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New Weighted Integral Inequalities and Its Application to Exponential Stability Analysis of Time-Delay Systems

CHENG GONG[1], GUOPU ZHU[2], (Senior Member, IEEE), AND LIGANG WU[3], (Senior Member, IEEE)

[1]School of Mathematical Science, Heilongjiang University, Harbin 150080, China
[2]Shenzhen Institutes of Advanced Technology, Chinese Academy of Sciences, Shenzhen 518055, China
[3]Research Institute of Intelligent Control and Systems, Harbin Institute of Technology, Harbin 150001, China

Corresponding author: G. Zhu (guopu.zhu@gmail.com)

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ABSTRACT This paper is concerned with the exponential stability analysis for time-delay systems. First, two new weighted integral inequalities are presented based on the auxiliary function-based integral inequalities. In the new weighted integral inequalities, unlike previous studies, exponentially weighted integral vectors are used to find the lower bounds of the weighted integral quadratic terms. Then, by utilizing the new weighted integral inequalities, a new linear matrix inequality (LMI) condition is derived for the exponential stability of the considered time-delay systems. Finally, the numerical examples are conducted to validate the effectiveness of the new LMI condition. The example results show that the LMI condition derived in this paper is less conservative than existing ones in analyzing exponential stability of the considered systems.

INDEX TERMS Time-delay systems, exponential stability analysis, Lyapunov–Krasovskii functional, linear matrix inequalities.

I. INTRODUCTION

Many practical systems, such as network control systems, chemical engineering systems, and biological systems, can be modeled as linear systems with time delay. The appearance of time delay in these systems may make them unstable. Hence, for time-delay systems, stability analysis is an important issue to be considered [1]–[27]. In general, the stability analysis can be grouped into two classes, namely the asymptotic stability analysis [1], [7], [18], [21]–[23] and the exponential stability analysis [2], [3], [5], [6], [19], [25], [28]–[30]. The goal of the asymptotic stability analysis is just to derive the sufficient conditions of the asymptotic stability of time-delay systems, whereas that of the exponential stability analysis is further to determine the decay rates of these systems.

This paper focuses on the issue of the exponential stability analysis of time-delay systems. With the help of linear matrix inequalities (LMIs), the Lyapunov–Krasovskii functional (LKF) approach has become one of the most effective ways to address this issue [2], [3], [6], [28]–[34]. In [2], Liu transformed the exponential stability analysis of a time-delay system into the asymptotic stability analysis of another time-delay system by using a state transformation. In [3], Mondie and Kharitonov introduced an exponential weighted functional into the LKF to investigate the exponential stability of time-delay systems. Later, Xu *et al.* [6] improved the results of [2] and [3] by using a more complicated LKF functional and the state transformation. Recently, Cao [29] introduced slack matrices for LMIs to obtain a novel exponential stability criterion. Very recently, based on Jensen's integral inequalities, Van Hien and Trinh [30] proposed two weighted integral inequalities for both single and double exponential weighted functionals. Then, these two integral inequalities were successfully applied to the exponential stability analysis of several kinds of time-delay systems.

In this paper, based on the auxiliary function-based integral inequalities given by Park *et al.* [23], we present two new weighted integral inequalities. Instead of the commonly used integral vector, exponentially weighted integral vectors are used to derive the new weighted integral inequalities. We then establish a new LMI condition of the exponential stability of time-delay systems by using the new weighted integral inequalities. It is also worth mentioning that to our knowledge, we are the first to introduce weighted integral states into the augmented state vector of LKF functional. Finally, we provide two numerical examples to demonstrate the effectiveness of the proposed exponential stability condition.

The rest of this paper is organized as follows. In Section II, we presented two new weighted integral inequalities, which will be applied to analyze the exponential stability of time-delay systems in Section III. Section IV reports the comparison results for two examples. Finally, the conclusion is given in Section V.

Notation: Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the n-dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively; I and 0 denote the identity matrix and zero matrix of appropriate dimensions, respectively; the superscript "T" denotes matrix transpose; $\|\cdot\|$ denotes the Euclidean vector norm; $\lambda_{min}(\cdot)$ and $\lambda_{max}(\cdot)$ denote the minimum and maximum eigenvalues of a symmetric matrix, respectively. Note that matrices are assumed to have compatible dimensions for algebraic operations, if not otherwise stated.

II. WEIGHTED INTEGRAL INEQUALITIES

In this section, based on the auxiliary function-based integral inequalities (AFIIs, for short) established by Park *et al.* [23], we presented two new weighted integral inequalities for exponential stability analysis. We first review the AFIIs with the following two lemmas.

