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Reaching Law Based Discrete Time Sliding Mode Inventory Management Strategy

ANDRZEJ BARTOSZEWICZ AND PAWEŁ LATOSIŃSKI
Institute of Automatic Control, Łódź University of Technology, 90-924 Łódź, Poland
Corresponding author: A. Bartoszewicz (andrzej.bartoszewicz@p.lodz.pl)

ABSTRACT In this paper, a new reaching law for uncertain discrete time systems is presented. The proposed reaching law is designed to ensure a limited sliding variable rate of change and at least asymptotic convergence of the variable to zero. The law is utilized to design a control strategy, which is then applied to an inventory management system with multiple suppliers and limited warehouse capacity. The strategy is proved to ensure all the desirable properties of the system. On the one hand, it guarantees that the suppliers are never forced to send more goods than they are capable of and that warehouse capacity is never exceeded. On the other hand, it ensures that the consumers' demand is always fully satisfied.

INDEX TERMS Discrete time sliding mode control, reaching law approach, inventory management systems.

I. INTRODUCTION

Sliding mode control of continuous time systems is an area of research with roots in late 1950s in Russia [1], [2]. Control algorithms it provides are characterized by high computational efficiency and complete rejection of matched disturbance [3]. However, the application of continuous time sliding mode controllers can result in an undesirable phenomenon called chattering, which can potentially damage the plant or cause energy loss [4]. The implementation of the aforementioned controllers was elaborated upon by various authors [5]–[8]. Since continuous states cannot be perfectly represented in digital environments, a natural development was the introduction of discrete time sliding mode control [9], [10]. Although they are not completely insensitive to disturbance, discrete time sliding mode controllers became attractive to the control engineering community [11]–[15] as they offer good robustness without requiring high frequency switching in the sliding phase.

The classic approach to sliding mode controller design involves stating the control law and then proving via Lyapunov analysis that it ensures stability of the sliding motion. However, in this paper an alternative method called the reaching law approach will be utilized. This method was first introduced for continuous time systems in [8] and for discrete time ones in [11], with further comments in [12]. Reaching law approach is based on stating the desired evolution of the sliding variable and then utilizing the evolution to derive the control law, thus bypassing the complex Lyapunov-based proof of stability. Various authors have proposed new reaching laws [16]–[22], which improve on the classic formula introduced by Gao *et al.*

In this paper, a new discrete time reaching law based control strategy will be presented and applied to an inventory management system [23]–[27] with multiple suppliers, limited warehouse capacity and an *a priori* unknown consumers' demand. For such a system, it is necessary for the strategy to ensure non-negative and upper bounded values of control signal, which denotes the amount of goods sent in each sampling period by all suppliers. Furthermore, system output representing the amount of merchandise stored in the warehouse has to be upper bounded due to limited storage space, and strictly positive to ensure that consumers' demand is fully satisfied. All of these properties are guaranteed by the proposed strategy and will be formally proven in this paper.

The remainder of the paper is organized in the following way. Section 2 describes the reaching law based sliding mode control strategy and demonstrates properties related to the sliding variable. In section 3, the inventory management system is presented and expressed with a delay free model in the extended state space. In the same section, all of the favorable properties described in the previous paragraph are formally proven. Section 4 contains simulation results, which illustrate the effectiveness of the proposed method. Section 5 gives concluding remarks.

II. CONTROL STRATEGY

In this paper, we propose a reaching law based sliding mode control strategy for a class of discrete time systems expressed in the state space as

$$\begin{aligned} \boldsymbol{x}[(k+1)T] &= \boldsymbol{A}\boldsymbol{x}(kT) + \widetilde{\boldsymbol{A}}(kT)\boldsymbol{x}(kT) \\ &\quad + \boldsymbol{b}u(kT) + \boldsymbol{p}d(kT) \\ y(kT) &= \boldsymbol{q}^{\mathrm{T}}\boldsymbol{x}(kT), \end{aligned} \tag{1}$$

where $\boldsymbol{A}$ is the $n \times n$ dimensional state matrix, $\widetilde{\boldsymbol{A}}(kT)$ is the matrix representing internal parameter uncertainties, $u(kT)$ and $d(kT)$ are scalars representing control signal and disturbance respectively and $\boldsymbol{b}, \boldsymbol{p}, \boldsymbol{q}$ are $n \times 1$ dimensional vectors. Since vectors $\boldsymbol{b}$ and $\boldsymbol{p}$ are not necessarily equal, the disturbance $d(kT)$ is not matched. First, the sliding variable is defined as

$$s(kT) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}_d - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(kT), \tag{2}$$

where $\boldsymbol{c}$ is a real vector such that $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b} \neq 0$ selected to ensure the stability of the system and $\boldsymbol{x}_d$ is the desired state of the system. In order to make the sliding mode control strategy applicable, it is assumed that there exist constants D_{min} and D_{max} such that for all k

$$D_{min} \leq \boldsymbol{c}^{\mathrm{T}}\widetilde{\boldsymbol{A}}(kT)\boldsymbol{x}(kT) + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{p}d(kT) \leq D_{max}. \tag{3}$$

