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The t/s-Diagnosability of Hypercube Networks Under the PMC and Comparison Models

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ABSTRACT A t/s-diagnosable system, a generalization of t/t-diagnosable system, refers to such a system that all the faulty nodes of the system can be isolated within a set of size at most s in the presence of at most t faulty nodes. In this paper, the t/s-diagnosability of the hypercubes under the PMC model (the comparison model) is evaluated. First, several novel properties of hypercube are proposed, which are previously unknown in the literatures. Second, based on the above properties of hypercubes, we show that an n-dimensional ($n \geqslant 5$) hypercube is $(kn - ((k(k+1))/2) + 1)/(kn - ((k(k+1))/2) + k - 1)$-diagnosable in terms of both the PMC and the comparison models, where $2 \leqslant k \leqslant n-2$. Furthermore, we introduce a fast diagnosis algorithm to isolate the faulty nodes in a subset of the system under the PMC model (the comparison model). And the time complexity of the algorithm is $O(n2^n)$ for an n-dimensional hypercube.

INDEX TERMS t/s-diagnosable systems, fault diagnosis, n-dimensional hypercube, isolation algorithm, PMC model, comparison model.

I. INTRODUCTION

With the rapid development of Multi-Processors, multiprocessor computer systems can now contain hundreds and thousands of processors. It is inevitable that some processors in such a system may fail. On one hand, in order to maintain the communication of the multiprocessor system, the system should have the capability of fault tolerance with a certain degree, which is relative to the structure of the system. Some studies on the fault tolerance properties of the system structures can be found in [14], [17], [18], [23], [27], [28]. On the other hand, to ensure reliability, the system should have the ability to identify the faulty processors. On fault identification, a realistic diagnosis method, called system-level diagnosis, is first proposed by Preparata et al. [1], [2]. In [1], Preparata et al. proposes the first system-level diagnosis model, namely the PMC model. In the PMC model, a system can be represented by a digraph $G(V, E)$, and the edge (i, j) means node i tests node j. The test outcome of node i testing node j is represented by $\omega(i, j)$. $\omega(i, j) = 1(0)$ implies that node j is faulty(fault-free) and the outcome of test (i, j) is reliable only if node i is fault-free. The PMC model has been adopted in [6]–[13], [15], [16]. Another practical system-level diagnosis model is the comparison model (also called MM model), proposed by Maeng and Malek [3], [4]. Sengupta and Dahbura [5] suggested a further modification, called the MM* model, in which any node has to test other two nodes if it is adjacent to them. Under the comparison model, for a system represented by an undirected graph $G(V, E)$, node k is a comparator for nodes i and j if and only if $(i, k) \in E$ and $(j, k) \in E$. The test outcome of comparator k testing i, j is denoted by $\omega(k : i, j)$. $\omega(k : i, j) = 1$ implies that at least one of nodes i and j is faulty and $\omega(k : i, j) = 0$ implies that nodes i and j are all fault-free. The test outcome $\omega(k : i, j)$ is reliable only if node k is fault-free. If node k is faulty, then $\omega(k : i, j)$ can be arbitrary. The comparison model has been adopted in [12], [19], [21], [23]–[26].

There are two fundamentally different strategies to system-level diagnosis: t-diagnosis [1] and t/t-diagnosis [6], [7]. A system is t-diagnosable if and only if the system can identify all the nodes within the system correctly in the presence of at most t faulty nodes. And a system is t/t-diagnosable if and only if it can isolate all of the faulty nodes to within a set of size at most t in the presence of at most t faulty nodes. For a system, the t/t-diagnosability of it is usually larger than it's t-diagnosability. For example, under the PMC model (the comparison model), the t/t-diagnosability of an n-dimensional hypercube is $2n-2$, which is larger than n, the t-diagnosability of it [14]. However, for a system, if the real number of faulty nodes is larger than it's t/t-diagnosability, then the above two diagnosis strategies can do little for diagnosis. As a generalization of the t/t-diagnosis, t/s-diagnosis ($s \geqslant t$) can be expected to diagnose the faulty nodes in the system when the number of faulty nodes of the system is larger than its t/t-diagnosability.

Hypercube is an important network topology in modeling multiprocessor systems, and has been applied to the parallel computer systems for commerce and research, such as Intel iPSC/2, nCUBE/10, CM-2, SGI/Cray Origin2000, may be a general method for designing nanocomputers, a future computer system. Hypercube has been widely studied (see [14], [15], [18], [27], [28]). However, the research about the t/s-diagnosability of the hypercube is unknown in the previous literatures. In this paper, we introduce some novel properties of the hypercube and apply them to discuss the t/s-diagnosability of the hypercube under the PMC model (the comparison model).

The rest of this paper is organized as follows: Section 2 outlines the preliminaries for a system-level diagnosis. In section 3, some novel important properties of hypercube are derived. In section 4, by using properties of hypercube proposed in section 3, the t/s-diagnosability of hypercube is presented under the PMC and comparison models. In section 5, an algorithm is proposed to locate a subset such that all faulty nodes can be isolated in this subset for an n-dimensional ($n \geqslant 5$) hypercube, whose time complexity is $O(n2^n)$. Section 6 draws a conclusion.

II. PRELIMINARIES

We begin our paper by doing the following preliminaries. For convenience, the following graphs Gs are all undirected graphs.

For a graph given by $G(V, E)$, let $V(G)$ be the set of all nodes of graph G. The connected subgraph set of G is denoted by $C_{sub}(G) = \{C_1, C_2, \ldots, C_k | C_i$ is connected$(1 \leqslant i \leqslant k)$ and C_i, C_j is disconnected$(1 \leqslant i, j \leqslant k)$ and $V(C_1) \cup V(C_2) \cup \ldots \cup V(C_k) = V(G)\}$. Especially, if G is connected, the $C_{sub}(G) = G$.

