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M/M/1 Multiple Vacation Queueing Systems With Differentiated Vacations and Vacation Interruptions

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ABSTRACT We consider an M/M/1 multiple vacation queueing system with two types of server vacations. Type 1 vacation is taken after the server has exhaustively served all the customers in the system, where the number of customers served is at least one. Type 2 vacation is taken when the server returns from a vacation and finds no customer waiting. Each type of vacation can be interrupted when the number of customers in the system reaches two predefined thresholds, where each vacation type has a different threshold. It is assumed that service times and vacation durations are exponentially distributed with different means. We present a steady-state solution of the system under two vacation interruption policies.

INDEX TERMS Complete vacation interruption, differentiated vacation system, multiple vacation queueing system, partial vacation interruption, vacation interruption.

I. INTRODUCTION

A server in a queueing system is said to be on vacation when it becomes unavailable for a period of time from its customers. The vacation could occur as a result of many factors like server breakdown, server maintenance or even when it is taken away from its primary customers to serve elsewhere. A lot of work has been done since the idea of vacation queueing system was first introduced by Levy and Yechiali in [1]. Different types of vacation models have been discussed by researchers like Doshi in [2] and [3] while books by Takagi [4] and Tian and Zhang [5] are dedicated to the topic.

In an earlier paper [6], we proposed a single-server multiple vacation queueing system with differentiated vacations, the M/M/1/DV queue, in which two types of vacation were defined. The first type of vacation, *type* 1 *vacation*, is taken after a busy period in which at least one customer is served; and the second type of vacation, *type* 2 *vacation*, is taken when the server returns from a vacation and finds the queue empty. The two types of vacation have different durations.

Queueing systems with differentiated vacations can be used to model many physical systems. One example is a hospital emergency room operation where a type 1 vacation is used to set up the room for the next wave of patients; getting the equipment ready and performing any cleanups and sterilization. Similarly, a type 2 vacation can be used to give the emergency room personnel some actual rest, given that the necessary cleanup and setup preparation for the room have been done.

In this paper, we extend the model to include vacation interruption by forcing the server to return from a vacation when the number of customers in the system reaches some predefined thresholds. Vacation queueing systems with vacation interruption can be used to model many real life situations. For example, a doctor's break (or vacation) can be interrupted by certain hospital emergency situations. Similarly, the vacation of an active-duty soldier can be interrupted by some pressing public defense needs.

When an interruption of a differentiated vacation queueing system is desired, it makes sense to interrupt a type 2 vacation before interrupting a type 1 vacation. Thus, we assume that a type 2 vacation may be interrupted when the number of customers in the system reaches the threshold value k_2, and a type 1 vacation can be interrupted when the number of customers in the system reaches $k_1 \geq k_2$. If we choose to interrupt the server's vacation only when a type 2 vacation is being taken, we refer to this as a *partial vacation interruption*. Similarly, if both types of vacation may be interrupted, we refer to this as a *complete vacation interruption*. The rationale for a partial vacation interruption is that if a type 1 vacation is used for service setup and preparation before service in the next busy period can begin, then it cannot be interrupted once it starts. A type 1 vacation may be interrupted only if it is not used to prepare the service area for the next wave of customers, and this leads to the complete vacation model.

Vacation queueing systems with vacation interruption was first introduced by Li and Tian in [7] where they studied the M/M/1 queue with working vacation and vacation interruption. M/M/1 queue with working vacation was introduced by Servi and Finn in [8]. Li and Tian also considered the discrete-time GI/Geo/1 queue with working vacations and vacation interruption in [9]. Li, Tian and Ma analyzed the GI/M/1 queue with working vacations and vacation interruption in [10]. Zhang and Hou analyzed the M/G/1 queue with working vacations and vacation interruption in [11]. Ayyappan, Sekar and Ganapathi considered an M/M/1 retrial queueing system with vacation interruption which uses an Erlang-k type distribution in [12]. In [13] Krishnamoorthy and Sreenivasan considered an M/M/2 queueing system with heterogeneous servers where one server is always available and the other goes on a working vacation when there are no customers in the system. In [14] Sreenivasan, Chakravarthy and Krishnamoorthy considered a single server queueing model in which customers arrive according to a Markovian arrival process. They also introduced a threshold, N, where $1 \leq N < \infty$, such that the server offering services (at a lower rate) during a vacation will have the vacation interrupted the moment the queue size reaches N.

The paper is organized as follows. The model is described in Section II. Steady-state analysis of the model is given in Section III. Numerical results are shown in section IV, and concluding remarks are made in section V.

II. SYSTEM MODEL

We consider a multiple vacation queueing system where customers arrive according to a Poisson process with rate λ. The time to serve a customer is assumed to be exponentially distributed with mean $1/\mu$, where $\mu > \lambda$. Two types of vacation may be taken: a type 1 vacation that may be taken after a busy period in which at least one customer is served, and a type 2 vacation that is taken after the server returns from any vacation and finds the system empty. In other words, a type 2 vacation is taken when the server returns from either a type 1 vacation or a type 2 vacation and finds the system empty. Thus, we say that a type 2 vacation is taken after a busy period of zero duration while a type 1 vacation is taken after a busy period of non-zero duration. The duration of a type 1 vacation is assumed to be exponentially distributed with a mean of $1/\gamma_1$, and the duration of a type 2 vacation is assumed to be exponentially distributed with a mean of $1/\gamma_2$. We define the state of the system by $(a,\ b)$, where a is the number of customers in the system and b is defined as follows:

$$b = \begin{cases} 0 & \text{if the server is actively serving customers} \\ 1 & \text{if the server is currently on a type 1 vacation} \\ 2 & \text{if the server is currently on a type 2 vacation} \end{cases}$$

Thus, the state transition-rate diagram of the original model is shown in Figure 1.

