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Precise Characterizations of the Stability Margin in Time-Domain Space for Planar Systems Undergoing Periodic Switching

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ABSTRACT This paper addresses the stability analysis problem for planar periodic switching systems. We characterize the stability margin in the space constituted by the dwell times of the subsystems, by which we can assess the asymptotic stability of the overall system in the necessary and sufficient sense. The mutual constraint conditions on the dwell times in nature depend on the type of equilibrium point of each subsystem. The stability conditions are expressed in terms of a family of transcendental inequalities, which can be numerically solved and precisely depicted in the time-domain space. An example is worked out in detail to illustrate the theoretical results.

INDEX TERMS Planar dynamical systems, periodic switching, asymptotic stability.

I. INTRODUCTION

The stability problem of switching systems has attracted considerable attention during the past two decades. A switching system is composed of several systems with a switching signal to orchestrate among them. In studying such systems, we have learned that switching can produce very interesting and complex dynamical behaviors that might be beyond our imagination.

The stability analysis problem of switching systems can be classified into two categories; see [7], [12]. Firstly, we are concerned with the so-called absolute stability problem that whether it is possible to construct a switching law to achieve destabilization. In the spirit of variational approach, the absolute stability problem requires to investigate if the extremal trajectory is convergent, which is constructed so that each point on it gets away from the origin as far as possible. If so, then we can confirm that the overall switching system will stay stable for an arbitrary switching signal. Secondly, the stability analysis problem is purely intended to characterize the evolution of switching signal over time and its effect on stability. An important aspect of the time-evolution of switching signal is the density of the switching points distributed within an interval of time, which can be captured in terms of average dwell-times. To guarantee the stability of the overall system, conventionally, the multiple Lyapunov functions approach is used to derive constraint conditions on the average dwell-time; see, e.g., [7], [11], [12], [14]. The mechanism of this methodology is to make the period between any two successive switching points long enough so as to allow the overshoot caused by switching to fade. However, this may lead to conservatism because without knowing the phase of the overshoot, we actually suppress it simply according to the worst case.

In the present paper, we shall investigate the stability problem for a switching system composed of two planar linear subsystems, which are triggered into activation alternatively and periodically. In the research field of switching systems, the planar case indeed constitutes the most developed branch because a set of dedicated methods allow us to gain deep insight into the stability problem beyond simply using a general method to address it; see [2]–[5], [9], [10]. Within the context of periodic switching, we shall establish the mutual constraint conditions on the dwell-times of the subsystems to ensure the stability of the overall system. An invariant that plays a key role in describing the interrelations of the subsystem matrices is involved in the stability criteria. This signifies that the overall switching dynamics in a strong way depends on the interrelation of the subsystem matrices as well as the types of their eigenvalues. Moreover, the derived stability criteria do not depend on the particular choice of the coordinates and, therefore, allow a system to speak for itself.

As we know, the stability conditions derived through the multiple Lyapunov functions approach usually account for the effect of average dwell-time on stability in a linear manner. The result in [8] showed that conditions that are linear in the average dwell-time may be very restricted. Actually, the results posed in the present paper demonstrate that the dwell-times of the subsystems actually influence the overall switching dynamics in a highly nonlinear manner. Therefore, we are able to precisely determine the stability margin in the space constituted by the dwell-times and hence assess stability in a necessary and sufficient sense. On the other hand, to ensure this stability, the existing methods usually require the stable subsystems to play the dominant role in the overall switching dynamics, among others. For the sake of generality, we will not suppose the subsystems to be stable because this assumption turns out to be restricted. To demonstrate the theoretical results, we shall construct an example to show that two unstable subsystems with quite long periods to stay active may generate stable dynamics.

The remainder of this paper is organized as follows. In Section 2, we introduce the basic definitions and formulate the problem. In Section 3, we present the stability conditions in the time-domain space and give some explanations to the matters relevant to the results. An example is included in Section 4 to illustrate the theoretical results. Finally, this work is briefly summarized in Section 5.

Notation: Throughout this paper, we use the following notations. Let $\det(X)$ and $\mathrm{tr}(X)$ be the determinant and trace of X, respectively. Let $[X, Y] = XY - YX$ be the Lie commutator of X and Y. Besides, we write I and j for the identity matrix and the unit of imaginary numbers, respectively.