For a subset $X \subset V$, the induced subgraph of X in graph G can be denoted by $G_{ind}(X) = (V', E')$ where $V' = X$ and $E' = \{(i, j) | i, j \in X$ and $(i, j) \in E\}$

Let $Card_k(C_{sub}(G)) = \{C_i | C_i \in C_{sub}(G)$ and $|V(C_i)| = k\}$ be the set of k nodes connected subgraph in G.

For example, consider a graph G shown in Figure 1, $C_{sub}(G) = \{C_1, C_2, C_3\}$ where $C_1 = G_{ind}(\{v_1, v_2\})$, $C_2 = G_{ind}(\{v_3, v_4, v_5\})$ and $C_3 = G_{ind}(\{v_6, v_7, v_8, v_9\})$. $Card_2(C_{sub}(G)) = \{C_1\}$, $Card_3(C_{sub}(G)) = \{C_2\}$ and $Card_4(C_{sub}(G)) = \{C_3\}$.

Definition 1: Under the PMC model, for a system given by $G(V, E)$ and a syndrome σ, a set $X \subset V$ is called an allowable fault set (AFS) of the system for syndrome σ if for any two nodes i, j such that $(i, j) \in E$,

i) If $i, j \in V - X$ then $\omega(i, j) = 0$, and

ii) If $i \in V - X$ and $j \in X$ then $\omega(i, j) = 1$.

Definition 2: Under the comparison model, for a system given by $G(V, E)$ and a syndrome σ, a set $X \subset V$ is called an allowable fault set (AFS) of the system for syndrome σ if for any three nodes i, j, k such that $(i, k) \in E$ and $(j, k) \in E$,

i) If $k \in V - X$ and $i, j \in V - X$ then $\omega(k : i, j) = 0$, and

ii) If $k \in V - X$ and $\{i, j\} \cap X \neq \varnothing$ then $\omega(k : i, j) = 1$.

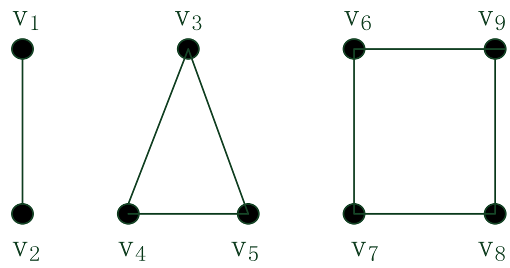

FIGURE 1. An interconnection topology graph G.

For a node $v \in V$, let $N(v) = \{u | (u, v) \in E\}$ denote the set of the all neighbors of node v. And for a subset $X \subseteq V$, let $N(X) = \cup_{v \in X} N(v) - X$.

Lemma 3: A system is given by undirected graph $G(V, E)$ with at most t faulty nodes. For a connected subgraph of G, say G', then following conditions hold:

i) Under the PMC model, if the test results are all 0 and $|V(G')| \geqslant t + 1$, then all the nodes of G' are fault-free.

ii) Under the comparison model, if the test results are all 0 and $|V(G')| \geqslant t + 1$ and each node of G' has at least two neighbors in G', then all the nodes of G' are fault-free.

Proof: For condition i), assume that there exists some node, say u, is faulty, then the nodes of $N(u)$ are all faulty. Similarly, $N(N(u))$ are all faulty. Furthermore, the nodes of G' are all faulty which implies that the system has at least $t + 1$ faulty nodes, which is a contracdiction to the assumption that the system has at most t faulty nodes.

For condition ii), assume that there exists some node, say u, is faulty. By assumption, for each neighbor of u, say v, there exists one node $u_1 \in V(G'), u_1 \neq u$, such that $\omega(v : u_1, u) = 0$, which implies that v is faulty. Furthermore, the nodes of $N(v)$ are all faulty and the nodes of $N(N(v))$ are also all faulty, which implies all nodes of G' are faulty. Therefore, the system has at least $t + 1$ faulty nodes, which is a contracdiction to the assumption that the system has at most t faulty nodes. □

III. PROPERTIES OF HYPERCUBE

An n-dimensional hypercube Q_n has 2^n nodes and each node is labeled by an n-bit binary string. Two nodes are adjacent if and only if their labels differ in exactly one bit position.

A 4-dimensional hypercube is shown in Figure 2.

Lemma 4 [20]: Let $G(V, E)$ be the graph of a hypercube of dimension n and $X \subset V$ with $|X| = k$, $0 < k \leqslant n+1$, then $N(X) \geqslant kn - k(k + 1)/2 + 1$.

Lemma 5: Let $G(V, E)$ be the graph of a hypercube of dimension n ($n \geqslant 5$) and $S = \{v_1, v_2, \ldots, v_k | v_1, v_2, \ldots, v_k \in V$ and $v_1, v_2, \ldots, v_k$ have one common neighbor$\}(k \leqslant n)$. Then $|N(S)| = kn - \frac{k(k+1)}{2} + 1$.

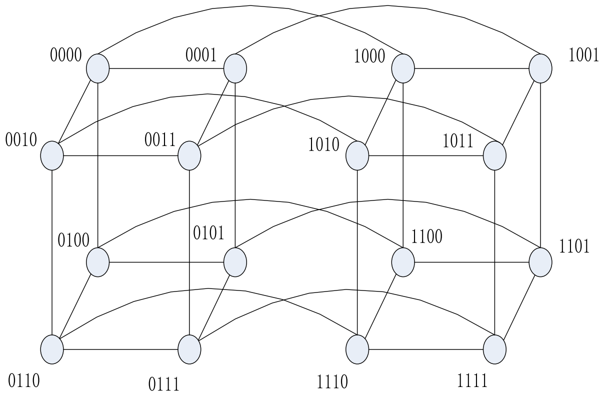

FIGURE 2. A 4-dimensional hypercube.