As discussed earlier, we extend the model by permitting the server's vacation to be interrupted. We consider two types of vacation interruption policies:

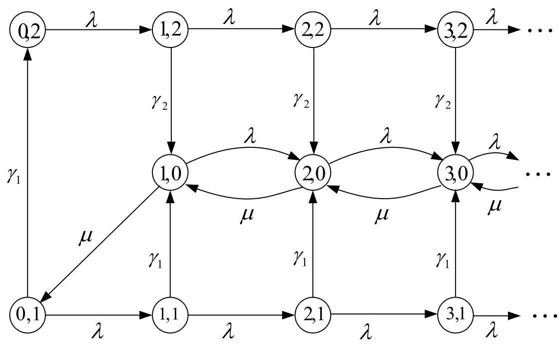

FIGURE 1. Original M/M/1/DV queue without vacation Interruption.

1) *Partial Interruption Policy*: Under this policy, the server cannot be interrupted when he is on a type 1 vacation; he can only be interrupted when he is on a type 2 vacation. The server is interrupted when he is on a type 2 vacation and the number of customers in the system reaches a predefined threshold, k_2.
2) *Complete Interruption Policy*: In this case, the server's vacation may be interrupted when he is on either type of vacation. The server's vacation is interrupted when the number of customers in the system reaches k_1 when he is on type 1 vacation and when the number reaches k_2 when he is on type 2 vacation, where we assume that $k_1 > k_2$ because we would like the server to be interrupted earlier when he takes a vacation without serving a customer than when he takes a vacation after having a non-zero busy period.

III. ANALYSIS OF THE MODELS

In this section we provide a steady-state analysis of the two vacation interruption policies. Let $P_{n,k}$ denote the limiting-state probability that the system is in state (n, k) where n denotes the number of customers in the system and k is the state of the server. Specifically, $k = 0$ when the server is actively serving customers, $k = 1$ when server is on a type 1 vacation and $k = 2$ when the server is on a type 2 vacation. In [6] it was shown that

$$P_{nk} = \begin{cases} \rho\left[\frac{\alpha_1\beta_1\left(\beta_1^{n-1}-\rho^{n-1}\right)}{\beta_1-\rho} + \frac{\alpha_2\beta_2\left(\beta_2^{n-1}-\rho^{n-1}\right)}{\beta_2-\rho} + \rho^{n-2}\right]P_{1,0} & k=0 \\ \alpha_1\beta_1^n P_{1,0} & k=1 \\ \alpha_2\beta_2^n P_{1,0} & k=2 \end{cases}$$

where

$$P_{1,0} = \frac{(1-\rho)(1-\beta_1)(1-\beta_2)}{(1-\beta_1)(1-\beta_2)+\alpha_1(1-\beta_2)\{1-\rho(1-\beta_1)\} + \alpha_2(1-\beta_1)\{1-\rho(1-\beta_2)\}}$$

$\rho = \lambda/\mu$ is the offered load, $\alpha_1 = \mu/(\lambda+\gamma_1)$, $\alpha_2 = \mu\gamma_1/\lambda(\lambda+\gamma_1)$, $\beta_1 = \lambda/(\lambda+\gamma_1) < 1$, and $\beta_2 = \lambda/(\lambda+\gamma_2) < 1$.

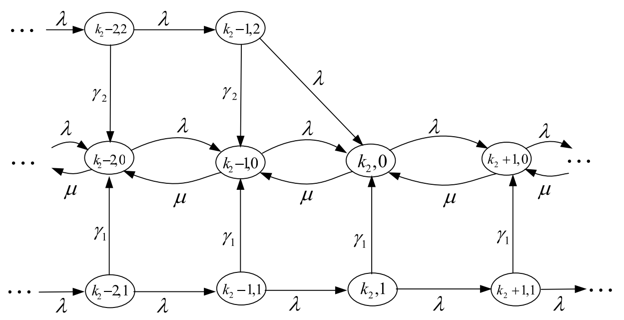

FIGURE 2. M/M/1/DV queue with partial vacation interruption.

A. ANALYSIS OF THE PARTIAL VACATION INTERRUPTION POLICY

A partial state transition-rate diagram for the model is shown in Figure 2. The diagram includes those states near where or at which a vacation interruption takes place. Specifically, when the server is on a type 2 vacation and there are k_2-1 customers in the system (i.e., the system is in state $(k_2-1, 2)$), the next customer arrival forces the vacation to end, and a transition to state $(k_2, 0)$ takes place.

Theorem 1: Let T denote the total time a customer spends in the system, which is also called the delay in the system. Also, let N denote the number of customers in the system. Then the mean delay in the system is given by

$$E[T] = \frac{E[N]}{\lambda}$$

where

$$\begin{aligned} E[N] &= [H_1 + H_2 + H_3 + H_4] P_{1,0} \\ H_1 &= \sum_{k=1}^{k_2-1} k\rho \left[\frac{\alpha_1\beta_1\left(\beta_1^{k-1}-\rho^{k-1}\right)}{\beta_1-\rho} + \frac{\alpha_2\beta_2\left(\beta_2^{k-1}-\rho^{k-1}\right)}{\beta_2-\rho} + \rho^{k-2} \right] \\ &= \frac{\alpha_1\beta_1\rho}{\beta_1-\rho}\left[\frac{1+\beta_1^{k_2}(k_2\beta_1-k_2-1)}{(1-\beta_1)^2} - \frac{1+\rho^{k_2}(k_2\rho-k_2-1)}{(1-\rho)^2}\right] + \frac{\alpha_2\beta_2\rho}{\beta_2-\rho} \\ &\quad \times \left[\frac{1+\beta_2^{k_2}(k_2\beta_2-k_2-1)}{(1-\beta_2)^2} - \frac{1+\rho^{k_2}(k_2\rho-k_2-1)}{(1-\rho)^2}\right] \\ &\quad + \left[\frac{1+\rho^{k_2}(k_2\rho-k_2-1)}{(1-\rho)^2}\right] \\ H_2 &= \sum_{k=k_2}^{\infty} k \left[\rho^{k-k_2} A(k_2,0) + \frac{\alpha_1\beta_1^{k_2}\rho\left(\beta_1^{k-k_2}-\rho^{k-k_2}\right)}{\beta_1-\rho}\right] \\ &= A(k_2,0)\left[\frac{\rho+k_2(1-\rho)}{(1-\rho)^2}\right] + \frac{\alpha_1\beta_1^{k_2}\rho}{\beta_1-\rho} \\ &\quad \times \left[\frac{\beta_1+k_2(1-\beta_1)}{(1-\beta_1)^2} - \frac{\rho+k_2(1-\rho)}{(1-\rho)^2}\right] \end{aligned}$$