II. PROBLEM FORMULATION

Consider the planar switching system described as follows

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad x \in \mathbf{R}^2. \tag{1}$$

This model is composed of two linear subsystems and the switching signal $\sigma(t)$ to orchestrate among them. Let $\Delta_i = \mathrm{tr}(A_i)^2 - 4\det(A_i)$ be the discriminant of the characteristic polynomial of A_i. Due to space limitations, we suppose $\Delta_1\Delta_2 \neq 0$ throughout the paper. According to the evolution over time, we can represent a switching signal as follows

$$\Big\{(\sigma(t_0), t_0 = 0), (\sigma(t_1), t_1), \cdots, (\sigma(t_k), t_k)\Big| \lim_{k\to+\infty} t_k = +\infty\Big\}. \tag{2}$$

This sequential form indicates that at the switching time t_k, the switching signal takes on a value $\sigma(t_k)$ in the binary set $\{1, 2\}$ to trigger the two subsystems into activation alternatively and successively. The dynamics of the system then is determined by the transition matrix as follows

$$e^{A_{\sigma(t_k)}(t-t_k)}e^{A_{\sigma(t_{k-1})}(t_k-t_{k-1})}\cdots e^{A_{\sigma(t_0)}(t_1-t_0)}, \quad t \in [t_k, t_{k+1}). \tag{3}$$

Indeed, switching system (1) is asymptotically stable if and only if the transition matrix in (3) has its eigenvalues inside the unit circle of the complex plane for all $t \geq 0$; see [15]. To characterize the distribution of the eigenvalues of the transition matrix, we suppose the switching signals to obey the following regular property.

Definition 1: A switching rule is said to be *periodic* if the interval between two successive switching points satisfies

$$t_{k+1} - t_k = \begin{cases} \tau_1, & \sigma(t_k) = 1 \\ \tau_2, & \sigma(t_k) = 2 \end{cases}, \quad k \geq 0$$

where $\tau_1 > 0$ and $\tau_2 > 0$ are referred to as the dwell-times of the individual subsystems. When undergoing periodic switching, system (1) is referred to as a *periodic switching system*.

Hereafter, the switching signals will be considered to be periodic, and the system in (1) will be considered to be a periodic switching system unless otherwise specified. Therefore, our first observation is that when driven by a periodic switching signal, system (1) is asymptotically stable if and only if the eigenvalues of $e^{A_1\tau_1}e^{A_2\tau_2}$ are located inside the unit circle of the complex plane. We now put this observation in the following way.

Definition 2: The periodic switching system in (1) is said to be asymptotically stable if the unit of its transition matrix $e^{A_1\tau_1}e^{A_2\tau_2}$ is Schur stable.

Within this context, the paper is devoted to characterize the joint effects of the dwell-times τ_1 and τ_2 on state-transitions over time and, therefore, deriving their mutual constraint relation guaranteeing system (1) to be asymptotically stable.

III. MAIN RESULTS

The following fact plays a key role in deriving our results, which can be directly checked.

Lemma 1 [4]: For two-dimensional square matrices X and Y, we have the following identity

$$\det(X + Y) = \det(X) + \det(Y) + \mathrm{tr}(X)\mathrm{tr}(Y) - \mathrm{tr}(XY).$$

Using Lemma 1 and the fact $\det(e^X) = e^{\mathrm{tr}(X)}$ yields

$$\begin{aligned} m(\mu) &:= \det(e^{A_1\tau_1}e^{A_2\tau_2} - \mu I) \\ &= \mu^2 - \mu\,\mathrm{tr}(e^{A_1\tau_1}e^{A_2\tau_2}) + e^{\mathrm{tr}(A_1)\tau_1}e^{\mathrm{tr}(A_2)\tau_2}. \end{aligned} \tag{4}$$

The discriminant of the parabolic function $m(\mu)$ is

$$[\mathrm{tr}(e^{A_1\tau_1}e^{A_2\tau_2})]^2 - 4e^{\mathrm{tr}(A_1)\tau_1}e^{\mathrm{tr}(A_2)\tau_2}.$$

By the distribution property of the roots of parabolic functions, we have the following result.

Lemma 2: The equation $m(\mu) = 0$ has both roots inside the unit circle of the complex plane (i.e., $e^{A_1\tau_1}e^{A_2\tau_2}$ is Schur stable) if and only if τ_1 and τ_2 satisfy

$$\begin{aligned} [\mathrm{tr}(e^{A_1\tau_1}e^{A_2\tau_2})]^2 - 4e^{\mathrm{tr}(A_1)\tau_1}e^{\mathrm{tr}(A_2)\tau_2} &< 0, \\ e^{\mathrm{tr}(A_1)\tau_1}e^{\mathrm{tr}(A_2)\tau_2} &< 1; \end{aligned} \tag{5}$$

or,

$$\begin{aligned}[\operatorname{tr}(e^{A_1\tau_1}e^{A_2\tau_2})]^2 - 4e^{\operatorname{tr}(A_1)\tau_1}e^{\operatorname{tr}(A_2)\tau_2} &\geq 0,\\ m(1) > 0,\ m(-1) &> 0,\\ -1 < \frac{1}{2}\operatorname{tr}(e^{A_1\tau_1}e^{A_2\tau_2}) &< 1. \end{aligned} \tag{6}$$