For the sake of convenience, constants representing the average perturbations and their maximum admissible deviation from the average are defined as

$$D_{avg} = \frac{D_{max} + D_{min}}{2}, \quad D_{\delta} = \frac{D_{max} - D_{min}}{2}. \tag{4}$$

The reaching law for the class of systems (1) is now introduced as

$$\begin{aligned} s[(k+1)T] &= s(kT) - s_0 f[s(kT)]\mathrm{sgn}[s(kT)] + D_{avg} \\ &\quad - \boldsymbol{c}^{\mathrm{T}}\widetilde{\boldsymbol{A}}(kT)\boldsymbol{x}(kT) - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{p}d(kT), \end{aligned} \tag{5}$$

where f is a function

$$f[s(kT)] = 1 - exp\left[-\frac{s(kT)^2}{s_0^2}\right] \tag{6}$$

with values from the interval $[0, 1)$ and s_0 is the design parameter. The reaching law designed in such a way ensures that the sliding variable converges to a certain vicinity of zero at least asymptotically and that its rate of change is bounded by s_0 plus the maximum influence of uncertainties. The reaching law will now be utilized to obtain a sliding mode control strategy. First $s[(k+1)T]$ is substituted from (2) into (5) and $\boldsymbol{x}[(k+1)T]$ is further substituted from (1) into (5) resulting in

$$\begin{aligned} &\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}_d - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}(kT) - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}u(kT) \\ &\quad = s(kT) - s_0 f[s(kT)]\mathrm{sgn}[s(kT)] + D_{avg}. \end{aligned} \tag{7}$$

Then, (7) is solved for $u(kT)$ giving the control strategy

$$\begin{aligned} u(kT) &= -(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b})^{-1}\{s(kT) - s_0 f[s(kT)]\mathrm{sgn}[s(kT)] \\ &\quad + D_{avg} + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}(kT) - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}_d\}. \end{aligned} \tag{8}$$

Remark 1: It is easy to verify that the expression $s(kT) - s_0 f[s(kT)]\mathrm{sgn}[s(kT)]$ in the reaching law (5) is strictly increasing for all real values of the sliding variable. Therefore, for any real R_1 and R_2 such that $R_1 < R_2$, we have

$$R_1 - s_0 f(R_1)\mathrm{sgn}(R_1) < R_2 - s_0 f(R_2)\mathrm{sgn}(R_2). \tag{9}$$

A. PROPERTIES OF THE PROPOSED STRATEGY

It will now be demonstrated that the reaching law (5) drives the sliding variable into a band

$$B = \left\{\boldsymbol{x} : |\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}_d - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}| \leq \beta = s_0\sqrt{ln\left(\frac{s_0}{s_0 - D_{\delta}}\right)}\right\} \tag{10}$$

around the sliding surface $s(kT) = 0$ and ensures that the variable is confined to the band (10) in all future steps. To that end, the following two theorems will be proven.

Theorem 1: If $s_0 > D_{\delta}$ and $\boldsymbol{x}(kT)$ is out of the band (10), then the system representative point will approach the band at least asymptotically.

Proof: Let us consider any state $\boldsymbol{x}(kT)$ out of the band (10). The proof will only be conducted for the case where $s(kT) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}_d - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(0) \geq \beta$, since the analysis for the case $s(kT) \leq -\beta$ is almost identical. First, we substitute β from (10) into (6) and get

$$s_0 f(\beta) = s_0\left[1 - exp\left(-\frac{\beta^2}{s_0^2}\right)\right] = s_0\left[1 - \frac{s_0 - D_{\delta}}{s_0}\right] = D_{\delta}. \tag{11}$$

Since $s(kT) \geq \beta$, then relation (9) gives $f[s(kT)] \geq f(\beta)$. Consequently, relations (5) and (11) imply

$$\begin{aligned} s[(k+1)T] &\leq s(kT) - s_0 f(\beta) - \boldsymbol{c}^{\mathrm{T}}\widetilde{\boldsymbol{A}}(kT)\boldsymbol{x}(kT) \\ &\quad + D_{avg} - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{p}d(kT) \\ &\leq s(kT) - D_{\delta} + D_{avg} - D_{min} \\ &= s(kT) - D_{\delta} + D_{\delta} = s(kT). \end{aligned} \tag{12}$$

