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Asymptotic Mean and Variance of Gini Correlation Under Contaminated Gaussian Model

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ABSTRACT This paper establishes the asymptotic closed forms of the expectation and variance of the Gini correlation (GC) under a particular type of bivariate contaminated Gaussian model emulating a frequently encountered scenario in statistical signal processing. Monte Carlo simulation results verify the correctness of the theoretical results established in this paper. In order to gain further insight into GC, we also compare GC to Pearson's product moment correlation coefficient, Kendall's tau, and Spearman's rho by means of root mean squared error. The newly explored theoretical and simulational findings not only deepen the understanding of the rather new GC, but also shed new light on the topic of correlation theory, which is widely applied in statistical signal processing.

INDEX TERMS Contaminated Gaussian model (CGM), correlation coefficient, Gini correlation (GC), Pearson's product moment correlation coefficient (PPMCC).

I. INTRODUCTION

Correlation coefficients have been the most popular statistics to quantify the strength of statistical relationship between random variables (signals/images) in many sub-areas of signal/image processing [1]–[5]. Among a multitude of methods prevailing in the literature, Pearson's product moment correlation coefficient (PPMCC), Kendall's tau (KT) and Spearman's rho (SR) are perhaps the most widely utilized [6]. PPMCC is appropriate mainly for quantifying linear associations, while KT and SR are invariant under increasing monotone transformations, thus often considered as robust alternatives to PPMCC. Based on Cauchy-Schwarz inequality, Daniels proposed a generalized correlation coefficient which embraces PPMCC, KT and SR as particular cases [7]. Besides these three conventional coefficients, other correlation coefficients based on order statistics have also been proposed, such as Gini correlation (GC) [8] and order statistics correlation coefficient (OSCC) [9]–[11]. Recently, Xu *et al.* have shown that OSCC, GC and SR can be linked together by another generalized correlation coefficient under various combinations of variates and ranks in its definition [12].

It is well known that PPMCC is an *optimal* estimator of the population correlation coefficient in the sense of unbiasedness and approaching the Cramer-Rao lower bound for large samples under bivariate normal model (BNM) [13]. On the other hand, GC, KT and SR are only suboptimal under BNM. By deriving the exact bounds of asymptotic relative efficiency (ARE) to PPMCC, Xu *et al.* have shown that GC outperforms KT outperforms SR in terms of ARE under BNM [12], [14]. Despite its optimality in BNM, empirical evidences have shown that PPMCC performs poorly in the following two scenarios [12], [15]:

1) the underlying data follows BNM, but one variable is attenuated by some monotone nonlinearity in the transfer characteristics of electronic devices [16]; and
2) the majority of the data follows BNM, but one variable is corrupted by a tiny fraction of outliers with very large variance (impulsive noise) [17]–[19].

These two scenarios are frequently encountered in radar and communication when measuring the intensity of association between a prescribed ''clean'' signal and a distorted version probably attenuated by the presence of receiver nonlinearity and/or environmental impulsive noise [20].

To account for these scenarios, one might stick to the conventional strategy, that is, ranking the cardinal variable(s) and employing afterwards the rank-based SR or KT, which are robust against both nonlinearity and impulsive noise [14]. However, using only ranks of the two variables, we unavoidably lose information embedded in the variates of the clean variable [8]. A better strategy would be to resort to GC [8], which makes full use of ordinal and cardinal information jointly provided by the two variables. Being invariant under monotone nonlinear transforms [8] and mathematically tractable under BNM [12], [28], GC has been shown to be an appropriate choice in Scenario 1 mentioned above [12], [15]. Now the question is, under a reasonable model emulating Scenario 2, does GC still possess mathematical tractability and a performance higher than KT and SR as in BNM and hence Scenario 1? Our purpose in this work is thus to answer this question, in both theoretical and empirical manners.

The organization of the rest part is arranged as follows. Section II lays the foundation of this work by presenting the definition of GC, a particular type of contaminated Gaussian model (CGM) that simulates Scenario 2, and two auxiliary lemmas which are mandatory for further theoretical developments. Section III derives the asymptotic closed form formulas concerning the mean and variance of GC under the specified CGM. In Section IV we verify our theoretical findings via Monte Carlo simulations. Finally, we conclude this paper by summarizing and our main findings of Gini correlation as well as the other three coefficients in Section V.

For convenience of following discussions, throughout we employ symbols $\mathbb{E}(\cdot)$, $\mathbb{V}(\cdot)$, $\mathbb{C}(\cdot,\cdot)$ and $\mathrm{corr}(\cdot,\cdot)$ to denote the mean, variance, covariance and correlation of (between) random variables, respectively. Symbols of $\mathcal{N}(\mu,\sigma^2)$ and $\mathcal{N}(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho)$ represent univariate and bivariate normal distributions, respectively. The sign $\simeq$ reads "is approximately equals to", whereas the sign $\triangleq$ stands for "is defined as". The notation $u(t)=\mathcal{O}(v(t))$, $t\to L$ (might be infinite), denotes that $|u(t)/v(t)|$ remains bounded as $t\to L$ [21]. All other notation is to be defined where it first occurs.

II. DEFINITIONS AND AUXILIARY RESULTS

In this section, we first present the definition of GC in terms of ranks and variates which, while equivalent to the original version proposed in [8], is more convenient for later analyses in this work. We then construct a particular type of CGM which simulates Scenario 2 remarked in the previous section. We also formulate two auxiliary lemmas based on which the major results in Theorem 1 are established.

A. DEFINITION OF GC

Let $\{(X_i,Y_i)\}_{i=1}^n$ denote n data pairs drawn from some continuous bivariate population. Rearranging $\{X_i\}_{i=1}^n$ in ascending order yields a new sequence of $X_{(1)}<\cdots<X_{(n)}$, which is termed the order statistics of X [22]–[24]. Suppose that X_j is at the kth position in the sorted sequence $\{X_{(i)}\}_{i=1}^n$, the number $k\in[1\quad n]$ is termed the rank of X_j and is denoted by P_j. Similarly we can also define the rank of Y_j which is denoted by Q_j. Then, as shown in [12], the Gini correlation with respect to $\{(X_i,Y_i)\}_{i=1}^n$ can be defined as

$$r_G(Y,X)\triangleq\frac{\frac{1}{n(n-1)}\sum_{i=1}^{n}(2P_i-1-n)Y_i}{\frac{1}{n(n-1)}\sum_{i=1}^{n}(2Q_i-1-n)Y_i}. \tag{1}$$

Swapping X and Y as well as P and Q in (1) gives the other version of $r_G(X,Y)$. In general $r_G(X,Y)\neq r_G(Y,X)$. The choice between $r_G(X,Y)$ and $r_G(Y,X)$ depends on different roles played by X and Y in Scenario 2.

B. CONTAMINATED GAUSSIAN MODEL

To simulate Scenario 2 mentioned in Section I, throughout this work we set the pdf of X and Y as follow CGM [25]

$$(1-\varepsilon)\mathcal{N}(\mu_x,\mu_y,\sigma_x^2,\sigma_y^2,\rho)+\varepsilon\mathcal{N}(\mu_x,\mu_y,\sigma_x'^2,\sigma_y'^2,\rho') \tag{2}$$

where $0<\varepsilon\ll 1$, $\sigma_y'=\sigma_y$ and $\sigma_x'\to\infty$. By such setup, the marginal distribution of Y is $\mathcal{N}(\mu_y,\sigma_y^2)$, representing the impulsive-noise-free (clean) variable; whereas the marginal distribution of X is $(1-\varepsilon)\mathcal{N}(\mu_x,\sigma_x^2)+\varepsilon\mathcal{N}(\mu_x,\sigma_x'^2)$, emulating a normal variable corrupted with a tiny fraction ε of impulsive noise whose variance σ_x' is very large ($\to\infty$). Under this configuration, the parameter ρ is of interest we seek to estimate, while the parameter ρ' is considered the undesirable interference. Our purpose throughout is then to investigate the influnce of both ε and ρ' on the performances of GC, KT, SR and PPMCC when estimating the value of ρ.

C. AUXILIARY RESULTS

Lemma 1: Let $[W_1\ W_2\ W_3\ W_4]^\mathsf{T}$ be a quadrivariate normal random vector with $\mathbb{E}(W_r)=0$, $\mathbb{V}(W_r)=\sigma_r^2$, and $\mathrm{corr}(W_r,W_s)=\varrho_{rs}$ for $r,s=1,\ldots,4$. Write $H(t)=1$ for $t>0$ and $H(t)=0$ for $t\le 0$. Then

$$\begin{aligned}\mathcal{I}&\triangleq\mathbb{E}\{H(W_1)H(W_2)W_3W_4\}\\&=\frac{1}{2\pi}\frac{\sigma_3\sigma_4}{\sqrt{1-\varrho_{12}^2}}\\&\quad\times[\varrho_{13}\varrho_{24}+\varrho_{14}\varrho_{23}-\varrho_{12}(\varrho_{13}\varrho_{14}+\varrho_{23}\varrho_{24})]\\&\quad+\sigma_3\sigma_4\varrho_{34}\left(\frac{1}{4}+\frac{\sin^{-1}\varrho_{12}}{2\pi}\right)\end{aligned} \tag{3}$$

$$\mathcal{K}\triangleq\mathbb{E}\{H(W_1)W_2W_3\}=\frac{\sigma_2\sigma_3\varrho_{23}}{2} \tag{4}$$

and

$$\mathcal{L}\triangleq\mathbb{E}\{H(W_1)W_2\}=\frac{\sigma_2\varrho_{12}}{\sqrt{2\pi}}. \tag{5}$$