$$\begin{aligned} H_3 &= \sum_{k=0}^{\infty} k\alpha_1\beta_1^k = \frac{\alpha_1\beta_1}{(1-\beta_1)^2} \\ H_4 &= \sum_{k=0}^{k_2-1} k\alpha_2\beta_2^k = \alpha_2\left[\frac{\beta_2-k_2\beta_2+(k_2-1)\beta_2^{k_2-1}}{(1-\beta_1)^2}\right] \\ P_{1,0} &= \frac{1}{A_1+A_2+A_3+A_4} \end{aligned}$$

where

$$\begin{aligned} A_1 &= \sum_{k=1}^{k_2-1} \rho \left[\frac{\alpha_1\beta_1\left(\beta_1^{k-1}-\rho^{k-1}\right)}{\beta_1-\rho} + \frac{\alpha_2\beta_2\left(\beta_2^{k-1}-\rho^{k-1}\right)}{\beta_2-\rho} + \rho^{k-2} \right] \\ &= \frac{\alpha_1\beta_1\rho}{\beta_1-\rho}\left[\frac{1-\beta_1^{k_2}}{1-\beta_1} - \frac{1-\rho^{k_2}}{1-\rho}\right] + \frac{\alpha_2\beta_2\rho}{\beta_2-\rho}\left[\frac{1-\beta_2^{k_2}}{1-\beta_2} - \frac{1-\rho^{k_2}}{1-\rho}\right] + \left[\frac{1-\rho^{k_2}}{1-\rho}\right] \\ A_2 &= \sum_{k=k_2}^{\infty}\left\{\rho^{k-k_2}A(k_2,0) + \alpha_1\beta_1^{k_2}\rho\left[\frac{\beta_1^{k-k_2}-\rho^{k-k_2}}{\beta_1-\rho}\right]\right\} \\ &= \frac{A(k_2,0)}{1-\rho} + \frac{\alpha_1\beta_1^{k_2}\rho}{(1-\beta_1)(1-\rho)} \\ A_3 &= \sum_{k=0}^{\infty}\alpha_1\beta_1^k = \frac{\alpha_1}{1-\beta_1} \\ A_4 &= \sum_{k=0}^{k_2-1} P_{k,2} = \sum_{k=0}^{k_2-1}\alpha_2\beta_2^k = \frac{\alpha_2\left(1-\beta_2^{k_2}\right)}{1-\beta_2} \\ A(k_2,0) &= \rho\left\{\frac{\rho\alpha_1\beta_1\left(\beta_1^{k_2-2}-\rho^{k_2-2}\right)}{\beta_1-\rho} + \frac{\rho\alpha_2\beta_2\left(\beta_2^{k_2-2}-\rho^{k_2-2}\right)}{\beta_2-\rho} + \rho^{k_2-2} + \alpha_1\beta_1^{k_2-1} + \alpha_2\beta_2^{k_2-1}\right\} \end{aligned}$$

The proof of this theorem is given in Appendix 1.

B. ANALYSIS OF THE COMPLETE VACATION INTERRUPTION POLICY

A partial state transition-rate diagram for the model is shown in Figure 3. As in Figure 2, the diagram includes those states near where or at which vacation interruptions take place. Specifically, when the server is on a type 2 vacation and there are k_2-1 customers in the system (i.e., the system is in state $(k_2-1, 2)$), the next customer arrival forces the vacation to end, and a transition to state $(k_2, 0)$ takes place. Similarly, when the server is on a type 1 vacation and there are k_1-1

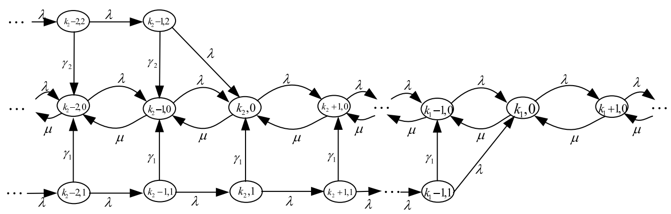

FIGURE 3. **M/M/1/DV queue with complete vacation interruption.**

customers in the system (i.e., the system is in state $(k_1 - 1, 1)$), the next customer arrival forces the vacation to end, and a transition to state $(k_1, 0)$ takes place.

Theorem 2: The mean delay in the system is given by

$$E[T] = \frac{E[N]}{\lambda}$$

where

$$E[N] = [J_1 + J_2 + J_3 + J_4 + J_5]P_{1,0}$$

$$J_1 = \sum_{k=1}^{k_2-1} k\rho \left[\frac{\alpha_1\beta_1\left(\beta_1^{k-1} - \rho^{k-1}\right)}{\beta_1 - \rho} + \frac{\alpha_2\beta_2\left(\beta_2^{k-1} - \rho^{k-1}\right)}{\beta_2 - \rho} + \rho^{k-2} \right]$$

$$= \frac{\alpha_1\beta_1\rho}{\beta_1 - \rho}\left[\frac{1 + \beta_1^{k_2}(k_2\beta_1 - k_2 - 1)}{(1-\beta_1)^2} - \frac{1 + \rho^{k_2}(k_2\rho - k_2 - 1)}{(1-\rho)^2}\right] + \frac{\alpha_2\beta_2\rho}{\beta_2 - \rho}\left[\frac{1 + \beta_2^{k_2}(k_2\beta_2 - k_2 - 1)}{(1-\beta_2)^2} - \frac{1 + \rho^{k_2}(k_2\rho - k_2 - 1)}{(1-\rho)^2}\right] + \left[\frac{1 + \rho^{k_2}(k_2\rho - k_2 - 1)}{(1-\rho)^2}\right]$$