Therefore, the sliding variable will either become smaller than β in finite time or approach a certain positive value s_+ asymptotically. In the former case, the system representative point enters the quasi-sliding mode band (10). Let us now consider the case where the sliding variable approaches s_+ as k tends to infinity. We have

$$\begin{aligned} \lim_{k\to\infty} s[(k+1)T] &\leq \lim_{k\to\infty}\{s(kT) - s_0 f[s(kT)] + D_{\delta}\} \\ &= s_+ - s_0 f(s_+) + D_{\delta}. \end{aligned} \tag{13}$$

Since $s[(k+1)T]$ also converges to s_+, relation (13) implies

$$s_+ \leq s_+ - s_0 f(s_+) + D_{\delta}. \tag{14}$$

By solving (14) for s_+, we obtain $s_+ \leq \beta$, which means that the sliding variable can only converge to a value inside the band (10). In conclusion, for any initial state the system representative point will either enter the aforementioned band or approach it asymptotically. ■

It will now be demonstrated that once the system representative point enters the quasi-sliding mode band (10), it will remain inside the band for all future sampling instants. This property will be proven in the following theorem.

Theorem 2: If $s_0 > D_\delta$ and $\boldsymbol{x}(kT)$ is inside the band (10), then $\boldsymbol{x}[(k+1)T]$ will also be confined to that band.

Proof: Let $\boldsymbol{x}(kT)$ be such a state that $-\beta \leq s(kT) \leq \beta$. It will be shown that $s[(k+1)T]$ also belongs to that interval. Since Remark 1 states that the expression $s(kT) - s_0 f[s(kT)]\text{sgn}[s(kT)]$ in the reaching law (5) is strictly increasing, only the cases of $s(kT) = \pm\beta$ will be considered. First, for $s(kT) = \beta$ substitution of (11) into (5) yields

$$\begin{aligned} s[(k+1)T] &= \beta - s_0 f[s(\beta)]\text{sgn}[s(\beta)] + D_{avg} \\ &\quad - \boldsymbol{c}^{\mathrm{T}}\widetilde{\boldsymbol{A}}(kT)\boldsymbol{x}(kT) - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{p}d(kT) \\ &\leq \beta - s_0 f[s(\beta)]\text{sgn}[s(\beta)] + D_{avg} - D_{min} \\ &= \beta - D_\delta + D_\delta = \beta. \end{aligned} \quad (15)$$

Now let $s(kT) = -\beta$. Since f is an even function, relations (5) and (11) give

$$\begin{aligned} s[(k+1)T] &= -\beta - s_0 f[s(-\beta)]\text{sgn}[s(-\beta)] + D_{avg} \\ &\quad - \boldsymbol{c}^{\mathrm{T}}\widetilde{\boldsymbol{A}}(kT)\boldsymbol{x}(kT) - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{p}d(kT) \\ &\geq -\beta + s_0 f[s(-\beta)]\text{sgn}[s(\beta)] + D_{avg} - D_{max} \\ &= -\beta + D_\delta - D_\delta = -\beta. \end{aligned} \quad (16)$$

Since the expression $s(kT) - s_0 f[s(kT)]\text{sgn}[s(kT)]$ is strictly increasing in the interval $[-\beta, \beta]$, relations (15) and (16) imply that $s[(k+1)T]$ is always confined to that interval. ■

Taking Theorems 1 and 2 into account, we conclude that the proposed reaching law ensures at least asymptotic convergence of the sliding variable to a band around the switching plane and guarantees that once the variable enters the band, it will remain inside for all subsequent sampling instants.

III. INVENTORY SUPPLY MODEL

The reaching law proposed in the previous section will now be applied to a specific plant with input and output constraints and its properties in this context will be further analyzed. Let us consider an inventory management system with *a priori* unknown consumers' demand and limited warehouse capacity. The current amount of goods stored at the warehouse is expressed by $y(kT)$, where T is the sampling period. Shipments are delivered to the warehouse by m providers with an assumption that a certain amount of goods is damaged during transport. Therefore, a commodity loss factor is denoted by $f_i(kT)$ for shipment sent by the i-th provider and delivered at time instant kT. Consequently, out of the original shipment, only the amount multiplied by $f_i(kT)$ will arrive at its destination. For each i, $f_i(kT)$ is always lower and upper bounded by $f_{min}^i > 0$ and $f_{max}^i \leq 1$.