Proof: See Appendix I. ■

Lemma 2: Let $\{[\xi_i^1\ \xi_i^2\ \xi_i^3\ \xi_i^4]^\mathsf{T}\}_{i=1}^{n_1}$ be n_1 i.i.d. quadruples drawn from a quadrivariate normal population $\mathcal{N}(\mathbf{0},\boldsymbol{\Sigma})$ with $\mathbf{0}\triangleq[0\ 0\ 0\ 0]^\mathsf{T}$, $\boldsymbol{\Sigma}\triangleq(\varsigma_r\varsigma_s\eta_{rs})_{4\times4}$, $\varsigma_r^2\triangleq\mathbb{V}(\xi_i^r)$ and $\eta_{rs}\triangleq\mathrm{corr}(\xi_i^r,\xi_i^s)$ for $r,s=1,\ldots,4$. Let $\{[\zeta_j^1\ \zeta_j^2\ \zeta_j^3\ \zeta_j^4]^\mathsf{T}\}_{j=1}^{n_2}$ be n_2 i.i.d. quadruples drawn from another quadrivariate normal population $\mathcal{N}(\mathbf{0},\boldsymbol{\Sigma}')$ with $\boldsymbol{\Sigma}'\triangleq(\varsigma_r'\varsigma_s'\eta_{rs}')_{4\times4}$, $\varsigma_r'^2\triangleq\mathbb{V}(\zeta_i^r)$

and $\eta'_{rs} \triangleq \mathrm{corr}(\zeta_i^r, \zeta_i^s)$ for $r, s = 1, \dots, 4$. Assume that the ξ- and ζ-vectors are mutually independent. Let $n = n_1 + n_2$. Denote by $\{[Z_k^1\ Z_k^2\ Z_k^3\ Z_k^4]^\intercal\}_{k=1}^n$ the union of $\{[\xi_i^1\ \xi_i^2\ \xi_i^3\ \xi_i^4]^\intercal\}_{i=1}^{n_1}$ and $\{[\zeta_j^1\ \zeta_j^2\ \zeta_j^3\ \zeta_j^4]^\intercal\}_{j=1}^{n_2}$. Let $\tilde{Z}_k^r$ denote respectively the ranks of Z_k^r, $k = 1, \dots, n$, for $r = 1, \dots, 4$. Write $n^{[m]} \triangleq n(n-1)\cdots(n-m+1)$ and $\lambda_p \triangleq \varsigma_p^2 + \varsigma_p'^2$ for $p = 1, 2$. Define

$$U \triangleq \frac{1}{n(n-1)} \sum_{k=1}^{n} (2\tilde{Z}_k^1 - 1 - n) Z_k^3 \tag{6}$$

and

$$V \triangleq \frac{1}{n(n-1)} \sum_{k=1}^{n} (2\tilde{Z}_k^2 - 1 - n) Z_k^4. \tag{7}$$

Then

$$\mathbb{E}(U) = \frac{1}{\sqrt{\pi}} \frac{2}{n^{[2]}} \left(n_1^{[2]} \frac{\varsigma_3 \eta_{13}}{2} + n_1 n_2 \frac{\varsigma_1 \varsigma_3 \eta_{13}}{\sqrt{2\lambda_1}} + n_1 n_2 \frac{\varsigma_1' \varsigma_3' \eta_{13}'}{\sqrt{2\lambda_1}} + n_2^{[2]} \frac{\varsigma_3' \eta_{13}'}{2} \right) \tag{8}$$

$$\mathbb{E}(V) = \frac{1}{\sqrt{\pi}} \frac{2}{n^{[2]}} \left(n_1^{[2]} \frac{\varsigma_4 \eta_{24}}{2} + n_1 n_2 \frac{\varsigma_2 \varsigma_4 \eta_{24}}{\sqrt{2\lambda_2}} + n_1 n_2 \frac{\varsigma_2' \varsigma_4' \eta_{24}'}{\sqrt{2\lambda_2}} + n_2^{[2]} \frac{\varsigma_2' \eta_{24}'}{2} \right) \tag{9}$$

and

$$\mathbb{C}(U, V) = \frac{4A - 2(n-1)B + (n-1)^2 C - D}{n^2 (n-1)^2} \tag{10}$$

where

$$D = n^2 (n-1)^2 \mathbb{E}(U) \mathbb{E}(V) \tag{11}$$

$$C = n_1 \varsigma_3 \varsigma_4 \eta_{34} + n_2 \varsigma_3' \varsigma_4' \eta_{34}' \tag{12}$$

$$B = (n-1) \left(n_1 \varsigma_3 \varsigma_4 \eta_{34} + n_2 \varsigma_3' \varsigma_4' \eta_{34}' \right) \tag{13}$$

$$A = \sum_{\ell=1}^{28} [\alpha_\ell(n_1, n_2) \mathcal{I}_\ell(\varsigma, \varsigma', \eta, \eta') + \alpha_\ell(n_2, n_1) \mathcal{I}_\ell(\varsigma', \varsigma, \eta', \eta)] \tag{14}$$

with α_ℓ being the number of terms (the third column) in Table 3, $\mathcal{I}_\ell$ the quantities obtained upon substitution into (3) of variances and correlation coefficients with respect to $[W_1\ W_2\ W_3\ W_4]^\intercal$ listed in Table 3.

Proof: See Appendix II. ■

III. ASYMPTOTIC MEAN AND VARIANCE OF GC IN CONTAMINATED GAUSSIAN MODEL

This section is devoted to derivations of the expectation and variance of GC for samples generated by CGM (2). For notational compactness, the argument of $r_G(Y, X)$ will be dropped in the sequel unless ambiguity happens.

Theorem 1: Let $\{(X_i, Y_i)\}_{i=1}^n$ be a union of $\{(\mathcal{X}_j, \mathcal{Y}_j)\}_{j=1}^{n_1}$ and $\{(\mathcal{X}'_{j'}, \mathcal{Y}'_{j'})\}_{j'=1}^{n_2}$, the former being n_1 i.i.d. data pairs following $\mathcal{N}(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$, and the latter being n_2 i.i.d. data pairs following $\mathcal{N}(\mu_x, \mu_y, \sigma_x'^2, \sigma_y'^2, \rho')$. Assume that $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{X}', \mathcal{Y}')$ are independent with each other. Assume also that $\sigma_x' \to \infty$. Denote by ε the ratio of n_2/n. Then, for large n and small ε, the expectation and variance of GC defined in (1) with respect to $\{(X_i, Y_i)\}_{i=1}^n$ are, respectively,

$$\begin{aligned}\mathbb{E}(r_G) \simeq{}& (1-2\varepsilon) \\ &\times \left\{ \rho + \frac{1}{n} \left[\left(\frac{\pi}{3} + 2\sqrt{3} \right) \rho - 2 \left(\sin^{-1} \frac{\rho}{2} + \rho \sqrt{4 - \rho^2} \right) \right] \right\} \\ &+ \sqrt{2}\varepsilon \\ &\times \left\{ \rho' + \frac{1}{n} \left[\left(\frac{\pi}{3} + 2\sqrt{3} - 2 \right) \rho' \right. \right. \\ &\left. \left. - \sqrt{2} \left(\sin^{-1} \frac{\rho'}{\sqrt{2}} + \rho' \sqrt{2 - \rho'^2} \right) \right] \right\}\end{aligned} \tag{15}$$

and

$$\begin{aligned}&\mathbb{V}(r_G) \\ &\simeq \frac{1}{n} \left[\frac{\pi}{3} + \left(\frac{\pi}{3} + 4\sqrt{3} \right) \rho^2 - 4\rho \sin^{-1} \frac{\rho}{2} - 4\rho^2 \sqrt{4 - \rho^2} \right] \\ &\quad + \frac{\varepsilon}{n} \left[16\rho^2 \sqrt{4 - \rho^2} + 16\rho \sin^{-1} \frac{\rho}{2} \right. \\ &\quad \left. - \left(\frac{4\pi}{3} + 4 + 14\sqrt{3} \right) \rho^2 \right] \\ &\quad - \frac{4}{n} \varepsilon \left(\rho \sin^{-1} \frac{\rho'}{\sqrt{2}} + \sqrt{2} \rho' \sin^{-1} \frac{\rho}{2} + \frac{\rho'^2}{2} \right) \\ &\quad + \frac{2\sqrt{2}}{n} \varepsilon \rho \rho' \left(\frac{\pi}{3} + 2\sqrt{3} + 2 - 2\sqrt{4 - \rho^2} - \sqrt{4 - 2\rho'^2} \right).\end{aligned} \tag{16}$$

Proof: Since r_G is shift invariant [8], we loss no generality by assuming $\mu_x = \mu_y = 0$ hereafter. Denote by U_0 and V_0 the numerator and denominator of (1), respectively. Then, by the well known *delta method* [26], it follows that

$$\mathbb{E}(r_G) \simeq \frac{\mathbb{E}(U_0)}{\mathbb{E}(V_0)} \left[1 + \frac{\mathbb{V}(V_0)}{\mathbb{E}^2(V_0)} - \frac{\mathbb{C}(U_0, V_0)}{\mathbb{E}(U_0)\mathbb{E}(V_0)} \right] \tag{17}$$

and

$$\mathbb{V}(r_G) \simeq \frac{\mathbb{E}^2(U_0)}{\mathbb{E}^2(V_0)} \left[\frac{\mathbb{V}(U_0)}{\mathbb{E}^2(U_0)} + \frac{\mathbb{V}(V_0)}{\mathbb{E}^2(V_0)} - \frac{2\mathbb{C}(U_0, V_0)}{\mathbb{E}(U_0)\mathbb{E}(V_0)} \right]. \tag{18}$$