$$J_2 = \sum_{k=k_2}^{k_1-1} kP_{k,0}$$

$$= \sum_{k=k_2}^{k_1-1} k\left[\rho^{k-k_2}A(k_2, 0) + \frac{\alpha_1\beta_1^{k_2}\rho\left(\beta_1^{k-k_2} - \rho^{k-k_2}\right)}{\beta_1 - \rho}\right]$$

$$= A(k_2, 0)\left[\frac{\rho - k_1\rho^{k_1} + (k_1 - 1)\rho^{k_1+1}}{(1-\rho)^2} + \frac{k_2\left(1 - \rho^{k_2}\right)}{1-\rho}\right] + \frac{\alpha_1\beta_1^{k_2}\rho}{\beta_1 - \rho}\left[\frac{\beta_1 - k_1\beta_1^{k_1} + (k_1 - 1)\beta_1^{k_1+1}}{(1-\beta_1)^2} + \frac{k_2\left(1 - \beta_1^{k_1}\right)}{1-\beta_1} - \frac{\rho - k_1\rho^{k_1} + (k_1 - 1)\rho^{k_1+1}}{(1-\rho)^2} - \frac{k_2\left(1-\rho^{k_2}\right)}{1-\rho}\right]$$

$$J_3 = \sum_{k=k_1}^{\infty} k\rho^{k-k_1}A(k_1, 0) = A(k_1, 0)\frac{\rho + k_1(1-\rho)}{(1-\rho)^2}$$

$$J_4 = \sum_{k=0}^{k_1-1} kP_{k,1} = \sum_{k=0}^{k_1-1} k\alpha_1\beta_1^k = \alpha_1\left[\frac{\beta_1 - k_1\beta_1^{k_1} + (k_1 - 1)\beta_1^{k_1+1}}{(1-\beta_1)^2}\right]$$

$$J_5 = \sum_{k=0}^{k_2-1} kP_{k,2} = \sum_{k=0}^{k_2-1} k\alpha_2\beta_2^k = \alpha_2\left[\frac{\beta_2 - k_2\beta_2^{k_2} + (k_2 - 1)\beta_2^{k_2+1}}{(1-\beta_2)^2}\right]$$

$$P_{1,0} = \frac{1}{B_1 + B_2 + B_3 + B_4 + B_5}$$

where

$$B_1 = \sum_{k=1}^{k_2-1}\rho\left[\frac{\alpha_1\beta_1\left(\beta_1^{k-1} - \rho^{k-1}\right)}{\beta_1 - \rho} + \frac{\alpha_2\beta_2\left(\beta_2^{k-1} - \rho^{k-1}\right)}{\beta_2 - \rho} + \rho^{k-2}\right]$$

$$= \frac{\alpha_1\beta_1\rho}{\beta_1 - \rho}\left[\frac{1 - \beta_1^{k_2}}{1 - \beta_1} - \frac{1 - \rho^{k_2}}{1-\rho}\right] + \frac{\alpha_2\beta_2\rho}{\beta_2 - \rho}\left[\frac{1 - \beta_2^{k_2}}{1 - \beta_2} - \frac{1 - \rho^{k_2}}{1-\rho}\right] + \left[\frac{1 - \rho^{k_2}}{1-\rho}\right]$$

$$B_2 = \sum_{k=k_2}^{k_1-1}\rho^{k-k_2}A(k_2, 0) + \alpha_1\beta_1^{k_2}\rho\left[\frac{\beta_1^{k-k_2} - \rho^{k-k_2}}{\beta_1 - \rho}\right]$$

$$= A(k_2, 0)\left[\frac{1 - \rho^{k_1}}{1-\rho}\right] + \frac{\alpha_1\beta_1^{k_2}\rho}{(\beta_1 - \rho)}\left[\frac{1 - \beta_1^{k_2}}{1 - \beta_1} - \frac{1 - \rho^{k_2}}{1-\rho}\right]$$

$$B_3 = \sum_{k=k_1}^{\infty}\rho^{k-k_1}A(k_1, 0) = \frac{A(k_1, 0)}{1-\rho}$$

$$B_4 = \sum_{k=0}^{k_1-1}\alpha_1\beta_1^k = \alpha_1\left[\frac{1 - \beta_1^{k_1}}{1 - \beta_1}\right]$$

$$B_5 = \sum_{k=0}^{k_2-1}\alpha_2\beta_2^k = \alpha_2\left[\frac{1 - \beta_2^{k_2}}{1 - \beta_2}\right]$$

$$A(k_1, 0) = \rho\left\{\rho^{k_1-k_2-1}A(k_2, 0) + \alpha_1\beta_1^{k_2}\rho\left[\frac{\beta_1^{k_1-k_2-1} - \rho^{k_1-k_2-1}}{\beta_1 - \rho}\right] + \alpha_1\beta_1^{k_1-1}\right\}$$

The proof of this theorem is given in Appendix 2.

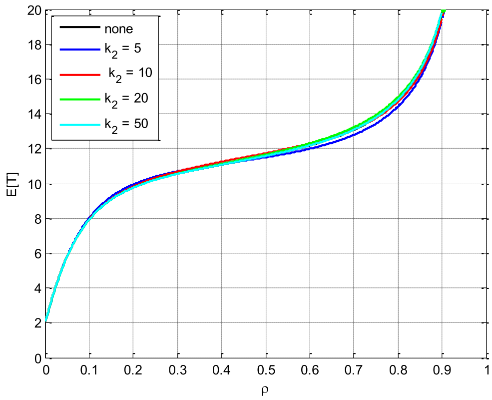

FIGURE 4. **Mean system delay versus offered load for $\gamma_1 = 0.1$ and $\gamma_2 = 1$.**

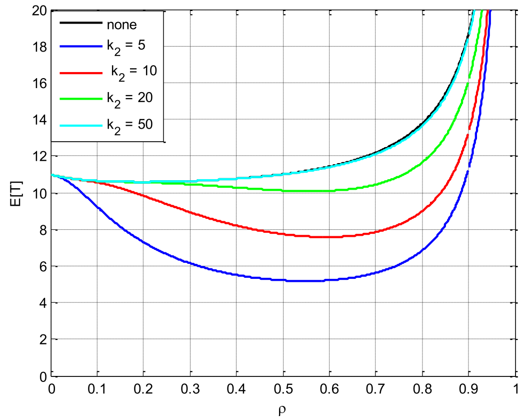

FIGURE 6. **Mean system delay versus offered load for $\gamma_1 = 1$ and $\gamma_2 = 0.1$.**