Lemma 1 [23]: For an $n \times n$ matrix $R > 0$ and a vector function $\omega \in C([a,b],\mathbb{R}^n)$, the following inequality holds:

$$\begin{aligned}(b-a)\int_a^b \omega^T(u)R\omega(u)\mathrm{d}u \\ \geq \Big(\int_a^b \omega^T(u)\mathrm{d}u\Big)R\Big(\int_a^b \omega(u)\mathrm{d}u\Big) \\ +3\Omega_1^T R\Omega_1+5\Omega_2^T R\Omega_2,\end{aligned} \tag{1}$$

where

$$\begin{aligned}\Omega_1 &= \int_a^b \omega(u)\mathrm{d}u - \frac{2}{b-a}\int_a^b\int_s^b \omega(u)\mathrm{d}u\mathrm{d}s,\\ \Omega_2 &= \int_a^b \omega(u)\mathrm{d}u - \frac{6}{b-a}\int_a^b\int_s^b \omega(u)\mathrm{d}u\mathrm{d}s \\ &\quad +\frac{12}{(b-a)^2}\int_a^b\int_v^b\int_s^b \omega(u)\mathrm{d}u\mathrm{d}s\mathrm{d}v.\end{aligned}$$

Lemma 2 [23]: For an $n \times n$ matrix $R > 0$ and a vector function $\omega \in C([a,b],\mathbb{R}^n)$, the following inequality holds:

$$\begin{aligned}\frac{(b-a)^2}{2}\int_a^b\int_s^b \omega^T(u)R\omega(u)\mathrm{d}u\mathrm{d}s \\ \geq \Big(\int_a^b\int_s^b \omega^T(u)\mathrm{d}u\mathrm{d}s\Big)R\Big(\int_a^b\int_s^b \omega(u)\mathrm{d}u\mathrm{d}s\Big) \\ +8\Omega_3^T R\Omega_3,\end{aligned} \tag{2}$$

where

$$\Omega_3 = \int_a^b\int_s^b \omega(u)\mathrm{d}u\mathrm{d}s - \frac{3}{b-a}\int_a^b\int_s^b\int_u^b \omega(v)\mathrm{d}v\mathrm{d}u\mathrm{d}s.$$

The above AFIIs were developed originally for the asymptotic stability analysis of time-delay systems. Next, we extend them for exponential stability analysis. To do so, two new weighted integral inequalities are established as follows:

Lemma 3: For an $n\times n$ matrix $R > 0$, a scalar $\beta \geq 0$, and a vector function $\omega \in C([a,b],\mathbb{R}^n)$, the following inequality holds:

$$\begin{aligned}(b-a)\int_a^b e^{\beta(u-a)}\omega^T(u)R\omega(u)\mathrm{d}u \\ \geq \Big(\int_a^b e^{\frac{\beta}{2}(u-a)}\omega^T(u)\mathrm{d}u\Big)R\Big(\int_a^b e^{\frac{\beta}{2}(u-a)}\omega(u)\mathrm{d}u\Big) \\ +3\Omega_4^T R\Omega_4+5\Omega_5^T R\Omega_5,\end{aligned} \tag{3}$$

where

$$\begin{aligned}\Omega_4 &= \int_a^b e^{\frac{\beta}{2}(u-a)}\omega(u)\mathrm{d}u \\ &\quad -\frac{2}{b-a}\int_a^b\int_s^b e^{\frac{\beta}{2}(u-a)}\omega(u)\mathrm{d}u\mathrm{d}s,\\ \Omega_5 &= \int_a^b e^{\frac{\beta}{2}(u-a)}\omega(u)\mathrm{d}u \\ &\quad -\frac{6}{b-a}\int_a^b\int_s^b e^{\frac{\beta}{2}(u-a)}\omega(u)\mathrm{d}u\mathrm{d}s \\ &\quad +\frac{12}{(b-a)^2}\int_a^b\int_v^b\int_s^b e^{\frac{\beta}{2}(u-a)}\omega(u)\mathrm{d}u\mathrm{d}s\mathrm{d}v.\end{aligned}$$

Proof: Let $\tilde{\omega}(u) = e^{\frac{\beta}{2}(u-a)}\omega(u)$, it follows that

$$\omega(u) = e^{-\frac{\beta}{2}(u-a)}\tilde{\omega}(u). \tag{4}$$

Then, substituting $e^{-\frac{\beta}{2}(u-a)}\tilde{\omega}(u)$ for $\omega(u)$ in the left side of inequality (3), we obtain

$$\begin{aligned}(b-a)\int_a^b e^{\beta(u-a)}\omega^T(u)R\omega(u)\mathrm{d}u \\ = (b-a)\int_a^b \tilde{\omega}^T(u)R\tilde{\omega}(u)\mathrm{d}u.\end{aligned} \tag{5}$$

By Lemma 1, we can rewrite Eq. (5) as

$$\begin{aligned}(b-a)\int_a^b e^{\beta(u-a)}\omega^T(u)R\omega(u)\mathrm{d}u \\ = (b-a)\int_a^b \tilde{\omega}^T(u)R\tilde{\omega}(u)\mathrm{d}u \\ \geq \Big(\int_a^b \tilde{\omega}^T(u)\mathrm{d}u\Big)R\Big(\int_a^b \tilde{\omega}(u)\mathrm{d}u\Big) \\ +3\tilde{\Omega}_1^T R\tilde{\Omega}_1+5\tilde{\Omega}_2^T R\tilde{\Omega}_2,\end{aligned} \tag{6}$$

where

$$\begin{aligned}\tilde{\Omega}_1 &= \int_a^b \tilde{\omega}(u)\mathrm{d}u - \frac{2}{b-a}\int_a^b\int_s^b \tilde{\omega}(u)\mathrm{d}u\mathrm{d}s,\\ \tilde{\Omega}_2 &= \int_a^b \tilde{\omega}(u)\mathrm{d}u - \frac{6}{b-a}\int_a^b\int_s^b \tilde{\omega}(u)\mathrm{d}u\mathrm{d}s \\ &\quad +\frac{12}{(b-a)^2}\int_a^b\int_v^b\int_s^b \tilde{\omega}(u)\mathrm{d}u\mathrm{d}s\mathrm{d}v.\end{aligned}$$