To work out (17) and (18), it suffices to find $\mathbb{E}(U_0)$, $\mathbb{E}(V_0)$, $\mathbb{V}(U_0)$, $\mathbb{V}(V_0)$ and $\mathbb{C}(U_0, V_0)$. It is easy to verify that U_0 and V_0 are two particular cases with ξ- and ζ-terms of

$$U : \begin{cases} [\xi^1\ \xi^2\ \xi^3\ \xi^4]^\intercal = [\mathcal{X}\ \mathcal{Y}\ \mathcal{Y}\ \mathcal{Y}]^\intercal \\ [\zeta^1\ \zeta^2\ \zeta^3\ \zeta^4]^\intercal = [\mathcal{X}'\ \mathcal{Y}'\ \mathcal{Y}'\ \mathcal{Y}']^\intercal \end{cases}$$

and

$$V : \begin{cases} [\xi^1\ \xi^2\ \xi^3\ \xi^4]^\intercal = [\mathcal{Y}\ \mathcal{Y}\ \mathcal{Y}\ \mathcal{Y}]^\intercal \\ [\zeta^1\ \zeta^2\ \zeta^3\ \zeta^4]^\intercal = [\mathcal{Y}'\ \mathcal{Y}'\ \mathcal{Y}'\ \mathcal{Y}']^\intercal \end{cases}$$

in Lemma 2, respectively. Substituting the associated variance and correlation coefficient terms

$$\begin{cases} U_0 : \varsigma_1 = \sigma_x, \varsigma_2 = \varsigma_3 = \varsigma_4 = \sigma_y, \eta_{13} = \rho\ , \eta_{24} = 1 \\ V_0 : \varsigma_1' = \sigma_x', \varsigma_2' = \varsigma_3' = \varsigma_4' = \sigma_y', \eta_{13}' = \rho', \eta_{24}' = 1 \end{cases}$$

into (8) and (9) yields

$$\mathbb{E}(U_0) = \frac{1}{\sqrt{\pi}}\frac{2}{n^{[2]}}\left[\frac{n_1^{[2]}\sigma_y\rho}{2} + \frac{n_2^{[2]}\sigma_y'\rho'}{2} + \frac{n_1n_2\sigma_x\sigma_y\rho}{\sqrt{2(\sigma_x^2+\sigma_x'^2)}} + \frac{n_1n_2\sigma_x'\sigma_y'\rho'}{\sqrt{2(\sigma_x^2+\sigma_x'^2)}}\right]$$

and

$$\mathbb{E}(V_0) = \frac{1}{\sqrt{\pi}}\frac{2}{n^{[2]}}\left[\frac{n_1^{[2]}\sigma_y}{2} + \frac{n_2^{[2]}\sigma_y'}{2} + \frac{n_1n_2\sigma_y^2}{\sqrt{2(\sigma_y^2+\sigma_y'^2)}} + \frac{n_1n_2\sigma_y'^2}{\sqrt{2(\sigma_y^2+\sigma_y'^2)}}\right]$$

which degenerate respectively to

$$\mathbb{E}(U_0) = \frac{\sigma_y}{\sqrt{\pi}}\frac{2}{n^{[2]}}\left(\frac{n_1^{[2]}\rho}{2} + \frac{n_1n_2\rho'}{\sqrt{2}} + \frac{n_2^{[2]}\rho'}{2}\right) \tag{19}$$

and

$$\mathbb{E}(V_0) = \frac{\sigma_y}{\sqrt{\pi}} \tag{20}$$

by letting $\sigma_y = \sigma_y'$ and $\sigma_x' \to \infty$.

Now we proceed to carrying out the second moments. Write $Z^1 = Z^2 = X$ and $Z^3 = Z^4 = Y$. Then $\mathbb{V}(U_0) = \mathbb{C}(U_0, U_0)$ becomes a particular case of $\mathbb{C}(U, V)$ in Lemma 2 with

$$U \text{ and } V : \begin{cases}[\xi^1\ \xi^2\ \xi^3\ \xi^4]^\intercal = [\mathcal{X}\ \ \mathcal{X}\ \ \mathcal{Y}\ \ \mathcal{Y}\ ]^\intercal \\ [\zeta^1\ \zeta^2\ \zeta^3\ \zeta^4]^\intercal = [\mathcal{X}'\ \mathcal{X}'\ \mathcal{Y}'\ \mathcal{Y}']^\intercal\end{cases}$$

where the associated variance and correlation coefficient terms are

$$\begin{cases}\varsigma_1 = \varsigma_2 = \sigma_x, & \varsigma_3 = \varsigma_4 = \sigma_y \\ \varsigma_1' = \varsigma_2' = \sigma_x', & \varsigma_3' = \varsigma_4' = \sigma_y' \\ \eta_{12} = \eta_{34} = 1, & \eta_{13} = \eta_{14} = \eta_{23} = \eta_{24} = \rho \\ \eta_{12}' = \eta_{34}' = 1, & \eta_{13}' = \eta_{14}' = \eta_{23}' = \eta_{24}' = \rho'.\end{cases}$$

Substituting these parameters into (10) and letting $\sigma_y = \sigma_y'$ and $\sigma_x' \to \infty$ thereafter, we have

$$\begin{aligned}\mathbb{V}(U_0) = \frac{\sigma_y^2}{n^2(n-1)^2}\Bigg\{&\frac{n(n^2-1)}{3} \\ &+ \frac{2\rho^2}{\pi}\left[\sqrt{3}n_1^{[3]} - n_1^{[2]}(2n_1-3)\right] \\ &+ \frac{2\rho'^2}{\pi}\Big[\sqrt{3}n_2^{[3]} - 2\sqrt{2}n_1n_2^{[2]} \\ &- n_2\left(n_1^2 - 2n_1n_2 + 2n_2^2 + 2n_1 - 5n_2 + 3\right)\Big]\Bigg\}.\end{aligned} \tag{21}$$

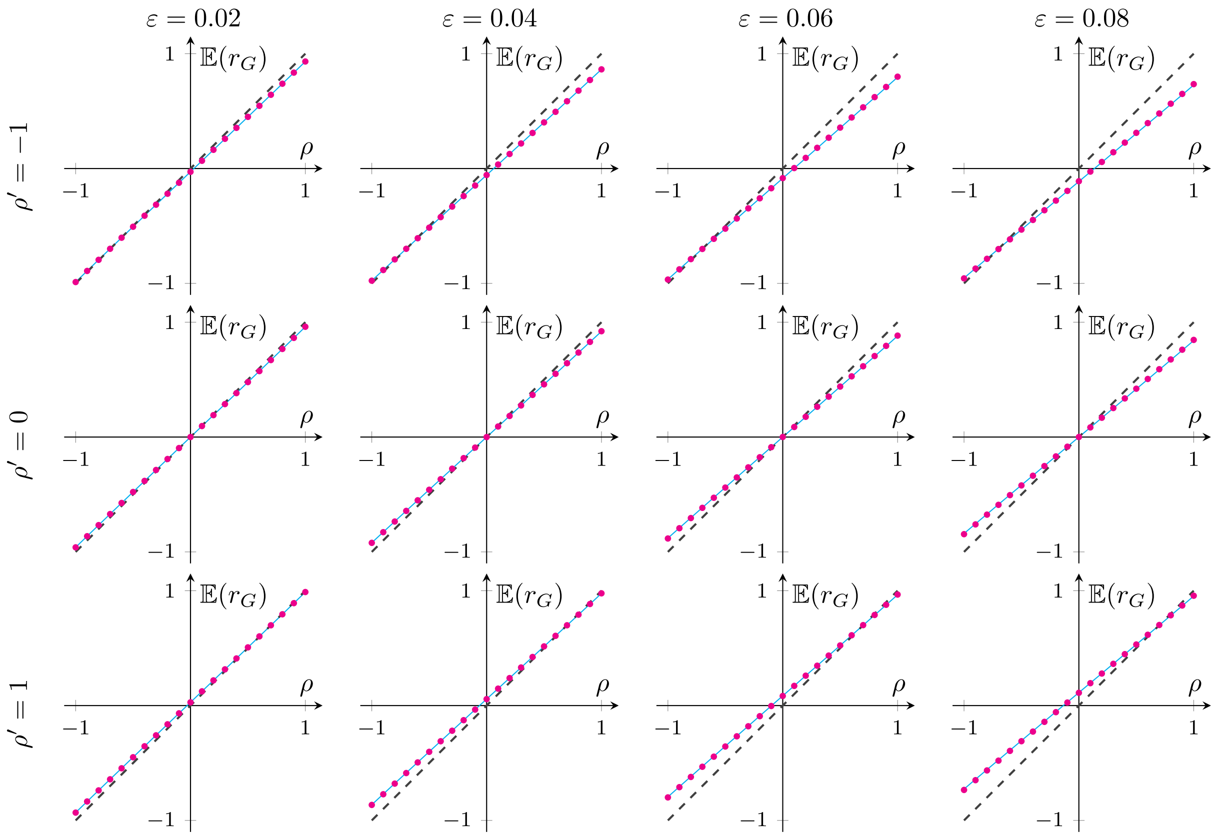

FIGURE 1. Verification of (15) concerning $\mathbb{E}(r_G)$ in Theorem 1 for $n = 100$. From top to bottom, each row corresponds to a different $\rho' \in \{-1, 0, 1\}$, respectively; whereas from left to right, each column corresponds to a different $\varepsilon \in \{0.02, 0.04, 0.06, 0.08\}$, respectively. Good agreements are observed between simulation results (circles) and corresponding theoretical counterparts (solid lines). For comparison, the contamination-free version (25) is also included in each subplot (see dashed curves).