Remark 1: The inequalities in (5) correspond to that $e^{A_1\tau_1}e^{A_2\tau_2}$ has a pair of complex eigenvalues located inside the unit circle of the complex plane, while the inequalities in (6) to that $e^{A_1\tau_1}e^{A_2\tau_2}$ has real eigenvalues within $(-1, 1)$. Additionally, the inequalities in (6) can be equivalently rewritten as

$$\begin{aligned}|\operatorname{tr}(e^{A_1\tau_1}e^{A_2\tau_2})| &\geq 2e^{\operatorname{tr}(A_1)\tau_1/2}e^{\operatorname{tr}(A_2)\tau_2/2},\\ \left|\operatorname{tr}(e^{A_1\tau_1}e^{A_2\tau_2})\right| &< 1 + e^{\operatorname{tr}(A_1)\tau_1}e^{\operatorname{tr}(A_2)\tau_2},\end{aligned}$$

and

$$\left|\operatorname{tr}(e^{A_1\tau_1}e^{A_2\tau_2})\right| < 2.$$

The compatibility of this set of inequalities is guaranteed by the elementary inequality $2|\alpha| < 1 + \alpha^2, \forall \alpha \in \mathbf{R}$.

One can straightforwardly derive the following conclusions.

Corollary 1: If $\operatorname{tr}(A_1) < 0$ and $\operatorname{tr}(A_2) < 0$ (a particular case is A_1 and A_2 both are Hurwitz stable), then system (1) is asymptotically stable if and only if τ_1 and τ_2 can satisfy

$$|\operatorname{tr}(e^{A_1\tau_1}e^{A_2\tau_2})| < 2e^{\operatorname{tr}(A_1)\tau_1/2}e^{\operatorname{tr}(A_2)\tau_2/2};$$

or

$$\begin{aligned}|\operatorname{tr}(e^{A_1\tau_1}e^{A_2\tau_2})| &\geq 2e^{\operatorname{tr}(A_1)\tau_1/2}e^{\operatorname{tr}(A_2)\tau_2/2},\\ \left|\operatorname{tr}(e^{A_1\tau_1}e^{A_2\tau_2})\right| &< 1 + e^{\operatorname{tr}(A_1)\tau_1}e^{\operatorname{tr}(A_2)\tau_2}.\end{aligned}$$

Corollary 2: There exist dwell-times τ_1 and τ_2 that guarantee system (1) to be asymptotically stable only if at least one of $\operatorname{tr}(A_1)$ and $\operatorname{tr}(A_2)$ is negative.

Proof: As $\operatorname{tr}(A_1) \geq 0$ and $\operatorname{tr}(A_2) \geq 0$, the inequalities in (5) will fail. At the same time, the inequalities in (6) become

$$2e^{\operatorname{tr}(A_1)\tau_1/2}e^{\operatorname{tr}(A_2)\tau_2/2} \leq |\operatorname{tr}(e^{A_1\tau_1}e^{A_2\tau_2})| < 2.$$

Obviously, this is infeasible. The assertion then is proven by this contradiction. □

Remark 2: From the perspective of assigning the distribution of the eigenvalues of $e^{A_1\tau_1}e^{A_2\tau_2}$, we now provide an explanation for the difference between the stability analysis problem and the absolute stability problem. When confined to periodic switching signals, the stability analysis problem requires us to determine the range of τ_1 and τ_2, for which $e^{A_1\tau_1}e^{A_2\tau_2}$ is Schur stable. Instead, the absolute stability problem requires us to establish some conditions for A_1 and A_2 so as to guarantee $e^{A_1\tau_1}e^{A_2\tau_2}$ to be Schur stable for any τ_1 and τ_2; see, e.g., [1], [6], [13], [16].

