IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received October 26, 2016, accepted December 1, 2016, date of publication December 7, 2016, date of current version January 27, 2017.

Digital Object Identifier 10.1109/ACCESS.2016.2636318

Approximate Analytic Quadratic-Optimization Solution for TDOA-Based Passive Multi-Satellite Localization With Earth Constraint

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This work was supported in part by the Open Research Fund of National Key Laboratory of Electromagnetic Environment, China Research Institute of Radiowave Propagation (No. 201500013), Qindao, China, and the National Natural Science Foundation of China (No. 61271230, and No. 61472190).

ABSTRACT In this paper, we make an investigation of the problem of passive multi-satellite localization based on time differences of arrival (TDOA) with Earth constraint (EC). By utilizing TDOA measurements and EC, the problem of estimating target position is formulated as a quadratically constrained quadratic optimization. Following this, the approximate analytic solution of target position is obtained by using the method of Lagrange multipliers and deleting the infeasible roots of polynomial in the Lagrange multiplier. Simulation results show that the proposed method can achieve the Cramer-Rao lower bound (CRLB) with EC for three typical scenarios, even in the worst case, e.g., in the presence of large TDOA measurement errors with even target being far from the subastral point. However, the existing TDOA localization methods will deviate from the CRLB with EC as the measurement error of TDOA increases. Thus, the proposed method is more robust compared with the existing methods. In addition, the EC has a significant impact on the TDOA localization performance. Compared with the case of no EC, the EC can make a one-order-magnitude improvement in localization precision.

INDEX TERMS Passive multi-satellite localization, time difference of arrival, quadratical optimization, least squares, earth constraint.

I. INTRODUCTION

Passive localization, referred to as determining the position of a target without emitting electromagnetic wave, has drawn significant attention in widespread applications such as sonar, navigation, tracking and wireless sensor networks (WSNs) [1]–[12]. Several measurement parameters like time differences of arrival (TDOA), frequency differences of arrival (FDOA), time of arrival (TOA), signal strength, and angle of arrival (AOA) are independently or partially jointly used to implement the target localization. The TDOA attracts heavy research activities due to its perfect localization performance. Several TDOA-based location methods have been proposed, such as spherical interpolation method (SI) [6], Taylor series method (TS) [7], least squares method (LS) [8], and approximate maximum likelihood method (AML) [13]. Based on the above methods, some improved algorithms are also presented. The SI estimator converts the nonlinear hyperbolic equations into a set of linear equations by introducing an intermediate variable, which is a function of the target position. But the SI approach solves the set of linear equations directly by LS without exploiting the known relation between the intermediate variable and the target position. To improve the location precision of the SI method and exploit the relation above, Chan and Ho [8] propose the two-stage weighted least square (TWLS) algorithm. Also, Huang *et al.* [14] propose the linear correction least squares (LCLS) method. These two methods can achieve the Cramer-Rao lower bound (CRLB) in the low TDOA-measurement-error region. TS and AML methods are iterative algorithms and they usually need a good initial value, which is close to the target position, to guarantee the convergence to globally optimal solution. However, it is most often difficult to obtain this initial value in practical localization scenarios. In [15], the optimal sensor placement for TDOA-based localization is studied by considering a distance-dependent noise model in WSNs.

In a passive multi-satellite localization scenario with Earth constraint (EC), how to achieve the CRLB of TDOA with EC is a challenging issue, in particular, by an analytic approach. The paper aims to present an algorithm which can obtain an approximate analytic solution of target position and provide a good localization performance even at high TDOA-measurement-error region. To make full use of EC, we model the TDOA localization as a problem of quadric optimization (QO). Via the method of Lagrange multipliers, the analytic solution of target position is also derived and proved. Computer simulations show that the proposed method can reach the CRLB with EC for three typical situations, even in the high TDOA-measurement-error region with target position being far from the subastral point.

The rest of the paper is organized as follows: Section II describes the system model. In Section III, considering EC, the TDOA localization problem is casted as a quadratic optimization with quadratic constraint. The approximate closed-form expression for QO is derived and proved by the method of Lagrange multipliers and deleting the infeasible roots of the polynomial in the Lagrange multiplier. Simulation results and analysis are presented in Section IV. Finally, we draw our conclusions in Section V .

Notations: Bold letters denote vectors and matrices. $(\cdot)^T$ and $(\cdot)^{-1}$ denote the transposition, and inverse of matrix, respectively. $\mathbf{I}_n$ represents the $n \times n$ identity matrix, and operator diag($\mathbf{v}$) puts the vector $\mathbf{v}$ on the main diagonal. $\|\bullet\|_2$ denotes the 2-norm of a vector. $\mathbf{0}_{m\times n}$ denotes the $m \times n$ zero matrix. Mathematic operation tr $\{\mathbf{A}\}$ denotes the trace of matrix $\mathbf{A}$.

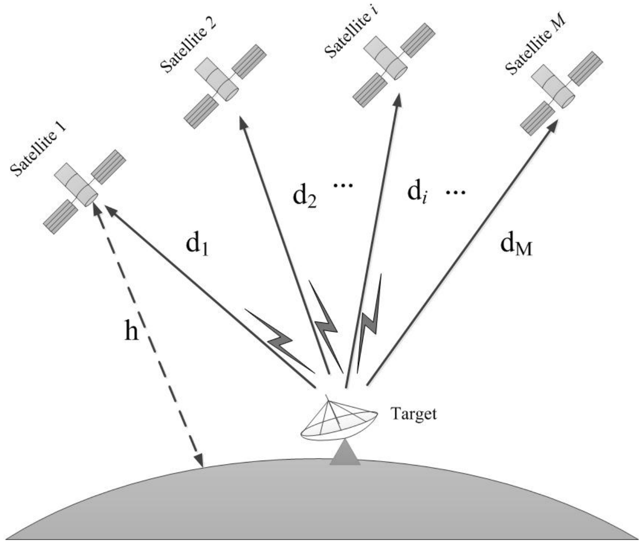

FIGURE 1. Passive multi-satellite localization system.

II. SYSTEM MODEL

In Fig. 1, we consider a 3D passive multi-satellite localization problem based on TDOA technique. The position of the unknown target on Earth surface position is denoted by $\mathbf{r} = (x, y, z)^T$. The satellite observation system consists of M satellites locating at $\mathbf{r}_i = (x_i, y_i, z_i)^T$, with $i = 1, 2, \ldots, M$. Without loss of generality, we choose satellite 1 as the primary one and the remaining $M-1$ satellites as the secondary ones.

Without channel noises, the TDOA from target to satellite 1 and satellite i is expressed as

$$\tau_{i1} = \frac{1}{c}\{\|\mathbf{r}_i - \mathbf{r}\|_2 - \|\mathbf{r}_1 - \mathbf{r}\|_2\} = \frac{d_i - d_1}{c}, \tag{1}$$

where c is the speed of light, and d_i denotes the distances between target and satellite i represented as

$$d_i = \|\mathbf{r} - \mathbf{r}_i\|_2, \quad i = 1, 2, \ldots, M. \tag{2}$$

Considering the measurement errors in practical environment, we obtain the TDOA from target to satellite 1 and satellite i

$$\hat{\tau}_{i1} = \tau_{i1} + \Delta\tau_{i1}, \quad i = 2, 3, \ldots, M. \tag{3}$$

where $\Delta\tau_{i1}$ is the TDOA measurement error and assumed to obey the Gaussian distribution with zero mean and variance σ^2 [8], [10], [13].