Proof: For convenience, let $add(v_0)$ be the address of node v_0. Without loss of generality, let node v_0 be their common neighbor and $add(v_0) = a_1a_2 \dots a_n (a_i = 0$ or 1 and if $a_i = 0(1)$, then $\overline{a_i} = 1(0)$). It is easily seen that the addresses of the nodes in S can be presented as follows: $add(v_1) = \overline{a_1}a_2 \dots a_n$, $add(v_2) = a_1\overline{a_2} \dots a_n, \dots, add(v_k) = a_1a_2 \dots \overline{a_k}a_{k+1} \dots a_n$. Now we consider the addresses of node $v_i's$ neighbors. The addresses of $N(v_i)$ can be represented by $\overline{a_1}a_2 \dots \overline{a_i} \dots a_n, a_1\overline{a_2}a_3 \dots \overline{a_i} \dots a_n, \dots, a_1a_2 \dots \overline{a_i} \dots \overline{a_n}$. Therefore, for any different i, j, $N(v_i) \cap N(v_j) = \{v_0, u\}$ where $add(u) = a_1a_2 \dots \overline{a_i} \dots \overline{a_j} \dots a_n$. Hence, $|N(S)| = k(n-1) - C_k^2 + 1 = kn - \frac{k(k+1)}{2} + 1$. □

Lemma 6 [22]: Let $G(V, E)$ be the graph of a hypercube of dimension n $(n \geqslant 3)$ and let S be a set of nodes $S \subset V$ with $n \leqslant |S| \leqslant 2(n-1) - 1$. Suppose that $G_{ind}(V - S)$ is disconnected and let $C_{sub}(G_{ind}(V-S)) = \{C_1, C_2, \dots, C_m\}$. Then following conditions hold:

i) $\Sigma_{i=0}^{1} i|Card_i(C_{sub}(G_{ind}(V - S)))| \leqslant 1$.

ii) There exists only one $C_i \in C_{sub}(G_{ind}(V - S))$ with $|V(C_i)| \geqslant 2$.

Lemma 7: Let $G(V, E)$ be the graph of a hypercube of dimension 5 and let S be a set of nodes $S \subset V$ with $|S| \leqslant 5k - \frac{k(k+1)}{2} (1 \leqslant k \leqslant 3)$. Suppose that $G_{ind}(V - S)$ is disconnected and let $C_{sub}(G_{ind}(V-S)) = \{C_1, C_2, \dots, C_m\}$. Then following conditions hold:

i) $\Sigma_{i=0}^{k-1} i|Card_i(C_{sub}(G_{ind}(V - S)))| \leqslant k - 1$.

ii) There exists only one $C_i \in C_{sub}(G_{ind}(V - S))$ with $|V(C_i)| \geqslant k$.

Proof: When $k = 1$, we have $|S| \leqslant 4$, therefore, the claim is true. According to Lemma 6, the claim also holds for $k = 2$. Next, we consider the situation that $k = 3$. Note that when $k = 3$, $|S| \leqslant 9$. For convenience, divide Q_5 to two 4-dimensional hypercubes denoted by Q_4^L and Q_4^R. Let $S_L = S \cap Q_4^L$ and $S_R = S \cap Q_4^R$. Consider the following cases:

Case 1: $|S_L| \geqslant 4k - \frac{k(k+1)}{2} = 6$ or $|S_R| \geqslant 4k - \frac{k(k+1)}{2} = 6$.

Without loss of generality, let $|S_L| \geqslant 4k - \frac{k(k+1)}{2} = 6$, then $|S_R| \leqslant 3$, which impiles that $Q_4^R - S_R$ is connected. Note that there exist $2^4 > 3$ edges between Q_4^L and Q_4^R. Then at most $(3 - 1)$ nodes can be surrounded by S. Hence the claim is true.

Case 2: $|S_L| < 4k - \frac{k(k+1)}{2} = 6$ and $|S_R| < 4k - \frac{k(k+1)}{2} = 6$.

Since $|S_L| \leqslant 5$ and $|S_R| \leqslant 5$. Let A (respectively, B) be the largest component of $Q_4^L - S_L$ (respectively, $Q_4^R - S_R$) and A' (respectively, B') be the union set of the remaining small components of $Q_4^L - S_L$ (respectively, $Q_4^R - S_R$). According to Lemma 6, we have that $|A'| \leqslant 1$ and $|B'| \leqslant 1$, which implies that the condition i) is true. Note that $|A| \geqslant 2^4 - 5 - 1 = 10$ and $|B'| + |S_R| \leqslant 5$, which implies there exists an edge from A to B. Therefore, condition ii) is true. □

Lemma 8: For any two positive integers a, b, $1 + ab \geqslant (a + b)$.

Proof: Let $f(a, b) = (1 + ab) - (a + b)$. After factorization, $f(a, b) = (a - 1)(b - 1)$, then $f(a, b) \geqslant 0$ when a, b are all positive integers. Therefore, for any two positive integers a, b, $1 + ab \geqslant (a + b)$. □

Lemma 9: Let $G(V, E)$ be the graph of a hypercube of $n (n \geqslant 5)$ dimension and let S be a set of nodes $S \subset V$ with $|S| \leqslant kn - \frac{k(k+1)}{2} (1 \leqslant k \leqslant n-2)$. Suppose that $G_{ind}(V-S)$ is disconnected and let $C_{sub}(G_{ind}(V-S)) = \{C_1, C_2, \dots, C_m\}$. Then following conditions hold:

i) $\Sigma_{i=0}^{k-1} i|Card_i(C_{sub}(G_{ind}(V - S)))| \leqslant k - 1$.

ii) There exists only one $C_i \in C_{sub}(G_{ind}(V - S))$ with $|V(C_i)| \geqslant k$.