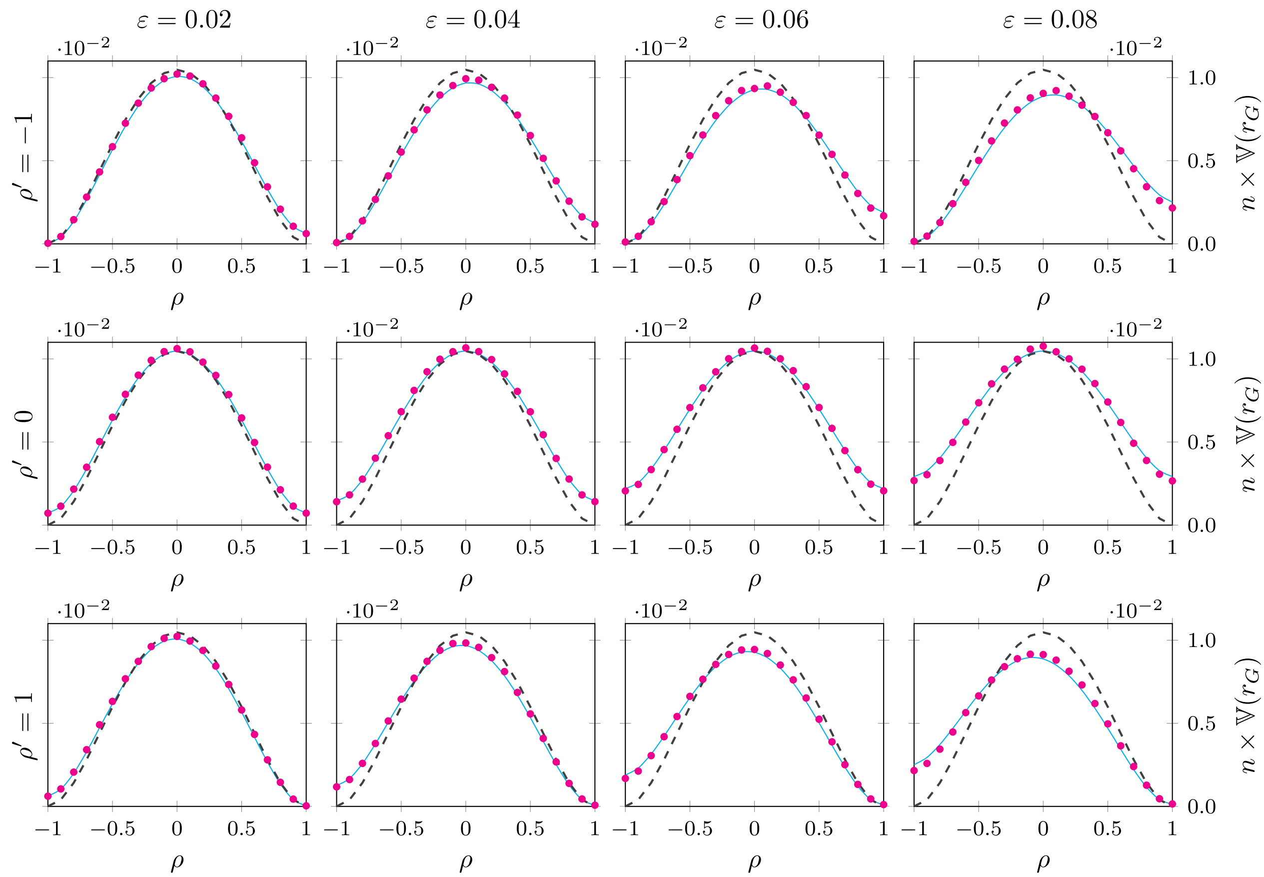

FIGURE 2. Verification of (16) concerning $\mathbb{V}(r_G)$ in Theorem 1 for $n = 100$. From top to bottom, each row corresponds to a different $\rho' \in \{-1, 0, 1\}$, respectively; whereas from left to right, each column corresponds to a different $\varepsilon \in \{0.02, 0.04, 0.06, 0.08\}$, respectively. For a better visual effect, all variances are scaled up by a factor of n. Good agreements are observed between simulation results (circles) and theoretical counterparts (solid lines). For comparison, the contamination-free version (26) is also included in each subplot (see dashed curves).

In a similar way we also obtain

$$\mathbb{V}(V_0) = \frac{\sigma_y^2}{n^{[2]}}\left[\frac{n+1}{3} + \frac{2(n-2)\sqrt{3}}{\pi} - \frac{2(2n-3)}{\pi}\right] \tag{22}$$

as well as

$$\mathbb{C}(U_0, V_0) = \frac{\sum_{k=1}^{9} \mathcal{C}_k}{n^2(n-1)^2} \tag{23}$$

where

$$\mathcal{C}_1 = \frac{2\sigma_y^2}{\pi} n_1^{[2]}\left[\rho\sqrt{1-\rho^2} + \sin^{-1}\rho\right]$$

$$\mathcal{C}_2 = \frac{2\sigma_y^2}{\pi} n_1^{[2]}(n-2)\left[\rho\sqrt{4-\rho^2} + \sin^{-1}\frac{\rho}{2}\right]$$

$$\mathcal{C}_3 = -\frac{2\sigma_y^2}{\pi} n_1^{[2]}(2n-3)\rho$$

$$\mathcal{C}_4 = \frac{2\sigma_y^2}{\pi} n_2^{[2]}\left(\rho'\sqrt{1-\rho'^2} + \sin^{-1}\rho'\right)$$

$$\mathcal{C}_5 = \frac{2\sigma_y^2}{\pi} n_2^{[2]}(n-2)\left(\rho'\sqrt{4-\rho'^2} + \sin^{-1}\frac{\rho'}{2}\right)$$

$$\mathcal{C}_6 = -\frac{2\sigma_y^2}{\pi} n_2^{[2]}(2n-3)\rho'$$

$$\mathcal{C}_7 = -\frac{2\sqrt{2}\sigma_y^2}{\pi} n_1 n_2 (n-1)\rho'$$

$$\mathcal{C}_8 = \frac{2\sigma_y^2}{\pi} n_1 n_2 (n-1)\rho'\sqrt{2-\rho'^2}$$

$$\mathcal{C}_9 = \frac{2\sigma_y^2}{\pi} n n_1 n_2 \sin^{-1}\frac{\rho'}{\sqrt{2}}.$$

Substituting (19)–(23) along with $\varepsilon = n_2/n$ into (17) and (18), tidying up, and omitting $\mathcal{O}(n^{-2})$ and $\mathcal{O}(\varepsilon^2)$ terms thereafter, we finally arrive at (15) and (16), respectively. Hence the theorem holds true. ■

Remark 1: Letting $n \to \infty$ in (15), $\mathbb{E}(r_G)$ simplifies to a neater form of

$$\mathbb{E}(r_G) = (1-2\varepsilon)\rho + \sqrt{2}\varepsilon\rho'. \tag{24}$$

Moreover, as $\varepsilon \to 0$, (15) and (16) reduce respectively to

$$\mathbb{E}(r_G) \simeq \rho + \frac{1}{n}\left[\left(\frac{\pi}{3} + 2\sqrt{3}\right)\rho - 2\left(\rho\sqrt{4-\rho^2} + \sin^{-1}\frac{\rho}{2}\right)\right] \tag{25}$$

TABLE 1. RMSE of four estimators for $n = 100$, $\varepsilon = \{0.02, 0.04, 0.06, 0.08\}$ and $\rho' = \{-1, 0\}$.