Finally, inequality (3) can be obtained directly by replacing $\tilde{\omega}(u)$ with $e^{\frac{\beta}{2}(u-a)}\omega(u)$ in (6). □

Lemma 4: For an $n \times n$ matrix $R > 0$, a scalar $\beta \geq 0$, and a vector function $\omega \in C([a, b], \mathbb{R}^n)$, the following inequality holds:

$$\frac{(b-a)^2}{2}\int_a^b\int_s^b e^{\beta(u-a)}\omega^T(u)R\omega(u)\mathrm{d}u\mathrm{d}s \geq \Omega_6^T R\Omega_6 + 8\Omega_7^T R\Omega_7, \tag{7}$$

where

$$\Omega_6 = \int_a^b\int_s^b e^{\frac{\beta}{2}(u-a)}\omega(u)\mathrm{d}u\mathrm{d}s$$
$$\Omega_7 = \int_a^b\int_s^b e^{\frac{\beta}{2}(u-a)}\omega(u)\mathrm{d}u\mathrm{d}s - \frac{3}{b-a}\int_a^b\int_s^b\int_u^b e^{\frac{\beta}{2}(u-a)}\omega(v)\mathrm{d}v\mathrm{d}u\mathrm{d}s.$$

Proof: Based on Lemma 2, we can obtain inequality (7) in a similar way as we did in the proof of Lemma 3. □

Remark 1: By setting $\beta = 0$, the new weighted integral inequalities (3) and (7) reduce to the AFIIs (1) and (2), respectively. That is to say, the AFIIs (1) and (2) are special cases of the new inequalities (3) and (7), respectively.

Remark 2: It is noted that in the weighted integral inequalities of [30], the integral vectors, such as $\int_a^b \omega(u)\mathrm{d}u$ and $\int_a^b\int_s^b \omega(u)\mathrm{d}u\mathrm{d}s$, were used to bound the weighted integral quadratic terms $\int_a^b e^{\beta(u-a)}\omega^T(u)R\omega(u)\mathrm{d}u$ and $\int_a^b\int_s^b e^{\beta(u-a)}\omega^T(u)R\omega(u)\mathrm{d}u\mathrm{d}s$; whereas, in the new weighted integral inequalities (3) and (7), the exponentially weighted integral vectors, such as $\int_a^b e^{\frac{\beta}{2}(u-a)}\omega(u)\mathrm{d}u$ and $\int_a^b\int_s^b e^{\frac{\beta}{2}(u-a)}\omega(u)\mathrm{d}u\mathrm{d}s$, are used instead. To our knowledge, this is the first time that such exponentially weighted integral vectors have been used in integral inequalities to find the lower bounds for the weighted integral quadratic terms.

III. EXPONENTIAL STABILITY ANALYSIS OF TIME-DELAY SYSTEMS

Consider a time-delay system of the following form:

$$\begin{cases}\dot{x}(t) = Ax(t) + A_d x(t-d), & t \geq 0,\\ x(t) = \phi(t), & t \in [-d, 0],\end{cases} \tag{8}$$

where $x(t) \in \mathbb{R}^n$ is the system state; $\phi(t) \in C([-d, 0], \mathbb{R}^n)$ is a continuous function called the initial function of x; d is the time delay that takes a positive value; A and $A_d \in \mathbb{R}^{n\times n}$ are constant matrices.

Let $x(t, \phi)$ denote the solution of system (8) with the initial condition ϕ, and let

$$\|\phi\|_d = \sup_{-d\leq\theta\leq 0}\|\phi(\theta)\|. \tag{9}$$

Now we give the definition of exponential stability for system (8) as below.

Definition 1: System (8) is said to be exponentially stable with a decay rate σ if there exist scalars $\sigma > 0$ and $\gamma \geq 0$ such that for every solution $x(t, \phi)$ of system (8), the following inequality holds:

$$\|x(t, \phi)\| \leq \gamma e^{-\sigma t}\|\phi\|_d, \ t \geq 0. \tag{10}$$

Note that the above definition is adapted from [3] and [6].

With the help of the proposed integral inequalities given by Lemmas 3 and 4, we obtain the following theorem.