The amount of goods the i-th provider can send at any moment is assumed to be upper bounded by constant u_{max}^i and the sum of u_{max}^i for all i is denoted as u_{max}. The time it takes for i-th supplier's shipments to arrive at the warehouse is expressed by T_i. The discretization period T is selected to ensure that every T_i is a multiple of it, i.e. for each i there exists a natural μ_i such that $T_i = \mu_i T$. The control signal $u(kT)$ determines the amount of goods that need to be sent at the moment kT and is distributed among the providers proportionally to their capabilities. Consequently, the i-th provider will be required to send the shipment equal to

$$u_i(kT) = u(kT)\frac{u_{max}^i}{u_{max}}. \quad (17)$$

The consumers' demand at each time instant kT is denoted by an *a priori* unknown function $d^*(kT)$. The amount of goods $d(kT)$ that are actually sold is equal to $d^*(kT)$ unless the stock at instant kT is insufficient to meet the demand, in which case $d(kT)$ is just large enough to empty the warehouse. The following relation holds

$$0 \leq d(kT) \leq d^*(kT) \leq d_{max}. \quad (18)$$

To make sure that it is possible for suppliers to satisfy the demand at any moment, the following assumption is made

$$\sum_{i=1}^{m} f_{min}^i u_{max}^i > d_{max}. \quad (19)$$

In conclusion, since the warehouse is assumed to be empty at the beginning of the control process, the amount of stored goods at any moment can be expressed as

$$y(kT) = \sum_{j=0}^{k-1}\sum_{i=1}^{m} f_i(jT)u_i(jT - T_i) - \sum_{j=0}^{k-1} d(jT). \quad (20)$$

Real life application of the proposed sliding mode control strategy to the inventory management system described above must adhere to several constraints. Since warehouse capacity is limited, the strategy must ensure that the system output $y(kT)$ is upper bounded for all time instants. Furthermore, the control signal should not require the suppliers to send more goods than they are capable of. Finally, to ensure that the consumers' demand is always satisfied, we require the strategy to maintain strictly positive stock levels after $\max(T_i/T)$ initial steps. In the following subsections the model of the inventory management system will be presented, after which the aforementioned properties will be formally proven.

A. SIMPLIFICATION OF THE MODEL

In order to simplify the controller design procedure, an n-dimensional delay-free model of the supply chain will be introduced, where $n = \max(T_i/T) + 1$. The current amount of stored goods (20) will be expressed by the first state variable of this model, while the remaining variables will represent the shipments that are already underway. It is further assumed that the amount of goods that reach their destination is always equal to f_{max}^i and any deviation from the maximum will be modeled via internal parameter uncertainties. The unpredictable consumers' demand will be expressed as external disturbance affecting the first state variable. First, the amount of goods arriving at the warehouse from all providers at the moment kT is denoted by $F_r(kT)u(kT - rT)$, where $r = 1, \ldots, n-1$ and

$$F_r(kT) = \sum_{i:T_i = rT} f_i(kT)\frac{u_{max}^i}{u_{max}}. \quad (21)$$

Then, the amount of stored goods (20) can be expressed as

$$y(kT)=\sum_{j=0}^{k-1}\sum_{r=1}^{n-1}F_r(jT)u_i(jT-rT)-\sum_{j=0}^{k-1}d(jT) \quad (22)$$

and consequently

$$y[(k+1)T]=y(kT)+\sum_{r=1}^{n-1}F_r(kT)u_i(kT-rT)-d(kT). \quad (23)$$

Next, F^r_{min} and F^r_{max} are defined in the following way

$$F^r_{min}=\sum_{i:T_i=rT}f^i_{min}\frac{u^i_{max}}{u_{max}},\quad F^r_{max}=\sum_{i:T_i=rT}f^i_{max}\frac{u^i_{max}}{u_{max}}. \quad (24)$$

With this in mind, the inventory management system can now be expressed as the plant (1), where the first state variable denotes the current stock level while the remaining ones represent shipments that are already underway. Consequently, state variables x_i for $i=2,\dots,n$ are lower bounded by zero and upper bounded by u_{max}, since suppliers can only send the amount of goods from that interval. Furthermore, $d(kT)\in[0,d_{max}]$ is the satisfied consumers' demand, the $n\times n$ dimensional state matrix

$$\boldsymbol{A}=\begin{bmatrix}1 & F^{n-1}_{max} & F^{n-2}_{max} & \cdots & F^1_{max}\\ 0 & 0 & 1 & & 0\\ \vdots & & & \ddots & \vdots\\ 0 & 0 & 0 & & 1\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}, \quad (25)$$

matrix representing parameter uncertainties

$$\widetilde{\boldsymbol{A}}(kT)=\begin{bmatrix}1 & F^*_{n-1}(kT) & \cdots & F^*_1(kT)\\ 0 & 0 & & 0\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}, \quad (26)$$

where $F^*_r(kT)=F_r(kT)-F^r_{max}$ for all r, and $n\times 1$ dimensional vectors

$$\boldsymbol{b}=\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix},\quad \boldsymbol{p}=\begin{bmatrix}-1\\ 0\\ \vdots\\ 0\end{bmatrix},\quad \boldsymbol{q}=\begin{bmatrix}1\\ 0\\ \vdots\\ 0\end{bmatrix}. \quad (27)$$

Taking the known system parameters into account, from relations (3) and (4), we obtain

$$D_{min}=-d_{max}-u_{max}\sum_{r=1}^{n-1}(F^r_{max}-F^r_{min}),\quad D_{max}=0,$$

$$D_{avg}=-D_\delta=-\frac{d_{max}}{2}-\frac{u_{max}}{2}\sum_{r=1}^{n-1}(F^r_{max}-F^r_{min}). \quad (28)$$

In the next section, a control strategy for the inventory management system defined in such a way will be presented and several of its advantageous properties will be proven.