	$\varepsilon = 0.02$				$\varepsilon = 0.04$				$\varepsilon = 0.06$				$\varepsilon = 0.08$			
ρ	$\hat{\rho}_G$	$\hat{\rho}_S$	$\hat{\rho}_K$	$\hat{\rho}_P$	$\hat{\rho}_G$	$\hat{\rho}_S$	$\hat{\rho}_K$	$\hat{\rho}_P$	$\hat{\rho}_G$	$\hat{\rho}_S$	$\hat{\rho}_K$	$\hat{\rho}_P$	$\hat{\rho}_G$	$\hat{\rho}_S$	$\hat{\rho}_K$	$\hat{\rho}_P$
-1.0	0.0131	0.0111	0.0007	0.8752	0.0244	0.0202	0.0023	0.8142	0.0350	0.0288	0.0047	0.7671	0.0452	0.0369	0.0078	0.7279
-0.9	0.0226	0.0255	0.0234	0.7761	0.0262	0.0278	0.0256	0.7139	0.0310	0.0307	0.0288	0.6673	0.0360	0.0336	0.0321	0.6287
-0.8	0.0385	0.0418	0.0404	0.6766	0.0381	0.0409	0.0400	0.6150	0.0382	0.0403	0.0400	0.5686	0.0385	0.0397	0.0399	0.5291
-0.7	0.0531	0.0563	0.0552	0.5769	0.0515	0.0545	0.0536	0.5157	0.0502	0.0531	0.0521	0.4693	0.0491	0.0520	0.0509	0.4299
-0.6	0.0660	0.0689	0.0681	0.4778	0.0643	0.0672	0.0662	0.4166	0.0634	0.0665	0.0652	0.3701	0.0628	0.0665	0.0647	0.3317
-0.5	0.0770	0.0797	0.0792	0.3790	0.0762	0.0791	0.0783	0.3187	0.0760	0.0794	0.0781	0.2726	0.0775	0.0818	0.0801	0.2346
-0.4	0.0859	0.0882	0.0881	0.2813	0.0864	0.0891	0.0887	0.2221	0.0884	0.0920	0.0912	0.1780	0.0920	0.0971	0.0961	0.1422
-0.3	0.0932	0.0954	0.0956	0.1857	0.0953	0.0981	0.0981	0.1300	0.0994	0.1035	0.1034	0.0933	0.1062	0.1120	0.1121	0.0718
-0.2	0.0988	0.1008	0.1014	0.0988	0.1030	0.1060	0.1065	0.0684	0.1101	0.1147	0.1155	0.0763	0.1203	0.1269	0.1282	0.1007
-0.1	0.1025	0.1046	0.1054	0.0704	0.1086	0.1121	0.1131	0.1110	0.1192	0.1245	0.1259	0.1515	0.1334	0.1410	0.1433	0.1877
0.0	0.1050	0.1072	0.1081	0.1417	0.1136	0.1175	0.1186	0.1999	0.1284	0.1344	0.1362	0.2451	0.1461	0.1546	0.1576	0.2829
0.1	0.1061	0.1086	0.1095	0.2349	0.1180	0.1224	0.1234	0.2964	0.1365	0.1431	0.1449	0.3423	0.1586	0.1677	0.1708	0.3807
0.2	0.1050	0.1079	0.1084	0.3320	0.1210	0.1259	0.1264	0.3942	0.1442	0.1515	0.1526	0.4411	0.1708	0.1804	0.1828	0.4795
0.3	0.1031	0.1064	0.1062	0.4306	0.1239	0.1292	0.1285	0.4928	0.1515	0.1591	0.1586	0.5397	0.1830	0.1930	0.1935	0.5788
0.4	0.0991	0.1028	0.1016	0.5293	0.1254	0.1310	0.1283	0.5924	0.1583	0.1658	0.1626	0.6388	0.1942	0.2039	0.2010	0.6780
0.5	0.0943	0.0983	0.0954	0.6282	0.1268	0.1322	0.1263	0.6916	0.1650	0.1721	0.1644	0.7383	0.2066	0.2156	0.2074	0.7774
0.6	0.0884	0.0925	0.0870	0.7280	0.1277	0.1326	0.1216	0.7911	0.1719	0.1779	0.1635	0.8381	0.2182	0.2257	0.2093	0.8774
0.7	0.0820	0.0856	0.0760	0.8274	0.1287	0.1321	0.1137	0.8908	0.1788	0.1831	0.1584	0.9376	0.2301	0.2352	0.2064	0.9771
0.8	0.0764	0.0787	0.0625	0.9271	0.1306	0.1321	0.1021	0.9901	0.1868	0.1882	0.1478	1.0372	0.2429	0.2447	0.1971	1.0767
0.9	0.0727	0.0727	0.0451	1.0269	0.1340	0.1321	0.0826	1.0899	0.1952	0.1923	0.1256	1.1373	0.2554	0.2525	0.1731	1.1762
1.0	0.0723	0.0658	0.0047	1.1261	0.1392	0.1297	0.0179	1.1898	0.2048	0.1933	0.0391	1.2371	0.2693	0.2568	0.0679	1.2765
-1.0	0.0480	0.0431	0.0028	1.0037	0.0873	0.0795	0.0095	1.0047	0.1255	0.1152	0.0195	1.0049	0.1629	0.1507	0.0328	1.0048
-0.9	0.0500	0.0507	0.0341	0.9051	0.0836	0.0820	0.0532	0.9047	0.1170	0.1135	0.0748	0.9054	0.1502	0.1453	0.0985	0.9054
-0.8	0.0571	0.0597	0.0507	0.8050	0.0831	0.0839	0.0677	0.8056	0.1107	0.1102	0.0879	0.8054	0.1396	0.1378	0.1107	0.8062
-0.7	0.0662	0.0696	0.0644	0.7065	0.0850	0.0874	0.0782	0.7070	0.1068	0.1081	0.0952	0.7066	0.1298	0.1303	0.1145	0.7062
-0.6	0.0755	0.0789	0.0760	0.6073	0.0886	0.0915	0.0863	0.6082	0.1045	0.1069	0.0997	0.6080	0.1227	0.1244	0.1154	0.6080
-0.5	0.0833	0.0865	0.0850	0.5095	0.0918	0.0949	0.0923	0.5093	0.1035	0.1064	0.1026	0.5099	0.1164	0.1188	0.1140	0.5100
-0.4	0.0904	0.0932	0.0928	0.4123	0.0960	0.0989	0.0979	0.4120	0.1031	0.1058	0.1042	0.4120	0.1114	0.1140	0.1119	0.4119
-0.3	0.0966	0.0991	0.0992	0.3160	0.0992	0.1018	0.1016	0.3165	0.1033	0.1059	0.1056	0.3168	0.1077	0.1102	0.1097	0.3162
-0.2	0.1001	0.1022	0.1027	0.2240	0.1013	0.1034	0.1039	0.2235	0.1028	0.1051	0.1055	0.2238	0.1054	0.1077	0.1080	0.2241
-0.1	0.1025	0.1047	0.1054	0.1420	0.1025	0.1047	0.1054	0.1416	0.1027	0.1048	0.1055	0.1416	0.1039	0.1060	0.1067	0.1419
0.0	0.1034	0.1053	0.1061	0.1007	0.1031	0.1051	0.1059	0.1004	0.1031	0.1051	0.1060	0.1003	0.1030	0.1049	0.1058	0.1004
0.1	0.1027	0.1048	0.1055	0.1418	0.1027	0.1048	0.1056	0.1417	0.1028	0.1047	0.1055	0.1417	0.1037	0.1058	0.1065	0.1413
0.2	0.0998	0.1021	0.1027	0.2236	0.1012	0.1036	0.1040	0.2238	0.1031	0.1056	0.1060	0.2240	0.1053	0.1077	0.1079	0.2241
0.3	0.0959	0.0984	0.0985	0.3163	0.0985	0.1011	0.1010	0.3160	0.1031	0.1057	0.1053	0.3168	0.1078	0.1104	0.1098	0.3159
0.4	0.0901	0.0930	0.0925	0.4116	0.0957	0.0985	0.0974	0.4124	0.1031	0.1058	0.1042	0.4121	0.1115	0.1142	0.1120	0.4123
0.5	0.0837	0.0869	0.0855	0.5100	0.0923	0.0954	0.0927	0.5096	0.1034	0.1061	0.1023	0.5093	0.1167	0.1190	0.1142	0.5100
0.6	0.0750	0.0782	0.0754	0.6071	0.0882	0.0913	0.0861	0.6078	0.1050	0.1073	0.1000	0.6087	0.1231	0.1247	0.1157	0.6082
0.7	0.0662	0.0696	0.0645	0.7063	0.0852	0.0875	0.0782	0.7069	0.1071	0.1084	0.0955	0.7070	0.1300	0.1304	0.1145	0.7066
0.8	0.0571	0.0597	0.0507	0.8053	0.0833	0.0843	0.0680	0.8056	0.1114	0.1108	0.0885	0.8063	0.1397	0.1379	0.1107	0.8064
0.9	0.0500	0.0507	0.0341	0.9045	0.0834	0.0819	0.0531	0.9051	0.1171	0.1136	0.0750	0.9049	0.1504	0.1452	0.0983	0.9054
1.0	0.0480	0.0431	0.0028	1.0037	0.0871	0.0793	0.0094	1.0038	0.1255	0.1153	0.0195	1.0044	0.1628	0.1506	0.0328	1.0043

In the upper panel are RMSEs for $\rho' = -1$; whereas in the lower panel are RMSEs for $\rho' = 0$. In each of the eight blocks, the minima of RMSE with respect to $\hat{\rho}_G$, $\hat{\rho}_S$, $\hat{\rho}_K$ and $\hat{\rho}_P$ are highlighted with gray areas in a rowwise manner.

and

$$\mathbb{V}(r_G) \simeq \frac{1}{n}\left[\frac{\pi}{3} + \left(\frac{\pi}{3} + 4\sqrt{3}\right)\rho^2 - 4\rho \sin^{-1}\frac{\rho}{2} - 4\rho^2\sqrt{4-\rho^2}\right] \quad (26)$$

which are consistent with the contamination-free versions that have been established in our previous work [12] ((36) and (37) therein).

IV. NUMERICAL RESULTS

In this section we verify the theoretical results established in Theorem 1 and compare the performances among four correlation coefficients by Monte Carlo simulations. Since all analytical results are derived under the assumptions of large n and small ε, we choose the sample size $n = 100$ and contamination fractions $0.02 \leq \varepsilon \leq 0.1$ in the following numerical study. The majority of the samples $\{(\mathcal{X}_i, \mathcal{Y}_i)\}_{i=1}^{n_1}$ are drawn from $\mathcal{N}(0, 0, 1, 1, \rho)$; whereas the minority of contaminating samples $\{(\mathcal{X}_i', \mathcal{Y}_i')\}_{i=1}^{n_2}$ are drawn from $\mathcal{N}(0, 0, 10^8, 1, \rho')$. All bivariate normal samples are generated by the function of `mvnrnd` in Matlab environment. For reason of accuracy, the number of Monte Carlo trials is set to be 10^5.

A. VERIFICATION OF $\mathbb{E}(r_G)$ AND $\mathbb{V}(r_G)$ IN THEOREM 1

Fig. 1 illustrates the simulation results (circles) superimposing on solid theoretical curves with respect to (15) under various circumstances. From top to bottom, each row corresponds to a different $\rho' \in \{-1, 0, 1\}$, respectively; whereas from left to right, each column corresponds to a different $\varepsilon \in \{0.02, 0.04, 0.06, 0.08\}$, respectively. Good agreements are observed between observed values and corresponding theoretical curves. Moreover, we can also observe that both

ε and ρ' contribute to the bias effect between $\mathbb{E}(r_G)$ and the ideal case ($\varepsilon = 0$; dashed lines) governed by (25).

Fig. 2 contains diagrams of the theoretical results (16) (solid lines) superimposed on simulation results (circles) in the same scenarios as in Fig. 1. For comparison, the contamination-free version (26) is also included in each subplot (dashed curves). For a better visual effect, all variances are scaled up by a factor of n. Again, good agreements are observed between simulation results and corresponding theoretical counterparts. It is also observed that, 1) for $|\rho|$ large, $\mathbb{V}(r_G)$ increases if ρ and ρ' have opposite signs; and it decreases if ρ and ρ' have the same signs, 2) ε increases $\mathbb{V}(r_G)$ for $|\rho|$ large, and 3) $\mathbb{V}(r_G)|_{\rho'<0}$ is the reversal of $\mathbb{V}(r_G)|_{\rho'>0}$ when ε is fixed.

B. COMPARISON OF RMSE AMONG FOUR ESTIMATORS

In order to gain further insight into GC, we make a comparison among GC, PPMCC(ρ_P), KT(ρ_K) and SR(ρ_S) by means of root mean squared error (RMSE). For a fair comparison, some transformations of KT and SR are necessary. From [14], when samples are drawn from CGM, $\varepsilon \to 0$ and $n \to \infty$, it follows:

$$\lim_{\substack{\varepsilon\to 0\\ n\to\infty}} \mathbb{E}(r_K) = \frac{2}{\pi}\sin^{-1}\rho \tag{27}$$

$$\lim_{\substack{\varepsilon\to 0\\ n\to\infty}} \mathbb{E}(r_S) = \frac{6}{\pi}\sin^{-1}\frac{\rho}{2}. \tag{28}$$

Inverting these two equations yields two asymptotic unbiased estimators of ρ

$$\hat{\rho}_K \triangleq \sin\left(\frac{\pi}{2}r_K\right)$$
$$\hat{\rho}_S \triangleq 2\sin\left(\frac{\pi}{6}r_S\right)$$

which are termed Fisher consistent versions of KT and SR under the "pure" bivariate normal models (for $\varepsilon = 0$). Since when $\varepsilon = 0$, both $\lim_{n\to\infty}\mathbb{E}(r_G) = \rho$ and $\lim_{n\to\infty}\mathbb{E}(r_P) = \rho$ hold true, their Fisher consistent versions are then defined to be $\hat{\rho}_P \triangleq r_P$ and $\hat{\rho}_G \triangleq r_G$, respectively. Based on these four unbiased estimators of ρ, we proceed to comparing their performances in the following. As customary in statistical signal processing, the RMSE, defined as

$$\mathrm{RMSE} \triangleq \sqrt{\mathbb{E}(\hat{\rho}-\rho)^2}$$

is chosen as our figure of merit in performance comparison.