Combining (1) and (3), we have,

$$c\hat{\tau}_{i1} = d_i - d_1 + c\Delta\tau_{i1}. \tag{4}$$

Moving the second term of the right-hand side of (4) to the left-hand side of (4) forms a new equation

$$c\hat{\tau}_{i1} + d_1 = d_i + c\Delta\tau_{i1}. \tag{5}$$

Then squaring both sides of (5) yields

$$\begin{aligned}\left(c\hat{\tau}_{i1}\right)^2 + 2\left(c\hat{\tau}_{i1}d_1\right) + d_1^2 &= (c\Delta\tau_{i1})^2 \\ &+ 2c\Delta\tau_{i1}d_i + d_i^2,\ i \in \{2, 3, \ldots, M\},\end{aligned} \tag{6}$$

and using (2), meanwhile, ignoring the first term $(c\Delta\tau_{i1})^2$ on the right-hand side of (6), due to its far smaller value compared with other terms, hence we obtain

$$\begin{aligned}\left(c\hat{\tau}_{i1}\right)^2 + \mathbf{r}_1^T\mathbf{r}_1 - \mathbf{r}_i^T\mathbf{r}_i &= 2\left(\mathbf{r}_1^T - \mathbf{r}_i^T\right)\mathbf{r} \\ &- 2\left(c\hat{\tau}_{i1}r_{10}\right) + 2c\Delta\tau_{i1}d_i,\ i \in \{2, 3, \ldots, M\},\end{aligned} \tag{7}$$

where $r_{10} = d_1$ represents the range between target and satellite 1. Formulating (7) as the following matrix form

$$\widetilde{\mathbf{b}} = \mathbf{A}\widetilde{\mathbf{w}} + \mathbf{n}, \tag{8}$$

where

$$\mathbf{A} = \begin{pmatrix} 2\mathbf{r}_1^T - 2\mathbf{r}_2^T & -2c\hat{\tau}_{21} \\ 2\mathbf{r}_1^T - 2\mathbf{r}_3^T & -2c\hat{\tau}_{31} \\ \vdots & \\ 2\mathbf{r}_1^T - 2\mathbf{r}_M^T & -2c\hat{\tau}_{M1} \end{pmatrix}, \tag{9}$$

$$\widetilde{\mathbf{w}} = (x,\ y,\ z,\ r_{10})^T, \tag{10}$$

$$\widetilde{\mathbf{b}} = \begin{pmatrix} c^2\hat{\tau}_{21}^2 + \mathbf{r}_1^T\mathbf{r}_1 - \mathbf{r}_2^T\mathbf{r}_2 \\ c^2\hat{\tau}_{31}^2 + \mathbf{r}_1^T\mathbf{r}_1 - \mathbf{r}_3^T\mathbf{r}_3 \\ \vdots \\ c^2\hat{\tau}_{M1}^2 + \mathbf{r}_1^T\mathbf{r}_1 - \mathbf{r}_M^T\mathbf{r}_M \end{pmatrix}, \tag{11}$$

and

$$\mathbf{n} = \left(2c\Delta\tau_{21}d_2,\ 2c\Delta\tau_{31}d_3,\ \cdots,\ 2c\Delta\tau_{M1}d_M\right)^T. \quad (12)$$

The CRLB provides the lowest bound on the estimated precision that an unbiased location estimator can achieve. It is usually used for evaluate the performance of estimation methods. The target is on Earth and r_e is the radius of Earth, then the EC [16] can be written as

$$\mathbf{r}^T\mathbf{r} = r_e^2. \quad (13)$$

Considering the EC, the CRLB of an unbiased estimator is given by [16]

$$\mathrm{tr}\left\{\mathbf{cov}\left(\mathbf{r}\right)\right\} = \mathrm{tr}\left\{\mathbf{J}_t^{-1} - \mathbf{J}_t^{-1}\mathbf{F}\left(\mathbf{F}^T\mathbf{J}_t^{-1}\mathbf{F}\right)^{-1}\mathbf{F}^T\mathbf{J}_t^{-1}\mid_{\mathbf{r}=\mathbf{r}^o}\right\}, \quad (14)$$

where matrix $\mathbf{F}$ is the gradient vector

$$\mathbf{F} = 2\left(x,\ y,\ z\right)^T \quad (15)$$

of EC with respect to the unknown target position, and the matrix $\mathbf{J}_t$ is the Fisher information matrix

$$\mathbf{J}_t = \mathbf{G}_t^T\mathbf{Q}^{-1}\mathbf{G}_t, \quad (16)$$

with $\mathbf{Q} = c^2\sigma^2\mathbf{I}_{M-1}$ being the range measurement error covariance matrix, matrix $\mathbf{G}_t$ is

$$\mathbf{G}_t = \begin{pmatrix} \frac{(\mathbf{r}-\mathbf{r}_2)^T}{d_2} - \frac{(\mathbf{r}-\mathbf{r}_1)^T}{d_1} \\ \frac{(\mathbf{r}-\mathbf{r}_3)^T}{d_3} - \frac{(\mathbf{r}-\mathbf{r}_1)^T}{d_1} \\ \vdots \\ \frac{(\mathbf{r}-\mathbf{r}_M)^T}{d_M} - \frac{(\mathbf{r}-\mathbf{r}_1)^T}{d_1} \end{pmatrix}. \quad (17)$$

Regardless of EC, the CRLB simply reduces to $\mathrm{tr}\left\{\mathbf{J}_t^{-1}\right\}$. Observing (14) and compared to $\mathrm{tr}\left\{\mathbf{J}_t^{-1}\right\}$, it is evident that introducing EC can lower location error. Additionally, a good satellite constellation requires that the choice of satellite positions should guarantee the non-singular property of the matrix product $\mathbf{A}^T\mathbf{A}$.

III. PROPOSED QUADRATIC OPTIMIZATION METHOD

In the section, we model the TDOA location problem as a quadratic optimization [17]. By introducing the method of Lagrange multipliers, the problem of TDOA localization with EC is transformed into a problem of finding a set of feasible roots of seven-degree polynomial in the Lagrange multiplier.

Firstly, let us define

$$\widetilde{\mathbf{r}}_1 = \left(x_1,\ y_1,\ z_1,\ 0\right)^T, \quad (18)$$
$$\mathbf{b} = \widetilde{\mathbf{b}} - \mathbf{A}\widetilde{\mathbf{r}}_1, \quad (19)$$

and

$$\mathbf{w} = \left(x - x_1,\ y - y_1,\ z - z_1,\ r_{10}\right)^T = \widetilde{\mathbf{w}} - \widetilde{\mathbf{r}}_1, \quad (20)$$

then, (8) can be rewritten as

$$\mathbf{b} = \mathbf{A}\mathbf{w} + \mathbf{n}. \quad (21)$$

The EC equation in (13) is represented in matrix form

$$\mathbf{w}^T\Sigma_e\mathbf{w} + 2\mathbf{w}^T\Sigma_e\widetilde{\mathbf{r}}_1 = \rho, \quad (22)$$

where

$$\Sigma_e = \mathrm{diag}\{1, 1, 1, 0\}, \quad (23)$$
$$\mathbf{q} = \Sigma_e\widetilde{\mathbf{r}}_1, \quad (24)$$

and

$$\rho = r_e^2 - \mathbf{r}_1^T\mathbf{r}_1. \quad (25)$$

In accordance with (21) and (22), the TDOA localization problem with EC is casted as the following quadratic optimization

$$\min_{\mathbf{w}}\ \left(\mathbf{b} - \mathbf{A}\mathbf{w}\right)^T\left(\mathbf{b} - \mathbf{A}\mathbf{w}\right)$$
$$\text{s. t. }\ \mathbf{w}^T\Sigma_e\mathbf{w} + 2\mathbf{q}^T\mathbf{w} = \rho. \quad (26)$$

