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# Master–Slave Control for Active Suspension Systems With Hydraulic Actuator Dynamics

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**ABSTRACT** In order to solve the input nonlinearity of the hydraulic active suspension system, a master–slave control law is proposed through a nonlinear separation strategy. A Robust $H_\infty$ control is used as the master controller and an adaptive backstepping control scheme is designed as the slave controller. The robust $H_\infty$ master controller is studied to deal with the problems of input delay, parameter uncertainties, and multi-objective optimization in the linear system. A desired active control force is calculated by the master controller to guarantee the performances of the closed-loop system within allowable constraint ranges. The slave controller is applied to solve the problems of nonlinearity and the time constant uncertainty of the hydraulic actuator, an actual control law is obtained in this step. A quarter-car model with the hydraulic active suspension system is considered and the effectiveness of the proposed approach is illustrated by a realistic design example.

**INDEX TERMS** Hydraulic active suspension system, input nonlinearity, master-slave controller.

## I. INTRODUCTION

The performance indicators for the vehicle suspension include ride comfort, suspension deflection and road holding, however, these three indexes are often conflicting. For instance, improving the ride comfort often increases the suspension deflection and decreases the tyre road holding. The active suspension system can balance the relationships among the above three performance indexes effectively [1]. The active suspension system can adjust the system energy to restrain the vibration of vehicle body, to keep the tyre road holding, to reduce influences of road disturbances and improper operations in braking and steering. Then the ride comfort and safety can be improved. Therefore, active suspension control technologies have become a hot research topic in recent years. Many control algorithms for active suspension systems have been presented by scholars, such as linear quadratic Gaussian control [2], [3], $H_\infty$ control, sliding mode control [4], [5], neural networks control [6], predictive control [7], genetic algorithm [8], singular perturbation theory [9], and so on.

The $H_\infty$ control has been widely discussed in active suspension system because of its robustness and disturbance attenuation, also it has been recognized as a convenient method to deal with constraint problems. The $H_\infty$ control can achieve maximum level of ride comfort improvement within acceptable time-domain hard constraints [10]. The work [11] proposed the $H_\infty$ control with constraints to realize the multi-objective control for the active suspension system, the controller can improve ride comfort and keep suspension stroke, tire deflection and actuator saturation within an acceptable level. The controller was solved by LMI. In [12], a multi-objective $H_\infty$ state feedback controller for the active suspension system based on LMI was designed by using the concepts of reachable set and state-space ellipsoid. The controller has solved the multi-objective optimization problem with time-domain hard constraints. The paper [13] applied robust $H_\infty$ control to solve the problems of parameter uncertainties and hard constraints on the foundation of establishing non-static road disturbances model. The controller they proposed can enhance ride comfort on the premise of meeting suspension deflection requirement. Weichao Sun proposed a limited frequency $H_\infty$ controller for a quarter-car active suspension system with input delay. In the process of controller design, the human body sensitive frequency range and input delay in actuator were taken into account. The controller can guarantee the closed-loop stability and the limited frequency $H_\infty$ performance index even if the input delay existed [14], [15]. Furthermore, in order to dispose the problems which the system state variables are not fully measurable, Weichao Sun et al presented the dynamic output feedback limited frequency $H_\infty$ controller to reduce the dependence on measured signals [16]. Wei *et al.* [17] investigated the problem of model reduction for a class of continuous-time Markovian jump linear systems with incomplete statistics of mode information, which simultaneously considers the exactly known, partially unknown and uncertain transition rates.

Although these existing works give excellent results in the active suspension control, in most works studying active suspensions, the dynamic characteristics of actuators were often ignored. Actuators were usually assumed as ideal force generators. In the active suspension design, electrohydraulic systems are usually used as the actuators to generate the forces to isolate the vibrations. This is because they are more powerful and less bulky than other actuators. However, hydraulic actuators are difficult to reach the idealization because of the strong nonlinearity and high uncertainty. An approach combining with constrained $H_\infty$ control and non-linear backstepping algorithm was presented in [18]. The approach can deal with non-linear dynamics of actuator, external disturbance and time-domain constraints simultaneously, while the input delay and parameters uncertainties were not contained in this work. The paper [19] used sliding-mode strategy to obtain both controller and observer for the active suspension with nonlinear actuator dynamics. Different observer forms were designed to solve the linear growth vanishing bounded uncertainties and non-vanishing bounded uncertainties. In [20], a filter-based adaptive control strategy was presented for nonlinear uncertain suspension systems to stabilize motions of the car and to overcome the phenomenon of "exploration of terms" in standard backstepping algorithm, however, this control strategy was complicated to deal with constraint problems.

In active suspension systems, parameter uncertainties are inevitable because of the change of carrying capacity, running velocity, tire wear and some other reasons. Parameters uncertainties also exit in the hydraulic actuator. The impacts of parameters uncertainties on active suspension system performances are significant [21]. Besides, in order to ensure the control effect, the input delay issue can not be neglected neither.

In the modeling process of this paper, the issues of input nonlinearity, parameters uncertainties and input delay are involved. The motivation of this paper is to design the controller to maintain the stability, safety and ride comfort of the active suspension system even if all the problems in modeling process exist. For this purpose, according to the massive construction of the hydraulic active suspension system and the nonlinear separation strategy, a master-slave control law based on a robust $H_\infty$ control and an adaptive backstepping algorithm is proposed. The robust $H_\infty$ master controller is designed to dispose the problems of input delay, parameters uncertainties and constraints in the linear system. A desired active control force is calculated by the master controller to guarantee performance indexes of the active suspension system within the allowable constraints. Next step is applying an adaptive backstepping algorithm to solve the problems of nonlinearity and the uncertain time constant of the hydraulic actuator, an actual control law is got in this step.

The remainder of this paper is organized as follows. A quarter-car active suspension model with actuator dynamic characteristics is formulated in section 2. The Master-slave controller design is presented in Section 3. Section 4 provides a design example to illustrate the effectiveness of the proposed master-slave controller. Some concluding remarks are given in section 5.

## II. PROBLEM FORMATION

A quarter-car active suspension model with actuator dynamics is investigated in this brief, as shown in Fig. 1. Although this model has a simple structure, it can reflect the actual acceleration and suspension deflection characteristics of the active suspension system accurately. In this model, passengers are viewed as a part of body system, the influences of engine and transmission mechanism on body are neglected, and the body system is regarded as a rigid spring mass. Besides, the actuator dynamics are considered in the model.