Table 1 lists the RMSEs with respect to $\hat{\rho}_G$, $\hat{\rho}_S$, $\hat{\rho}_K$ and $\hat{\rho}_P$ for $n = 100$, $\varepsilon \in \{0.02, 0.04, 0.06, 0.08\}$ and $\rho' \in \{-1, 0\}$, respectively. Due to symmetry as well as space constraint, the RMSE values for $\rho' = 1$ are not shown here. Within each of the eight blocks, the minima of RMSE with respect to $\hat{\rho}_G$, $\hat{\rho}_S$, $\hat{\rho}_K$ and $\hat{\rho}_P$ are highlighted with gray areas in a rowwise manner. It appears that 1) the conventional $\hat{\rho}_P$, which has maximal RMSE in most cases, performs far more worst than the other three estimators, 2) except for some rare cases, $\hat{\rho}_G$ outperforms $\hat{\rho}_K$ when $|\rho|$ is of small to medium magnitudes, 3) $\hat{\rho}_K$ outperforms $\hat{\rho}_G$ when $|\rho|$ falls around the neighborhood of 1, and 4) $\hat{\rho}_S$ plays an intermediate role between $\hat{\rho}_G$ and $\hat{\rho}_K$, which is manefested by the phenomenon that its RMSE lies always in between those of $\hat{\rho}_G$ and $\hat{\rho}_K$ in Table 1.

V. CONCLUDING REMARKS

In this paper, we have derived the asymptotic closed form formulas (Theorem 1) concerning the expectation and variance of GC under the specified CGM of (2). This model simulates reasonably the frequently encountered Scenario 2 in radar and communication, where one variable is clean and the other corrupted by a tiny fraction of impulsive noise with very large variance [20]. Theoretical calculations and simulation results suggest that Gini correlation exhibits robust behaviors under the scenario where one channel contains impulsive noise. The mathematical tractability and empirical findings revealed in this work not only deepen the understanding of the rather new Gini correlation, but also shed new light on the topic of correlation analysis, which is widely applied in the area of statistical statistical processing.

APPENDIX I
PROOF OF LEMMA 1

Proof: The first two statements (3) and (4) follow readily from [12]. By the definition of $\mathcal{L}$, it follows that

$$\mathcal{L} = \int_0^\infty dw_1 \int_{-\infty}^\infty w_2\phi(w_1, w_2; \varrho_{12})dw_2 \tag{29}$$

with

$$\phi = \frac{1}{2\pi\sigma_1\sigma_2}\frac{1}{\sqrt{1-\varrho_{12}^2}} \times \exp\left\{\frac{-1}{2(1-\varrho_{12}^2)}\left[\frac{w_1^2}{\sigma_1^2} - \frac{2\varrho_{12}w_1w_2}{\sigma_1\sigma_2} + \frac{w_2^2}{\sigma_2^2}\right]\right\} \tag{30}$$

being the pdf of the bivariate normal random vector $[W_1\ W_2]^\mathsf{T}$ with mean $[0\ 0]^\mathsf{T}$. Differentiating both side of (29) with respect to ϱ_{12} yields

$$\begin{aligned}
\frac{d\mathcal{L}}{d\varrho_{12}} &= \sigma_1\sigma_2\int_0^\infty dw_1\int_{-\infty}^\infty w_2\frac{\partial}{\partial w_2}\left(\frac{\partial\phi}{\partial w_1}\right)dw_2\\
&= \sigma_1\sigma_2\int_0^\infty dw_1\left[\underbrace{w_2\frac{\partial\phi}{\partial w_1}\Big|_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^\infty \frac{\partial\phi}{\partial w_1}dw_2\right]\\
&= -\sigma_1\sigma_2\int_{-\infty}^\infty dw_2\int_0^\infty \frac{\partial\phi}{\partial w_1}dw_1\\
&= -\sigma_1\sigma_2\int_{-\infty}^\infty dw_2\left[\phi\Big|_0^\infty\right]\\
&= \sigma_1\sigma_2\int_{-\infty}^\infty \phi(0, w_2; \varrho_{12})dw_2.
\end{aligned}\tag{31}$$

where the first step follows from the well known relationship

$$\frac{d\phi}{d\varrho_{12}} = \sigma_1\sigma_2 \frac{\partial^2\phi}{\partial w_1 \partial w_2}.$$

From (30) it follows that

$$\phi(0, w_2; \varrho_{12}) = \frac{1}{2\pi\sigma_1\sigma_2} \frac{1}{\sqrt{1-\varrho_{12}^2}} \exp\left\{-\frac{1}{2(1-\varrho_{12}^2)} \frac{w_2^2}{\sigma_2^2}\right\}. \tag{32}$$

A substitution of (32) into (31) gives $d\mathcal{L}/d\varrho_{12} = \sigma_2/\sqrt{2\pi}$, and hence $\mathcal{L} = \sigma_2\varrho_{12}/\sqrt{2\pi}$ +const., where the constant term is null by noticing that $\mathbb{E}\{H(W_1)W_2\} = 0$ for $\varrho_{12} = 0$. This completes the proof of (5). ■

APPENDIX II
SKETCH PROOF OF LEMMA 2

Proof: Due to space limitation, we only provide a sketch of the proof here. We first derive $\mathbb{E}(U)$ and $\mathbb{E}(V)$. From definition (6),

$$\mathbb{E}(U) = \frac{1}{n^{[2]}} \mathbb{E}\left\{\sum_{i=1}^{n} (2\tilde{Z}_i^1 - 1 - n) Z_i^3\right\}. \tag{33}$$

Substituting $\tilde{Z}_i = \sum_{k=1}^{n} H(Z_i - Z_k) + 1$ [27] into (33), expanding and recalling that $\mathbb{E}(Z) = 0$, it follows that

$$\mathbb{E}(U) = \frac{2}{n^{[2]}} \sum_{i=1}^{n} \sum_{k=1}^{n} \mathbb{E}\left\{H(Z_i^1 - Z_k^1) Z_i^3\right\}. \tag{34}$$

Since, by definition, $\{[Z_k^1\ Z_k^2\ Z_k^3\ Z_k^4]^\mathsf{T}\}_{k=1}^{n}$ is a mixture of $\{[\xi_i^1\ \xi_i^2\ \xi_i^3\ \xi_i^4]^\mathsf{T}\}_{i=1}^{n_1}$ and $\{[\zeta_j^1\ \zeta_j^2\ \zeta_j^3\ \zeta_j^4]^\mathsf{T}\}_{j=1}^{n_2}$, (34) can be written as

$$\begin{aligned}\mathbb{E}(U) = &\frac{2}{n^{[2]}} \sum_{i=1}^{n} \sum_{k=1}^{n} \mathbb{E}\left\{H(\xi_i^1 - \xi_k^1) \xi_i^3\right\} \\ &+ \frac{2}{n^{[2]}} \sum_{i=1}^{n_1} \sum_{k'=1}^{n_2} \mathbb{E}\left\{H(\xi_i^1 - \zeta_{k'}^1) \xi_i^3\right\} \\ &+ \frac{2}{n^{[2]}} \sum_{i'=1}^{n_2} \sum_{k=1}^{n_1} \mathbb{E}\left\{H(\zeta_{i'}^1 - \xi_k^1) \zeta_{i'}^3\right\} \\ &+ \frac{2}{n^{[2]}} \sum_{i'=1}^{n_2} \sum_{k'=1}^{n_2} \mathbb{E}\left\{H(\zeta_{i'}^1 - \zeta_{k'}^1) \zeta_{i'}^3\right\}\end{aligned} \tag{35}$$

which becomes the first lemma statement (8) after some straightforward algebra along with the assistance of (5) in Lemma 1. In a similar manner we also have the second lemma statement (9) for $\mathbb{E}(V)$.

Next we deal with $\mathbb{C}(U, V)$. It follows from [12] that

$$\mathbb{C}(U, V) = \frac{4A - 2(n-1)B + (n-1)^2 C - D}{n^2(n-1)^2} \tag{36}$$

where

$$D = n^2(n-1)^2 \mathbb{E}(U)\mathbb{E}(V) \tag{37}$$

$$C = \sum_{i,j}^{n} \mathbb{E}(Z_i^3 Z_j^4) \tag{38}$$

$$\begin{aligned}B = &\sum_{i,j,k}^{n} \mathbb{E}\left\{H(Z_i^1 - Z_k^1) Z_i^3 Z_j^4\right\} \\ &+ \sum_{i,j,l}^{n} \mathbb{E}\left\{H(Z_j^2 - Z_l^2) Z_i^3 Z_j^4\right\}\end{aligned} \tag{39}$$

and

$$A = \sum_{i,j,k,l}^{n} \mathbb{E}\left\{H(Z_i^1 - Z_k^1) H(Z_j^2 - Z_l^2) Z_i^3 Z_j^4\right\}. \tag{40}$$

The expression of D is easily obtained by substituting (8) and (9) into (37).

For convenience, denote by B_1 and B_2 the two triple summations of B in (39). Then it follows that B_1 is decomposable into eight sub-triple summations which can be further partitioned into 16 disjoint and exhaustive subsets that listed in Table 2. An application of (13) to Table 2 leads directly to

$$\begin{aligned}B_1 = &\, n_1^{[2]} \frac{\eta_{34}\varsigma_3\varsigma_4}{2} + n_1 n_2 \frac{\eta_{34}\varsigma_3\varsigma_4}{2} \\ &+ n_1 n_2 \frac{\eta'_{34}\varsigma'_3\varsigma'_4}{2} + n_2^{[2]} \frac{\eta'_{34}\varsigma'_3\varsigma'_4}{2}.\end{aligned}$$

Similarly we also have

$$\begin{aligned}B_2 = &\, n_1^{[2]} \frac{\eta_{34}\varsigma_3\varsigma_4}{2} + n_1 n_2 \frac{\eta_{34}\varsigma_3\varsigma_4}{2} \\ &+ n_1 n_2 \frac{\eta'_{34}\varsigma'_3\varsigma'_4}{2} + n_2^{[2]} \frac{\eta'_{34}\varsigma'_3\varsigma'_4}{2}.\end{aligned}$$

Thus

$$B = B_1 + B_2 = (n-1)(n_1\eta_{34}\varsigma_3\varsigma_4 + n_2\eta'_{34}\varsigma'_3\varsigma'_4)$$

which is the result of (13).