To solve the above optimization problem, using the method of Lagrange multipliers, we define the Lagrange function

$$L\left(\mathbf{w}, \lambda\right) = \left(\mathbf{b} - \mathbf{A}\mathbf{w}\right)^T\left(\mathbf{b} - \mathbf{A}\mathbf{w}\right) + \lambda\left(\mathbf{w}^T\Sigma_e\mathbf{w} + 2\mathbf{q}^T\mathbf{w} - \rho\right). \quad (27)$$

Differentiating $L\left(\mathbf{w}, \lambda\right)$ with respect to $\mathbf{w}$ and then equating the result to zero lead to the solution of $\widehat{\mathbf{w}}$

$$\widehat{\mathbf{w}} = \left(\mathbf{A}^T\mathbf{A} + \lambda\Sigma_e\right)^{-1}\left(\mathbf{A}^T\mathbf{b} - \lambda\mathbf{q}\right), \quad (28)$$

where λ is needed to be determined. So we substitute (28) into the equation constraint of (26) as shown in (22), then we have

$$\left(\mathbf{b}^T\mathbf{A} - \lambda\mathbf{q}^T\right)\left(\mathbf{A}^T\mathbf{A} + \lambda\Sigma_e\right)^{-1}\Sigma_e \left(\mathbf{A}^T\mathbf{A} + \lambda\Sigma_e\right)^{-1}\left(\mathbf{A}^T\mathbf{b} - \lambda\mathbf{q}\right) + 2\mathbf{q}^T\left(\mathbf{A}^T\mathbf{A} + \lambda\Sigma_e\right)^{-1}\left(\mathbf{A}^T\mathbf{b} - \lambda\mathbf{q}\right) - \rho = 0, \quad (29)$$

which is an equation of the unknown Lagrange multiplier λ. Below, we will show how to compute the parameter λ. The key is to convert the left-hand side of equation (29) into a seven-degree polynomial of λ by using the following theorem.

Theorem 1: If rank($\mathbf{A}$) $= 4$, and $\Sigma_e = \mathrm{diag}\{1, 1, 1, 0\}$, where the measurement matrix $\mathbf{A} \in R^{(M-1)\times 4}$, then the matrix product $\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\Sigma_e$ is diagonalizable.

Proof: Please see Appendix A. ■

Since the $\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\Sigma_e$ is diagonalizable, assuming Λ is the corresponding diagonal matrix after diagonalization, then the $\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\Sigma_e$ can be diagonalized in the following form

$$\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\Sigma_e = \mathbf{U}\Lambda\mathbf{U}^{-1}. \quad (30)$$

where $\mathbf{U}$ is an 4×4 nonsingular matrix. Hence, we get

$$\Sigma_e = \left(\mathbf{A}^T\mathbf{A}\right)\mathbf{U}\Lambda\mathbf{U}^{-1}, \tag{31}$$

which yields

$$\left(\mathbf{A}^T\mathbf{A} + \lambda\Sigma_e\right)^{-1} = \mathbf{U}\left(\mathbf{I} + \lambda\Lambda\right)^{-1}\mathbf{U}^{-1}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}. \tag{32}$$

Placing (32) in (29), and letting $\mathbf{R} = \mathbf{I} + \lambda\Lambda$, we have

$$\begin{aligned}\rho = &\left(\mathbf{b}^T\mathbf{A} - \lambda\mathbf{q}^T\right)\\ &\times\left[\mathbf{U}\mathbf{R}^{-1}\Lambda\mathbf{R}^{-1}\mathbf{U}^{-1}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\right]\left(\mathbf{A}^T\mathbf{b} - \lambda\mathbf{q}\right)\\ &+2\mathbf{q}^T\left[\mathbf{U}\mathbf{R}^{-1}\mathbf{U}^{-1}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\right]\left(\mathbf{A}^T\mathbf{b} - \lambda\mathbf{q}\right).\end{aligned} \tag{33}$$

To simplify the above equation, let us define

$$\mathbf{c}^T = \mathbf{q}^T\mathbf{U} = (c_1, c_2, c_3, c_4), \tag{34}$$

$$\mathbf{e}^T = \mathbf{b}^T\mathbf{A}\mathbf{U} = (e_1, e_2, e_3, e_4), \tag{35}$$

$$\mathbf{g} = \mathbf{U}^{-1}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{q} = (g_1, g_2, g_3, g_4)^T, \tag{36}$$

and

$$\mathbf{f} = \mathbf{U}^{-1}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b} = (f_1, f_2, f_3, f_4)^T. \tag{37}$$

Substituting the above equations from (34) to (37) in (33) yields

$$\begin{aligned}&2\mathbf{c}^T\mathbf{R}^{-1}\mathbf{f} - 2\lambda\mathbf{c}^T\mathbf{R}^{-1}\mathbf{g} + \mathbf{e}^T\mathbf{R}^{-1}\Lambda\mathbf{R}^{-1}\mathbf{f}\\ &\quad-\lambda\mathbf{e}^T\mathbf{R}^{-1}\Lambda\mathbf{R}^{-1}\mathbf{g} - \lambda\mathbf{c}^T\mathbf{R}^{-1}\Lambda\mathbf{R}^{-1}\mathbf{f}\\ &\quad+\lambda^2\mathbf{c}^T\mathbf{R}^{-1}\Lambda\mathbf{R}^{-1}\mathbf{g} - \rho = 0.\end{aligned} \tag{38}$$

Considering that the rank of matrix $\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\Sigma_e$ is 3, and there exists only one zero eigenvalue. Assuming $\Lambda = \text{diag}\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$, and γ_i, $i = 1, 2, \ldots, 4$, is the corresponding eigenvalue. Let $\gamma_4 = 0$, then the equation in (38) can be further simplified as

$$\begin{aligned}&2c_4f_4 - \rho - 2\lambda c_4g_4 + 2\sum_{i=1}^{3}\frac{c_if_i}{1+\lambda\gamma_i}\\ &-2\lambda\sum_{i=1}^{3}\frac{c_ig_i}{1+\lambda\gamma_i} + \sum_{i=1}^{3}\frac{e_if_i\gamma_i}{(1+\lambda\gamma_i)^2} - \lambda\sum_{i=1}^{3}\frac{e_ig_i\gamma_i}{(1+\lambda\gamma_i)^2}\\ &-\lambda\sum_{i=1}^{3}\frac{c_if_i\gamma_i}{(1+\lambda\gamma_i)^2} + \lambda^2\sum_{i=1}^{3}\frac{c_ig_i\gamma_i}{(1+\lambda\gamma_i)^2} = 0,\end{aligned} \tag{39}$$