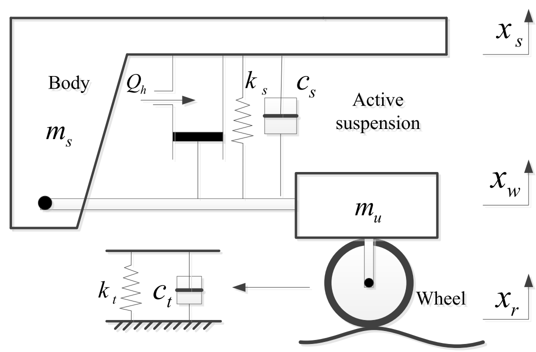


**FIGURE 1.** Quarter-car active suspension model with hydraulic actuator.

In Fig. 1, $m_s$ and $m_u$ represent the 1/4 body weight and 1/2 axle weight, $c_t$ and $k_t$ are damping and rigidity coefficients of the pneumatic tyre, $c_s$ and $k_s$ are the damping and stiffness coefficients of the suspension system, $x_s$ and $x_w$ stand for the displacements of the body and axle, $x_r$ is the road input, $Q_h$ is the load-flow of the hydraulic actuator. The desired dynamic equations of the 1/4 vehicle active suspension system are described as:

$$\begin{aligned} &m_s\ddot{x}_s + c_s[\dot{x}_s - \dot{x}_w] + k_s[x_s - x_w] = u_a \\ &m_u\ddot{x}_w + c_s[\dot{x}_w - \dot{x}_s] + c_t[\dot{x}_w - \dot{x}_r] \\ &\qquad + k_s[x_w - x_s] + k_t[x_w - x_r] = -u_a \end{aligned} \tag{1}$$

In above equations, $u_a$ is the active control force generated by the hydraulic actuator, $u_a$ can be given by:

$$\boldsymbol{u}_a = \boldsymbol{S} \cdot \boldsymbol{P_L} \tag{2}$$

where, $S$ is the piston area, $P_L$ is the pressure drop when fluid flows through the piston. The hydraulic actuator considered here contains a three-position four-way servo valve and a hydraulic cylinder. The hydraulic cylinder is the drive part of the actuator, inputs are flow and pressure of the fluid, outputs are force and linear velocity. The servo valve is an electro-hydraulic switch amplifier which can transform a low-power imported electrical signal into a high-power hydraulic output [22], [23]. The displacement of the main spool in servo valve varied with the magnitude and direction of current, then the flow and pressure to the hydraulic cylinder can be controlled. The dynamic equation of the hydraulic actuator can be described as

$$\dot{P}_L = -\beta P_L - \alpha S(\dot{x}_s - \dot{x}_w) + x_v \varsigma w_0 \tag{3}$$

where, $\beta = \alpha C_{tp}$, $\varsigma = \alpha C_d w_s \sqrt{\frac{1}{\rho}}$, $\alpha = \frac{4\beta_e}{V_t}$, $w_0 = \sqrt{P_s - sgn(x_v)P_L} = \sqrt{P_s - sgn(x_6)\frac{x_5}{\eta}}$, $\beta_e$ is the effective elastic modulus, $V_t$ is the volume of actuator, $C_{tp}$ is the piston leakage coefficient, $C_d$ is the fluid coefficient, $\rho$ is the fluid density, $w$ is the regional gradient of the slide valve, $P_s$ is the supply pressure drop, and $x_v$ stands for the displacement of the servo valve. $x_v$ is controlled by the voltage or current input signal $u$. In general, the relationship between $x_v$ and $u$ can be approximated by the liner filter with time constant $\tau$:

$$\dot{x}_v = \frac{1}{\tau}(-x_v + u) \tag{4}$$

Define the following state variables:

$$\begin{aligned} &x_1 = x_s - x_w, \quad x_2 = x_w - x_r, \; x_3 = \dot{x}_s, \\ &x_4 = \dot{x}_w, \quad x_5 = \eta P_L, \; x_6 = x_v \end{aligned} \tag{5}$$

Then the state equations obtaining from equations (1) - (3) can be described as:

$$\begin{aligned} \dot{x}_1 &= x_3 - x_4 \\ \dot{x}_2 &= x_4 - \dot{x}_r \\ \dot{x}_3 &= -\frac{1}{m_s}\left[k_s x_1 - c_s(x_3 - x_4) + \frac{S}{\eta}x_5\right] \\ \dot{x}_4 &= \frac{1}{m_u}\left[\begin{array}{l} k_s x_1 - k_t x_2 + c_s(x_3 - x_4) \\ -c_t(x_4 - \dot{x}_\gamma) - \frac{S}{\eta}x_5 \end{array}\right] \\ \dot{x}_5 &= -\eta\alpha S x_3 + \eta\alpha S x_4 - \beta x_5 + \eta\varsigma x_6 w_0 \\ \dot{x}_6 &= \frac{1}{\tau}(-x_6 + u) \end{aligned} \tag{6}$$

Besides, because $P_s$ is so large that $\eta = 10^{-7}$ is always used to readjust $P_L$ to improve digital accuracy.

From the state equations, we can get that there is a nonlinear term in the active suspension model because of the hydraulic actuator. Then the linear theory can't be applied to design the controller. Although the backstepping control can solve the above trouble, it is complicated to handle the constrained problems and it is easy to generate the "exploration of terms". From analyzing of the active suspension model, we can find the main part of the model is still linear, the nonlinear term $x_6 w_0$ just exists in the hydraulic actuator. Therefore, the master-slave control can be applied in active suspension system with the hydraulic actuator. The active control force (the desired intermediate variable) can be estimated according to the performance requirements of the active suspension system. Then, the actual control action can be deduced on the basis of the input nonlinear model and the desired intermediate variable. Better than entirety solving approach, the master-slave control has the following advantages in disposing an input nonlinear problem: first, it avoids the difficulty in directly solving nonlinear control law effectively. Then, it is easier and more efficient by attributing the main controller design to the linear control. The active suspension system can be decomposed into two parts, linear control technique is used to design the master controller, while the backstepping control algorithm is applied to compensate the nonlinear term.

The nonlinear term $x_6 w_0$ is considered as the virtual control input. By defining $X = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^{\mathrm{T}}$, the main linear structure can be described as the following state-space form:

$$\dot{x}(t) = Ax + B\hat{u} + B_1 w \tag{7}$$

where,

$$A = \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{k_s}{m_s} & 0 & -\frac{c_s}{m_s} & \frac{c_s}{m_s} & \frac{S}{m_s\eta} \\ \frac{k_s}{m_u} & -\frac{k_t}{m_u} & \frac{c_s}{m_u} & -\frac{c_s + c_t}{m_u} & -\frac{S}{m_u\eta} \\ 0 & 0 & -\eta\alpha s & \eta\alpha s & -\beta \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \eta r \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ \frac{c_t}{m_u} \\ 0 \end{bmatrix},$$