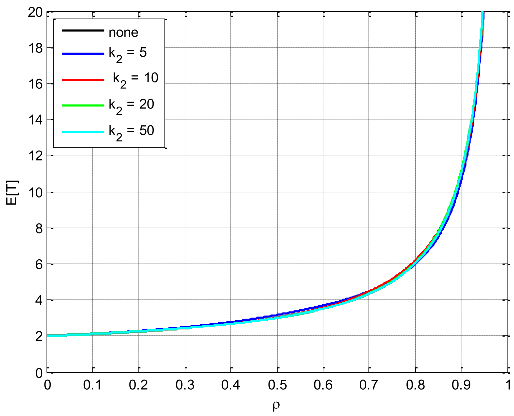

FIGURE 5. **Mean system delay versus offered load for $\gamma_1 = 1$ and $\gamma_2 = 1$.**

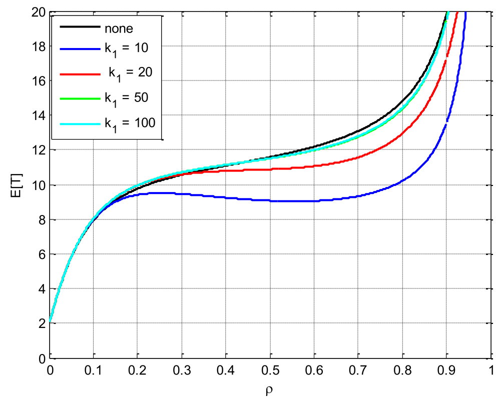

FIGURE 7. **Mean system delay versus offered load for $\gamma_1 = 0.1$, $\gamma_2 = 1$ and $k_2 = 5$.**

IV. NUMERICAL RESULTS

As in the original model in [6], we make the following assumptions: $\mu = 1$, which implies that $\rho = \lambda/\mu = \lambda$. Also,

$$\alpha_1 = \frac{\mu}{\lambda + \gamma_1} = \frac{1}{\rho\left(1 + \gamma_1/\rho\right)} = \frac{1}{\rho + \gamma_1}$$

$$\alpha_2 = \left(\frac{\gamma_1}{\lambda}\right)\alpha_1 = \frac{\gamma_1}{\rho^2\left(1 + \gamma_1/\rho\right)} = \frac{\gamma_1}{\rho\,(\rho + \gamma_1)}$$

$$\beta_1 = \frac{\lambda}{\lambda\,(1 + \gamma_1/\lambda)} = \frac{1}{1 + \gamma_1/\lambda} = \frac{\rho}{\rho + \gamma_1}$$

$$\beta_2 = \frac{\lambda}{\lambda\,(1 + \gamma_2/\lambda)} = \frac{1}{1 + \gamma_2/\lambda} = \frac{\rho}{\rho + \gamma_2}.$$

We first consider the effect of different thresholds on the mean time spent in the system.

A. PARTIAL VACATION INTERRUPTION POLICY

In this case only a type 2 vacation can be interrupted and this occurs when the number of customers in the system reaches k_2. We consider the following values: $k_2 = 5$, 10, 20 and 50 in three different cases: when type 1 vacation duration is longer than type 2 vacation, that is, $\gamma_1 < \gamma_2$; when type 1 vacation is shorter than type 2 vacation, that is, $\gamma_1 > \gamma_2$; and when both vacation types are of the same duration, that is $\gamma_1 = \gamma_2$. For comparison, we have included the results from original model with the label "none" in the legend, which means that the server had no interruption while on vacation.

Figures 4 and 5 show that there is no significant impact to the mean delay as k_2 is varied. However, the value of the expected delay is higher in Figure 4 when type 1 vacation is much longer than type 2 vacation compared to the case when both vacation durations are the same in Figure 5. Thus, when $\gamma_1 \leq \gamma_2$, interrupting a vacation of longer or equal duration does not have a significant impact on the mean delay. However, in Figure 6 where the type 2 vacation duration is much longer than the type 1 vacation duration we notice that the mean delay decreases as k_2 decreases. This makes sense since interrupting a longer vacation has more impact on the mean delay than interrupting a shorter vacation.

Observe that the behavior of the system at very high value of ρ tends to be identical because at that load level the server is more likely to be busy serving customers than to be on any type of vacation.

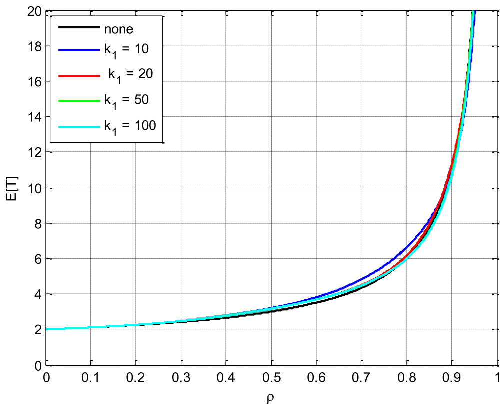

FIGURE 8. **Mean system delay versus offered load for $\gamma_1 = 1$, $\gamma_2 = 1$ and $k_2 = 5$.**

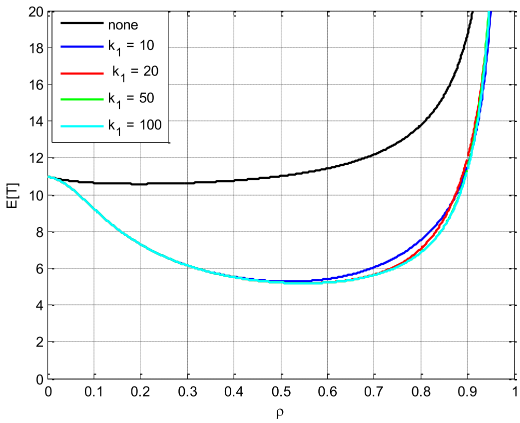

FIGURE 9. **Mean system delay versus offered load for $\gamma_1 = 1$, $\gamma_2 = 0.1$ and $k_2 = 5$.**