which can be expanded as a polynomial of degree 7 in λ as follows

$$\begin{aligned}f(\lambda) = &P_7\lambda^7 + P_6\lambda^6 + P_5\lambda^5\\ &+P_4\lambda^4 + P_3\lambda^3 + P_2\lambda^2 + P_1\lambda^1 + P_0,\end{aligned} \tag{40}$$

where all coefficients of the polynomial in (40) are given in Appendix B. The roots of equation in (40) can be obtained by using the numerical methods in [18], like the Newton-Raphson method in [19]. Notice that in a real optimization problems, the target position is a real three-dimensional vector, thus the Lagrange multiplier should be also a real value. So, all complex roots of λ should be discarded. Then, substituting the remaining roots into (28), we get the set S_w of estimated values of $\widehat{\mathbf{w}}_\lambda$ of $\mathbf{w}$. Combining (20), (18), and (21), the S_r set of estimated candidate positions of target is obtained as $S_r = \{\widehat{\mathbf{r}}_\lambda | \widehat{\mathbf{r}}_\lambda = \widehat{\mathbf{w}}_\lambda(1:3) + \mathbf{r}_1, \widehat{\mathbf{w}}_\lambda \in S_w\}$. Finally, we attain the set $S_{o,r}$ of the optimal target positions $\widehat{\mathbf{r}}$ of minimizing

$$\sum_{i=2}^{M}\{(\|\widehat{\mathbf{r}}_\lambda - \mathbf{r}_i\| - \|\widehat{\mathbf{r}}_\lambda - \mathbf{r}_1\|) - c\widehat{\tau}_{i1}\}^2. \tag{41}$$

over all $\widehat{\mathbf{r}}_\lambda \in S_r$. To clearify, the above operation procedure is summarized in Algorithm 1 below, and the corresponding detailed flow-chart graph is also plotted in Fig. 2. It is particularly noted that it is possible to still exist several optimal feasible solutions in set $S_{o,\ r}$ achieving the same minimum value in (41). If this situation appears, any element of set $S_{o,\ r}$ is randomly chosen as the optimal solution to target position.

Algorithm 1 Proposed Quadratic Optimization TDOA Localization Method

Input: $\mathbf{A}$, $\mathbf{b}$, Σ_e, $\mathbf{q}$, and ρ.

Output: Target position $\widehat{\mathbf{r}}$.

Steps:

1) Calculate matrices Λ, and $\mathbf{U}$ by using (30), and $\mathbf{c}$, $\mathbf{e}$, $\mathbf{g}$ and $\mathbf{f}$ by using (34), (35), (36), and (37).
2) Get the seven roots of equation (40) by the Newton-Raphson method.
3) Discard the complex roots for a real optimization problem.
4) Obtain the corresponding set S_w of all $\widehat{\mathbf{w}}_\lambda^{\prime}s$ by substituting the rest roots into (28).
5) Compute the set S_r of all candidate target positions $\widehat{\mathbf{r}}_\lambda = \widehat{\mathbf{w}}_\lambda(1:3) + \mathbf{r}_1$ using (20).
6) Choose the best estimated target position by minimizing the following cost function $\sum_{i=2}^{M}\{(\|\widehat{\mathbf{r}}_\lambda - \mathbf{r}_i\| - \|\widehat{\mathbf{r}}_\lambda - \mathbf{r}_1\|) - c\widehat{\tau}_{i1}\}^2$ over the set S_r of candidate target positions, which is expressed as: $$\widehat{\mathbf{r}} = \arg\min_{\widehat{\mathbf{r}}_\lambda \in S_r}\sum_{i=2}^{M}\{(\|\widehat{\mathbf{r}}_\lambda - \mathbf{r}_i\| - \|\widehat{\mathbf{r}}_\lambda - \mathbf{r}_1\|) - c\widehat{\tau}_{i1}\}^2.$$

IV. SIMULATIONS AND DISCUSSIONS

In this section, simulations are carried out to evaluate the performance of the proposed method compared with the TWLS in [8], the AML in [13], and CRLBs. Below, the root of mean square errors (RMSE) is adopted as the metric for localization performance which is defined as

$$RMSE(\mathbf{r}) = \sqrt{\frac{1}{N}\sum_{k=1}^{N}\left(\widehat{\mathbf{r}}^n - \mathbf{r}\right)^T\left(\widehat{\mathbf{r}}^n - \mathbf{r}\right)}, \tag{42}$$

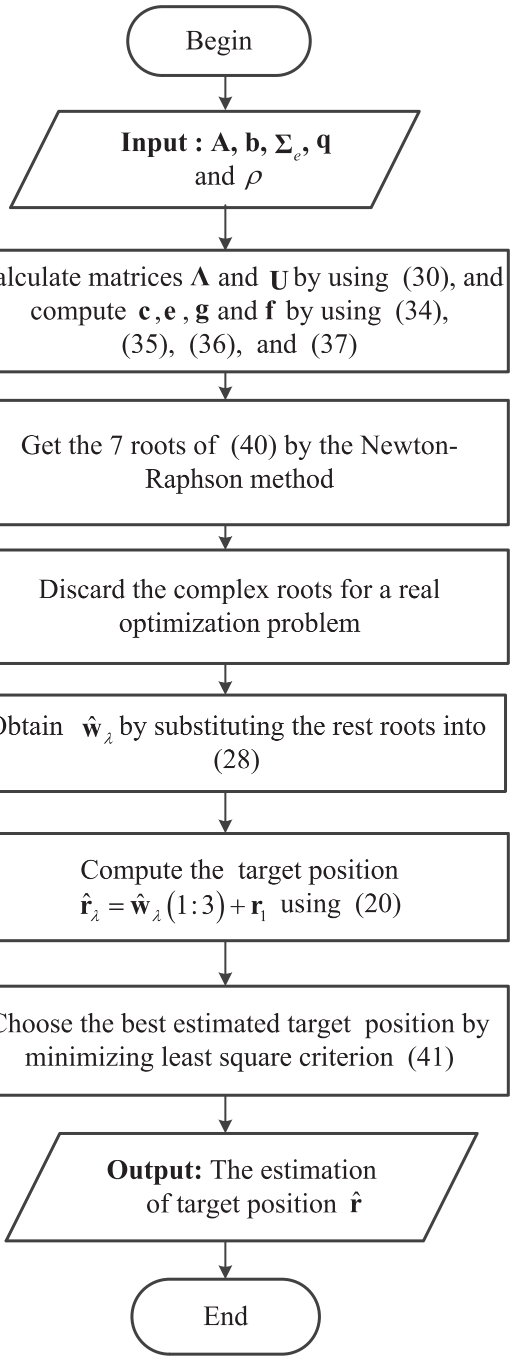

FIGURE 2. Block diagram for the proposed method.

where N is the total times of simulation loops for every given measurement error and $\widehat{\mathbf{r}}^n$ represents the estimation of target position in the nth time of Monte Carlo simulation. The random TDOA measurement error obeys the Gaussian distribution with zero mean and its standard deviation σ varies from 0.01ns to 100ns. The subastral point is the intersection point between the Earth surface and the line segment of connecting the primary satellite and Earth center.

According to [10], we also use a symmetric satellite constellation for the convenience of computation.