Then, in order to meet the requirements of the riding comfort, suspension stroke and vehicle maneuvering stability, the following control outputs are defined:

$$\begin{aligned} z_1(t) &= \ddot{z}_s(t) \\ z_2(t) &= \begin{bmatrix} \frac{z_s(t) - z_u(t)}{z_{\max}} & \frac{k_t(z_u(t) - z_r(t))}{(m_s + m_u)g} \end{bmatrix}^T \end{aligned} \tag{8}$$

where, $z_1(t)$ shows the body acceleration and it is used to evaluate the ride comfort. One of our main purposes is to reduce the body acceleration. $z_2(t)$ denotes the suspension stroke constraint and the vehicle maneuvering safety constraint, $z_{\max}$ is the maximum suspension deflection, and the suspension stroke should not surpass the allowable maximum because of the limitation of the mechanical structure, as showed in the first term of $z_2(t)$. The second term of $z_2(t)$ ensures the firm uninterrupted contact of wheels to road, that is the dynamic load between the wheels and road must be less than the static load. Then, the main linear structure in the suspension control system can be written as:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\hat{u}(t) + B_1 w(t) \\ z_1(t) &= C_1 x(t) + D_1 \hat{u}(t) \\ z_2(t) &= C_2 x(t) \end{aligned} \tag{9}$$

$$\Pi_k = \begin{bmatrix} PA_k + A_k^T P + 2M_{1k} & PB_k K - M_{1k} + M_{2k} & PB_{1k} + M_{3k} \\ * & -2M_{2k} & -M_{3k} \\ * & * & -\gamma^2 I \end{bmatrix} < 0$$
$$\Gamma_k = \begin{bmatrix} A_k & B_k K & B_{1k} \end{bmatrix}$$
$$N_k = \begin{bmatrix} C_{1k} & D_{1k}K & 0 \end{bmatrix}$$

where,

$$C_1 = \begin{bmatrix} -\frac{k_s}{m_s} & 0 & -\frac{c_s}{m_s} & \frac{c_s}{m_s} \end{bmatrix}, \quad D_1 = \frac{1}{m_s},$$
$$C_2 = \begin{bmatrix} \frac{1}{z_{\max}} & 0 & 0 & 0 \\ 0 & \frac{k_t}{(m_s+m_u)g} & 0 & 0 \end{bmatrix}.$$

In addition, considering the limited power of the hydraulic actuator, the actuator output force should not exceed the maximum, that is

$$\left|\hat{u}(t)\right| \le u_{\max} \tag{10}$$

Uncertainties are inevitable in active suspension model because of the change of body mass. Besides, the delay problem generally exists in the control input side because of the mechanical properties in actuators. In order to improve the control effect and the reliability, the uncertainties and the input delay are taken into account in the controller design of this paper. The main linear structure in the suspension control system can be rewritten as:

$$\begin{aligned} \dot{x}(t) &= A(\mu)x(t) + B(\mu)\hat{u}(t-\tau) + B_1(\mu)w(t) \\ z_1(t) &= C_1(\mu)x(t) + D_1(\mu)\hat{u}(t-\tau) \\ z_2(t) &= C_2(\mu)x(t) \end{aligned} \tag{11}$$

where, $\tau$ is the input delay, $\mu$ represents the modelling uncertainty, and it is assumed that $\mu$ varies in a convex polyhedron. $\mu_k$ shows the kth vertex of the polyhedron, the assembly of vertices constitutes a convex set, and the boundary of the convex set is $\Delta$:

$$\Delta \triangleq \{\Omega | \Omega = \sum_{k=1}^{r} \mu_k \Omega_k; \sum_{k=1}^{r} \mu_k = 1, \mu_k \ge 0\} \tag{12}$$

where $\Omega_k \triangleq (A_k, B_k, B_{1k}, C_{1k}, C_{2k})$ illustrates the vertices of the convex polytope.

## III. CONTROLLER DESIGN

### A. ROBUST $H_\infty$ MASTER CONTROLLER DESIGN

Known from the analysis of the second section, the main structure of the active suspension system can be regarded as a multi-objectives optimization problem including uncertainties and input delay [24], the robust $H_\infty$ controller is designed to solve this problem. Then, one of our objectives of this brief is to determine a state feedback controller with the form of $\hat{u}(t-\tau) = Kx(t-\tau)$, such that:

1) The closed-loop system is asymptotically stable.
2) Under zero initial condition, the closed-loop system guarantees that $\|z\|_2 < \gamma \|w\|_2$ for all nonzero $w \in L_2[0,\infty)$ and a prescribed scalar $\gamma > 0$.
3) Suspension stroke and vehicle safety performance constraints need be met, $\left|\{z_2(t)\}_j\right| \le \left\{z_{2,\max}\right\}_j, j = 1, 2$.
4) The input constraint $\left|\hat{u}(t-\tau)\right| \le u_{\max}$ is guaranteed.

In the following, we will study how to design a desired master controller based on the linear main structure of the active suspension system in (11).

*Theorem 1:* For given scalars $\gamma > 0$, $\tau > 0$, $\varepsilon > 0$, if there exists matrix $P = P^{\mathrm{T}} > 0$, $Q = Q^{\mathrm{T}} > 0$, and $M_k$ satisfying

$$\Psi = \begin{bmatrix} \Pi_k & \sqrt{\tau}\Gamma_k^T & \sqrt{\tau}M_k & N_k^{\mathrm{T}} \\ * & -Q^{-1} & 0 & 0 \\ * & * & -Q & 0 \\ * & * & * & -I \end{bmatrix} < 0,$$
$$k = 1, \cdots, r \tag{13}$$
$$\begin{bmatrix} -I & \sqrt{\varepsilon}\{C_{2k}\}_j \\ * & -P \end{bmatrix} < 0, \quad j = 1, 2 \tag{14}$$
$$\begin{bmatrix} -I & \sqrt{\varepsilon}K \\ * & -u_{\max}^2 P \end{bmatrix} < 0 \tag{15}$$

where, we have the equations at the top of the page, then a controller with the form of $\hat{u}(t-\tau(t)) = Kx(t-\tau(t))$ exists and the conditions 1)-4) can be guaranteed by the controller.