Theorem 1: For a given $\beta > 0$, system (8) is exponentially stable with a decay rate $\sigma = \beta/2$ if there exist symmetric positive definite matrices $P \in \mathbb{R}^{4n\times 4n}$ and $R_i \in \mathbb{R}^{n\times n}$, $i = 0, 1, 2$, such that the following LMI holds:

$$\begin{aligned}\Theta = &\left(\Sigma_1^T P\Sigma_2 + \Sigma_2^T P\Sigma_1 + \beta\Sigma_1^T P\Sigma_1\right)\\ &+\left[e_1^T\left(e^{\beta d}R_0 + e^{\beta d}dR_1 + e^{\beta d}\frac{d^2}{2}R_2\right)e_1 - e_2^T R_0 e_2\right]\\ &-\left[\frac{1}{d}e_3^T R_1 e_3 + \frac{3}{d}\left(e_3 - \frac{2}{d}e_4\right)^T R_1\left(e_3 - \frac{2}{d}e_4\right)\right]\\ &-\left[\frac{5}{d}\left(e_3 - \frac{6}{d}e_4 + \frac{12}{d^2}e_5\right)^T R_1\left(e_3 - \frac{6}{d}e_4 + \frac{12}{d^2}e_5\right)\right]\\ &-\left[\frac{2}{d^2}e_4^T R_2 e_4 + \frac{16}{d^2}\left(e_4 - \frac{3}{d}e_5\right)^T R_2\left(e_4 - \frac{3}{d}e_5\right)\right]\\ <\ &0,\end{aligned} \tag{11}$$

where

$$e_i = [0_{n\times(i-1)n}\ \ I_n\ \ 0_{n\times(5-i)n}], \quad i = 1, 2, ..., 5,$$
$$\delta_i = e^{\frac{\beta d}{2}}\frac{d^{i-1}}{(i-1)!}e_1 - e_{i+1} - \frac{\beta}{2}e_{i+2}, \quad i = 1, 2, 3,$$
$$\Xi = Ae_1 + A_d e_2,$$
$$\Sigma_1 = [e_1^T, e_3^T, e_4^T, e_5^T]^T,$$
$$\Sigma_2 = [\Xi^T, \delta_1^T, \delta_2^T, \delta_3^T]^T.$$

Proof: Let

$$\eta(t) = \begin{bmatrix} x(t)\\ \int_{t-d}^t e^{\frac{\beta}{2}(s-t+d)}x(s)\mathrm{d}s\\ \int_{-d}^0\int_{t+\theta}^t e^{\frac{\beta}{2}(s-t+d)}x(s)\mathrm{d}s\mathrm{d}\theta\\ \int_{-d}^0\int_\beta^0\int_{t+\theta}^t e^{\frac{\beta}{2}(s-t+d)}x(s)\mathrm{d}s\mathrm{d}\theta\mathrm{d}\beta\end{bmatrix}, \tag{12}$$

and

$$\varphi(t) = \begin{bmatrix} x(t)\\ x(t-d)\\ \int_{t-d}^t e^{\frac{\beta}{2}(s-t+d)}x(s)\mathrm{d}s\\ \int_{-d}^0\int_{t+\theta}^t e^{\frac{\beta}{2}(s-t+d)}x(s)\mathrm{d}s\mathrm{d}\theta\\ \int_{-d}^0\int_\beta^0\int_{t+\theta}^t e^{\frac{\beta}{2}(s-t+d)}x(s)\mathrm{d}s\mathrm{d}\theta\mathrm{d}\beta\end{bmatrix}. \tag{13}$$

Note that from (12) and (13), we have

$$\eta(t) = \Sigma_1\varphi(t). \tag{14}$$

$$\dot{\eta}(t)=\begin{bmatrix} Ax(t)+A_dx(t-d) \\ e^{\frac{\beta d}{2}}x(t)-x(t-d)-\frac{\beta}{2}\int_{t-d}^{t}e^{\frac{\beta}{2}(s-t+d)}x(s)\mathrm{d}s \\ de^{\frac{\beta d}{2}}x(t)-\int_{t-d}^{t}e^{\frac{\beta}{2}(s-t+d)}x(s)\mathrm{d}s-\frac{\beta}{2}\int_{-d}^{0}\int_{t+\theta}^{t}e^{\frac{\beta}{2}(s-t+d)}x(s)\mathrm{d}s\mathrm{d}\theta \\ \frac{d^2}{2}e^{\frac{\beta d}{2}}x(t)-\int_{-d}^{0}\int_{t+\theta}^{t}e^{\frac{\beta}{2}(s-t+d)}x(s)\mathrm{d}s\mathrm{d}\theta-\frac{\beta}{2}\int_{-d}^{0}\int_{\beta}^{0}\int_{t+\theta}^{t}e^{\frac{\beta}{2}(s-t+d)}x(s)\mathrm{d}s\mathrm{d}\theta\mathrm{d}\beta \end{bmatrix}=\Sigma_2\varphi(t). \tag{16}$$

To prove the theorem, we first construct an LKF as follows:

$$V=V_0+W_0+W_1+W_2, \tag{15}$$

where

$$\begin{aligned} V_0&=\eta^T(t)P\eta(t),\\ W_0&=\int_{t-d}^{t}e^{\beta(s-t+d)}x^T(s)R_0x(s)\mathrm{d}s,\\ W_1&=\int_{-d}^{0}\int_{t+\theta}^{t}e^{\beta(s-t+d)}x^T(s)R_1x(s)\mathrm{d}s\mathrm{d}\theta,\\ W_2&=\int_{-d}^{0}\int_{\theta_2}^{0}\int_{t+\theta_1}^{t}e^{\beta(s-t+d)}x^T(s)R_2x(s)\mathrm{d}s\mathrm{d}\theta_1\mathrm{d}\theta_2. \end{aligned}$$