TABLE 1. Satellite Positions (E: East, N: North)

Satellite	height h	longitude β	latitude α
1	1000	130^{o}E	0.0^o
2	894.71	130.59^{o}E	0.5908^{o}N
3	894.71	130.59^{o}E	0.5908^{o}S
4	894.71	129.41^{o}E	0.5908^{o}N
5	894.71	129.41^{o}E	0.5908^{o}S

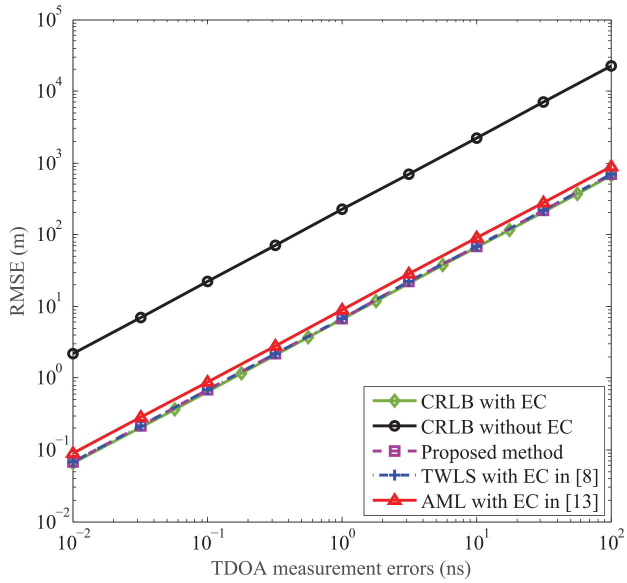

FIGURE 3. RMSE versus TDOA measurement error σ far from the subastral point at r = (0, 138°*E*, 8°*N*).

The localization system consists of five satellites formed as a rectangular pyramid, and their positions are listed in Table 1 above, where h denotes the height away from the Earth in terms of kilometers (km), β denotes longitude, and α denotes latitude of the satellites. With the definition of $\mathbf{r}_m = (x_m, y_m, z_m)^T$, we have

$$\begin{aligned} x_m &= (r_e + h_m)\cos\alpha_m \cos\beta_m, \\ y_m &= (r_e + h_m)\cos\alpha_m \sin\beta_m, \\ z_m &= (r_e + h_m)\sin\alpha_m. \end{aligned} \tag{43}$$

Fig. 3 plots the curves of the RMSEs versus standard deviation σ of TDOA measurement error for the proposed method, TWLS in [8], AML in [13] , where the target is on Earth and sited at 138°E (E: East longitude) and 8°N (N: North latitude), which is approximately 1000 km away from the subastral point. In the region near the subastral point, localization errors are small, the proposed method and TWLS algorithm can both achieve CRLBs with EC, and outperform AML slightly.

In Fig. 4, we move the target a little farther away from the subastral point and make a performance comparison similar to Fig. 3, where the target is located at 112°E and 8°N and is about 2000 km from the subastral point. From this figure, it is very obvious that the proposed method performs much better than AML for the entire measurement error region.

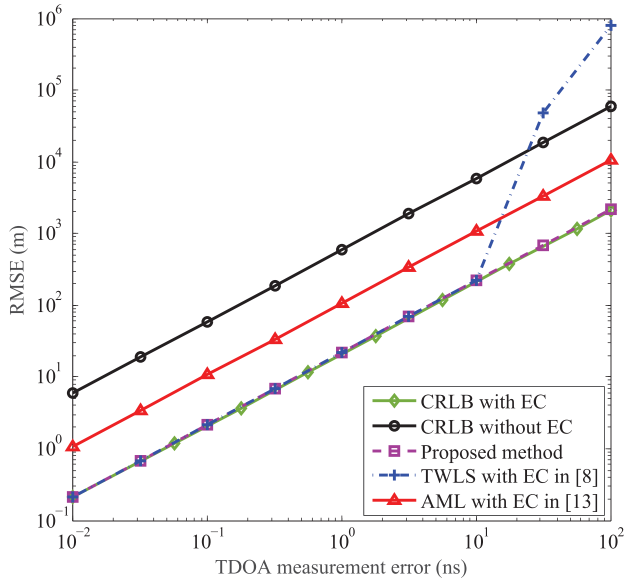

FIGURE 4. RMSE versus TDOA measurement error σ far from the subastral point at $\mathbf{r} = (0, 112°E, 8°N)$.

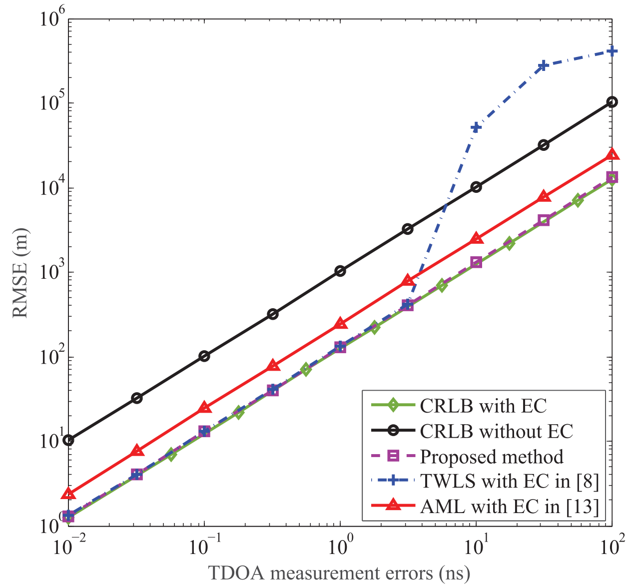

FIGURE 5. RMSE versus TDOA measurement error σ far from the subastral point at $\mathbf{r} = (0, 155°E, 20°S)$.

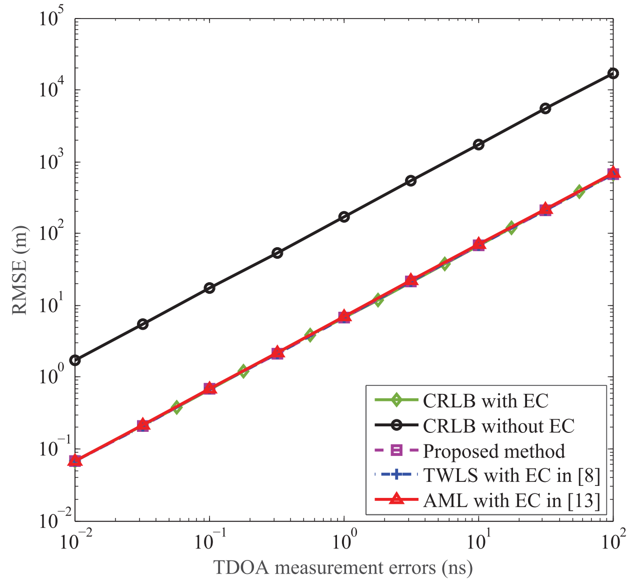

FIGURE 6. RMSE versus TDOA measurement error σ far from the subastral point at $\mathbf{r} = (0, 138°E, 8°N)$.

Compared with Fig. 3, the localization performance difference becomes large. For example, at the TDOA measurement error $\sigma = 10ns$, the location error of the AML is 1 km while that of the proposed method is only 0.2 km. And when $\sigma \leq 10ns$, the performance of the TWLS and proposed schemes is almost the same, they are both close to the CRLB with EC. However, as the TDOA measurement error increases to $\sigma > 10ns$, the performance degradation of TWLS becomes obvious, while our proposed method still achieves the CRLB with EC.