B. PROPERTIES OF THE INVENTORY MANAGEMENT STRATEGY

The reaching law based control strategy will now be applied to the inventory management system described above. The strategy will lead the system state $\boldsymbol{x}(kT)$ to a desired value $\boldsymbol{x}_d=[y_d\ 0\dots 0]^{\mathrm{T}}$, where y_d is a positive constant selected by the designer. Thus, the sliding variable becomes

$$s(kT)=\boldsymbol{c}^{\mathrm{T}}[\boldsymbol{x}_d-\boldsymbol{x}(kT)]=c_1y_d-\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(kT). \quad (29)$$

Vector $\boldsymbol{c}$ is selected to ensure a dead-beat response of the system [19]. Therefore, it must satisfy the following equation

$$\det\{\lambda\boldsymbol{I}_n-\boldsymbol{A}+\boldsymbol{b}(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b})^{-1}\boldsymbol{c}^{\mathrm{T}}\boldsymbol{A}\}=\lambda^n, \quad (30)$$

where $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{b}\neq 0$. Considering relations (25) and (30), elements of vector $\boldsymbol{c}$ are obtained as

$$c_i=\begin{cases}1 & \text{for } i=1\\ \sum_{r=1}^{i-1}F^{n-r}_{max} & \text{for } i=2,\dots,n.\end{cases} \quad (31)$$

The control signal (8) can now be simplified taking the known system parameters into account. First, relation (25) implies that

$$\boldsymbol{c}^{\mathrm{T}}(\boldsymbol{A}-\boldsymbol{I}_n)=[0\quad c_1F^{n-1}_{max}-c_2\quad \dots\quad c_1F^1_{max}+c_{n-1}-c_n] \quad (32)$$

and substitution of (31) into (32) yields

$$\boldsymbol{c}^{\mathrm{T}}(\boldsymbol{A}-\boldsymbol{I}_n)=[0\quad 0\quad 0\quad \dots\quad 0]. \quad (33)$$

From (33) we further obtain

$$\boldsymbol{c}^{\mathrm{T}}(\boldsymbol{A}-\boldsymbol{I}_n)\boldsymbol{x}(kT)+\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(kT)=\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(kT). \quad (34)$$

Taking relations (31) and (34) into account, control law (8) can be rewritten in a simplified form

$$u(kT)=\left(\sum_{r=1}^{n-1}F^r_{max}\right)^{-1}\{s_0f[s(kT)]\mathrm{sgn}[s(kT)]-D_{avg}\}. \quad (35)$$

When applied to the considered inventory management system, the control signal must always be non-negative and it should never force the suppliers to exceed their total capabilities u_{max}. These properties will be proven in the following theorems.

Theorem 3: Control signal (35) is always non-negative.

Proof: It is known that the state vector $\boldsymbol{x}$ is equal to zero at the beginning of the control process, $c_1=1$ and $y_d>0$. Consequently $s(0)=c_1y_d-\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}(0)>0$. If $s(0)>\beta$, then Theorem 1 states that the sliding variable will either arrive inside the interval $[-\beta,\beta]$ in finite time or approach β asymptotically. On the other hand, if the considered state belongs to the band (10), i.e. if the variable belongs to the interval $[-\beta,\beta]$, then Theorem 2 ensures that the state will remain inside the band in all subsequent sampling instants. In conclusion, for the considered system the sliding variable will never assume values smaller than $-\beta$. Since the expression $s_0 f(s)\mathrm{sgn}(s)$ in the control law (35) is strictly increasing with respect to s, we obtain

$$\begin{aligned} u(kT) &= \left(\sum_{r=1}^{n-1} F_{max}^r\right)^{-1} \{s_0 f[s(kT)]\mathrm{sgn}[s(kT)] - D_{avg}\} \\ &\geq \left(\sum_{r=1}^{n-1} F_{max}^r\right)^{-1} [-s_0 f(-\beta) - D_{avg}]. \end{aligned} \tag{36}$$

Furthermore, relations (11) and (28) give

$$\begin{aligned} u(kT) &\geq \left(\sum_{r=1}^{n-1} F_{max}^r\right)^{-1} (-D_\delta - D_{avg}) \\ &= \left(\sum_{r=1}^{n-1} F_{max}^r\right)^{-1} (D_{avg} - D_{avg}) = 0 \end{aligned} \tag{37}$$

for every k. Therefore, the control signal for the considered system is always non-negative. ■

Next, it will be demonstrated that with the right choice of parameter s_0, the proposed strategy never requires the suppliers to exceed their manufacturing capabilities.