Then, to obtain $\dot{V}+\beta V$, in the following we calculate $\dot{V}_0+\beta V_0$ and $\dot{W}_i+\beta W_i$, $i=0,1,2$, separately. Considering that $\eta(t)=\Sigma_1\varphi(t)$ (see Eq. (14)) and $\dot{\eta}(t)=\Sigma_2\varphi(t)$ (see Eq. (16), which is on the top of this page), we have

$$\dot{V}_0=\varphi^T(t)\Big(\Sigma_1^TP\Sigma_2+\Sigma_2^TP\Sigma_1\Big)\varphi(t). \tag{17}$$

From Eq. (17), it follows that

$$\dot{V}_0+\beta V_0=\varphi^T(t)\Big(\Sigma_1^TP\Sigma_2+\Sigma_2^TP\Sigma_1+\beta\Sigma_1^TP\Sigma_1\Big)\varphi(t). \tag{18}$$

For W_0, we get its derivative by

$$\begin{aligned} \dot{W}_0&=e^{\beta(s-t+d)}x^T(s)R_0x(s)\Big|_{t-d}^{t}\\ &\quad-\beta\int_{t-d}^{t}e^{\beta(s-t+d)}x^T(s)R_0x(s)\mathrm{d}s\\ &=e^{\beta(s-t+d)}x^T(s)R_0x(s)\Big|_{t-d}^{t}-\beta W_0. \end{aligned}$$

Further, we have

$$\begin{aligned} \dot{W}_0+\beta W_0&=e^{\beta(s-t+d)}x(s)^TR_0x(s)\Big|_{t-d}^{t}\\ &=e^{\beta d}x(t)^TR_0x(t)-x(t-d)^TR_0x(t-d)\\ &=\varphi^T(t)\Big(e^{\beta d}e_1^TR_0e_1-e_2^TR_0e_2\Big)\varphi(t). \end{aligned} \tag{19}$$

Next, the derivative of W_1 is calculated as follows:

$$\begin{aligned} \dot{W}_1&=\int_{-d}^{0}\Big[e^{\beta(s-t+d)}x^T(s)R_1x(s)\Big|_{t+\theta}^{t}\\ &\quad-\beta\int_{t+\theta}^{t}e^{\beta(s-t+d)}x^T(s)R_1x(s)\mathrm{d}s\Big]\mathrm{d}\theta\\ &=\int_{-d}^{0}\Big[e^{\beta(s-t+d)}x^T(s)R_1x(s)\Big|_{t+\theta}^{t}\Big]\mathrm{d}\theta\\ &\quad-\beta\int_{-d}^{0}\int_{t+\theta}^{t}e^{\beta(s-t+d)}x^T(s)R_1x(s)\mathrm{d}s\mathrm{d}\theta\\ &=\int_{-d}^{0}e^{\beta d}x^T(t)R_1x(t)\mathrm{d}\theta\\ &\quad-\int_{-d}^{0}e^{\beta(\theta+d)}x^T(t+\theta)R_1x(t+\theta)\mathrm{d}\theta-\beta W_1\\ &=de^{\beta d}x^T(t)R_1x(t)\\ &\quad-\int_{t-d}^{t}e^{\beta(s-t+d)}x^T(s)R_1x(s)\mathrm{d}s-\beta W_1. \end{aligned} \tag{20}$$

Based on (20), we have

$$\begin{aligned} \dot{W}_1+\beta W_1&=\varphi^T(t)\Big(de^{\beta d}e_1^TR_1e_1\Big)\varphi(t)\\ &\quad-\int_{t-d}^{t}e^{\beta(s-t+d)}x^T(s)R_1x(s)\mathrm{d}s. \end{aligned} \tag{21}$$

Similarly, the derivative of W_2 is given by

$$\begin{aligned} \dot{W}_2&=\int_{-d}^{0}\int_{\theta_2}^{0}\Big[e^{\beta(s-t+d)}x^T(s)R_2x(s)\Big|_{t+\theta_1}^{t}\\ &\quad-\beta\int_{t+\theta_1}^{t}e^{\beta(s-t+d)}x^T(s)R_2x(s)\mathrm{d}s\Big]\mathrm{d}\theta_1\mathrm{d}\theta_2\\ &=\int_{-d}^{0}\int_{\theta_2}^{0}\Big[e^{\beta(s-t+d)}x^T(s)R_2x(s)\Big|_{t+\theta_1}^{t}\Big]\mathrm{d}\theta_1\mathrm{d}\theta_2\\ &\quad-\beta\int_{-d}^{0}\int_{\theta_2}^{0}\int_{t+\theta_1}^{t}e^{\beta(s-t+d)}x^T(s)R_2x(s)\mathrm{d}s\mathrm{d}\theta_1\mathrm{d}\theta_2\\ &=\int_{-d}^{0}\int_{\theta_2}^{0}e^{\beta d}x^T(t)R_2x(t)\mathrm{d}\theta_1\mathrm{d}\theta_2\\ &\quad-\int_{-d}^{0}\int_{\theta_2}^{0}e^{\beta(\theta_1+d)}x^T(t+\theta_1)R_2x(t+\theta_1)\mathrm{d}\theta_1\mathrm{d}\theta_2\\ &\quad-\beta W_2\\ &=\varphi^T(t)\left(e^{\beta d}\frac{d^2}{2}e_1^TR_2e_1\right)\varphi(t)\\ &\quad-\int_{-d}^{0}\int_{t+\theta}^{t}e^{\beta(s-t+d)}x^T(s)R_2x(s)\mathrm{d}s\mathrm{d}\theta-\beta W_2. \end{aligned} \tag{22}$$