Proof: We prove the claim by induction on n. According to Lemma 7, we have that the claim holds for $n = 5$. And assume that the claim holds for some $(n - 1)$, $n - 1 \geqslant 5$. Next, we will show that it holds for n. For convenience, divide Q_n to two $(n - 1)$-dimensional hypercubes denoted by Q_{n-1}^L and Q_{n-1}^R. Let $S_L = S \cap Q_{n-1}^L$ and $S_R = S \cap Q_{n-1}^R$. Let A (respectively, B) be the largest component of $Q_{n-1}^L - S_L$ (respectively, $Q_{n-1}^R - S_R$) and A' (respectively, B') be the union set of the remaining small components of $Q_{n-1}^L - S_L$ (respectively, $Q_{n-1}^R - S_R$). We consider following cases:

Case 1: $|S_L| \geqslant k(n-1) - \frac{k(k+1)}{2}$ or $|S_R| \geqslant k(n-1) - \frac{k(k+1)}{2}$.

Without loss of generality, let $|S_L| \geqslant k(n - 1) - \frac{k(k+1)}{2}$, then $|S_R| \leqslant k \leqslant n - 2$, which implies that $Q_{n-1}^R - S_R$ is connected. Note that there exist $2^{n-1} > k - 1 (n \geqslant 5)$ edges between Q_{n-1}^L and Q_{n-1}^R. Then at most k nodes can be surrounded by S. According to Lemma 4, we have that if there are k nodes surrounded by S, then $|S| \geqslant kn - \frac{k(k+1)}{2} + 1$, which is a contradiction to that $|S| \leqslant kn - \frac{k(k+1)}{2}$. Hence the claim is true.

Case 2: $|S_L| < k(n-1) - \frac{k(k+1)}{2}$ and $|S_R| < k(n-1) - \frac{k(k+1)}{2}$.

Case 2.1: $k < n - 2$.

Since $k \leqslant (n - 1) - 2$, according to the induction hypothesis, $|A'| \leqslant k - 1$ and $|B'| \leqslant k - 1$. We claim that A and B are connected. According to the induction hypothesis, $|A| \geqslant 2^{n-1} - [k(n-1) - \frac{k(k+1)}{2}] - (k-1)$, similarly, $|Q_{n-1}^R - B| \leqslant k(n-1) - \frac{k(k+1)}{2} + (k-1)$. Note that $|A| - |Q_{n-1}^R - B| \geqslant 2^{n-1} - 2kn + 2 + k(k+1) \geqslant 2^{n-1} - n^2 + n + 8 > 0 (n \geqslant 5)$. Therefore, there exists at least one edge from A to B, which implies A and B are connected. Let $|A'| = k_a \leqslant k - 1$,

$|B'| = k_b \leqslant k-1$. Suppose, to the contrary, $|A' \cup B'| \geqslant k$. Then $|S| = |S_L| + |S_R| \geqslant (k_a + k_b)(n-1) - \frac{k_a(k_a+1)}{2} - \frac{k_b(k_b+1)}{2} + 2$. We claim that $k_a, k_b \geqslant 1$. Otherwise, $|A' \cup B'| < k$ which is a contradiction to the hypothesis. According to Lemma 8, $|S| = |S_L| + |S_R| \geqslant (k_a + k_b)(n-1) - \frac{k_a(k_a+1)}{2} - \frac{k_b(k_b+1)}{2} + 2 \geqslant (k_a + k_b)n - \frac{(k_a+k_b)(k_a+k_b+1)}{2} + 1$. Let $k'_b = k - k_a$. Note that $k'_b \leqslant k_b$ and $k'_b(n-1) - \frac{k'_b(k'_b+1)}{2} \leqslant k_b(n-1) - \frac{k_b(k_b+1)}{2}$. Therefore, $|S| = |S_L| + |S_R| \geqslant (k_a + k_b)(n-1) - \frac{k_a(k_a+1)}{2} - \frac{k_b(k_b+1)}{2} + 2 \geqslant (k_a + k'_b)(n-1) - \frac{k_a(k_a+1)}{2} - \frac{k'_b(k'_b+1)}{2} + 2 \geqslant (k_a + k'_b)n - \frac{(k_a+k'_b)(k_a+k'_b+1)}{2} + 1 = kn - \frac{k(k+1)}{2} + 1$, which is contradiction to the hypothesis that $|S| \leqslant kn - \frac{k(k+1)}{2}$.

Case 2.2: $k = n-2$.