Theorem 4: If

$$s_0 \leq u_{max} \left(\sum_{r=1}^{n-1} F_{max}^r\right) + D_{avg}, \tag{38}$$

then for all time instants the control signal is not greater than u_{max}.

Proof: It is known that f in (35) is a non-negative function upper bounded by 1. Consequently

$$\begin{aligned} u(kT) &= \left(\sum_{r=1}^{n-1} F_{max}^r\right)^{-1} \{s_0 f[s(kT)]\mathrm{sgn}[s(kT)] - D_{avg}\} \\ &\leq \left(\sum_{r=1}^{n-1} F_{max}^r\right)^{-1} (s_0 - D_{avg}). \end{aligned} \tag{39}$$

Substitution of (38) into (39) yields

$$\begin{aligned} u(kT) &\leq \left(\sum_{r=1}^{n-1} F_{max}^r\right)^{-1} \\ &\quad \times \left\{u_{max} \left(\sum_{r=1}^{n-1} F_{max}^r\right) + D_{avg} - D_{avg}\right\} \\ &= u_{max}. \end{aligned} \tag{40}$$

■

In conclusion, the control signal generated by the proposed strategy satisfies input constraints for the considered inventory management system. However, the inventory management system further requires that the consumers' demand is always satisfied and that the result is achieved with finite storage space. These properties will be demonstrated in the next two theorems.

Theorem 5: If

$$y_d > \beta + u_{max} \sum_{i=2}^{n} c_i, \tag{41}$$

then for every $k > \max(T_i/T)$ the amount of stored goods $y(kT)$ is strictly positive.

Proof: Let $k > \max(T_i/T)$. First let us consider the case where $s(kT) \leq \beta$. Relation (29) implies

$$y_d - y(kT) - \sum_{i=2}^{n} c_i u[(k-n+i-1)T] \leq \beta. \tag{42}$$

Since Theorem 4 ensures that the control signal is upper bounded by u_{max}, relations (41) and (42) give

$$\begin{aligned} y(kT) &\geq y_d - \sum_{i=2}^{n} c_i u[(k-n+i-1)T] - \beta \\ &\geq y_d - u_{max} \sum_{i=2}^{n} c_i - \beta > 0. \end{aligned} \tag{43}$$

Now let us consider the case where $s(kT) > \beta$. Theorems 1 and 2 imply that $s(lT) > \beta$ for l smaller than k. Therefore, for each $l = 0, \ldots, k-1$

$$\begin{aligned} u(lT) &= \left(\sum_{r=1}^{n-1} F_{max}^r\right)^{-1} \{s_0 f[s(lT)] - D_{avg}\} \\ &> \left(\sum_{r=1}^{n-1} F_{max}^r\right)^{-1} [s_0 f(\beta) - D_{avg}] \\ &= \left(\sum_{r=1}^{n-1} F_{max}^r\right)^{-1} 2D_\delta. \end{aligned} \tag{44}$$

Relation (22), together with (44), gives

$$\begin{aligned} y(kT) &= y[(k-1)T] - d[(k-1)T] \\ &\quad + \sum_{r=1}^{n-1} F_r(kT) u(kT - rT) \\ &> y[(k-1)T] - d[(k-1)T] \\ &\quad + \left[\sum_{r=1}^{n-1} F_r(kT)\right] \left(\sum_{r=1}^{n-1} F_{max}^r\right)^{-1} 2D_\delta. \end{aligned} \tag{45}$$

It is known that the disturbance is upper bounded by d_{max}, the amount of stored goods $y[(k-1)T]$ is non-negative and $F_r(kT) \geq F_{min}^r$ for each r. Therefore, relation (45) gives

$$y(kT) > -d_{max} + 2D_\delta \left(\sum_{r=1}^{n-1} F_{min}^r\right) \left(\sum_{r=1}^{n-1} F_{max}^r\right)^{-1}. \tag{46}$$

Substitution of D_δ from (28) into (46) yields

$$y(kT) > -d_{max} + d_{max}\left(\sum_{r=1}^{n-1} F^r_{min}\right)\left(\sum_{r=1}^{n-1} F^r_{max}\right)^{-1} + u_{max}\left(\sum_{r=1}^{n-1} F^r_{max} - F^r_{min}\right) \times \left(\sum_{r=1}^{n-1} F^r_{min}\right)\left(\sum_{r=1}^{n-1} F^r_{max}\right)^{-1}. \quad (47)$$