*Proof:* In order to prove the stability of system by asking the disturbance as zero, provided that $P$ and $Q$ are positive definite matrix to be solved, choose a Lyapunov functional candidate as

$$V(t) = x^T(t)Px(t) + \int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)Q\dot{x}(s)dsd\theta \tag{16}$$

Then the differential coefficient of $V(t)$ is,

$$\begin{aligned} \dot{V}(t) &= 2x^{\mathrm{T}}(t)P\dot{x}(t) + \tau\dot{x}^{\mathrm{T}}(t)Q_k\dot{x}(t) \\ &\quad - \int_{t-d}^{t} \dot{x}^T(s)Q\dot{x}(s)ds \end{aligned} \tag{17}$$

According to the Leibniz-Newton formula, for any appropriate dimensioned matrices $M(\mu) = \sum_{k=1}^{r} \mu_k M_k = \sum_{k=1}^{r} \mu_k \begin{bmatrix} M_{1k}^T & M_{2k}^T \end{bmatrix}^T$ and $k = 1, 2, 3, 4$, we can get:

$$\begin{aligned} \dot{V}(t) &\le 2x^{\mathrm{T}}(t)P\dot{x}(t) - \int_{t-d}^{t} \dot{x}^T(s)Q\dot{x}(s)ds \\ &\quad + d\dot{x}^{\mathrm{T}}(t)Q_k\dot{x}(t) + 2[x^{\mathrm{T}}(t)M_1 + x^{\mathrm{T}}(t-\tau)M_2] \end{aligned}$$

$$\times [x(t) - x(t-\tau) - \int_{t-\tau}^{t} \dot{x}(s)ds] \tag{18}$$

Let $\chi(t) = \left[x^{\mathrm{T}}(t)\ x^{\mathrm{T}}(t-\tau)\right]$, then

$$\dot{V}(t) \le \chi^T(t)R(\mu)\chi(t) - \int_{t-\tau}^{t} \left[\chi^T(t)M(\mu) + x^{\mathrm{T}}(s)Q\right] \times Q^{-1}\left[M^T(\mu)\chi(t) + Qx^{\mathrm{T}}(s)\right]ds \tag{19}$$

where,

$$R(\mu) = \begin{bmatrix} PA(\mu) + A^T(\mu)P & PB(\mu)K \\ * & 0 \end{bmatrix} + \begin{bmatrix} M(\mu) & -M(\mu) \end{bmatrix} + \begin{bmatrix} M(\mu) & -M(\mu) \end{bmatrix}^T + \tau\begin{bmatrix} A(\mu) & B(\mu)K \end{bmatrix}^T Q \begin{bmatrix} A(\mu) & B(\mu)K \end{bmatrix} + \tau M(\mu)Q^{-1}M^T(\mu)$$

As $Q > 0$, we have

$$\int_{t-\tau}^{t} \left[\chi^T(t)M(\mu) + x^{\mathrm{T}}(s)Q\right] Q^{-1}\left[M^T(\mu)\chi(t) + Qx^{\mathrm{T}}(s)\right]ds > 0 \tag{20}$$

Moreover, the equation can be obtained from theorem 1 by using Schur complement,

$$R_k = \begin{bmatrix} PA_k + A_k^T P & PB_kK \\ * & 0 \end{bmatrix} + \begin{bmatrix} M_k & -M_k \end{bmatrix} + \begin{bmatrix} M_k & -M_k \end{bmatrix}^T + \tau\begin{bmatrix} A_k & B_kK \end{bmatrix}^T Q \begin{bmatrix} A_k & B_kK \end{bmatrix} + \tau M_k Q^{-1} M_k^T \tag{21}$$

Then according to the uncertainty properties of polyhedron, $A(\mu) = \sum_{k=1}^{r} \mu_k A_k$,

$$B(\mu) = \sum_{k=1}^{r} \mu_k B_k, \quad M(\mu) = \sum_{k=1}^{r} \mu_k M_k,$$

$R(\mu) < 0$ can be got, and then we have $\dot{V}(t) < 0$, thus the stability of system is proven.

Next, our target is establishing the $H_\infty$ performance index, under the zero initial condition and considering the disturbance effect, the time derivative of $V(t)$ has the following form:

$$\dot{V}(t) \le \tilde{\chi}^T(t)\tilde{R}(\mu)\tilde{\chi}(t) - \int_{t-\tau}^{t} \left[\tilde{\chi}^T(t)\tilde{M}(\mu) + x^{\mathrm{T}}(s)Q\right] \times Q^{-1}\left[\tilde{M}^T(\mu)\tilde{\chi}(t) + Qx^{\mathrm{T}}(s)\right]ds \tag{22}$$

$$\tilde{R}(\mu) = \begin{bmatrix} PA(\mu)\ C\ A^T(\mu)P & PB(\mu)K & PB_1(\mu) \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} + \begin{bmatrix} M(\mu) & -M(\mu) \end{bmatrix} + \begin{bmatrix} M(\mu) & -M(\mu) \end{bmatrix}^T + \tau\begin{bmatrix} A(\mu) & B(\mu)K & B_1(\mu) \end{bmatrix}^T \times Q\begin{bmatrix} A(\mu) & B(\mu)K\ B_1(\mu) \end{bmatrix} + \tau M(\mu)Q^{-1}M^T(\mu) \tag{23}$$

where,

$$\tilde{\chi}(t) = \begin{bmatrix} x^{\mathrm{T}}(t) & x^{\mathrm{T}}(t-\tau) & w^T(t) \end{bmatrix}^T,$$
$$\tilde{M}(\mu) = \begin{bmatrix} \tilde{M}_1 & \tilde{M}_2 & \tilde{M}_3 \end{bmatrix}.$$

Thus, we can obtain:

$$\dot{V}(t) + z_1^T(t)z_1(t) - \gamma^2 w^T(t)w(t) \le \tilde{\chi}^T(t)L(\mu)\tilde{\chi}(t) - \int_{t-\tau}^{t} \begin{matrix}\left[\tilde{\chi}^T(t)\tilde{M}(\mu) + x^{\mathrm{T}}(s)Q\right] \\ Q^{-1}\left[\tilde{M}^T(\mu)\tilde{\chi}(t) + Qx^{\mathrm{T}}(s)\right]\end{matrix} ds \tag{24}$$

where

$$L(\mu) = \tilde{R}(\mu) + \begin{bmatrix} C_1(\mu) & 0 & 0 \end{bmatrix}^T \times \begin{bmatrix} C_1(\mu) & 0 & 0 \end{bmatrix} + diag\left\{0 \quad 0 \quad -\gamma^2 I\right\}$$

Through Schur complement and internal properties of convex uncertainty, (12) and (13) guarantee $L(\mu) < 0$.

By (24), for $w \in L_2\,[0, \infty]$ except zero, we can conclude,

$$\dot{V}(t) + z_1^T(t)z(t) - \gamma^2 w^T(t)w(t) < 0 \tag{25}$$

Then, on the condition of $V(0) = 0$ and $V(\infty) > 0$, for all nonzero $w \in L_2\,[0, \infty]$, $\|z_1\|_2 < \gamma\,\|w\|_2$ can be got, then $H_\infty$ performance index is established.

In addition, we will explain the input constraints are satisfied. Equation(25) ensures $\dot{V}(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) < 0$. Integrating the above inequality from zero to any $t > 0$, we can get

$$V(t) - V(0) < \gamma^2 \int_0^t w^{\mathrm{T}}(t)w(t)dt < \gamma^2\|w\|_2^2 \tag{26}$$

Noting that the integral terms of (16) is more than zero, we obtain $x^{\mathrm{T}}(t)Px(t)$, with $\varepsilon = \gamma^2 x_{\max} + V(0)$, and we can also obtain $x^{\mathrm{T}}(t-\tau)Px(t-\tau) < \varepsilon$ with $t > \tau$.