In Fig. 5, the target position is moved to the point $\mathbf{r} = (0, 155^0 E, 20°S)$ on Earth surface, more than 3000 km distance from the subastral point. Observing this figure, the TWLS scheme performs worse than that in Fig. 4. It can be seen that the RMSE value of TWLS scheme increases significantly and deviates from the CRLB curve with EC at $\sigma \geq 5ns$. Here, the AML method outperforms the TWLS scheme. Unlike the TWLS algorithm, the proposed method can still achieve the CRLBs even in the large TDOA-measurement-error region. Also, the proposed method still performs better than AML algorithm in the regions far away from the subastral point.

From the Fig. 3, Fig. 4 and Fig. 5, it can be seen that the TWLS algorithm can achieve the CRLB with EC in the small TDOA measurement error region. However, the TWLS performance degrades quickly when TDOA measurement error becomes larger. It means that this method is not robust. The proposed method can always achieve CRLB with EC, which shows a good robustness. Meanwhile, the curve of CRLB with EC is below the curve of CRLB without EC and the value of the former value is about one tenth of that of the latter. It is evident that, exploiting EC can significantly improve the localization accuracy, which is in good agreement with theoretical analysis.

TABLE 2. Satellite Positions (E: East, N: North)

Satellite	height h	longitude β	latitude α
1	1000	130^oE	0.0^o
2	903.38	130.6342^oE	0.6342^oN
3	903.38	130.6342^oE	0.6342^oS
4	903.38	129.3658^oE	0.6342^oN
5	903.38	129.3658^oE	0.6342^oS

To further verify the localization performance of the proposed method, all satellite positions are moved to the new positions, which are indicated in Table 2 with the same primary satellite position as Table 1, but the angles between the

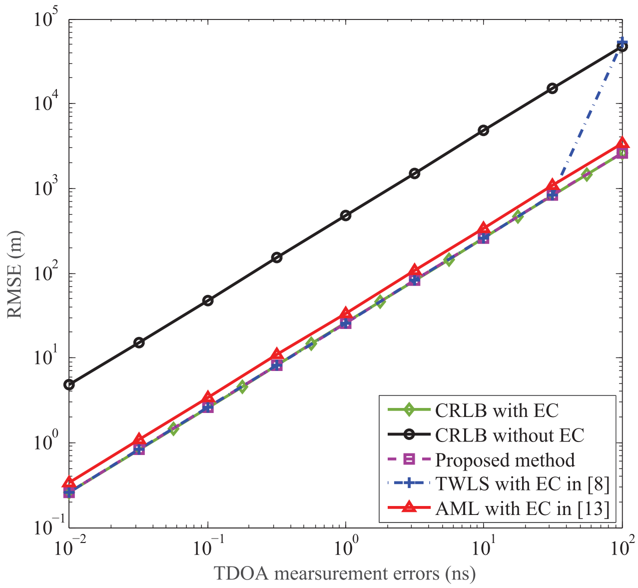

FIGURE 7. RMSE versus TDOA measurement error σ far from the subastral point at $\mathbf{r} = (0, 148°E, 10°N)$.

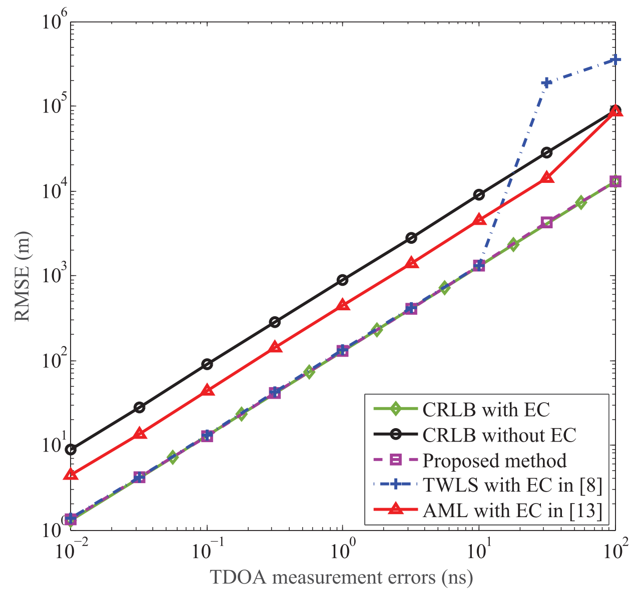

FIGURE 8. RMSE versus TDOA measurement error σ far from the subastral point at $\mathbf{r} = (0, 155°E, 20°N)$.

lines of connecting the primary satellite to the secondary ones and the line of connecting the Earth center and the primary satellite are greater than those angles shown in Table 1. Similar to Fig. 3, Fig. 4, and Fig. 5, Fig. 6, Fig. 7, and Fig. 8 plot the corresponding curves of RMSE versus σ for three different scenarios: near, medium, and far, respectively. Clearly, they show the same performance trend as Fig. 3, Fig. 4, and Fig. 5. EC still makes an one-order-magnitude performance improvement over no EC case. The localization performance of the proposed method is better than or equal to those of AML and TWLS in three different situations (distance: close, medium, and far).

V. CONCLUSION

In this paper, a quadratically constrained quadratic programming is developed for passive multi-satellite TDOA localization with EC. By adopting the method of Lagrange multipliers, the TDOA localization problem is converted into a problem of computing all roots of seven-degree polynomial in the Lagrange multiplier λ. Then, the corresponding analytic solution of target position is readily attained by using the optimal value of λ. From simulations and analysis, we find the proposed algorithm can achieve CRLB with EC for three typical scenarios, even when TDOA measurement error is large and the target is far away the subastral point. Here, EC plays a significant role in improving the TDOA localization precision and makes an one-order-magnitude improvement in localization precision of TDOA compared to the case of no EC.

APPENDIX A
PROOF OF THEOREM 1

Proof: The matrix $\mathbf{A} \in R^{(M-1)\times 4}$, where $R^{(M-1)\times 4}$ denotes a space of $(M-1)\times 4$ real matrices, and $\text{rank}(\mathbf{A}) = 4$. Since $(\mathbf{A}^T\mathbf{A})^T = \mathbf{A}^T\mathbf{A}$, and $((\mathbf{A}^T\mathbf{A})^{-1})^T = (\mathbf{A}^T\mathbf{A})^{-1}$, we can conclude that both matrices $(\mathbf{A}^T\mathbf{A})$ and $(\mathbf{A}^T\mathbf{A})^{-1}$ are real symmetric matrices, which can be diagonalized.

If $\mathbf{Av} = 0$, the vector $\mathbf{v}$ must be a zero vector because of $\text{rank}(\mathbf{A}) = 4$. Hence, for any $\mathbf{v} \in R^4$ and $\mathbf{v} \neq 0$, where R^4 denotes the space of four-dimensional real vectors, we have

$$\mathbf{v}^T(\mathbf{A}^T\mathbf{A})\mathbf{v} = (\mathbf{Av})^T(\mathbf{Av}) = (\mathbf{Av}, \mathbf{Av}) > 0. \quad (44)$$

The matrix $\mathbf{A}^T\mathbf{A}$ is positive definite, which means that all eigenvalues of $\mathbf{A}^T\mathbf{A}$ are positive and the matrix $(\mathbf{A}^T\mathbf{A})^{-1}$ is also a positive-definite matrix.