From the Cayley-Hamilton theorem, one can deduce the following expansions (cf. [8]):

$$e^{A_1\tau_1} = f_1(\tau_1)I + g_1(\tau_1)A_1,$$

and

$$e^{A_2\tau_2} = f_2(\tau_2)I + g_2(\tau_2)A_2.$$

Therefore, we get that

$$\begin{aligned}e^{A_1\tau_1}e^{A_2\tau_2} &= f_1(\tau_1)f_2(\tau_2)I + f_2(\tau_2)g_1(\tau_1)A_1\\ &\quad + f_1(\tau_1)g_2(\tau_2)A_2 + g_1(\tau_1)g_2(\tau_2)A_1A_2,\end{aligned} \tag{7}$$

and hence

$$\begin{aligned}\operatorname{tr}(e^{A_1\tau_1}e^{A_2\tau_2}) &= 2f_1(\tau_1)f_2(\tau_2) + f_2(\tau_2)g_1(\tau_1)\operatorname{tr}(A_1)\\ &\quad + f_1(\tau_1)g_2(\tau_2)\operatorname{tr}(A_2) + g_1(\tau_1)g_2(\tau_2)\operatorname{tr}(A_1A_2).\end{aligned} \tag{8}$$

According to the type of the equilibrium point of each subsystem, for the function $g_i(s)$, we have

$$g_i(s) = \begin{cases}\dfrac{1}{\sqrt{\Delta_i}/2}e^{\operatorname{tr}(A_i)s/2}\sinh(\sqrt{\Delta_i}s/2), & \Delta_i > 0,\\ \dfrac{1}{\sqrt{-\Delta_i}/2}e^{\operatorname{tr}(A_i)s/2}\sin(\sqrt{-\Delta_i}s/2), & \Delta_i < 0.\end{cases} \tag{9}$$

Correspondingly, for the function $f_i(s)$, we have

$$f_i(s) = \dot{g}_i(s) - \operatorname{tr}(A_i)g_i(s) = e^{\operatorname{tr}(A_i)s/2} \times \begin{cases}\left[-\dfrac{\operatorname{tr}(A_i)}{\sqrt{\Delta_i}}\sinh(\sqrt{\Delta_i}s/2) + \cosh(\sqrt{\Delta_i}s/2)\right],\\ \qquad \Delta_i > 0,\\ \left[-\dfrac{\operatorname{tr}(A_i)}{\sqrt{-\Delta_i}}\sin(\sqrt{-\Delta_i}s/2) + \cos(\sqrt{-\Delta_i}s/2)\right],\\ \qquad \Delta_i < 0.\end{cases} \tag{10}$$

Let $\lambda_{1,2}^i$ denote the eigenvalues of A_i; then,

$$\lambda_{1,2}^i = \begin{cases}(\operatorname{tr}(A_i) \pm \sqrt{\Delta_i})/2, & \Delta_i > 0,\\ (\operatorname{tr}(A_i) \pm \mathrm{j}\sqrt{-\Delta_i})/2, & \Delta_i < 0.\end{cases} \tag{11}$$

Therefore, by Euler's formula, we can rewrite the expressions of $g_i(s)$ and $f_i(s)$ as follows

$$g_i(s) = (e^{\lambda_1^i s} - e^{\lambda_2^i s})/\sqrt{\Delta_i},$$

and

$$f_i(s) = \frac{1}{2}\left[-\frac{\operatorname{tr}(A_i)}{\sqrt{\Delta_i}}(e^{\lambda_1^i s} - e^{\lambda_2^i s}) + (e^{\lambda_1^i s} + e^{\lambda_2^i s})\right].$$

We shall derive the constraint conditions on the dwell-times τ_1 and τ_2 by exhaustively categorizing all possible combinations of the subsystems. To this end, we now present the parameter

$$\mathcal{K} := 2\frac{\operatorname{tr}(A_1A_2) - \frac{1}{2}\operatorname{tr}(A_1)\operatorname{tr}(A_2)}{\sqrt{|\Delta_1\Delta_2|}}. \tag{12}$$

which remains invariant for changing coordinates. As shown in [3] and [4], $\mathcal{K}$ contains important information about the interrelation of A_1 and A_2. In particular, the following fact exposes its connection with the Lie commutator of A_1 and A_2.

Lemma 3 [4]:

$$\det([A_1, A_2]) = \begin{cases} \frac{1}{4}\left(1-\mathcal{K}^2\right)\Delta_1\Delta_2, & \Delta_1\Delta_2 > 0 \\ \frac{1}{4}\left(1+\mathcal{K}^2\right)\Delta_1\Delta_2, & \Delta_1\Delta_2 < 0. \end{cases}$$

In particular, if $\Delta_1, \Delta_2 < 0$, then there must be $|\mathcal{K}| \geq 1$.

In what follows, we shall show that $\mathcal{K}$ is naturally involved in the switching dynamics. In light of Lemma 2 and Remark 2, it turns out to be a fundamental problem to calculate $\mathrm{tr}(e^{A_1\tau_1}e^{A_2\tau_2})$ for assessing the stability of system (1). To this end, we apply the formulation in (8), whose expressions rely on the specified eigenstructure of each subsystem. Furthermore, letting the dwell-times τ_1 and τ_2 be subject to the inequalities derived from (5) and (6) then is equivalent to confining them to the mutual constraint conditions guaranteeing system (1) to be asymptotically stable.