Then the equation can be obtained as,

$$\begin{aligned}
\max_{t>0}\left|\{z_2(t)\}_j\right|^2 &= \max_{t>0}\left\|x^T(t)\{C_{2k}\}_j^T\{C_{2k}\}_j x(t)\right\|_2 \\
&= \max_{t>0}\left\|\begin{matrix} x^T(t)P^{\frac{1}{2}}P^{-\frac{1}{2}}\{C_{2k}\}_j^T\{C_{2k}\}_j \\ P^{-\frac{1}{2}}P^{\frac{1}{2}}x(t)\end{matrix}\right\|_2 \\
&< \varepsilon\cdot\theta_{\max}(P^{\frac{1}{2}}\{C_{2k}\}_j^T\{C_{2k}\}_j P^{-\frac{1}{2}}), \\
&\qquad k = 1,\cdots,r; j = 1, 2 \\
\max_{t>0}\left|\hat{u}(t)\right|^2 &= \max_{t>\tau}\left\|x^{\mathrm{T}}(t-\tau)K^{\mathrm{T}}Kx(t-\tau)\right\|_2 \\
&= \max_{t>\tau}\left\|\begin{matrix} x^{\mathrm{T}}(t-\tau)P^{1/2}P^{-1/2}K^{\mathrm{T}} \\ KP^{-1/2}P^{1/2}x(t-\tau)\end{matrix}\right\|_2 \\
&< \varepsilon\cdot\theta_{\max}(P^{-1/2}K^{\mathrm{T}}KP^{-1/2})
\end{aligned}$$

where, $\theta_{\max}(\cdot)$ shows maximal eigenvalue. From the above inequality, the vehicle safety performance constraints and the input constraint (10) are established

$$\Pi_k = \begin{bmatrix} \bar{P}A_k + A_k^T\bar{P} + 2\bar{M}_{1k} & B_k\bar{K} - \bar{M}_{1k} + \bar{M}_{2k} & B_{1k} + \bar{M}_{3k} \\ * & -2\bar{M}_{2k} & -\bar{M}_{3k} \\ * & * & -\delta^2 I \end{bmatrix} < 0$$

$$\Gamma_k = \begin{bmatrix} A_k\bar{P} & B_k\bar{K} & B_{1k} \end{bmatrix}, \quad N_k = \begin{bmatrix} C_k\bar{P} & 0 & 0 \end{bmatrix}$$

if $\varepsilon \cdot P^{-1/2}\{C_{2k}\}_j^{\mathrm{T}}\{C_{2k}\}_j P^{-1/2} < \{z_{2,\max}\}_j^2 I$, $\varepsilon \cdot P^{-1/2}K^{\mathrm{T}}KP^{-1/2} < u_{\max}^2 I$. By Schur complements, the above inequalities are equivalent to (14) and (15). This finishes the proof.

It is noted that the matrix condition (13)-(15) in Theorem 1 is not a standard LMI which can be solved by Matlab. Therefore, we provide the following theorem to transform the condition of Theorem 1 into a linear version.

*Theorem 2:* Given scalars $\gamma > 0$, $\tau > 0$ and $\varepsilon > 0$, if there exist positive definite matrices $\bar{P}$, $\bar{Q}$ and matrices with appropriate dimension $\bar{M}_k$, $\bar{K}$ satisfying

$$\bar{\Psi} = \begin{bmatrix} \bar{\Pi}_k & \sqrt{\tau}\bar{\Gamma}_k^T & \sqrt{\tau}\bar{M}_k & \bar{N}_k^{\mathrm{T}} \\ * & \bar{Q} - 2\bar{P} & 0 & 0 \\ * & * & -\bar{Q} & 0 \\ * & * & * & -I \end{bmatrix} < 0 \tag{27}$$

$$\begin{bmatrix} -I & \sqrt{\varepsilon}\{C_{2k}\}_j \\ * & -P \end{bmatrix} < 0, \quad j = 1, 2 \tag{28}$$

$$\begin{bmatrix} -I & \sqrt{\varepsilon}\bar{K} \\ * & -u_{\max}^2\bar{P} \end{bmatrix} < 0 \tag{29}$$

where, we have the equation at the top of the page.

Then the closed loop system is stable, however, for all the nonzero $w \in L_2[0, \infty]$, the closed loop system meets $\|z_1\|_2 < \gamma\|w\|_2$, at the same time, when disturbance meets the condition of $w_{\max} = (\varepsilon - V(0))/\gamma^2$, the constraints of control input can be guaranteed.

In addition, if the above inequalities have a feasible solution, the $H_\infty$ state feedback controller can be given by:

$$\hat{u}(t-\tau) = \bar{K}\bar{P}^{-1}x(t-\tau) \tag{30}$$

### B. ADAPTIVE BACKSTEPPING SLAVE CONTROLLER DESIGN

Backstepping control is an effective algorithm to deal with the control problem in master-slave nonlinear systems. A virtual control is introduced to track the desired intermediate variable calculated by the master controller, so as to realize the control objectives. In this brief, the time constant uncertainty of the hydraulic actuator is taken into account in designing the adaptive backstepping control. Virtual control variable $x_6w_0$ is used to approximate the desired intermediate variable $\hat{u}$ generated in robust $H_\infty$ master controller. The error variable $e = x_6w_0 - \hat{u}$ will approach zero gradually. The time derivative of error variable is given by

$$\dot{e} = \frac{1}{\bar{\tau}}(-x_6 + u)w_0 - \frac{1}{2|w_0|}|x_6| \times \left[-\frac{\beta}{\eta}x_5 - \alpha S(x_2 - x_4) + \gamma x_6 w_0\right] - \dot{\hat{u}} \tag{31}$$

In order to solve the time constant uncertainty of the hydraulic actuator, an adaptive backstepping is introduced to design the actual control law $u$. The designed $u$ guarantees that $x_6w_0$ can still trace the desired intermediate variable even if there exists uncertain time constant $\hat{\bar{\tau}}$ in the hydraulic actuator. In order to realize the control goals, the actual control law $u$ is designed as:

$$u = x_6 + \frac{\hat{\bar{\tau}}}{w_0}\left\{-h_1 e + \frac{1}{2|w_0|}|x_6| \times \left[-\frac{\beta}{\eta}x_5 - \alpha S(x_2 - x_4) + \gamma x_6 w_0\right] + \dot{\hat{u}}\right\} \tag{32}$$

where, $\hat{\bar{\tau}}$ is the estimated value of the time constant $\bar{\tau}$, $\hat{\bar{\tau}}$ is given by

$$\dot{\hat{\bar{\tau}}} = -\dot{\tilde{\bar{\tau}}} = -\gamma_{\bar{\tau}} e\left\{-h_1 e + \frac{1}{2|w_0|}|x_6| \cdot \left[-\frac{\beta}{\eta}x_5 - \alpha S(x_2 - x_4) + \gamma x_6 w_0\right] + \dot{\hat{u}}\right\} \tag{33}$$

