m(t)$ is nonnegative.

Taking $\mathcal{L}\{\cdot\}$ of (35), one has

$$
s^{\alpha}\mathcal{V}_2(s)-s^{\alpha-1}V_2(0)+\mathcal{M}(s)=l_2\mathcal{V}_2(s), \qquad (36)
$$

where $\mathcal{V}_2(s)=\mathcal{L}\{V_2(t)\}$, $\mathcal{M}(s)=\mathcal{L}\{m(t)\}$.

Thus, we obtain

$$
\mathcal{V}_2(s)=\frac{s^{\alpha-1}}{s^{\alpha}-l_2}V_2(0)-\mathcal{M}(s)\frac{1}{s^{\alpha}-l_2}. \qquad (37)
$$

Applying the Laplace inverse transform to (37), we have

$$
V_2(t)=V_2(0)E_{\alpha}(l_2t^{\alpha})-m(t)*t^{\alpha-1}E_{\alpha,\alpha}(l_2t^{\alpha}), \qquad (38)
$$

where $*$ represents the convolution operator.

Since $t^{\alpha-1}$ and $E_{\alpha,\alpha}(l_2t^{\alpha})$ are nonnegative functions, it follows from (33) and (38) that

$$
\lambda_m(\hat{P})e^T(t)e(t)\leqslant V_2(t)\leqslant V_2(0)E_{\alpha}(l_2t^{\alpha}), \qquad (39)
$$

which leads to that

$$
0\leqslant e^T(t)e(t)\leqslant \frac{V_2(0)}{\lambda_m(\hat{P})}E_{\alpha}(l_2t^{\alpha}). \qquad (40)
$$

It should be pointed out that $E_{\alpha}(l_2t^{\alpha})$ is completely monotonic and decreases much faster than the function e^{l_2t}, for $0<\alpha<1$ and $l_2<0$. Therefore, we have a conclusion that $\lim_{t\to+\infty}e^T(t)e(t)=0$, that is $\lim_{t\to+\infty}\|e(t)\|_2=0$. Therefore, the FCNNs (23) realizes robust synchronization.

B. ROBUST PINNING SYNCHRONIZATION FOR FCNNs

In the preceding discussion, we have investigated robust synchronization for FCNNs with parameteric uncertainties. Howbeit, the considered network cannot realize robust synchronization by themselves sometimes. Therefore, in this subsection, pinning controllers are applied to first r nodes in the network so as to ensure that the considered network achieves robust pinning synchronization.

A FCNNs with parameteric uncertainties consisting of N identical nodes is described as

$$
\begin{cases}
D_t^{\alpha}w_i(t)=-Cw_i(t)+Bf(w_i(t))+d\sum_{j=1}^{N}A_{ij}\Gamma w_j(t),\\
\qquad +J+u_i(t),\quad i=1,2,\ldots,r,\\
D_t^{\alpha}w_i(t)=-Cw_i(t)+Bf(w_i(t))+d\sum_{j=1}^{N}A_{ij}\Gamma w_j(t),\\
\qquad +J,\quad i=r+1,r+2,\ldots,N,\\
u_i(t)=-k_i\left(w_i(t)-\frac{1}{N}\sum_{j=1}^{N}w_j(t)\right),
\end{cases} \qquad (41)
$$

where $u_i(t)$ denotes the feedback controller; $\alpha, w_i(t), f(w_i(t))$, C, B, Γ, J, d and $A = (A_{ij})_{N\times N}$ have the same definitions as in (23); $k_i > 0$ is the control gain.