Further substituting F^r_{min} from (24) into (47) gives

$$y(kT) > -d_{max} + d_{max}\left(\sum_{r=1}^{n-1} F^r_{min}\right)\left(\sum_{r=1}^{n-1} F^r_{max}\right)^{-1} + \left(\sum_{r=1}^{n-1} f^i_{min}u^i_{max}\right)\left(\sum_{r=1}^{n-1} F^r_{max} - F^r_{min}\right) \times \left(\sum_{r=1}^{n-1} F^r_{max}\right)^{-1}. \quad (48)$$

Finally, substitution of (19) into (48) results in

$$y(kT) > -d_{max} + d_{max}\left(\sum_{r=1}^{n-1} F^r_{min}\right)\left(\sum_{r=1}^{n-1} F^r_{max}\right)^{-1} + d_{max}\left(\sum_{r=1}^{n-1} F^r_{max} - F^r_{min}\right)\left(\sum_{r=1}^{n-1} F^r_{max}\right)^{-1} = -d_{max} + d_{max}\left(\sum_{r=1}^{n-1} F^r_{max}\right)\left(\sum_{r=1}^{n-1} F^r_{max}\right)^{-1} = 0. \quad (49)$$

Therefore, after the initial $\max(T_i/T)$ steps, the amount of stored goods is strictly positive, which implies that customers' demand is fully satisfied. ■

Finally, it will be demonstrated that the proposed strategy imposes an upper bound on the system output, thus ensuring that the result is achieved with limited storage space.

Theorem 6: The amount of stored goods $y(kT)$ is always smaller than $y_d + \beta$.

Proof: Theorems 1 and 2 imply that the variable $s(kT)$ is always greater than $-\beta$. Consequently, relation (29) gives

$$y_d - y(kT) - \sum_{i=2}^{n} c_i u[(k-n+i-1)T] \geq -\beta. \quad (50)$$

Theorem 3 further states that $u(kT)$ is always non-negative. Therefore

$$y(kT) \leq y_d + \beta - \sum_{i=2}^{n} c_i u[(k-n+i-1)T] \leq y_d + \beta. \quad (51)$$

■

TABLE 1. Characteristics of the suppliers.

i	u^i_{max}	T_i	f^i_{min}	f^i_{max}
Supplier 1	20	1	0.9	1
Supplier 2	25	1	0.9	0.94
Supplier 3	25	2	0.84	0.9
Supplier 4	30	2	0.8	1
Supplier 5	50	4	0.75	0.9

IV. SIMULATION RESULTS

The proposed strategy will now be applied to an example inventory management system with five suppliers and its effectiveness will be demonstrated in the presence of an unpredictable, rapidly changing consumers' demand. Characteristics of the suppliers are presented in Table 1. The sampling period T is assumed to be 1 day and the simulation encompasses 6 months (180 days). The available warehouse can contain 600 tons of the product. The consumers' demand is assumed to change according to the following formula

$$d(kT) = 60 + 60 \cdot (-1)^{\lfloor 1+k/30 \rfloor} \quad (52)$$

with maximum demand being $d_{max} = 120$. The commodity loss coefficient for i-th supplier changes in the following way

$$f_i(kT) = \frac{f^i_{max} + f^i_{min}}{2} + \frac{f^i_{max} - f^i_{min}}{2} \cdot (-1)^{\lfloor k/30 \rfloor}. \quad (53)$$

TABLE 2. Supplier groups.

r	F^r_{min}	F^r_{max}
Group 1	0.27	0.29
Group 2	0.3	0.35
Group 3	0	0
Group 4	0.25	0.3

In other words, relations (52) and (53) imply that every month the unpredictable losses and consumers' demand shift between the two extreme values. Suppliers with the same delivery time are now grouped together according to their delivery times and coefficients (24) for each group are displayed in Table 2. Since the maximum delivery time is 4, matrix $\boldsymbol{A}$ defined in (25) is 5×5 dimensional. Vector $\boldsymbol{c}$ is calculated according to (31) and equals

$$\boldsymbol{c}^{\mathrm{T}} = [1 \quad 0.3 \quad 0.3 \quad 0.65 \quad 1]. \quad (54)$$

Next, coefficients D_{avg} and D_δ are obtained from (28) and

$$D_{avg} = -D_\delta = -\frac{d_{max}}{2} - \frac{u_{max}}{2}\sum_{r=1}^{n-1}(F^r_{max} - F^r_{min}) = -69. \quad (55)$$

Design parameter s_0 is determined according to (38) and equals 72 tons. Consequently, the width of the band (10) is $\beta = 128.3551$ tons. Theorem 5 states that the demand value y_d must be greater than 465.8551 tons. On the other hand, Theorem 6 states that the warehouse capacity must be no lesser than $y_d + \beta$. In conclusion, since the available