In equation (33), $h_1$ is the adaptive gain constant greater than zero, $\hat{\bar{\tau}}$ is the estimation for an unknown parameter, $\tilde{\bar{\tau}} = \tau - \hat{\bar{\tau}}$ is the error of estimate. The stability proof is showed as follows, according to equations (32) and (33), we can get:

$$\begin{aligned} \dot{e} &= \frac{\hat{\bar{\tau}}}{\bar{\tau}}\left\{-h_1 e + \frac{1}{2|w_0|}|x_6| \times \left[-\frac{\beta}{\eta}x_5 - \alpha S(x_2 - x_4) + \gamma x_6 w_0\right] + \dot{\hat{u}}\right\} - \frac{1}{2|w_0|} \\ &\quad \cdot |x_6|\left[-\frac{\beta}{\eta}x_5 - \alpha S(x_2 - x_4) + \gamma x_6 w_0\right] - \dot{\hat{u}} \\ &= -h_1 e - \frac{\tilde{\bar{\tau}}}{\bar{\tau}}\left\{-h_1 e + \frac{1}{2|w_0|}|x_6| \times \left[-\frac{\beta}{\eta}x_5 - \alpha S(x_2 - x_4) + \gamma x_6 w_0\right] + \dot{\hat{u}}\right\} \end{aligned} \tag{34}$$

The time constant greater than zero $\bar{\tau}$ is introduced into the Lyapunov function, and the Lyapunov functional candidate is chose as

$$V = \frac{1}{2}(e^2 + \frac{\gamma_{\bar{\tau}}}{\bar{\tau}}\tilde{\bar{\tau}}^2) > 0 \tag{35}$$

The derivative of $V$ satisfies:

$$\dot{V} = e\dot{e} + \frac{\gamma_{\bar{\tau}}^{-1}}{\bar{\tau}}\tilde{\bar{\tau}}\dot{\tilde{\bar{\tau}}} = -h_1 e^2 - \frac{\tilde{\bar{\tau}}}{\bar{\tau}}\gamma_{\bar{\tau}}^{-1}\gamma_{\bar{\tau}} e$$

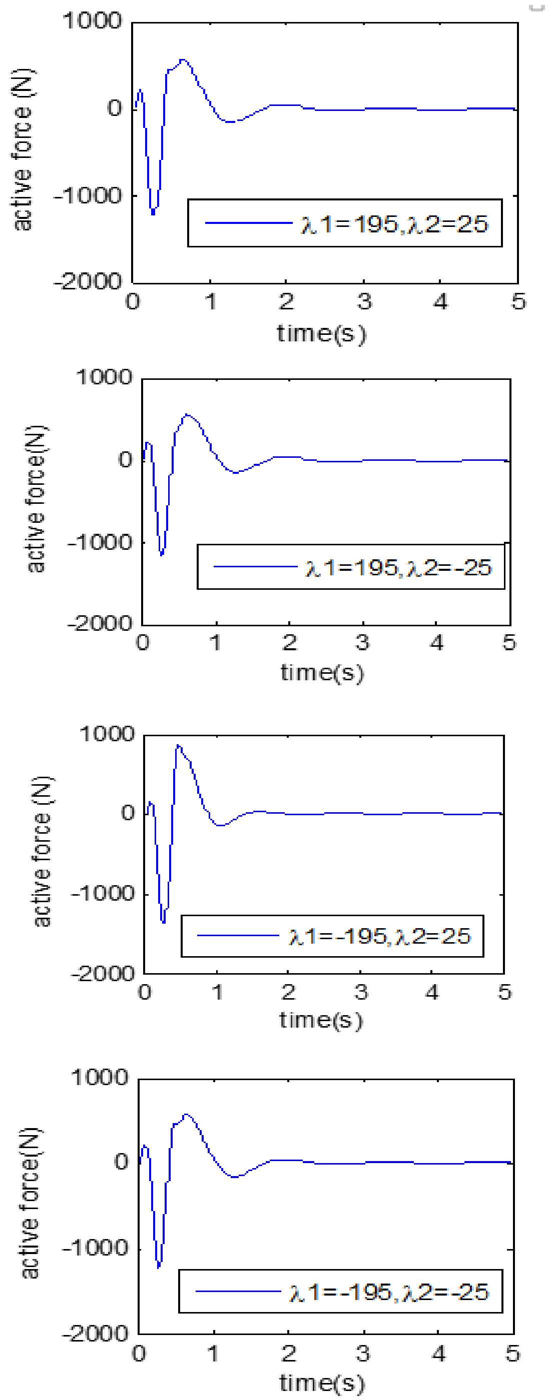


**FIGURE 2.** Active forces at four vertexes of the convex polyhedron.

$$\times\left\{-h_1e+\frac{1}{2\,|w_0|}\,|x_6|\,[-\frac{\beta}{\eta}x_5-\alpha S(x_2-x_4)\right.$$
$$\left.+\gamma x_6w_0]+\dot{\hat{u}}\right\}-\frac{\tilde{\tau}}{\bar{\tau}}\gamma_{\bar{\tau}}^{-1}\dot{\hat{\tau}}$$

Plug the equation (33) into the above equation, we can get

$$\dot{V}=-h_1e^2\leq 0 \tag{36}$$

Then, the error variable is asymptotically stable, the adaptive backstepping controller we design satisfies the requirements.

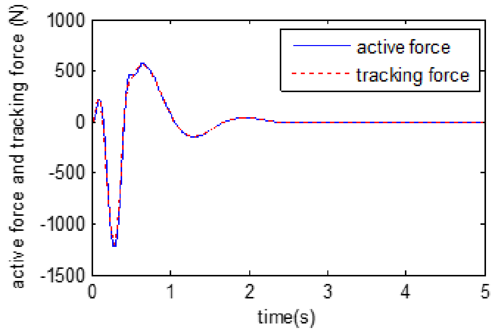


**FIGURE 3.** Active forces and tracking force at $\bar{\mu}_1=195$ and $\bar{\mu}_2=25$.

## IV. SIMULATION ANALYSIS

In this section, a simulation example is provided to illustrate the effectiveness of the proposed master-slave controller design method in the situations of parameters uncertainties, input delay and disturbances. A quarter-car parameters [20] are listed as follows. $m_s = 973$kg, $k_s = 42720$N/m, $k_t = 101115$N/m, $c_s = 1095$Ns/m, $c_t = 14.6$Ns/m, $c_t = 14.6$Ns/m, $m_u = 114$kg, $P_s = 10342500$Pa, $\alpha = 4.5\times10^{13}$N/m$^5$, $S = 3.35\times10^{-4}$m$^2$. Suppose that the body weight $m_s$ and the axle weight $m_u$ include uncertainties $\mu_1$ and $\mu_2$ because of the changes of passenger number and weight, where $\mu_1$ and $\mu_2$ satisfy $|\mu_1|\leq\bar{\mu}_1$, $|\mu_2|\leq\bar{\mu}_2$. Then the active suspension system is described as a convex polyhedron with four vertexes, and the theorem 2 is used to solve this convex optimization problem. The parameters in theorem 2 are given as: $\bar{\mu}_1 = 195$, $\bar{\mu}_2 = 25$, $z_{\max} = 0.08$m, $u_{\max} = 1500$N, $\varepsilon = 1$, $\tau = 5$ms, then the minimum guaranteed closed-loop $H_\infty$ performance indicator is $\gamma_{\min} = 7.6986$. An admissible state feedback control gain matrix is given by:

$$K=\bar{K}\bar{P}^{-1}$$
$$=\begin{bmatrix}19.2347 & 66.7809 & 2.2915 & -4.8484 & -2.3538\end{bmatrix}$$