Since $|S_L| \leqslant k(n-1) - \frac{k(k+1)}{2} - 1 = (n-2)(n-1) - \frac{(n-1)(n-2)}{2} - 1 = \frac{(n-1)(n-2)}{2} - 1 = \frac{n^2-3n}{2} = (n-3)(n-1) - \frac{(n-3)(n-2)}{2}$. According to the induction hypothesis, $|A'| \leqslant k-1$ and $|B'| \leqslant k-2 = n-4$. We claim that A and B are connected. According to the induction hypothesis, $|A| \geqslant 2^{n-1} - \frac{n^2-3n}{2} - (n-4)$, similarly, $|Q^R_{n-1} - B| \leqslant \frac{n^2-3n}{2} + (n-4)$. Note that $|A| - |Q^R_{n-1} - B| \geqslant 2^{n-1} - 2[\frac{n^2-3n}{2} - (n-4)] = 2^{n-1} - n^2 + 8 + n > 0 (n \geqslant 5)$. Therefore, there exists at least one edge from A to B, which implies A and B are connected. Let $|A'| = k_a \leqslant k-2$, $|B'| = k_b \leqslant k-2$. Suppose, to the contrary, $|A' \cup B'| \geqslant k$. Then $|S| = |S_L| + |S_R| \geqslant (k_a + k_b)(n-1) - \frac{k_a(k_a+1)}{2} - \frac{k_b(k_b+1)}{2} + 2$. We claim that $k_a, k_b \geqslant 1$. Otherwise, $|A' \cup B'| < k$ which is a contradiction to the hypothesis. Then according to Lemma 8, $|S| = |S_L| + |S_R| \geqslant (k_a + k_b)(n-1) - \frac{k_a(k_a+1)}{2} - \frac{k_b(k_b+1)}{2} + 2 \geqslant (k_a + k_b)n - \frac{(k_a+k_b)(k_a+k_b+1)}{2} + 1$. Let $k'_b = k - k_a$. Note that $k'_b \leqslant k_b$ and $k'_b(n-1) - \frac{k'_b(k'_b+1)}{2} \leqslant k_b(n-1) - \frac{k_b(k_b+1)}{2}$. Therefore, $|S| = |S_L| + |S_R| \geqslant (k_a+k_b)(n-1) - \frac{k_a(k_a+1)}{2} - \frac{k_b(k_b+1)}{2} + 2 \geqslant (k_a + k'_b)(n-1) - \frac{k_a(k_a+1)}{2} - \frac{k'_b(k'_b+1)}{2} + 2 \geqslant (k_a + k'_b)n - \frac{(k_a+k'_b)(k_a+k'_b+1)}{2} + 1 = kn - \frac{k(k+1)}{2} + 1$ which is contradiction to the hypothesis that $|S| \leqslant kn - \frac{k(k+1)}{2}$. And our proof is done. □

With above preliminaries, we shall discuss the diagnosability of t/s-diagnosable of the n-dimensional hypercube.

IV. DIAGNOSABILITY OF T/S-DIAGNOSABLE OF THE HYPERCUBE

Definition 10: A system S is t/s-diagnosable ($s \geqslant t$) if and only if all the faulty nodes can be isolated within a set of size at most s in the presence of at most t faulty nodes.

Lemma 11: For a system given by $G(V, E)$, let $X \subseteq V$ with $X = \{v_1, v_2, \ldots, v_k\}$ and $(v_i, v_j) \notin E$ ($1 \leqslant i, j \leqslant k, i \neq j$) ($k \geqslant 3$). Let $\alpha = |N(X)|$, then the system is not $(\alpha+1)/[\alpha + (k-1)]$-diagnosable under the PMC model.

Proof: Let $F = N(X) \cup \{v_1\}$ be the real fault set in the system, then $|F| = \alpha + 1$ and $|N(X) \cup X| = \alpha + k$. Consider the following syndrome σ for each pair of nodes $i, j \in V$ such that $(i, j) \in E$(Shown in Figure 3).

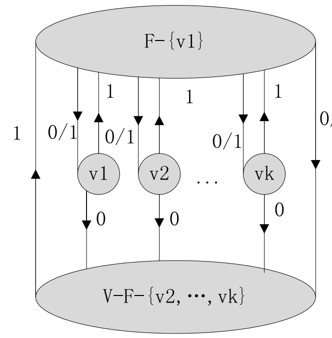

FIGURE 3. A syndrome σ of Theorem 13 under the PMC model.

i) If $i, j \in V - F - \{v_2, \ldots, v_k\}$, then $\omega(i, j) = 0$.

ii) If $i \in V - F - \{v_2, \ldots, v_k\}$ and $j \in F - \{v_2, \ldots, v_k\}$, then $\omega(i, j) = 1$.

iii) The test results from X to $F - \{v_2, \ldots, v_k\}$ are 1s.

iv) The test results from X to $V - F - \{v_2, \ldots, v_k\}$ are 0s.

v) The other possible test results are arbitrary.

For above syndrome σ, $N(X) \cup \{v_i\}(1 \leqslant i \leqslant k)$ are all allowable fault sets. Since $|\cup_{i=1}^{k}[N(X) \cup \{v_i\}]| = \alpha + k > \alpha + k - 1$, we conclude that the system is not $(\alpha+1)/[\alpha+(k-1)]$-diagnosable. □

Lemma 12: For a system given by $G(V, E)$, let $X \subseteq V$ with $X = \{v_1, v_2, \ldots, v_k\}$ and $(v_i, v_j) \notin E$ ($1 \leqslant i, j \leqslant k, i \neq j$) ($k \geqslant 3$). Let $\alpha = |N(X)|$, then the system is not $(\alpha+1)/[\alpha + (k-1)]$-diagnosable under the comparison model.

Proof: Let $F = N(X) \cup \{v_1\}$ be the real fault set in the system, then $|F| = \alpha + 1$ and $|N(X) \cup X| = \alpha + k$. Consider following syndrome σ, for three nodes $i, j, k \in V$ such that $(i, k), (j, k) \in E$:

i) If $i, j, k \in V - F - \{v_2, \ldots, v_k\}$, then $\omega(k : i, j) = 0$.

ii) If $i, k \in V - F - \{v_2, \ldots, v_k\}$ and $j \in F - \{v\}$, then $\omega(k : i, j) = 1$.

iii) If $j, k \in V - F - \{v_2, \ldots, v_k\}$ and $i \in F - \{v\}$, then $\omega(k : i, j) = 1$.

iv) If $k \in X$, and $i, j \in F - \{v\}$, then $\omega(k : i, j) = 1$.

v) The other possible test results are arbitrary.