Then, we obtain from (22) that

$$\begin{aligned} \dot{W}_2+\beta W_2&=\varphi^T(t)\left(e^{\beta d}\frac{d^2}{2}e_1^TR_2e_1\right)\varphi(t)\\ &\quad-\int_{-d}^{0}\int_{t+\theta}^{t}e^{\beta(s-t+d)}x^T(s)R_2x(s)\mathrm{d}s\mathrm{d}\theta. \end{aligned} \tag{23}$$

Combining (15), (18), (19), (21), and (23) yields

$$\begin{aligned}\dot{V}+\beta V &= \varphi^T(t)\Big(\Sigma_1^T P\Sigma_2+\Sigma_2^T P\Sigma_1+\beta\Sigma_1^T P\Sigma_1\Big)\varphi(t)\\ &\quad+\varphi^T(t)\left[e_1^T\Big(e^{\beta d}R_0+e^{\beta d}dR_1+e^{\beta d}\frac{d^2}{2}R_2\Big)e_1\right]\varphi(t)\\ &\quad-\varphi^T(t)\Big(e_2^T R_0 e_2\Big)\varphi(t)\\ &\quad-\int_{t-d}^{t} e^{\beta(s-t+d)}x^T(s)R_1x(s)\mathrm{d}s\\ &\quad-\int_{-d}^{0}\int_{t+\theta}^{t} e^{\beta(s-t+d)}x^T(s)R_2x(s)\mathrm{d}s\mathrm{d}\theta. \quad (24)\end{aligned}$$

Furthermore, by Lemma 3, we have

$$\int_{t-d}^{t} e^{\beta(s-t+d)}x^T(s)R_1x(s)\mathrm{d}s \geq \varphi^T(t)\Gamma\varphi(t), \quad (25)$$

where

$$\begin{aligned}\Gamma &= \frac{1}{d}e_3^T R_1 e_3+\frac{3}{d}\left(e_3-\frac{2}{d}e_4\right)^T R_1\left(e_3-\frac{2}{d}e_4\right)\\ &\quad+\frac{5}{d}\left(e_3-\frac{6}{d}e_4+\frac{12}{d^2}e_5\right)^T R_1\left(e_3-\frac{6}{d}e_4+\frac{12}{d^2}e_5\right),\end{aligned}$$

and by Lemma 4, we also have

$$\begin{aligned}&\int_{-d}^{0}\int_{t+\theta}^{t} e^{\beta(s-t+d)}x^T(s)R_2x(s)\mathrm{d}s\mathrm{d}\theta\\ &= \int_{-d}^{0}\int_{\theta}^{0} e^{\beta(u+d)}x^T(t+u)R_2x(t+u)\mathrm{d}u\mathrm{d}\theta\\ &\geq \varphi^T(t)\left(\frac{2}{d^2}e_4^T R_2 e_4\right)\varphi(t)\\ &\quad+\varphi^T(t)\left[\frac{16}{d^2}\Big(e_4-\frac{3}{d}e_5\Big)^T R_2\Big(e_4-\frac{3}{d}e_5\Big)\right]\varphi(t). \quad (26)\end{aligned}$$

Substituting (25) and (26) into (24), we obtain the following inequality

$$\dot{V}(t)+\beta V(t) \leq \varphi^T(t)\Theta\varphi(t), \quad (27)$$

where Θ is defined in (11).

Let

$$\begin{aligned}\alpha_1 &= \lambda_{min}(P),\\ \alpha_2 &= \Big[1+(d^2+d^4+d^6)e^{\beta d}\Big]\lambda_{max}(P)+de^{\beta d}\lambda_{max}(R_0)\\ &\quad+d^2e^{\beta d}\lambda_{max}(R_1)+d^3e^{\beta d}\lambda_{max}(R_2).\end{aligned}$$

If inequality (11) holds, then we obtain from (27) that $\dot{V}(t)+\beta V(t)\leq 0$, which yields

$$V(t) \leq e^{-\beta t}V(\phi) \leq \alpha_2 e^{-\beta t}\|\phi\|_d^2, \quad t\geq 0. \quad (28)$$