CASE I: $\Delta_1 > 0$ and $\Delta_2 > 0$. This implies that both A_1 and A_2 have distinct real eigenvalues. Accordingly, we have

$$\begin{aligned} \mathrm{tr}(e^{A_1\tau_1}e^{A_2\tau_2}) = {} & 2e^{\mathrm{tr}(A_1)\tau_1/2}e^{\mathrm{tr}(A_2)\tau_2/2} \\ & \times\big[\mathcal{K}\sinh(\sqrt{\Delta_1}\tau_1/2)\sinh(\sqrt{\Delta_2}\tau_2/2) \\ & + \cosh(\sqrt{\Delta_1}\tau_1/2)\cosh(\sqrt{\Delta_2}\tau_2/2)\big]. \end{aligned} \tag{13}$$

Proposition 1: If both A_1 and A_2 have distinct eigenvalues, the periodic switching system in (1) is asymptotically stable if and only if the dwell-times τ_1 and τ_2 satisfy the following inequalities

$$\begin{aligned} \big|\mathcal{K}\sinh(\sqrt{\Delta_1}\tau_1/2)\sinh(\sqrt{\Delta_2}\tau_2/2) & \\ +\cosh(\sqrt{\Delta_1}\tau_1/2)\cosh(\sqrt{\Delta_2}\tau_2/2)\big| & < 1, \end{aligned} \tag{14}$$

$$e^{\mathrm{tr}(A_1)\tau_1}e^{\mathrm{tr}(A_2)\tau_2} < 1 \tag{15}$$

or

$$\begin{aligned} \big|\mathcal{K}\sinh(\sqrt{\Delta_1}\tau_1/2)\sinh(\sqrt{\Delta_2}\tau_2/2) & \\ + \cosh(\sqrt{\Delta_1}\tau_1/2)\cosh(\sqrt{\Delta_2}\tau_2/2)\big| & \geq 1, \end{aligned} \tag{16}$$

$$\begin{aligned} & \big|\mathcal{K}\sinh(\sqrt{\Delta_1}\tau_1/2)\sinh(\sqrt{\Delta_2}\tau_2/2) \\ & \quad +\cosh(\sqrt{\Delta_1}\tau_1/2)\cosh(\sqrt{\Delta_2}\tau_2/2)\big| \\ & < \cosh(\mathrm{tr}(A_1)\tau_1/2 + \mathrm{tr}(A_2)\tau_2/2), \end{aligned} \tag{17}$$

$$\begin{aligned} & \big|\mathcal{K}\sinh(\sqrt{\Delta_1}\tau_1/2)\sinh(\sqrt{\Delta_2}\tau_2/2) \\ & \quad + \cosh(\sqrt{\Delta_1}\tau_1/2)\cosh(\sqrt{\Delta_2}\tau_2/2)\big| \\ & < e^{-\mathrm{tr}(A_1)\tau_1/2}e^{-\mathrm{tr}(A_2)\tau_2/2}. \end{aligned} \tag{18}$$

CASE II: $\Delta_1 < 0$ and $\Delta_2 < 0$. This implies that both A_1 and A_2 have a pair of conjugate complex eigenvalues. Accordingly, we have

$$\begin{aligned} \mathrm{tr}(e^{A_1\tau_1}e^{A_2\tau_2}) = {} & 2e^{\mathrm{tr}(A_1)\tau_1/2}e^{\mathrm{tr}(A_2)\tau_2/2} \\ & \times\big[\mathcal{K}\sin(\sqrt{-\Delta_1}\tau_1/2)\sin(\sqrt{-\Delta_2}\tau_2/2) \\ & + \cos(\sqrt{-\Delta_1}\tau_1/2)\cos(\sqrt{-\Delta_2}\tau_2/2)\big]. \end{aligned} \tag{19}$$