Let $\bar{w}(t) = \frac{1}{N}\sum_{j=1}^{N} w_j(t)$. Then, it gives

$$D_t^\alpha \bar{w}(t) = -\frac{C}{N}\sum_{j=1}^{N} w_j(t) + \frac{1}{N}\sum_{j=1}^{N} Bf(w_j(t)) + J - \frac{1}{N}\left(\sum_{j=1}^{r} u_j(t)\right).$$

The error vector $e_i(t) = w_i(t) - \bar{w}(t)$ can be presented as

$$D_t^\alpha e_i(t) = -Ce_i(t) + Bf(w_i(t)) - \frac{1}{N}\sum_{j=1}^{N} Bf(w_j(t)) + d\sum_{j=1}^{N} A_{ij}\Gamma e_j(t) - k_i e_i(t) + \frac{1}{N}\sum_{j=1}^{N} k_j e_j(t), \quad (42)$$

where $i = 1, 2, \ldots, N$; $k_i > 0$ for $i = 1, 2, \ldots, r$ and $k_i = 0$ for $i = r+1, r+2, \ldots, N$.

Theorem 13: The FCNNs (41) is robustly synchronized if there exists a matrix $0 < \hat{P} = \mathrm{diag}(\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n)$ such that

$$I_N \otimes \hat{M} - 2K \otimes \hat{P} < 0, \quad (43)$$

where $\hat{M} = -2\hat{P}C^- + (\xi_B + d^+\xi_A)\hat{P}^2 + \Psi + d^+\xi_\Gamma I_n$, $K = \mathrm{diag}(k_1, k_2, \ldots, k_N)$, $\xi_B = \sum_{i=1}^{N}\sum_{j=1}^{N}\hat{b}_{ij}^2$, $\xi_A = \sum_{i=1}^{N}\sum_{j=1}^{N}\hat{A}_{ij}^2$ and $\xi_\Gamma = \sum_{i=1}^{N}\sum_{j=1}^{N}\hat{\gamma}_{ij}^2$.

Proof: For error system (42), take $V_2(t)$ as in (27). Then, one has

$$\begin{aligned}
D_t^\alpha V_2(t) &\leqslant 2\sum_{i=1}^{N} e_i^T(t)\hat{P}D_t^\alpha e_i(t) \\
&= 2\sum_{i=1}^{N} e_i^T(t)\hat{P}\big[-Ce_i(t) + Bf(w_i(t)) - k_i e_i(t) \\
&\quad + Bf(\bar{w}(t)) - Bf(\bar{w}(t)) - \frac{1}{N}\sum_{j=1}^{N} Bf(w_j(t)) \\
&\quad + d\sum_{j=1}^{N} A_{ij}\Gamma e_j(t) + \frac{1}{N}\sum_{j=1}^{N} k_j e_j(t)\big] \\
&= -2\sum_{i=1}^{N} e_i^T(t)PCe_i(t) - 2\sum_{i=1}^{N} k_i e_i^T(t)\hat{P}e_i(t) \\
&\quad + 2\sum_{i=1}^{N} e_i^T(t)\hat{P}B\big[f(\bar{w}(t)) - \frac{1}{N}\sum_{j=1}^{N} f(w_j(t))\big] \\
&\quad + 2\sum_{i=1}^{N} e_i^T(t)\hat{P}B\big[f(w_i(t)) - f(\bar{w}(t))\big] \\
&\quad + 2\sum_{i=1}^{N} e_i^T(t)P\left(\frac{1}{N}\sum_{j=1}^{N} k_j e_j(t)\right) \\
&\quad + 2d\sum_{i=1}^{N}\sum_{j=1}^{N} e_i^T(t)A_{ij}\hat{P}\Gamma e_j(t) \\
&\leqslant -2e^T(t)(K\otimes\hat{P})e(t) \\
&\quad - 2e^T(t)\big[I_N \otimes (\hat{P}C^-)\big]e(t) \\
&\quad + e^T(t)\big[I_N \otimes (\Psi + \xi_B\hat{P}^2)\big]e(t) \\
&\quad + d^+\xi_\Gamma e^T(t)e(t) + d^+\xi_A e^T(t)\big(I_N \otimes \hat{P}^2\big)e(t) \\
&= e^T(t)\big[I_N \otimes \hat{M} - 2K \otimes \hat{P}\big]e(t) \\
&\leqslant \zeta_4 e^T(t)e(t),
\end{aligned} \quad (44)$$

where $\zeta_4 = \lambda_M\big(I_N \otimes \hat{M} - 2K \otimes \hat{P}\big) < 0$.