At the same time, it follows from (15) that

$$V(t) \geq \alpha_1\|\eta(t)\|^2 \geq \alpha_1\|x(t,\phi)\|^2, \quad t \geq 0. \quad (29)$$

Then, the combination of (28) and (29) leads to

$$\|x(t,\phi)\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}}e^{-\frac{\beta}{2}t}\|\phi\|_d, \quad t\geq 0. \quad (30)$$

Finally, by Definition 1 and inequality (30), we have that system (8) is exponentially stable with a decay rate $\sigma = \beta/2$. This completes the proof. □

Remark 3: As discussed in Remark 2, unlike previous studies, the new weighted integral inequalities (3) and (7) use the exponentially weighted integral vectors instead of the commonly used integral vectors. We find that the use of the exponentially weighted integral vectors poses a challenging problem in establishing the exponential stability condition. Fortunately, this problem is successfully solved by the introduction of the quadratic form $V_0 = \eta^T(t)P\eta(t)$, in which the augmented state vector $\eta(t)$ consists of not only the system state $x(t)$ but also the weighted integral states, such as $\int_{-d}^{0}\int_{t+\theta}^{t} e^{\frac{\beta}{2}(s-t+d)}x(s)\mathrm{d}s\mathrm{d}\theta$. Note that the use of the weighted integral states in V_0 is also different from previous studies, but is consistent with the new inequalities (3) and (7) that utilize the exponentially weighted integral vectors.

Remark 4: It is worth mentioning that the way to derive Theorem 1 is clearly different from the state transformation approach applied in [2] and [6]. To establish an exponential stability condition, in [2] and [6], the authors first set $z(t) = e^{\beta t}x(t)$, and then transformed system (8) into a new system as follows:

$$\dot{z}(t) = (A+\beta I_n)z(t)+e^{\beta d}A_d z(t-d). \quad (31)$$

As stated in [30], the state transformation approach usually introduces conservatism in exponential stability conditions. This is because that the state transformation approach can only obtain the asymptotic stability condition of (31), which is more restrictive than the boundedness condition of (31), i.e., the exponential stability condition of (8). The derivation of Theorem 1 is based on system (8), not on (31), thus can avoid introducing such extra conservatism.

IV. EXAMPLES

In this section, we present two numerical examples to show the effectiveness of the new exponential stability condition given by Theorem 1. In both examples, the proposed stability condition is compared with those obtained in [2], [3], [6], [29], and [30] in terms of the decay rate for various time delays. The solutions of the examples are achieved by using the YALMIP toolbox of MATLAB.

Example 1: Consider the system (8) with

$$A = \begin{bmatrix} -3 & -2\\ 1 & 0\end{bmatrix}, \quad A_d = \begin{bmatrix} -0.5 & 0.1\\ 0.3 & 0\end{bmatrix}.$$

This example was used in [6] and [29]. As done in [6] and [29], the time delay d investigated here ranges from 0.8 to 2.0. The compared results of this example are listed in Table 1. It is shown from Table 1 that among the six compared conditions, Theorem 1 achieves the largest decay rate in all cases except for $d = 0.8$ and $d = 1.0$. Note that when $d = 0.8$ and $d = 1.0$, only the decay rate of [29] is slightly larger than that obtained by Theorem 1.

Example 2: Consider the system (8) with

$$A = \begin{bmatrix} 0 & 1\\ -2 & 0.1\end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0\\ 1 & 0\end{bmatrix}.$$

TABLE 1. Decay rate σ for various d for Example 1.

d	0.8	1.0	1.2	1.4	1.6	1.8	2.0
[2]	0.9366	0.5903	0.3400	0.1813	0.0752	0.0014	-
[3]	0.7344	0.6715	0.6145	0.5642	0.5202	0.4818	0.4481
[6]	0.9366	0.9192	0.8990	0.8115	0.6990	0.6148	0.5494
[29]	0.9429	0.9220	0.8894	0.7951	0.7043	0.6313	0.5719
[30]	0.9366	0.9192	0.8990	0.8712	0.7493	0.6573	0.5856
Theorem 1	0.9366	0.9192	0.8990	0.8734	0.7511	0.6589	0.5870

TABLE 2. Decay rate σ for various d for Example 2.

d	0.3	0.5	0.8	1.0	1.5	1.6
[2]	-	-	-	-	-	-
[3]	-	-	-	-	-	-
[6]	-	-	-	-	-	-
[29]	-	-	-	-	-	-
[30]	0.1022	0.2148	0.4195	0.4978	0.1108	0.0496
Theorem 1	0.1022	0.2156	0.4322	0.5210	0.1151	0.0556

This example was used in [30]. Also, as done in [30], the time delay d investigated in this example ranges from 0.3 to 1.6. The results are reported in Table 2. It is surprising to see that the stability conditions of [2], [3], [6], and [29] cannot obtain any result when solving their LMIs for each time delay. Note that while in Example 1, the condition of [6] obtain the best results for $d = 0.8$ and $d = 1.0$. The significantly different performance of the condition [6] in these two examples indicates that it is, in fact, sensitive to different examples. However, both the stability conditions of Theorem 1 and [30] still work well in the example. Furthermore, the decay rate of Theorem 1 is not smaller than that of [30] in all cases. Especially, the decay rate of Theorem 1 is 4.66% and 12.09% larger than that of [30] for $d = 1.0$ and $d = 1.6$, respectively.