Proposition 2: If both A_1 and A_2 have complex eigenvalues, the periodic switching system in (1) is asymptotically stable if and only if the dwell-times τ_1 and τ_2 satisfy the following inequalities

$$\begin{aligned} \big|\mathcal{K}\sin(\sqrt{-\Delta_1}\tau_1/2)\sin(\sqrt{-\Delta_2}\tau_2/2) & \\ +\cos(\sqrt{-\Delta_1}\tau_1/2)\cos(\sqrt{-\Delta_2}\tau_2/2)\big| & < 1, \end{aligned} \tag{20}$$

$$e^{\mathrm{tr}(A_1)\tau_1}e^{\mathrm{tr}(A_2)\tau_2} < 1 \tag{21}$$

or

$$\begin{aligned} \big|\mathcal{K}\sin(\sqrt{-\Delta_1}\tau_1/2)\sin(\sqrt{-\Delta_2}\tau_2/2) & \\ +\cos(\sqrt{-\Delta_1}\tau_1/2)\cos(\sqrt{-\Delta_2}\tau_2/2)\big| & \geq 1, \end{aligned} \tag{22}$$

$$\begin{aligned} & \big|\mathcal{K}\sin(\sqrt{-\Delta_1}\tau_1/2)\sin(\sqrt{-\Delta_2}\tau_2/2) \\ & \quad +\cos(\sqrt{-\Delta_1}\tau_1/2)\cos(\sqrt{-\Delta_2}\tau_2/2)\big| \\ & < \cosh(\mathrm{tr}(A_1)\tau_1/2 + \mathrm{tr}(A_2)\tau_2/2), \end{aligned} \tag{23}$$

$$\begin{aligned} & \big|\mathcal{K}\sin(\sqrt{-\Delta_1}\tau_1/2)\sin(\sqrt{-\Delta_2}\tau_2/2) \\ & \quad +\cos(\sqrt{-\Delta_1}\tau_1/2)\cos(\sqrt{-\Delta_2}\tau_2/2)\big| \\ & < e^{-\mathrm{tr}(A_1)\tau_1/2}e^{-\mathrm{tr}(A_2)\tau_2/2}. \end{aligned} \tag{24}$$

CASE III: $\Delta_1 > 0$ and $\Delta_2 < 0$. This implies that A_1 has a pair of distinct eigenvalues and that A_2 has a pair of conjugate complex eigenvalues. Accordingly, we have

$$\begin{aligned} \mathrm{tr}(e^{A_1\tau_1}e^{A_2\tau_2}) = {} & 2e^{\mathrm{tr}(A_1)\tau_1/2}e^{\mathrm{tr}(A_2)\tau_2/2} \\ & \times\big[\mathcal{K}\sinh(\sqrt{\Delta_1}\tau_1/2)\sin(\sqrt{-\Delta_2}\tau_2/2) \\ & + \cosh(\sqrt{\Delta_1}\tau_1/2)\cos(\sqrt{-\Delta_2}\tau_2/2)\big]. \end{aligned} \tag{25}$$

Proposition 3: If A_1 has distinct eigenvalues and A_2 has complex eigenvalues, the periodic switching system in (1) is asymptotically stable if and only if the dwell-times τ_1 and τ_2 satisfy the following inequalities

$$\begin{aligned} \big|\mathcal{K}\sinh(\sqrt{\Delta_1}\tau_1/2)\sin(\sqrt{-\Delta_2}\tau_2/2) & \\ +\cosh(\sqrt{\Delta_1}\tau_1/2)\cos(\sqrt{-\Delta_2}\tau_2/2)\big| & < 1, \end{aligned} \tag{26}$$

$$e^{\mathrm{tr}(A_1)\tau_1}e^{\mathrm{tr}(A_2)\tau_2} < 1 \tag{27}$$

or

$$\begin{aligned} \big|\mathcal{K}\sinh(\sqrt{\Delta_1}\tau_1/2)\sin(\sqrt{-\Delta_2}\tau_2/2) & \\ + \cosh(\sqrt{\Delta_1}\tau_1/2)\cos(\sqrt{-\Delta_2}\tau_2/2)\big| & \geq 1, \end{aligned} \tag{28}$$

$$\begin{aligned} & \big|\mathcal{K}\sinh(\sqrt{\Delta_1}\tau_1/2)\sin(\sqrt{-\Delta_2}\tau_2/2) \\ & \quad + \cosh(\sqrt{\Delta_1}\tau_1/2)\cos(\sqrt{-\Delta_2}\tau_2/2)\big| \\ & < \cosh(\mathrm{tr}(A_1)\tau_1/2 + \mathrm{tr}(A_2)\tau_2/2), \end{aligned} \tag{29}$$

$$\begin{aligned} & \big|\mathcal{K}\sinh(\sqrt{\Delta_1}\tau_1/2)\sin(\sqrt{-\Delta_2}\tau_2/2) \\ & \quad + \cosh(\sqrt{\Delta_1}\tau_1/2)\cos(\sqrt{-\Delta_2}\tau_2/2)\big| \\ & < e^{-\mathrm{tr}(A_1)\tau_1/2}e^{-\mathrm{tr}(A_2)\tau_2/2}. \end{aligned} \tag{30}$$

When both subsystems are stable, the inequalities in Propositions 1-3 can be reduced according to Corollary 1. Moreover, to complete this section, we emphasize that instead of solving these inequalities directly, an intuitive way to present the mutual constraint relation between τ_1 and τ_2 is to depict the stability margin on their plane, which is formed by the corresponding equalities.