Let us define

$$\mathbf{J} = (\mathbf{A}^T\mathbf{A})^{-1}\Sigma_e, \quad (45)$$

and

$$\mathbf{K} = \mathbf{J}(1:3, 1:3). \quad (46)$$

If we rewrite $(\mathbf{A}^T\mathbf{A})^{-1}$ as follows

$$(\mathbf{A}^T\mathbf{A})^{-1} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \quad (47)$$

then matrices $\mathbf{J}$ and $\mathbf{K}$ have the following relationship

$$\mathbf{J} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{K} & \mathbf{0}_{3\times 1} \\ \mathbf{s}^T & 0 \end{pmatrix}, \quad (48)$$

where $\mathbf{s} = (a_{41}, a_{42}, a_{43})^T$, $\mathbf{0}_{3\times 1}$ is a three-dimensional column vector of all zeros, and

$$\mathbf{K} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \quad (49)$$

Because matrix $(\mathbf{A}^T\mathbf{A})^{-1}$ is real symmetric and positive definite, its leading principal minors are positive due to the nature of positive-definite matrix. Thus, we conclude that matrix $\mathbf{K}$ is a real symmetric matrix with rank($\mathbf{K}$) = 3. The matrix $\mathbf{K}$ can be diagonalizable, there exists three linearly independent eigenvectors, i.e., $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ associated with its three eigen-values λ_1, λ_2, and λ_3. This implies that $\mathbf{K}\mathbf{x}_i = \lambda_i \mathbf{x}_i$ for $i \in \{1, 2, 3\}$. The characteristic polynomial of $\mathbf{K}$ is

$$p(t) = \det(t\mathbf{I} - \mathbf{K}) = (t - \lambda_1)(t - \lambda_2)(t - \lambda_3), \tag{50}$$

where λ_1, λ_2, and λ_3 are the eigenvalues of matrix $\mathbf{K}$ and all are non-zeros. The characteristic polynomial of $\mathbf{J}$ can be expressed as

$$q(t) = \det(t\mathbf{I} - \mathbf{J}) = t(t - \lambda_1)(t - \lambda_2)(t - \lambda_3), \tag{51}$$

which means that the set of all non-zero eigen-values of matrix $\mathbf{J}$ is the same as that of $\mathbf{K}$. Below, we construct the corresponding eigen-vectors of $\mathbf{J}$ from all eigen-vectors of matrix $\mathbf{K}$.

Firstly, we find that the eigen-vector corresponding to the zero-eigen-value of matrix $\mathbf{J}$ is

$$\mathbf{y}_0 = (0, 0, 0, 1)^T \tag{52}$$

by observing the detailed structure of matrix $\mathbf{J}$ in (48). For the second eigen-value of $t = \lambda_1$, we assume the corresponding eigenvector is the form $\mathbf{y}_1 = (\mathbf{x}_1^T, y_{14})^T$, where $\mathbf{x}_1$ is known to be the eigen-vector of matrix $\mathbf{K}$ associated with the first eigen-value $t = \lambda_1$, and the value of y_{14} is unknown. Considering $\mathbf{y}_1$ is the eigen-vector of matrix $\mathbf{J}$ associated with the eigen-value $t = \lambda_1$, we have

$$\mathbf{J}\mathbf{y}_1 = \begin{pmatrix} \mathbf{K} & \mathbf{0}_{3\times 1} \\ \mathbf{s}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ y_{14} \end{pmatrix} = \begin{pmatrix} \mathbf{K}\mathbf{x}_1 \\ \mathbf{s}^T\mathbf{x}_1 \end{pmatrix} \tag{53}$$

Placing $\mathbf{K}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$ into the above equation, we have

$$\mathbf{J}\mathbf{y}_1 = \begin{pmatrix} \lambda_1\mathbf{x}_1 \\ \mathbf{s}\mathbf{x}_1 \end{pmatrix} = \lambda_1 \begin{pmatrix} \mathbf{x}_1 \\ \frac{\mathbf{s}^T\mathbf{x}_1}{\lambda_1} \end{pmatrix} \tag{54}$$

which yields

$$y_{14} = \frac{\mathbf{s}^T\mathbf{x}_1}{\lambda_1}, \tag{55}$$

which results in the fact that $\mathbf{y}_1$ is the eigen-vector associated with the eigen-value λ_1 of matrix $\mathbf{J}$ with

$$\mathbf{y}_1 = \begin{pmatrix} \mathbf{x}_1 \\ \frac{\mathbf{s}^T\mathbf{x}_1}{\lambda_1} \end{pmatrix}. \tag{56}$$

In a similar manner, we get the remaining two eigen-vectors

$$\mathbf{y}_2 = \begin{pmatrix} \mathbf{x}_2 \\ \frac{\mathbf{s}^T\mathbf{x}_2}{\lambda_2} \end{pmatrix}, \tag{57}$$

and

$$\mathbf{y}_3 = \begin{pmatrix} \mathbf{x}_3 \\ \frac{\mathbf{s}^T\mathbf{x}_3}{\lambda_3} \end{pmatrix} \tag{58}$$

of matrix $\mathbf{J}$ corresponding to the eigen-values λ_2 and λ_3. We infer that $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$ are three linearly independent eigen-vectors because of linear independence of $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$.

Below we will prove by contraction the fact that four eigen-vectors $\mathbf{y}_0$, $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ of matrix $\mathbf{J}$ are linearly independent. Suppose $\mathbf{y}_0$, $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are linearly dependent, then we have

$$\alpha_0\mathbf{y}_0 + \alpha_1\mathbf{y}_1 + \alpha_2\mathbf{y}_2 + \alpha_3\mathbf{y}_3 = 0, \tag{59}$$

where α_0, α_1, α_2, and α_3 are not all zeroes. It means at least one $\alpha_i \neq 0$ for $i \in \{0, 1, 2, 3\}$. Equation (59) can be expressed in the block-matrix form

$$\alpha_0 \begin{pmatrix} \mathbf{0}_{3\times 1} \\ 1 \end{pmatrix} + \alpha_1 \begin{pmatrix} \mathbf{x}_1 \\ y_{14} \end{pmatrix} + \alpha_2 \begin{pmatrix} \mathbf{x}_2 \\ y_{24} \end{pmatrix} + \alpha_3 \begin{pmatrix} \mathbf{x}_3 \\ y_{34} \end{pmatrix} = \begin{pmatrix} \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3 \\ \alpha_0 + \alpha_1\frac{\mathbf{s}^T\mathbf{x}_1}{\lambda_1} + \alpha_2\frac{\mathbf{s}^T\mathbf{x}_2}{\lambda_2} + \alpha_3\frac{\mathbf{s}^T\mathbf{x}_3}{\lambda_3} \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{3\times 1} \\ 0 \end{pmatrix}, \tag{60}$$

which yields

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3 = \mathbf{0}_{3\times 1}, \tag{61}$$

and

$$\alpha_0 + \alpha_1\frac{\mathbf{s}^T\mathbf{x}_1}{\lambda_1} + \alpha_2\frac{\mathbf{s}^T\mathbf{x}_2}{\lambda_2} + \alpha_3\frac{\mathbf{s}^T\mathbf{x}_3}{\lambda_3} = 0. \tag{62}$$