$\hat{u}(t) = \bar{K}\bar{P}^{-1}x(t)$ is the desired active control force which satisfies the performance requirements under the situation of 5ms input delay and the uncertain $m_s$, $m_u$. $\hat{u}(t)$ at four vertexes of convex polyhedron are depicted in Fig. 2. As shown in Fig. 2, the desired active control forces at four vertexes of convex polyhedron satisfy the output force constraint condition (10). It means that any desired active control force satisfies the constraint condition (10) as long as the parameters uncertainties are within the convex polyhedron. Next, the actual control force generated by equation (32) is used to trace the desired active control force, and Fig. 3 shows the desired and tracking control force at $\bar{\mu}_1 = 195$ and $\bar{\mu}_2 = 25$, we can obtain that $u(t)$ can follow $\hat{u}(t)$ well. The bump responses of the body acceleration, the suspension deflections and the safety constraints with different uncertainties $\mu_1$ and $\mu_2$ under the disturbance of the isolated bump are shown in Fig. 4-Fig. 6. Assume that the height of the bump is 60mm, the frequency is 8HZ, and the vehicle forward velocity is 45 km/h.

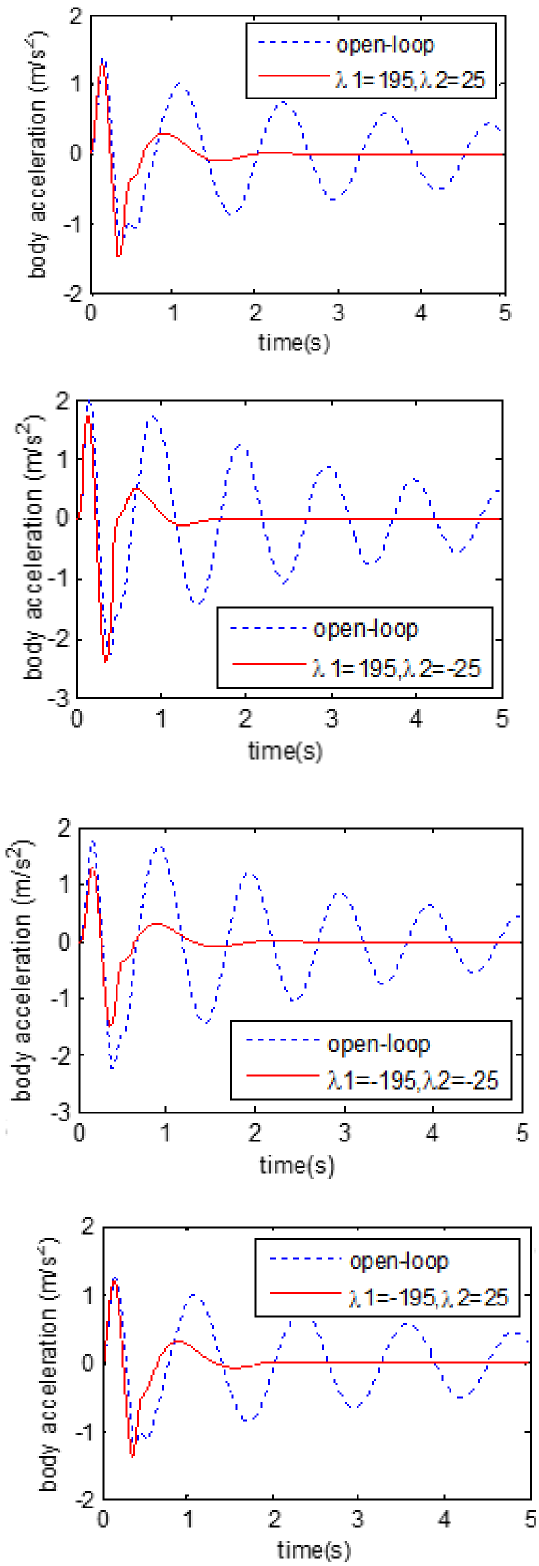


**FIGURE 4.** Body accelerations at four vertexes of the convex polyhedron.

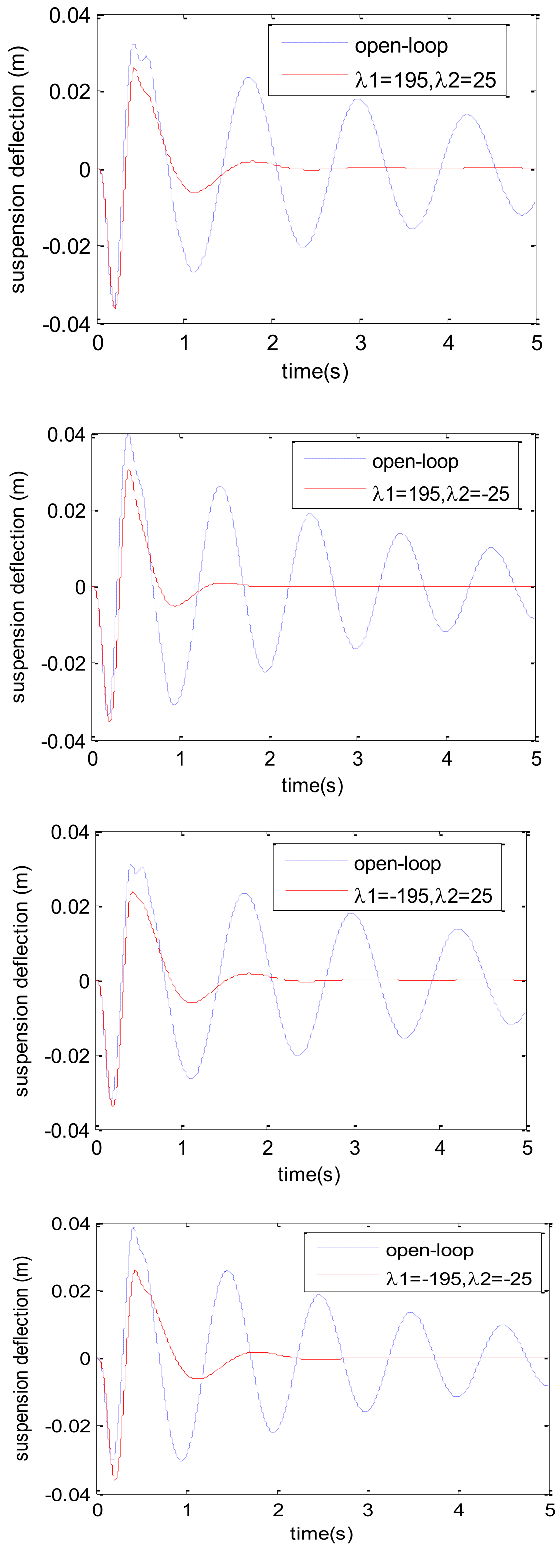


**FIGURE 5.** Suspension deflections at four vertexes of the convex polyhedron.

From Fig. 4-Fig. 6, we can see that the master slave controller yield smaller body acceleration and shorter setting time at every vertex of the convex polyhedron, compared with the open-loop system. The values of the body acceleration are related to the ride comfort, so the controller we design can improve the ride comfort. Meanwhile, the master slave controller we designed can satisfy all constraints of the active

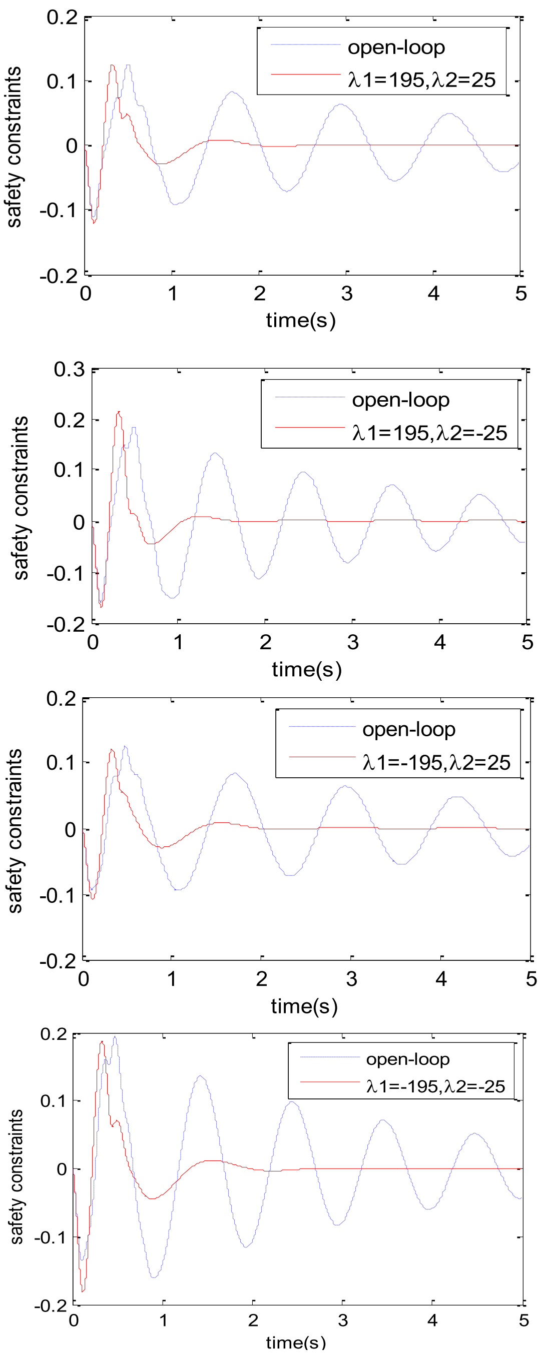


**FIGURE 6.** Safety constraints.

suspension system. Under the actions of uncertainties and input delay, the maximum of the suspension deflection is 0.028m, and it is less than the maximum allowed $z_{\max} = 0.08$. The maximal ration of the dynamic tire load and the static tire load is 0.205, and it is less than 1. That is, the dynamic tire load is less than the static tire load, then the firm