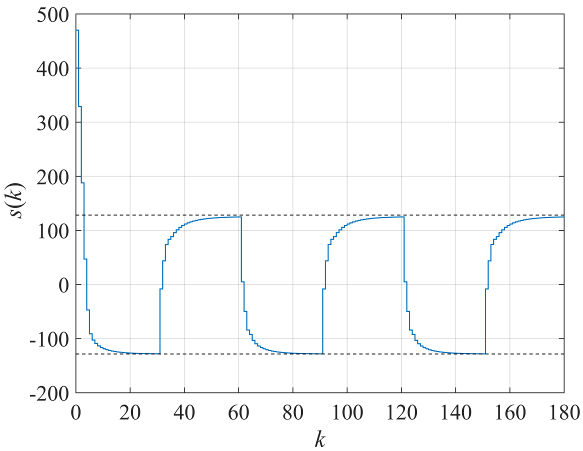

FIGURE 1. Sliding variable.

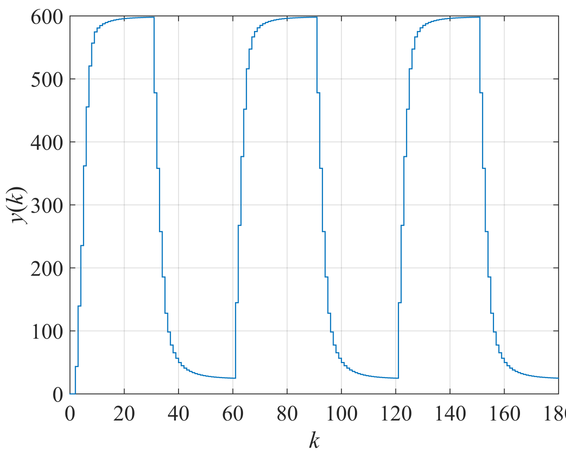

FIGURE 2. Amount of stored goods.

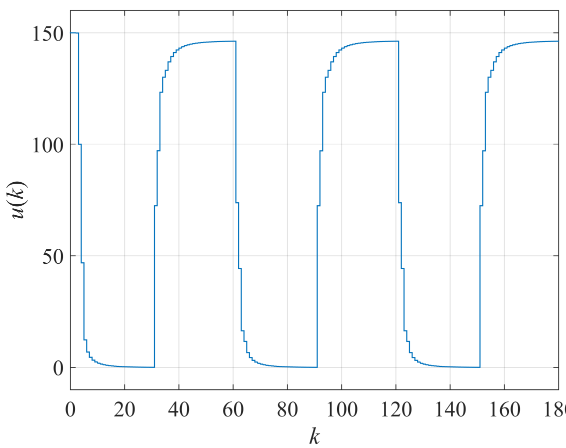

FIGURE 3. Amount of ordered goods.

warehouse capacity is 600 tons, the parameter y_d is set to 470 tons. Figure 1 shows the evolution of the sliding variable, Figure 2 depicts the amount of goods stored in the warehouse and Figure 3 shows the amount of ordered goods. Figure 1 demonstrates that the sliding variable quickly enters the band (10) illustrated by black dashed lines and remains confined to the band in all subsequent steps. It can further be seen from Figure 2 that the warehouse capacity is never exceeded and that, after initial deliveries are completed, the warehouse is never empty. Finally, Figure 3 demonstrates that the amount of ordered goods is always non-negative and never exceeds the capabilities of the suppliers.

V. CONCLUSIONS

A new reaching law based sliding mode control strategy has been proposed and applied to an inventory management system. The strategy for such plants must abide by several constraints. First of all, the control signal must never require the suppliers to send a larger shipment than they are capable of. Naturally, the signal also cannot be negative, as sending goods back to the suppliers is not feasible. Another important constraint is the warehouse capacity, which must never be exceeded. Finally, at any moment the warehouse stock level should be large enough to satisfy consumers' demand. In this paper, all of the properties described above have been formally proven. Simulation results further demonstrate the effectiveness of the proposed method. The reaching law approach - as proposed in this paper - can be seen as a solution alternative to the application of time-varying sliding hyperplanes [28]–[30] in variable structure systems.

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ANDRZEJ BARTOSZEWICZ received the Ph.D. degree in 1993. He is currently a Professor with the Łódź University of Technology, the Head of the Electric Drive and Industrial Automation Unit, and a Director of the Institute of Automatic Control. He has authored three monographs and over 300 papers, primarily in the field of discrete time sliding mode control and inventory management.

PAWEŁ LATOSIŃSKI received the M.Sc. degree in applied mathematics in 2013. He is currently pursuing the Ph.D. degree with the Łódź University of Technology. His main area of interest is sliding mode control of discrete time systems.