TABLE 2. **Quantities for evaluation of B_1 in Lemma 2.**

		Rep. terms			Correlation coefficients†		
Subsets	# of terms⋆	W_1	W_3	W_4	ϱ_{13}	ϱ_{14}	ϱ_{34}
$i{\neq}k{=}j$	$n_1^{[2]}$	$\xi_1^1 - \xi_2^1$	ξ_1^3	ξ_2^4	$\eta_{13}/\sqrt{2}$	$-\eta_{14}/\sqrt{2}$	0
$i{=}j{\neq}k$	$n_1^{[2]}$	$\xi_1^1 - \xi_2^1$	ξ_1^3	ξ_1^4	$\eta_{13}/\sqrt{2}$	$\eta_{14}/\sqrt{2}$	η_{34}
$i{\neq}j{\neq}k$	$n_1^{[3]}$	$\xi_1^1 - \xi_3^1$	ξ_1^3	ξ_2^4	$\eta_{13}/\sqrt{2}$	0	0
$i{=}j, k'$	$n_1 n_2$	$\xi_1^1 - \zeta_1^1$	ξ_1^3	ξ_1^4	$\eta_{13}\varsigma_1/\sqrt{\lambda_1}$	$\eta_{14}\varsigma_1/\sqrt{\lambda_1}$	η_{34}
$i{\neq}j, k'$	$n_1^{[2]} n_2$	$\xi_1^1 - \zeta_1^1$	ξ_1^3	ξ_2^4	$\eta_{13}\varsigma_1/\sqrt{\lambda_1}$	0	0
$i{\neq}k, j'$	$n_1^{[2]} n_2$	$\xi_1^1 - \xi_2^1$	ξ_1^3	ζ_1^4	$\eta_{13}\sqrt{2}$	0	0
$j'{=}k', i$	$n_1 n_2$	$\xi_1^1 - \zeta_1^1$	ξ_1^3	ζ_1^4	$\eta_{13}\varsigma_1/\sqrt{\lambda_1}$	$-\eta'_{14}\varsigma'_1/\sqrt{\lambda_1}$	0
$j'{\neq}k', i$	$n_1 n_2^{[2]}$	$\xi_1^1 - \zeta_1^1$	ξ_1^3	ζ_2^4	$\eta_{13}\varsigma_1/\sqrt{\lambda_1}$	0	0
$j{=}k, i'$	$n_1 n_2$	$\zeta_1^1 - \xi_1^1$	ζ_1^3	ξ_1^4	$\eta'_{13}\varsigma'_1/\sqrt{\lambda_1}$	$-\eta_{14}\varsigma_1/\sqrt{\lambda_1}$	0
$j{\neq}k, i'$	$n_1^{[2]} n_2$	$\zeta_1^1 - \xi_1^1$	ζ_1^3	ξ_2^4	$\eta'_{13}\varsigma'_1/\sqrt{\lambda_1}$	0	0
$i'{\neq}k', j$	$n_1 n_2^{[2]}$	$\zeta_1^1 - \zeta_2^1$	ζ_1^3	ξ_1^4	$\eta'_{13}/\sqrt{2}$	0	0
$i'{=}j', k$	$n_1 n_2$	$\zeta_1^1 - \xi_1^1$	ζ_1^3	ζ_1^4	$\eta'_{13}\varsigma'_1/\sqrt{\lambda_1}$	$\eta'_{14}\varsigma'_1/\sqrt{\lambda_1}$	η'_{34}
$i'{\neq}k'{=}j'$	$n_2^{[2]}$	$\zeta_1^1 - \zeta_2^1$	ζ_1^3	ζ_2^4	$\eta'_{13}/\sqrt{2}$	$-\eta'_{14}/\sqrt{2}$	0
$i'{=}j'{\neq}k'$	$n_2^{[2]}$	$\zeta_1^1 - \zeta_2^1$	ζ_1^3	ζ_1^4	$\eta'_{13}/\sqrt{2}$	$\eta'_{14}/\sqrt{2}$	η'_{34}
$i'{\neq}j'{\neq}k'$	$n_2^{[3]}$	$\zeta_1^1 - \zeta_2^1$	ζ_1^3	ζ_3^4	$\eta'_{13}/\sqrt{2}$	0	0

⋆ $n_p^{[m]} \triangleq n_p(n_p-1)\ldots(n_p-m+1)$ for $p = 1, 2$.
† $\lambda_1 \triangleq \varsigma_1^2 + \varsigma_1'^2$.

TABLE 3. Quantities for evaluating $A = \sum_{i,j,k,l}^{n} \mathbb{E}\{H(Z_i^1 - Z_k^1)H(Z_j^2 - Z_l^2)Z_i^3 Z_j^4\}$ in Lemma 2.