IV. AN ILLUSTRATIVE EXAMPLE

We now provide an example to typically illustrate Proposition 3.

Example 1: $A_1 = \begin{bmatrix} -2 & 1 \\ 0.5 & 0.5 \end{bmatrix}$ and $A_2 = \begin{bmatrix} -0.5 & -1 \\ 3 & 1 \end{bmatrix}$.

We have $\mathrm{tr}(A_1) = -1.5$, $\mathrm{tr}(A_1) = 0.5$, and $\Delta_1 = 8.25$, $\Delta_2 = -9.75$. Then, the eigenvalues of A_1 and A_2 are $\{-2.2967, 0.6967\}$ and $\{0.25 \pm \mathrm{j}1.5612\}$, respectively. In addition, $\mathcal{K} = 0.9756$. Therefore, by Proposition 3, the system is asymptotically stable for all τ_1 and τ_2 satisfying

$$\begin{aligned} &\big|0.9756\sinh(1.4361\tau_1)\sin(1.5612\tau_2) \\ &\quad + \cosh(1.4361\tau_1)\cos(1.5612\tau_2)\big| < 1, \\ &\quad e^{-1.5\tau_1}e^{0.5\tau_2} < 1 \end{aligned}$$

or

$$\begin{aligned} &\big|0.9756\sinh(1.4361\tau_1)\sin(1.5612\tau_2) \\ &\quad + \cosh(1.4361\tau_1)\cos(1.5612\tau_2)\big| \geq 1, \\ &\big|0.9756\sinh(1.4361\tau_1)\sin(1.5612\tau_2) \\ &\quad + \cosh(1.4361\tau_1)\cos(1.5612\tau_2)\big| \\ &< \cosh(-0.75\tau_1 + 0.25\tau_2), \\ &\big|0.9756\sinh(1.4361\tau_1)\sin(1.5612\tau_2) \\ &\quad + \cosh(1.4361\tau_1)\cos(1.5612\tau_2)\big| \\ &< e^{0.75\tau_1}e^{-0.25\tau_2}. \end{aligned}$$

Next, we need to depict the following implicit functions on the (τ_1, τ_2) plane:

$$\begin{aligned} &\big|0.9756\sinh(1.4361\tau_1)\sin(1.5612\tau_2) \\ &\quad + \cosh(1.4361\tau_1)\cos(1.5612\tau_2)\big| = 1, \qquad (31) \\ &\quad -1.5\tau_1 + 0.5\tau_2 = 0, \qquad (32) \\ &\big|0.9756\sinh(1.4361\tau_1)\sin(1.5612\tau_2) \\ &\quad + \cosh(1.4361\tau_1)\cos(1.5612\tau_2)\big| \\ &= \cosh(-0.75\tau_1 + 0.25\tau_2), \qquad (33) \\ &\big|0.9756\sinh(1.4361\tau_1)\sin(1.5612\tau_2) \\ &\quad + \cosh(1.4361\tau_1)\cos(1.5612\tau_2)\big| \\ &= e^{0.75\tau_1}e^{-0.25\tau_2}. \qquad (34) \end{aligned}$$

Doing this, we can use software such as Maple or Matlab, which includes tools to depict the implicit functions on the plane.

In Figure 1, the equations in (31) and (32) are depicted and colored in black and red, respectively. They form the boundaries of the regions, which correspond to the complex eigenvalues of $e^{A_1\tau_1}e^{A_2\tau_2}$ located inside the unit circle of the complex plane. For example, the point corresponding to $\tau_1 = 0.5$ and $\tau_2 = 0.9$ is located inside a branch of these regions and makes the eigenvalues of $e^{0.5A_1}e^{0.9A_2}$ be $\{0.8273 \pm \mathrm{j}0.2375\}$. Meanwhile, the point corresponding to

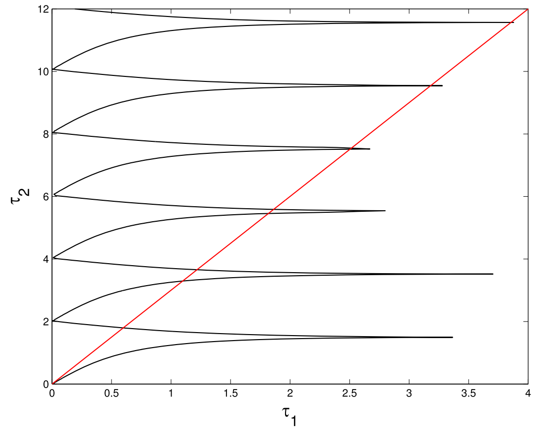

FIGURE 1. The stability margin corresponding to the complex eigenvalues of $e^{A_1\tau_1}e^{A_2\tau_2}$.