The rest of the proof for $\lim_{t\to+\infty}\|e(t)\|_2 = 0$ run as in Theorem 12. Therefore, the network (41) realizes robust synchronization.

V. NUMERICAL EXAMPLES

Two numerical examples are provided to confirm the correctness for the obtained synchronization criteria in this section.

Example 14: A FCNNs consisting of five identical 2-D fractional-order neural network is considered in the following:

$$D_t^\alpha w_i(t) = -Cw_i(t) + Bf(w_i(t)) + d\sum_{j=1}^{5} A_{ij}\Gamma w_j(t) + J - k_i\left(w_i(t) - \frac{1}{5}\sum_{j=1}^{5} w_j(t)\right), \quad (45)$$

where $i = 1, 2, \ldots, 5$, $\alpha = 0.97$, $f_j(\xi) = \frac{|\xi+1| - |\xi-1|}{4}$ $(j = 1, 2)$, $d = 0.7$, $k_i = 0.8i$ for $i = 1, 2$ and $k_i = 0$ for $i = 3, 4, 5$. We take C, B, A, Γ and J as follows:

$$C = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad B = \begin{pmatrix} 0.1 & 0.2 \\ 0.1 & 0.1 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0.5 & 0.6 \\ 0.7 & 0.3 \end{pmatrix},$$

$$A = \begin{pmatrix} -0.4 & 0.2 & 0.1 & 0 & 0.1 \\ 0.2 & -0.5 & 0 & 0.3 & 0 \\ 0.1 & 0 & -0.5 & 0.3 & 0.1 \\ 0 & 0.3 & 0.3 & -0.6 & 0 \\ 0.1 & 0 & 0.1 & 0 & -0.2 \end{pmatrix}, \quad J = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It is clear that $f_j(\cdot)(j = 1, 2)$ satisfy **Assumption 8** with $\psi_j = 0.5$.

By using the MATLAB, the following matrix P satisfying (19) can be found

$$P = \begin{pmatrix} 0.4619 & -0.0570 \\ -0.0570 & 0.5006 \end{pmatrix}.$$

Therefore, it can be got that the network (17) achieve pinning synchronization from Theorem 11 (see Figure 1).

Example 15: A FCNNs with parameteric uncertainties consisting of five identical 2-D fractional-order neural network is considered in the following:

$$D_t^\alpha w_i(t) = -Cw_i(t) + Bf(w_i(t)) + d\sum_{j=1}^{5} A_{ij}\Gamma w_j(t) + J - k_i\left(w_i(t) - \frac{1}{5}\sum_{j=1}^{5} w_j(t)\right), \quad (46)$$

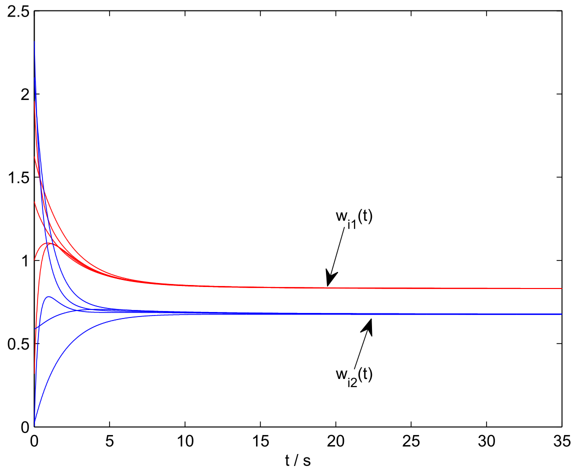

FIGURE 1. The change processes of $w_{i1}(t)$, $w_{i2}(t)$, $i = 1, 2, \ldots, 5$.