		terms*	Representative terms				Correlation coefficients between W_r and W_s†					
ℓ	Subsets	α	W_1	W_2	W_3	W_4	ϱ_{12}	ϱ_{13}	ϱ_{14}	ϱ_{23}	ϱ_{24}	ϱ_{34}
1	$i=j\neq k=l$	$n_1^{[2]}$	$\xi_1^1-\xi_2^1$	$\xi_1^2-\xi_2^2$	ξ_1^3	ξ_1^4	η_{12}	$\eta_{13}/\sqrt{2}$	$\eta_{14}/\sqrt{2}$	$\eta_{23}/\sqrt{2}$	$\eta_{24}/\sqrt{2}$	η_{34}
2	$l=i\neq j=k$	$n_1^{[2]}$	$\xi_1^1-\xi_2^1$	$\xi_2^2-\xi_1^2$	ξ_1^3	ξ_2^4	$-\eta_{12}$	$\eta_{13}/\sqrt{2}$	$-\eta_{14}/\sqrt{2}$	$-\eta_{23}/\sqrt{2}$	$\eta_{24}/\sqrt{2}$	0
3	$i=j\neq k\neq l$	$n_1^{[3]}$	$\xi_1^1-\xi_2^1$	$\xi_1^2-\xi_3^2$	ξ_1^3	ξ_1^4	$\eta_{12}/2$	$\eta_{13}/\sqrt{2}$	$\eta_{14}/\sqrt{2}$	$\eta_{23}/\sqrt{2}$	$\eta_{24}/\sqrt{2}$	η_{34}
4	$k=j\neq i\neq l$	$n_1^{[3]}$	$\xi_1^1-\xi_2^1$	$\xi_2^2-\xi_3^2$	ξ_1^3	ξ_2^4	$-\eta_{12}/2$	$\eta_{13}/\sqrt{2}$	$-\eta_{14}/\sqrt{2}$	0	$\eta_{24}/\sqrt{2}$	0
5	$l=i\neq j\neq k$	$n_1^{[3]}$	$\xi_1^1-\xi_3^1$	$\xi_2^2-\xi_1^2$	ξ_1^3	ξ_2^4	$-\eta_{12}/2$	$\eta_{13}/\sqrt{2}$	0	$-\eta_{23}/\sqrt{2}$	$\eta_{24}/\sqrt{2}$	0
6	$l=k\neq i\neq j$	$n_1^{[3]}$	$\xi_1^1-\xi_3^1$	$\xi_2^2-\xi_3^2$	ξ_1^3	ξ_2^4	$\eta_{12}/2$	$\eta_{13}/\sqrt{2}$	0	0	$\eta_{24}/\sqrt{2}$	0
7	$i\neq j\neq k\neq l$	$n_1^{[4]}$	$\xi_1^1-\xi_3^1$	$\xi_2^2-\xi_4^2$	ξ_1^3	ξ_2^4	0	$\eta_{13}/\sqrt{2}$	0	0	$\eta_{24}/\sqrt{2}$	0
8	$i=j\neq k,l'$	$n_1^{[2]}n_2$	$\xi_1^1-\xi_2^1$	$\xi_1^2-\zeta_1^2$	ξ_1^3	ξ_1^4	$\eta_{12}\varsigma_2/\sqrt{2\lambda_2}$	$\eta_{13}/\sqrt{2}$	$\eta_{14}/\sqrt{2}$	$\eta_{23}\varsigma_2/\sqrt{\lambda_2}$	$\eta_{24}\varsigma_2/\sqrt{\lambda_2}$	η_{34}
9	$i\neq j=k,l'$	$n_1^{[2]}n_2$	$\xi_1^1-\xi_2^1$	$\xi_2^2-\zeta_1^2$	ξ_1^3	ξ_2^4	$-\eta_{12}\varsigma_2/\sqrt{2\lambda_2}$	$\eta_{13}/\sqrt{2}$	$-\eta_{14}/\sqrt{2}$	0	$\eta_{24}\varsigma_2/\sqrt{\lambda_2}$	0
10	$i\neq j\neq k,l'$	$n_1^{[3]}n_2$	$\xi_1^1-\xi_3^1$	$\xi_2^2-\zeta_1^2$	ξ_1^3	ξ_2^4	0	$\eta_{13}/\sqrt{2}$	0	0	$\eta_{24}\varsigma_2/\sqrt{\lambda_2}$	0
11	$i=j\neq l,k'$	$n_1^{[2]}n_2$	$\xi_1^1-\zeta_1^1$	$\xi_1^2-\xi_2^2$	ξ_1^3	ξ_1^4	$\eta_{12}\varsigma_1/\sqrt{2\lambda_1}$	$\eta_{13}\varsigma_1/\sqrt{\lambda_1}$	$\eta_{14}\varsigma_1/\sqrt{\lambda_1}$	$\eta_{23}/\sqrt{2}$	$\eta_{24}/\sqrt{2}$	η_{34}
12	$i=l\neq j,k'$	$n_1^{[2]}n_2$	$\xi_1^1-\zeta_1^1$	$\xi_2^2-\xi_1^2$	ξ_1^3	ξ_2^4	$-\eta_{12}\varsigma_1/\sqrt{2\lambda_1}$	$\eta_{13}\varsigma_1/\sqrt{\lambda_1}$	0	$-\eta_{23}/\sqrt{2}$	$\eta_{24}/\sqrt{2}$	0
13	$i\neq j\neq l,k'$	$n_1^{[3]}n_2$	$\xi_1^1-\zeta_1^1$	$\xi_2^2-\xi_3^2$	ξ_1^3	ξ_2^4	0	$\eta_{13}\varsigma_1/\sqrt{\lambda_1}$	0	0	$\eta_{24}/\sqrt{2}$	0
14	$j=k\neq l,i'$	$n_1^{[2]}n_2$	$\zeta_1^1-\xi_1^1$	$\xi_1^2-\xi_2^2$	ζ_1^3	ξ_1^4	$-\eta_{12}\varsigma_1/\sqrt{2\lambda_1}$	$\eta'_{13}\varsigma'_1/\sqrt{\lambda_1}$	$-\eta_{14}\varsigma_1/\sqrt{\lambda_1}$	0	$\eta_{24}/\sqrt{2}$	0
15	$j\neq k=l,i'$	$n_1^{[2]}n_2$	$\zeta_1^1-\xi_2^1$	$\xi_1^2-\xi_2^2$	ζ_1^3	ξ_1^4	$\eta_{12}\varsigma_1/\sqrt{2\lambda_1}$	$\eta'_{13}\varsigma'_1/\sqrt{\lambda_1}$	0	0	$\eta_{24}/\sqrt{2}$	0
16	$j\neq k\neq l,i'$	$n_1^{[3]}n_2$	$\zeta_1^1-\xi_2^1$	$\xi_1^2-\xi_3^2$	ζ_1^3	ξ_1^4	0	$\eta'_{13}\varsigma'_1/\sqrt{\lambda_1}$	0	0	$\eta_{24}/\sqrt{2}$	0
17	$i\neq k=l,j'$	$n_1^{[2]}n_2$	$\xi_1^1-\xi_2^1$	$\zeta_1^2-\xi_2^2$	ξ_1^3	ζ_1^4	$\eta_{12}\varsigma_2/\sqrt{2\lambda_2}$	$\eta_{13}/\sqrt{2}$	0	0	$\eta'_{24}\varsigma'_2/\sqrt{\lambda_2}$	0
18	$i=k\neq l,j'$	$n_1^{[2]}n_2$	$\xi_1^1-\xi_2^1$	$\zeta_1^2-\xi_1^2$	ξ_1^3	ζ_1^4	$-\eta_{12}\varsigma_2/\sqrt{2\lambda_2}$	$\eta_{13}/\sqrt{2}$	0	$-\eta_{23}\varsigma_2/\sqrt{\lambda_2}$	$\eta'_{24}\varsigma'_2/\sqrt{\lambda_2}$	0
19	$i\neq k\neq l,j'$	$n_1^{[3]}n_2$	$\xi_1^1-\xi_2^1$	$\zeta_1^2-\xi_3^2$	ξ_1^3	ζ_1^4	0	$\eta_{13}/\sqrt{2}$	0	0	$\eta'_{24}\varsigma'_2/\sqrt{\lambda_2}$	0
20	$i=j,k'=l'$	n_1n_2	$\xi_1^1-\zeta_1^1$	$\xi_1^2-\zeta_1^2$	ξ_1^3	ξ_1^4	$\frac{\eta_{12}\varsigma_1\varsigma_2+\eta'_{12}\varsigma'_1\varsigma'_2}{\sqrt{\lambda_1\lambda_2}}$	$\eta_{13}\varsigma_1/\sqrt{\lambda_1}$	$\eta_{14}\varsigma_1/\sqrt{\lambda_1}$	$\eta_{23}\varsigma_2/\sqrt{\lambda_2}$	$\eta_{24}\varsigma_2/\sqrt{\lambda_2}$	η_{34}
21	$i=j,k'\neq l'$	n_1n_2	$\xi_1^1-\zeta_1^1$	$\xi_1^2-\zeta_2^2$	ξ_1^3	ξ_1^4	$\eta_{12}\varsigma_1\varsigma_2/\sqrt{\lambda_1\lambda_2}$	$\eta_{13}\varsigma_1/\sqrt{\lambda_1}$	$\eta_{14}\varsigma_1/\sqrt{\lambda_1}$	$\eta_{23}\varsigma_2/\sqrt{\lambda_2}$	$\eta_{24}\varsigma_2/\sqrt{\lambda_2}$	η_{34}
22	$i\neq j,k'=l'$	$n_1^{[2]}n_2$	$\xi_1^1-\zeta_1^1$	$\xi_2^2-\zeta_1^2$	ξ_1^3	ξ_2^4	$\eta'_{12}\varsigma'_1\varsigma'_2/\sqrt{\lambda_1\lambda_2}$	$\eta_{13}\varsigma_1/\sqrt{\lambda_1}$	0	0	$\eta_{24}\varsigma_2/\sqrt{\lambda_2}$	0
23	$i\neq j,k'\neq l'$	$n_1^{[2]}n_2^{[2]}$	$\xi_1^1-\zeta_1^1$	$\xi_2^2-\zeta_2^2$	ξ_1^3	ξ_2^4	0	$\eta_{13}\varsigma_1/\sqrt{\lambda_1}$	0	0	$\eta_{24}\varsigma_2/\sqrt{\lambda_2}$	0
24	$i'\neq k',j\neq l$	$n_1^{[2]}n_2^{[2]}$	$\zeta_1^1-\zeta_2^1$	$\xi_1^2-\xi_2^2$	ζ_1^3	ξ_1^4	0	$\eta'_{13}/\sqrt{2}$	0	0	$\eta_{24}/\sqrt{2}$	0
25	$i'=l',j=k$	n_1n_2	$\zeta_1^1-\xi_1^1$	$\xi_1^2-\zeta_1^2$	ζ_1^3	ξ_1^4	$-\frac{\eta_{12}\varsigma_1\varsigma_2+\eta'_{12}\varsigma'_1\varsigma'_2}{\sqrt{\lambda_1\lambda_2}}$	$\eta'_{13}\varsigma'_1\sqrt{\lambda_1}$	$-\eta_{14}\varsigma_1/\sqrt{\lambda_1}$	$-\eta'_{23}\varsigma'_2/\sqrt{\lambda_2}$	$\eta_{24}\varsigma_2/\sqrt{\lambda_2}$	0
26	$i'\neq l',j=k$	$n_1n_2^{[2]}$	$\zeta_1^1-\xi_1^1$	$\xi_1^2-\zeta_2^2$	ζ_1^3	ξ_1^4	$-\eta_{12}\varsigma_1\varsigma_2/\sqrt{\lambda_1\lambda_2}$	$\eta'_{13}\varsigma'_1/\sqrt{\lambda_1}$	$-\eta_{14}\varsigma_1/\sqrt{\lambda_1}$	0	$\eta_{24}\varsigma_2/\sqrt{\lambda_2}$	0
27	$i'=l',j\neq k$	$n_1^{[2]}n_2$	$\zeta_1^1-\xi_2^1$	$\xi_1^2-\zeta_1^2$	ζ_1^3	ξ_1^4	$-\eta'_{12}\varsigma'_1\varsigma'_2/\sqrt{\lambda_1\lambda_2}$	$\eta'_{13}\varsigma'_1/\sqrt{\lambda_1}$	0	$-\eta'_{23}\varsigma'_2/\sqrt{\lambda_2}$	$\eta_{24}\varsigma_2/\sqrt{\lambda_2}$	0
28	$i'\neq l',j\neq k$	$n_1^{[2]}n_2^{[2]}$	$\zeta_1^1-\xi_1^1$	$\xi_2^2-\zeta_2^2$	ζ_1^3	ξ_2^4	0	$\eta'_{13}\varsigma'_1/\sqrt{\lambda_1}$	0	0	$\eta_{24}\varsigma_2/\sqrt{\lambda_2}$	0

* $n_p^{[m]} \triangleq n_p(n_p-1)\ldots(n_p-m+1)$ for $p=1,2$.
† $\lambda_p \triangleq \varsigma_p^2 + \varsigma_p'^2$ for $p=1,2$.

By (38), it follows that

$$\begin{aligned} C &= \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}(Z_i^3 Z_j^4) \\ &= \sum_{i=1}^{n_1}\sum_{j=1}^{n_1}\mathbb{E}(\xi_i^3\xi_j^4) + \sum_{i=1}^{n_1}\sum_{j'=1}^{n_2}\mathbb{E}(\xi_i^3\zeta_{j'}^4) \\ &\quad + \sum_{i'=1}^{n_2}\sum_{j=1}^{n_1}\mathbb{E}(\zeta_{i'}^3\xi_j^4) + \sum_{i'=1}^{n_2}\sum_{j'=1}^{n_2}\mathbb{E}(\zeta_{i'}^3\zeta_{j'}^4) \end{aligned}$$

which simplifies to the statement of (12) after some elementary derivations.

The quadruple summation in (40) can be decomposed as

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} &= \left(\sum_{i=1}^{n_1}+\sum_{i'=1}^{n_2}\right)\left(\sum_{j=1}^{n_1}+\sum_{j'=1}^{n_2}\right) \\ &\quad \times \left(\sum_{k=1}^{n_1}+\sum_{k'=1}^{n_2}\right)\left(\sum_{l=1}^{n_1}+\sum_{l'=1}^{n_2}\right) \end{aligned} \quad (41)$$

where, in the right side, $(i \;\; j \;\; k \;\; l)$ and $(i' \;\; j' \;\; k' \;\; l')$ are suffixes corresponding to ξ- and ζ-terms, respectively. This means that A contains 16 sub-quadruple summations which can be further partitioned into 56 disjoint and exhaustive subsets. In other words, A is a summation of 56 integrals of the form $\mathbb{E}\{H(W_1)H(W_2)W_3W_4\}$, i.e., the $\mathcal{I}_\ell$-terms, weighted by corresponding subset cardinality, i.e., the α_ℓ-terms. The statement (14) thus follows by substituting into (3) the corresponding parameters tabulated in Table 3 as well as exploiting the symmetry of (41). ∎

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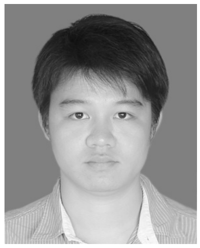

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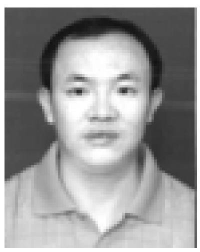

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