$\tau_1 = 1.88$ and $\tau_2 = 5.54$ is located inside another branch of these regions and makes the eigenvalues of $e^{1.88A_1}e^{5.54A_2}$ be $\{-0.2717 \pm \mathrm{j}0.9367\}$. In addition, the corresponding state response is shown in Figure 2.

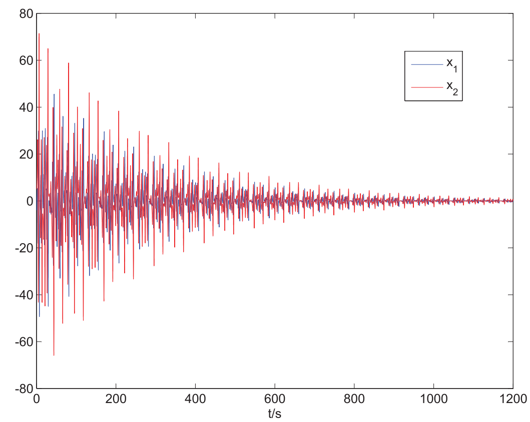

FIGURE 2. The state-response for $\tau_1 = 1.88s$ and $\tau_2 = 5.54s$.

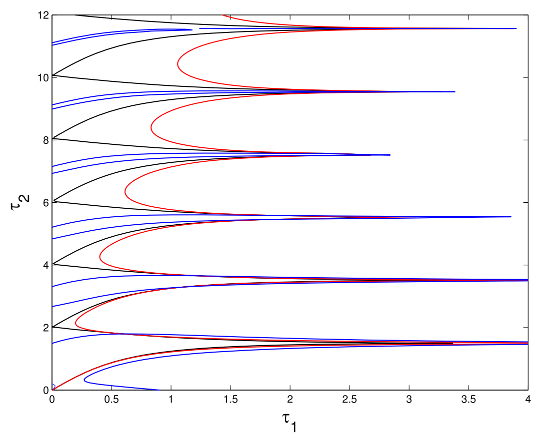

FIGURE 3. The stability margin corresponding to the real eigenvalues of $e^{A_1\tau_1}e^{A_2\tau_2}$.

In Figure 3, the equations in (31), (33), and (34) are depicted and colored in black, red, and blue, respectively.

These curves form the boundaries of the regions, which correspond to the eigenvalues of $e^{A_1\tau_1}e^{A_2\tau_2}$ within $(-1, 1)$. For example, the point corresponding to $\tau_1 = 1.6$ and $\tau_2 = 1.6$ is located in a branch of these regions and makes the eigenvalues of $e^{1.6A_1}e^{1.6A_2}$ be $\{-0.2678, -0.7540\}$. The corresponding state response is shown in Figure 4.

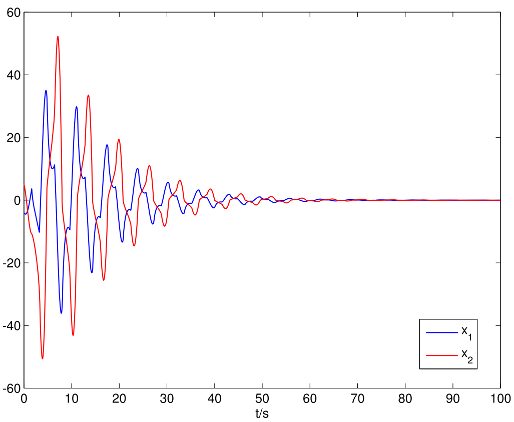

FIGURE 4. **The state-response for $\tau_1 = \tau_2 = 1.6s$.**

As shown in Figures 2 and 4, this example exposes a little surprising phenomenon that two unstable subsystems with quite long periods to stay active can generate a stable state-trajectory.

V. CONCLUSION

We considered the stability analysis problem for the planar linear systems undergoing periodic switching. We characterized the stability margin that the dwell-times of subsystems are confined to so that the overall system is asymptotically stable. The main technical preliminary includes a key lemma along with the distribution property of the roots of parabolic functions. Indeed, the expansion of the transition matrix of each subsystem up to the first order of its generator (i.e., the subsystem matrix) enables us to express the stability margin analytically and, moreover, compute it numerically. Finally, an example was worked in detail to illustrate the theoretical results.

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