For above syndrome σ, $N(X) \cup \{v_i\}(1 \leqslant i \leqslant k)$ are all allowable fault sets. Since $|\cup_{i=1}^{k}[N(X) \cup \{v_i\}]| = \alpha + k > \alpha + k - 1$, we conclude that the system is not $(\alpha+1)/[\alpha+(k-1)]$-diagnosable. □

Next, we present the diagnosability of t/s-diagnosable of the n-dimensional hypercube by the following theorems.

Theorem 13: Under both the PMC model and comparison model, an n-dimensional ($n \geqslant 5$) hypercube is not $kn - \frac{k(k+1)}{2} + 2/kn - \frac{k(k+1)}{2} + k$-diagnosable ($2 \leqslant k \leqslant n$).

Proof: According to Lemma 5, 11, 12, the claim is true. □

Theorem 14: Under the PMC model , an n-dimensional ($n \geqslant 5$) hypercube is $[kn - \frac{k(k+1)}{2} + 1]/[kn - \frac{k(k+1)}{2} + k - 1]$-diagnosable ($2 \leqslant k \leqslant n-2$ and $kn - \frac{k(k+1)}{2} + 1 \leqslant \frac{n^2-n}{2} - 1$).

Proof: Let $F \subseteq V$ be the real fault set with $|F| \leqslant kn - \frac{k(k+1)}{2} + 1$ in the system and

$C_{sub}(G_{ind}(V - F)) = \{C_1, C_2, \ldots, C_m\}$. According to Lemma 9, the following conditions hold:

i) $\Sigma_{i=0}^{k} i|Card_i(C_{sub}(G_{ind}(V - F)))| \leqslant k$.

ii) There exists only one $C_i \in C_{sub}(G_{ind}(V - F))$ with $|V(C_i)| \geqslant k + 1$.

For convenience, let $S = \cup_{i=0}^{k} Card_i(C_{sub}(G_{ind}(V - F)))$ and $\alpha = |S| \leqslant k$. It is obvious that $V - F - S$ is connected since $|V-F-S| \geqslant 2^n-[kn-\frac{k(k+1)}{2}+1]-k > kn-\frac{k(k+1)}{2}+1$ where $n \geqslant 5$.

According to Lemma 3, the test results of $V-F-S$ are all **0**s, which implies that all nodes of $V - F - S$ are fault-free. Consider the following cases:

Case 1: $\alpha < k - 1$.

Note that $|F \cup S| \leqslant kn - \frac{k(k+1)}{2} + k - 1$. Therefore, the all faulty nodes can be isolated within $F \cup S$.

Case 2: $\alpha = k - 1$.

We claim that there exists no such a node $v \in F$ such that $N(v) \subseteq F \cup S$, otherwise, there exists such one node v that $N(S \cup \{v\}) \subseteq F - \{v\}$. Therefore, $|N(S \cup \{v\})| \leqslant kn - \frac{k(k+1)}{2}$ which is a contradiction to Lemma 4. For each node $u \in F$, there always exists an edge from some node $w \in V - F - S$ to u, which implies that each node can be identified as faulty correctly

Case 3: $\alpha = k$.

We claim that there exists no such a node $v \in F$ that $N(v) \subseteq F \cup S$. Otherwise, there exists such one node v that $N(S \cup \{v\}) \subseteq F - \{v\}$. Therefore, $|N(S \cup \{v\})| \leqslant kn - \frac{k(k+1)}{2}$. According to Lemma 4, $|N(S\cup\{v\})| \geqslant (k+1)n-\frac{(k+1)(k+2)}{2}+1$. Note that $(k + 1)n - \frac{(k+1)(k+2)}{2} + 1 - [kn - \frac{k(k+1)}{2}] = n - k > 0$ which is a contradiction. Therefore, for each node $u \in F$, there always exists an edge from some node $w \in V - F - S$ to u, which implies that each node can be identified as faulty correctly. □

Theorem 15: Under the comparison model, an n-dimensional($n \geqslant 5$) hypercube is $[kn - \frac{k(k+1)}{2} + 1]/[kn - \frac{k(k+1)}{2}+k-1]$-diagnosable($2 \leqslant k \leqslant n-2$ and $kn-\frac{k(k+1)}{2}+1 \leqslant \frac{n^2-n}{2} - 1$).

Proof: Let $F \subseteq V$ be the real fault set with $|F| \leqslant kn - \frac{k(k+1)}{2} + 1$ in the system and $C_{sub}(G_{ind}(V - F)) = \{C_1, C_2, \ldots, C_m\}$. According to Lemma 9, the following conditions hold:

i) $\Sigma_{i=0}^{k} i|Card_i(C_{sub}(G_{ind}(V - F)))| \leqslant k$.

ii) There exists only one $C_i \in C_{sub}(G_{ind}(V - F))$ with $|V(C_i)| \geqslant k + 1$.

For convenience, let $S = \cup_{i=0}^{k} Card_i(C_{sub}(G_{ind}(V - F)))$ and $\alpha = |S| \leqslant k$. It is obvious that $V - F - S$ is connected since $|V-F-S| \geqslant 2^n-[kn-\frac{k(k+1)}{2}+1]-k > kn-\frac{k(k+1)}{2}+1$ where $n \geqslant 5$.