continuously contact of wheels to road can be ensured, and the driving safety will be guaranteed. The designed controller can meet the performance requirements of the active suspension system at all the four vertexes of the convex polyhedron. Then, known from the property of the convex polyhedron, the controller we design can satisfy the performance requirements of the active suspension system at any point in the convex polyhedron. Therefore, the master slave controller can still restrain the body acceleration and keep the driving security even if the parameters uncertainties and input delay exist.

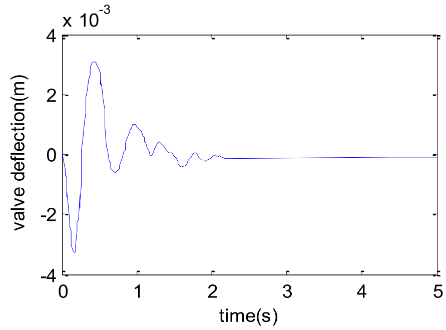


**FIGURE 7.** Servo valve displacement response.

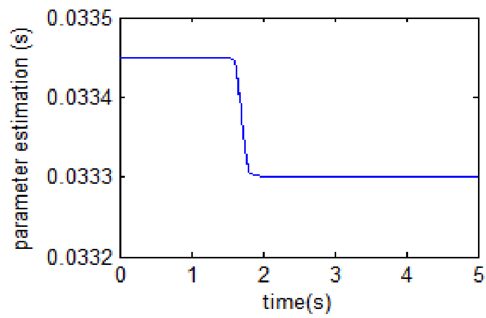


**FIGURE 8.** Estimation of the actuator time constant.

Fig. 7 depicts the servo valve displacement, and the servo valve moves in the limited region (-0.01m-0.01m). It means that the hydraulic actuator can provide the output active control force normally. Fig. 8 shows the estimated value of the hydraulic actuator time constant. It can be seen that the time constant of the actuator changes under the action of isolated bump, and as the disturbance is overcome, the time constant stays in the set value (0.0333s).

## V. CONCLUSION

In this brief, the master-slave controller based on robust $H_\infty$ control and adaptive backstepping control has been investigated through nonlinear separation strategy. By using the robust $H_\infty$ control, the constraints problems in the active suspension system with parameters uncertainties and input delay have been solved. The adaptive backstepping control has been used to solve the estimation problem of time constant and the nonlinear problem caused by the hydraulic actuator. Compared with the entirety solving method, the master-slave control has easier design process by attributing the design of master controller to the controller design in a linear system. A quarter-car model with the hydraulic actuator has been considered and the effectiveness of the proposed approach has been illustrated by a practical design example. In the future research, we will focus on the impacts of actuator faults on active suspension systems.

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