B. COMPLETE VACATION INTERRUPTION POLICY

In this case both types of vacation can be interrupted once the number of customers in the system reaches k_1 and k_2 for type 1 vacation and type 2 vacation, respectively, where $k_1 > k_2$. We consider the following values: $k_2 = 5$ and $k_1 = 10, 20, 50$ and 100 in three different cases: when type 1 vacation duration is longer than type 2 vacation, that is, $\gamma_1 < \gamma_2$; when both types of vacation are of the same duration, that is, $\gamma_1 = \gamma_2$; and when type 1 vacation duration is shorter than type 2 vacation, that is, $\gamma_1 > \gamma_2$. We also include the results from the original model where no vacation interruption occurs with the label "none" in the legend.

When a type 1 vacation duration is longer than that of a type 2 vacation, Figure 7 shows that $E[T]$ decreases as k_1 decreases. Thus, the mean delay increases as k_1 increases with the limiting case being when there is no vacation interruption. In Figure 8, where both types of vacation duration are the same, there is no significant improvement in the mean delay with varying k_1. Finally, Figure 9 shows the result when type 2 vacations are longer than type 1 vacations. In this case there is a significant reduction in the mean delay at medium to high values of ρ. This is due to the fact that we are interrupting a vacation whose duration is longer than the other vacation type.

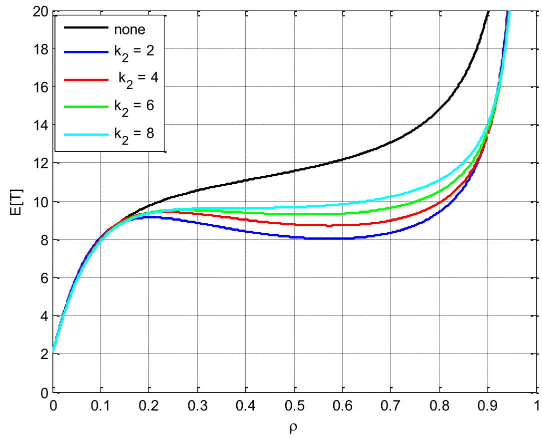

FIGURE 10. **Mean system delay versus offered load for $\gamma_1 = 0.1$, $\gamma_2 = 1$ and $k_1 = 10$.**

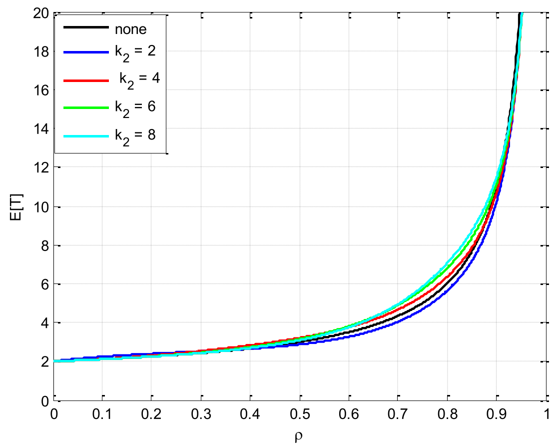

FIGURE 11. **Mean system delay versus offered load for $\gamma_1 = 1$, $\gamma_2 = 1$ and $k_1 = 10$.**

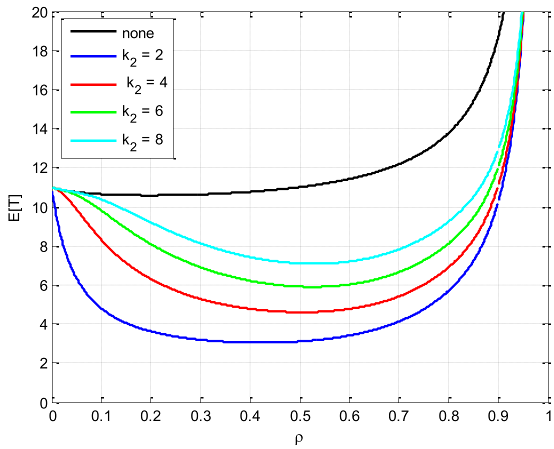

FIGURE 12. **Mean system delay versus offered load for $\gamma_1 = 1$, $\gamma_2 = 0.1$ and $k_1 = 10$.**

Figures 10, 11 and 12 show the performance of the system for different values of k_2 when $k_1 = 10$. Similarly, Figures 13, 14 and 15 show the performance of the system for different values of k_2 when $k_1 = 50$. As can be seen from the figures, early interruption of both types of vacation has a

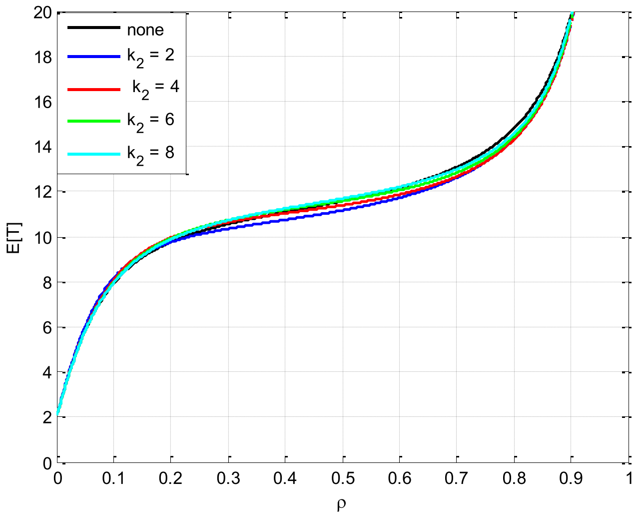

FIGURE 13. Mean system delay versus offered load for $\gamma_1 = 0.1$, $\gamma_2 = 1$ and $k_1 = 50$.