We shall show that the nodes of $V - F - S$ can be identified as being fault-free correctly by the following cases.

Case 1: Each node of $V-F-S$ has at least two neighbors in $V - F - S$.

According to Lemma 3, the test results of $V - F - S$ are all 0, which implies that all nodes of $V-F-S$ are fault-free.

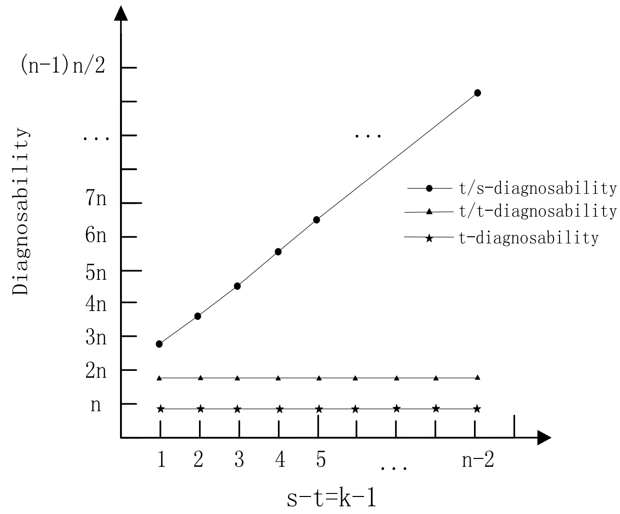

FIGURE 4. The comparison among three kinds of diagnosabilities on *n*-dimensional hypercubes.

Case 2: The other possible situations.

Let $X \subseteq V - F - S$ be the susbet such that each node of X has at least $n - 1$ neighbors in F and $|X| = \beta$. We claim that $|V - F - S| - \beta \geqslant kn - \frac{k(k+1)}{2} + 1 + 1$. Otherwise, $|V - F - S| - \beta < kn - \frac{k(k+1)}{2} + 1 + 1$. Note that $\beta \geqslant |V - F - S| - |F| \geqslant 2^n - [kn - \frac{k(k+1)}{2} + 1] - k - [kn - \frac{k(k+1)}{2} + 1] = 2^n - 2 - 2kn + k^2 \geqslant 2^n - n^2 - 1$. Since each node of X has at least $(n - 1)$ neighbors in F and for any two nodes $u, v \in V$, $|N(u) \cap N(v)| \leqslant 2$, $|N(X) \cap F| \geqslant (n-1)(2^n-n^2-1)-2(2^n-n^2-1) = (n-3)(2^n-n^2-1)$, which implies that $|F| \geqslant (n - 3)(2^n - n^2 - 1)$. Note that $(n-3)(2^n-n^2-1)-[kn-\frac{k(k+1)}{2}+1] \leqslant (n-3)(2^n-n^2-1)-[\frac{n(n-1)}{2}+1] \geqslant 3$ ($n \geqslant 5$) which is a contradiction to the hypothesis that $|F| \leqslant kn - \frac{k(k+1)}{2} + 1$. Therefore, $|V - F - S| - \beta \geqslant [kn - \frac{k(k+1)}{2} + 1] + 1$.

Let $M = V - F - S - X$. Then $|M| \geqslant [kn - \frac{k(k+1)}{2} + 1] + 1$ and each node of M has at least 2 neighbors in $V - F - S$, which implies that M can be identified as fault-free correctly by Lemma 3. Next we shall show that each node of X can be identified as fault-free correctly. Let $C_{sub}(G_{ind}(X)) = \{X_1, X_2, \ldots, X_l\}$. Note that $M \cup X$ is connected. Therefore, for each X_i ($1 \leqslant i \leqslant l$), there exist nodes $v_i \in X_i$ and $u \in M$ such that $(u, v_i) \in E$, which implies that v_i can be identified as fault-free correctly. And the neighbors of v_i in X_i can be identified as fault-free correctly. Furthermore, each node of X_i can be identified as fault-free correctly.

The remaining argument is similar to the proof of Case 1, 2, 3 of the Theorem 14. □

It is known that an n-dimensional hypercube is n-diagnosable and $(2n-2/2n-2)$-diagnosable. Previous study shows that an n-dimensional hypercube is $kn - \frac{k(k+1)}{2} + 1/kn - \frac{k(k+1)}{2} + k - 1$-diagnosable where $2 \leqslant k \leqslant n - 1$. As a comparison, Figure 4 shows the relationship of their diagnosabilities.

V. A FAST T/S-DIAGNOSABLE ALGORITHM FOR HYPERCUBE NETWORKS

In this section, we shall propose a fast t/s-diagnosable algorithm to isolate all faulty nodes within a subset under the PMC model (the comparison model).

This algorithm consists of two parts. The first part, Depth-First search (DFS), can find the largest fault-free component of the system (see Algorithm 1). The second part, Fast Isolation, uses the largest fault-free component to identify the remaining nodes in the system (see Algorithm 2).

Algorithm 1 Depth-First Search(PMC)

Input:
A system given by undirected graph $G(V, E)$ with 2^n nodes and a node $v \in V$. Let $W = \{v\}$.
Output:
A subset $W \subseteq V$.
1: $DFS(v)$:
for each $u \in N(v)$
if $\omega(u, v) = \omega(v, u) = 0$.
$W = W \cup \{u\}$ and DFS(u).
2: Output the nodes set W.