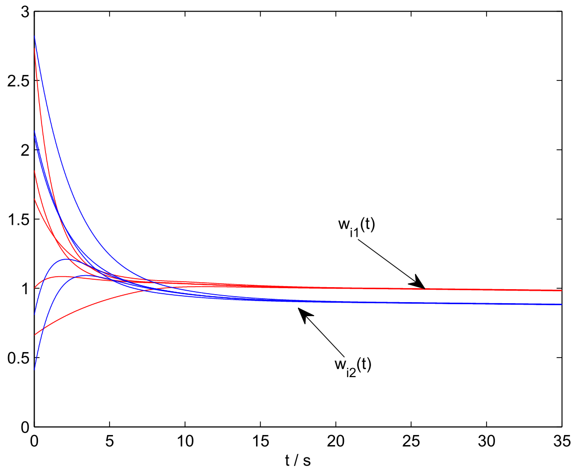

FIGURE 2. The change processes of $w_{i1}(t)$, $w_{i2}(t)$, $i = 1, 2, \ldots, 5$.

where $i = 1, 2, \ldots, 5$, $\alpha = 0.997$, $f_j(\xi) = \frac{|\xi+1|-|\xi-1|}{4}$ $(j = 1, 2)$, $k_i = 0.2i$ for $i = 1, 2, 3$ and $k_i = 0$ for $i = 4, 5$. The parameters d, C, B, Γ and A in the network (46) can be changed in some given precision described as:

$$\begin{cases} d^{(I)} := \{0.001 \leqslant d \leqslant 0.01, \forall d \in d^{(I)}\}; \\ C^{(I)} := \{C = \mathrm{diag}(c_i) : C^- \leqslant C \leqslant C^+, i.e., \frac{1}{i+2} + 0.01 \\ \qquad \leqslant c_i \leqslant \frac{1}{i+2} + 0.115, i = 1, 2, \ldots, n, \forall C \in C^{(I)}\}; \\ B^{(I)} := \{B = (b_{ij})_{n\times n} : \frac{1}{i+j} + 0.01 \leqslant b_{ij} \leqslant \frac{1}{i+j} + 0.05, \\ \qquad i = 1, 2, \ldots, n, j = 1, 2, \ldots, n, \forall B \in B^{(I)}\}; \\ \Gamma^{(I)} := \{\Gamma = (\gamma_{ij})_{n\times n} : \frac{1}{i+j} + 0.01 \leqslant \gamma_{ij} \leqslant \frac{1}{i+j} + 0.1, \\ \qquad i = 1, 2, \ldots, n, j = 1, 2, \ldots, n, \forall \Gamma \in \Gamma^{(I)}\}; \\ A^{(I)} := \{A = (A_{ij})_{N\times N} : \frac{1}{i+j} + 0.02 \leqslant A_{ij}(i \neq j) \leqslant \frac{1}{i+j} \\ \qquad + 0.05, i = 1, 2, \ldots, N, j = 1, 2, \ldots, N, \forall A \in A^{(I)}\}. \end{cases}$$

It is clear that $f_j(\cdot)(j = 1, 2)$ satisfy **Assumption 8** with $\psi_j = 0.5$.

By using the MATLAB, the following matrix $\hat{P}$ satisfying (43) can be found

$$\hat{P} = \begin{pmatrix} 0.6975 & 0 \\ 0 & 0.7140 \end{pmatrix}.$$

Therefore, it can be got that the network (41) achieve robust pinning synchronization from Theorem 13 (see Figure 2).

VI. CONCLUSION

In this paper, we have paid our attentions to investigating the FCNNs. On the one hand, with the help of pinning control technique and Lyapunov functional method, some criteria of synchronization and pinning synchronization for considered networks have been established. On the other hand, thanks to some inequality methods, the authors dealt with the problems of robust synchronization and robust pinning synchronization for FCNNs with parametric uncertainties. In the end, for confirming the correctness of the theoretical results, two numerical examples have been given.

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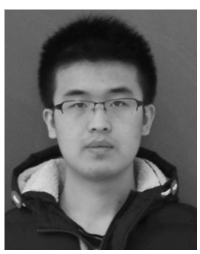

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