In terms of the linear independence of $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$, the equation in (61) reduces to

$$\alpha_1 = \alpha_2 = \alpha_3 = 0. \tag{63}$$

Substituting the above condition (63) in (62) gives

$$\alpha_0 = -\alpha_1\frac{\mathbf{s}^T\mathbf{x}_1}{\lambda_1} - \alpha_2\frac{\mathbf{s}^T\mathbf{x}_2}{\lambda_2} - \alpha_3\frac{\mathbf{s}^T\mathbf{x}_3}{\lambda_3} = 0. \tag{64}$$

Until now, we have proved the fact that α_0, α_1, α_2, and α_3 are all zeros, which contradicts our assumption. Thus, we can conclude that $\mathbf{y}_0$, $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are four linearly independent eigen-vectors. According to the [20, Th. 1.3.7], we claim that $\mathbf{J}$ or $(\mathbf{A}^T\mathbf{A})^{-1}\Sigma_e$ is diagonalizable. This completes the proof of Theorem 1. ■

APPENDIX B
COEFFICIENTS OF SEVEN-DEGREE POLYNOMIAL (40)

Proof: Define $S = \{1, 2, 3\}$, and S_{-i} represents the remaining set after excluding the *ith* element with any $i \in S$. Let $q, n, m \in S, j, k \in S_{-i}$, and

$$p = 2c_4f_4 - \rho, \tag{65}$$

all coefficients of polynomial (40) are the following

$$P_7 = -2c_4g_4(\prod \gamma_n^2), \tag{66}$$

$$P_6 = p(\prod \gamma_n^2) - 2c_4g_4(\sum_{m\neq n\neq q} \gamma_q\gamma_m^2\gamma_n^2), \tag{67}$$

$$
\begin{aligned}
P_5 = p(&\sum_{m\neq n\neq q} \gamma_q\gamma_m^2\gamma_n^2) + 2\sum_{i=1}^{3} c_i f_i(\gamma_i \prod_j \gamma_j^2) \\
&- 2c_4 g_4(\sum_{m\neq n\neq q} 2\gamma_q\gamma_m^2\gamma_n^2 + \frac{1}{2}\gamma_m^2\gamma_n^2) \\
&- 2\sum_{i=1}^{3} c_i g_i(\sum_{j\neq k} \frac{1}{2}\gamma_j^2\gamma_k^2 + 2\gamma_i\gamma_j\gamma_k^2) \\
&- \sum_{i=1}^{3} e_i g_i \gamma_i(\prod_j \gamma_j^2) - \sum_{i=1}^{3} c_i f_i \gamma_i(\prod_j \gamma_j^2) \\
&+ \sum_{i=1}^{3} c_i g_i \gamma_i(\sum_{j\neq k} 2\gamma_j\gamma_k^2),
\end{aligned} \tag{68}
$$

$$
\begin{aligned}
P_4 = p(&\sum_{m\neq n\neq q} 2\gamma_q\gamma_m\gamma n^2 + \frac{1}{2}\gamma_m^2\gamma_n^2) \\
&- 2c_4 g_4(\sum_{m\neq n\neq k} 2\gamma_m\gamma_n^2 + \frac{4}{3}\gamma_q\gamma_m\gamma_n) \\
&+ 2\sum_{i=1}^{3} c_i f_i(\sum_{j\neq k} \frac{1}{2}\gamma_j^2\gamma_k^2 + 2\gamma_i\gamma_j\gamma_k^2) \\
&- 2\sum_{i=1}^{3} c_i g_i(\sum_{j\neq k} 2\gamma_i\gamma_j\gamma_k + \gamma_i\gamma_j^2 + 2\gamma_j\gamma_k^2) \\
&+ \sum_{i=1}^{3} e_i f_i \gamma_i(\prod_j \gamma_j^2) - \sum_{i=1}^{3} e_i g_i \gamma_i(\sum_{j\neq k} 2\gamma_j\gamma_k^2) \\
&- \sum_{i=1}^{3} c_i f_i \gamma_i(\sum_{j\neq k} 2\gamma_j\gamma_k^2) \\
&+ \sum_{i=1}^{3} c_i g_i \gamma_i(\sum_{j\neq k} 2\gamma_j\gamma_k + \gamma_j^2),
\end{aligned} \tag{69}
$$

$$
\begin{aligned}
P_3 = p(&\sum_{m\neq n\neq q} 2\gamma_m\gamma_n^2 + \frac{4}{3}\gamma_q\gamma_m\gamma_n) \\
&- 2c_4 g_4(\sum_{m\neq n} \gamma_n^2 + 2\gamma_m\gamma_n) \\
&+ 2\sum_{i=1}^{3} c_i f_i(\sum_{j\neq k} 2\gamma_i\gamma_j\gamma_k + \gamma_i\gamma_j^2 + 2\gamma_j\gamma_k^2) \\
&- 2\sum_{i=1}^{3} c_i g_i(\sum_{j\neq k} 2\gamma_i\gamma_j + 2\gamma_j\gamma_k + \gamma_j^2) \\
&+ \sum_{i=1}^{3} e_i f_i \gamma_i(\sum_{j\neq k} 2\gamma_j\gamma_k^2) - \sum_{i=1}^{3} e_i g_i \gamma_i(\sum_{j\neq k} 2\gamma_j\gamma_k + \gamma_j^2) \\
&- \sum_{i=1}^{3} c_i f_i \gamma_i(\sum_{j\neq k} 2\gamma_j\gamma_k + \gamma_j^2) + \sum_{i=1}^{3} c_i g_i \gamma_i(\sum_j 2\gamma_j),
\end{aligned} \tag{70}
$$

$$
\begin{aligned}
P_2 = p(&\sum_{m\neq n} \gamma_n^2 + 2\gamma_m\gamma_n) - 2c_4 g_4(\sum 2\gamma_n) \\
&+ 2\sum_{i=1}^{3} c_i f_i(\sum_{j\neq k} 2\gamma_i\gamma_j + 2\gamma_j\gamma_k + \gamma_j^2) \\
&- 2\sum_{i=1}^{3} c_i g_i(\gamma_i + \sum_j 2\gamma_j) + \sum_{i=1}^{3} e_i f_i \gamma_i(\sum_{j\neq k} 2\gamma_j\gamma_k + \gamma_j^2) \\
&- \sum_{i=1}^{3} e_i g_i \gamma_i(\sum_j 2\gamma_j) - \sum_{i=1}^{3} c_i f_i \gamma_i(\sum_j 2\gamma_j) + \sum_{i=1}^{3} c_i g_i \gamma_i,
\end{aligned} \tag{71}
$$

$$
\begin{aligned}
P_1 = p(&\sum 2\gamma_n) - 2c_4 g_4 \\
&+ 2\sum_{i=1}^{3} c_i f_i(\gamma_i + \sum_j 2\gamma_j) - 2\sum_{i=1}^{3} c_i g_i \\
&+ \sum_{i=1}^{3} e_i f_i \gamma_i(\sum_j 2\gamma_j) - \sum_{i=1}^{3} e_i g_i \gamma_i - \sum_{i=1}^{3} c_i f_i \gamma_i,
\end{aligned} \tag{72}
$$

and

$$
P_0 = p + 2\sum_{i=1}^{3} c_i f_i + \sum_{i=1}^{3} e_i f_i \gamma_i. \tag{73}
$$

This completes the representation of all coefficients of the polynomial in (40). ■

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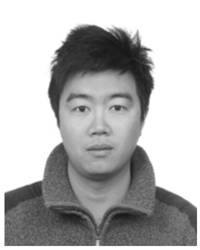

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