A. T/S-DIAGNOSABLE ALGORITHM UNDER THE PMC MODEL

Lemma 9 and Theorem 14 can be used to prove that the DFS can always output unique set S with $|S| \geqslant t+1$ for an n-dimensional hypercube provided the number of faulty nodes does not exceed t ($t \leqslant \frac{n^2-n}{2} - 1, n \geqslant 5$). Next, for a t/s-diagnosable hypercube of n dimension ($t \leqslant \frac{n^2-n}{2} - 1, n \geqslant 5$), an algorithm called Fast Isolation is introduced to locate a subset with size at most s such that the all faulty nodes are isolated in this subset (see Algorithm 2).

Algorithm 2 Fast Isolation(PMC)

Input:
A system given by undirected graph $G(V, E)$ with 2^n nodes and a fault bound $t(t \leqslant n^2 - \frac{n(n+1)}{2} - 1)$ and a syndrome σ.
Output:
A faulty nodes set F, a fault-free nodes set T and a possible faulty nodes set S with $F \cup T \cup S = V$.
1: Let $S_i = \varnothing (1 \leqslant i \leqslant 2^n)$, $T = F = S = \varnothing$.
For each node $u_i \in V - \cup_{j=1}^{i} S_i (1 \leqslant i \leqslant 2^n)$, do $DFS(u_i)$ and $S_i = DFS(u_i)$.
If $|S_i| \geqslant t+1$, then $T = T \cup S_i$ and $F = F \cup N(T)$.
2: If $V = T \cup F_c$, then output the fault-free nodes set T and faulty nodes set F_c. Otherwise go to step 3).
3: If $|F| \geqslant t$, then $T = V - F$ and output the fault-free nodes set T and faulty nodes set F_c. Otherwise,
$S = V - T - F$ and output sets T, F, S.

Theorem 16: The algorithm Fast Isolation has a time complexity $O(Nlog_2N)$, where $N = 2^n$.

Proof: For convenience, let F be the real fault set in the system and X be the largest component of $V - F$ and $Y = V - F - X$. If the selected node (u_i) belongs to the largest component of $V - F$, then step 1) costs $O(Nlog_2N)$ time. If the selected node(u_i) always belongs to $Y \cup F$, then step 1) costs $O(\frac{n^2+n}{2} + Nlog_2N) = O(Nlog_2N)$ time. Hence step 1) costs $O(Nlog_2N)$ time. step 2) and step 3) cost $O(N)$ time. Hence the total time of Fast Isolation is $O(Nlog_2N)$. □

B. T/S-DIAGNOSABLE ALGORITHM UNDER THE COMPARISON MODEL

Similar to the part A, we present the t/s-diagnosable algorithm under the comparison model for an n-dimensional hypercube network as follows (see Algorithm 3, 4).

Algorithm 3 Depth-First Search(MM)

Input:
A system given by undirected graph $G(V, E)$ with N nodes and a node $v \in V$. Let $W = \{v\}$.
Output:
A subset $W \subseteq V$.
1: $DFS(v)$:
for any two nodes $u_0, u_1 \in N(v)$ with $u_0 \notin S$ or $u_1 \notin S$.
if $\omega(v : u_0, u_1) = 0$.
$W = W \cup \{u_0\}$ and $DFS(u_0)$.
2: Output the nodes set W.

Algorithm 4 Fast Isolation(MM)

Input:
A system given by undirected graph $G(V, E)$ with 2^n nodes and a fault bound $t(t \leqslant n^2 - \frac{n(n+1)}{2} - 1)$ and a syndrome σ.
Output:
A faulty nodes set F, a fault-free nodes set T and a possible faulty nodes set P with $F \cup T \cup P = V$.
1: Let $S_i = \varnothing (1 \leqslant i \leqslant 2^n)$, $T = F = P = \varnothing$.
For each node $u_i \in V - \cup_{j=1}^{i} S_i (1 \leqslant i \leqslant 2^n)$, do $DFS(u_i)$ and $S_i = DFS(u_i)$.
If $|S_i| \geqslant t+1$, then $T = T \cup S_i$. Go to step 2).
2: Identify each node $w \in N(S_i)$ by using test result $\omega(w_0 : w, w_1)$ where $w_0, w_1 \in S_i$.
If w is faulty $F = F \cup \{w\}$, otherwise $S_i = S_i \cup \{w\}$.
Repeat step 2), until $N(S_i) \subseteq F$.
3: If $V = T \cup F$, then output the fault-free nodes set T and faulty nodes set F. Otherwise go to step 4).
4: If $|F| \geqslant t$, then $T = V - F$ and output the fault-free nodes set T and faulty nodes set F. Otherwise,
$P = V - T - F$ and output sets T, F, P.

Similar to Theorem 16, we have that the algorithm Fast Isolation has a time complexity $O(n2^n)$ for an n-dimensional hypercube.

VI. CONCLUSIONS

In this paper, we do a further study on the n-dimensional hypercube ($n \geqslant 5$). Some new important properties of hypercube are proposed. Based on the novel properties of hypercube, we show that an n-dimensional hypercube is $(kn - \frac{k(k+1)}{2} + 1)/(kn - \frac{k(k+1)}{2} + k - 1)$-diagnosable under the PMC model (the comparison model), which is much larger than n, the classical diagnosability of n-dimensional hypercube ($2 \leqslant k \leqslant n-2$ and $kn - \frac{k(k+1)}{2} + 1 \leqslant \frac{n^2-n}{2} - 1$). Furthermore, an $O(n2^n)$ algorithm is proposed to isolate faulty nodes within a set for an n-dimensional hypercube.

A future addition to this work would be to develop the t/s-diagnosability of other interconnection networks such as n-dimensional star graph, the exchanged hypercube, and their identification algorithms.

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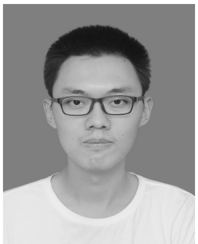

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