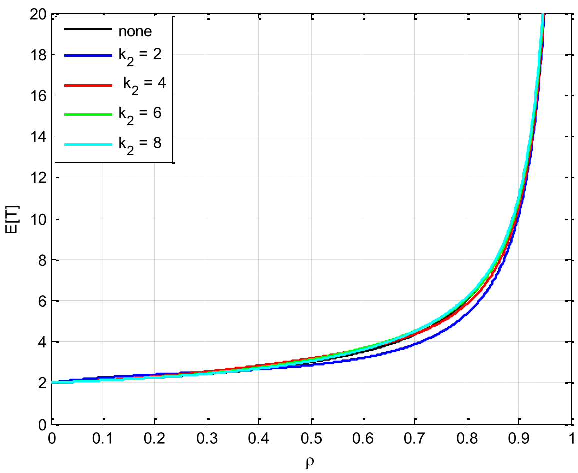

FIGURE 14. Mean system delay versus offered load for $\gamma_1 = 1$, $\gamma_2 = 1$ and $k_1 = 50$.

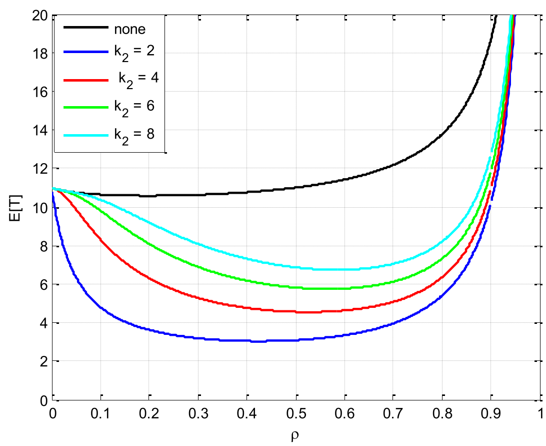

FIGURE 15. Mean system delay versus offered load for $\gamma_1 = 1$, $\gamma_2 = 0.1$ and $k_1 = 50$.

significant impact on the system delay. Also, k_2 has the most significant impact on the system delay when type 2 vacation duration is much greater than of type 1 vacation.

V. CONCLUSION

We have considered an M/M/1 multiple vacation queueing system with two types of vacations, which we defined as type 1 vacation and type 2 vacation. A type 1 vacation is taken when the server has served at least one customer and there is no other customer in the system; that is, it is taken after a busy period that involves the service of at least one customer. A type 2 vacation is taken when the server returns from a vacation and there is no customer waiting to be served. In addition to these differentiated vacations the server's vacation can be interrupted when the number of waiting customers reaches some predefined thresholds k_1, when the server is on a type 1 vacation, and k_2, when the server is on a type 2 vacation, where it is assumed that $k_1 > k_2$.

We also defined two types of vacation interruptions: *partial vacation interruption* in which the server can be interrupted only during a type 2 vacation, and *complete vacation interruption* in which the server can be interrupted during any vacation. We carried out the steady-state analysis of the system to investigate the impact vacation interruptions have on the system for different values of vacation durations and different values of k_1 and k_2 compared to the case of no vacation interruption.

The results indicate that for partial interruption, the mean time a customer spends in the system, which we define as the mean system delay, is sensitive to early vacation termination when type 2 vacation duration is much larger than that of type 1. Also as the threshold increases, the vacation termination has no significant advantage over when there was no termination. For complete interruption, the results also show that the mean system delay is greatly impacted by vacation termination when type 2 vacation duration is larger than that type 1, modestly impacted by vacation termination when type 1 vacation duration is larger than that of type 2. However, when type 1 vacation duration is equal to that of type 2, the system is not very sensitive to vacation interruptions. Thus, the greatest improvement in mean system delay tends to be obtained when the vacation that has a longer duration is interrupted very early.

APPENDIX 1

From global balance in Figure 1 we have that

$$\begin{aligned} \mu P_{1,0} &= (\lambda + \gamma_1) P_{0,1} \\ \gamma_1 \cdot P_{0,1} &= \lambda P_{0,2} \\ \lambda P_{k-1,2} &= (\lambda + \gamma_2) P_{k,2} \quad k = 1, 2, \ldots, k_2 - 1 \end{aligned}$$

Thus,

$$\begin{aligned} P_{0,1} &= \frac{\mu}{\lambda + \gamma_1} P_{1,0} = \alpha_1 P_{1,0} \\ P_{0,2} &= \frac{\gamma_1}{\lambda} P_{0,1} = \frac{\gamma_1}{\lambda} \alpha_1 P_{1,0} = \alpha_2 P_{1,0} \\ P_{k,2} &= \frac{\lambda}{\lambda + \gamma_2} P_{k-1,2} \quad k = 1, 2, \ldots, k_2 - 1 \end{aligned}$$

where

$$\alpha_1 = \frac{\mu}{\lambda+\gamma_1}$$
$$\alpha_2 = \frac{\gamma_1}{\lambda}\alpha_1 = \frac{\gamma_1\mu}{\lambda(\lambda+\gamma_1)}$$

From this we obtain,

$$P_{1,2} = \frac{\lambda}{\lambda+\gamma_2}P_{0,2}$$
$$P_{2,2} = \frac{\lambda}{\lambda+\gamma_2}P_{1,2} = \left(\frac{\lambda}{\lambda+\gamma_2}\right)^2 P_{0,2}$$
$$\vdots$$
$$P_{k,2} = \left(\frac{\lambda}{\lambda+\gamma_2}\right)^k P_{0,2} = \alpha_2\beta_2^k P_{1,0} \quad k=1,2,\ldots,k_2-1$$

where

$$\beta_2 = \frac{\lambda}{\lambda+\gamma_2}$$

Similarly, from global balance we have that

$$\lambda P_{k,1} = (\lambda+\gamma_1)P_{k+1,1} \quad k=0,1,\ldots$$

From this we obtain

$$P_{1,1} = \frac{\lambda}{\lambda+\gamma_1}P_{0,1} = \beta_1 P_{0,1} = \alpha_1\beta_1 P_{1,0}$$
$$P_{2,1} = \frac{\lambda}{\lambda+\gamma_1}P_{1,1} = \left(\frac{\lambda}{\lambda+\gamma_1}\right)^2 P_{1,1} = \alpha_1\beta_1^2 P_{1,0}$$
$$\vdots$$
$$P_{k,1} = \frac{\lambda}{\lambda+\gamma_1}P_{k-1,1} = \left(\frac{\lambda}{\lambda+\gamma_1}\right)^k P_{0,1} = \beta_1^k$$
$$P_{0,1} = \alpha_1\beta_1^k P_{1,0} \quad k=0,1,\ldots$$

where

$$\beta_1 = \frac{\lambda}{\lambda+\gamma_1}$$

Also, from local balance, we obtain

$$\lambda P_{k,1} + \lambda P_{k,2} + \lambda P_{k,0} = \mu P_{k+1,0} \quad k=1,2,\ldots,k_2-1$$