To sum up, the above two examples show that the condition of Theorem 1 is, on the whole, less conservative than those of [2], [3], [6], [29], and [30].

V. CONCLUSION

In this paper, we have studied the exponential stability of time-delay systems. The main contributions of this paper are as follows:

(1) Based on the AFIIs (1) and (2) [23], two new weighted integral inequalities have been derived for exponential stability analysis. To improve the lower bounds of the weighted integral quadratic terms, the exponentially weighted integral vectors (such as $\int_a^b e^{\frac{\beta}{2}(u-a)}\omega(u)\mathrm{d}u$ and $\int_a^b\int_s^b e^{\frac{\beta}{2}(u-a)}\omega(u)\mathrm{d}u\mathrm{d}s$), instead of the integral vectors (such as $\int_a^b \omega(u)\mathrm{d}u$ and $\int_a^b\int_s^b \omega(u)\mathrm{d}u\mathrm{d}s$), are used in the new inequalities (3) and (7). As far as we know, this is the fist time that such exponentially weighted integral vectors have been used in integral inequalities. Besides, note that the AFIIs were originally presented for asymptotic stability analysis, and can be regarded as special cases of the new weighted integral inequalities.

(2) With the help of the inequalities (3) and (7), a new LMI condition has been established for the exponential stability of time-delay systems. To solve the problem raised by the use of the exponentially weighted integral vectors in (3) and (7), a new quadratic form given by $V_0 = \eta^T(t)P\eta(t)$ is introduced into our Lyapunov–Krasovskii functional. It is worth noting that in the augmented state vector $\eta(t)$, the weighted integral states (such as $\int_{-d}^0\int_{t+\theta}^t e^{\frac{\beta}{2}(s-t+d)}x(s)\mathrm{d}s\mathrm{d}\theta$) are used. To our knowledge, this has not been done before. Two examples have also been provided to show that the new stability condition is less conservative than existing ones in determining the decay rate for the considered systems.

In this paper, the new weighted integral inequalities have only been applied to the exponential stability analysis of the constant time-delay systems. Our future work is to extend the application of these inequalities to varying time-delay systems.

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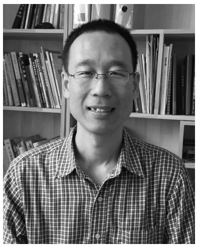
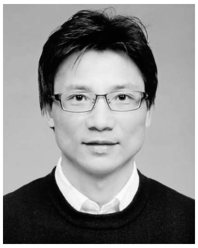


CHENG GONG received the B.S., M.S., and Ph.D. degrees from the Harbin Institute of Technology, Harbin, China, in 2002, 2004, and 2008, respectively. He is currently an Associate Professor with the School of Mathematics and Science, Heilongjiang University. His research interests mainly include stability analysis and stabilization for linear time-delay systems, H-infinity control and filtering, T-S fuzzy system with time delay, and integral inequality related to Lyapunov stability theory.

GUOPU ZHU (SM'14) received the B.S. degree from Jilin University, China, in 2002, and the M.S. and Ph.D. degrees from the Harbin Institute of Technology, China, in 2004 and 2007, respectively. He is currently a Professor with the Shenzhen Institutes of Advanced Technology, Chinese Academy of Sciences (CAS), China. He has authored or co-authored over 20 papers in the SCI journals. His main research areas are multimedia security, control theory, and image processing. He is a member of the Youth Innovation Promotion Association of CAS.



LIGANG WU (M'10–SM'12) received the B.S. degree in automation from the Harbin University of Science and Technology, China, in 2001, and the M.E. degree in navigation guidance and control and the Ph.D. degree in control theory and control engineering from the Harbin Institute of Technology, China, in 2003 and 2006, respectively. From 2006 to 2007, he was a Research Associate with the Department of Mechanical Engineering, The University of Hong Kong, Hong Kong. From 2007 to 2008, he was a Senior Research Associate with the Department of Mathematics, City University of Hong Kong, Hong Kong. From 2012 to 2013, he was a Research Associate with the Department of Electrical and Electronic Engineering, Imperial College London, London, U.K. In 2008, he joined the Harbin Institute of Technology, China, as an Associate Professor, and then was promoted to Professor in 2012. He was a recipient of the National Science Fund for Distinguished Young Scholars in 2015. He received the China Young Five Four Medal in 2016, and was named to the 2015 Thomson Reuters Highly Cited Researchers list.

His current research interests include switched systems, stochastic systems, computational and intelligent systems, multidimensional systems, sliding mode control, and flight control. He has authored five research monographs and over 120 research papers in the international referred journals. He currently serves as an Associate Editor of a number of journals, including the IEEE Transactions on Automatic Control, the IEEE/ASME Transactions on Mechatronics, Information Sciences, Signal Processing, and the *IET Control Theory and Applications*. He is also an Associate Editor of the Conference Editorial Board and the IEEE Control Systems Society.