That is,

$$P_{k+1,0} = \frac{\lambda}{\mu}(P_{k,0}+P_{k,1}+P_{k,2}) = \rho(P_{k,0}+P_{k,1}+P_{k,2}) \quad k=1,2,\ldots,k_2-1$$

where $\rho = \lambda/\mu < 1$. From this we obtain the following:

$$P_{2,0} = \rho(P_{1,0}+P_{1,1}+P_{1,2}) = \rho\{1+\alpha_1\beta_1+\alpha_2\beta_2\}P_{1,0}$$

Similarly,

$$P_{3,0} = \rho(P_{2,0}+P_{2,1}+P_{2,2})$$
$$= \rho\left\{\rho(1+\alpha_1\beta_1+\alpha_2\beta_2)+\alpha_1\beta_1^2+\alpha_2\beta_2^2\right\}P_{1,0}$$
$$P_{4,0} = \rho(P_{3,0}+P_{3,1}+P_{3,2})$$
$$= \rho\left\{\rho^2(1+\alpha_1\beta_1+\alpha_2\beta_2)+\rho\left(\alpha_1\beta_1^2+\alpha_2\beta_2^2\right) + \alpha_1\beta_1^3+\alpha_2\beta_2^3\right\}P_{1,0}$$

From this we obtain

$$P_{k,0} = \rho\left[\frac{\alpha_1\beta_1\left(\beta_1^{k-1}-\rho^{k-1}\right)}{\beta_1-\rho} + \frac{\alpha_2\beta_2\left(\beta_2^{k-1}-\rho^{k-1}\right)}{\beta_2-\rho} + \rho^{k-2}\right]P_{1,0} \quad k=1,2,\ldots,k_2-1$$

This implies that $\beta_1 > \rho$, and $\beta_2 > \rho$. That is,

$$\frac{\lambda}{\lambda+\gamma_1} > \frac{\lambda}{\mu} \Rightarrow \mu > \lambda+\gamma_1$$

and

$$\frac{\lambda}{\lambda+\gamma_2} > \frac{\lambda}{\mu} \Rightarrow \mu > \lambda+\gamma_2$$

Now, from local balance we have that

$$\mu P_{k_2,0} = \lambda\left(P_{k_2-1,0}+P_{k_2-1,1}+P_{k_2-1,2}\right)$$
$$\Rightarrow P_{k_2,0} = \rho\left(P_{k_2-1,0}+P_{k_2-1,1}+P_{k_2-1,2}\right)$$

That is,

$$P_{k_2,0} = \rho\left\{\left[\frac{\rho\alpha_1\beta_1\left(\beta_1^{k_2-2}-\rho^{k_2-2}\right)}{\beta_1-\rho} + \frac{\rho\alpha_2\beta_2\left(\beta_2^{k_2-2}-\rho^{k_2-2}\right)}{\beta_2-\rho} + \rho^{k_2-2}\right] + \alpha_1\beta_1^{k_2-1}+\alpha_2\beta_2^{k_2-1}\right\}P_{1,0}$$
$$= A(k_2,0)P_{1,0}$$

where

$$A(k_2,0) = \rho\left\{\frac{\rho\alpha_1\beta_1\left(\beta_1^{k_2-2}-\rho^{k_2-2}\right)}{\beta_1-\rho} + \frac{\rho\alpha_2\beta_2\left(\beta_2^{k_2-2}-\rho^{k_2-2}\right)}{\beta_2-\rho} + \rho^{k_2-2}+\alpha_1\beta_1^{k_2-1}+\alpha_2\beta_2^{k_2-1}\right\}$$

Finally, from local balance in Figure 2 we obtain

$$\lambda P_{k,0} + \lambda P_{k,1} = \mu P_{k+1,0} k = k_2, \quad k_2+1,\ldots$$

Thus,

$$P_{k+1,0} = \rho\left(P_{k,0}+P_{k,1}\right) k = k_2, \quad k_2+1,\ldots$$

From this we have that

$$
\begin{aligned}
P_{k_2+1,0} &= \rho\left(P_{k_2,0}+P_{k_2,1}\right) = \left(\rho A(k_2,0)+\rho\alpha_1\beta_1^{k_2}\right)P_{1,0}\\
P_{k_2+2,0} &= \rho\left(P_{k_2+1,0}+P_{k_2+1,1}\right)\\
&= \rho\left\{\rho\left(A(k_2,0)+\alpha_1\beta_1^{k_2}\right)+\alpha_1\beta_1^{k_2+1}\right\}P_{1,0}\\
&= \rho\left\{\rho A(k_2,0)+\rho\alpha_1\beta_1^{k_2}+\alpha_1\beta_1^{k_2+1}\right\}P_{1,0}\\
&= \rho\left\{\rho A(k_2,0)+\alpha_1\beta_1^{k_2}\left(\rho+\beta_1\right)\right\}P_{1,0}\\
&= \left\{\rho^2 A(k_2,0)+\rho\alpha_1\beta_1^{k_2}\left[\frac{\beta_1^2-\rho^2}{\beta_1-\rho}\right]\right\}P_{1,0}\\
P_{k_2+3,0} &= \rho\left(P_{k_2+2,0}+P_{k_2+2,1}\right)\\
&= \rho\left[\rho\left\{\rho\left(A(k_2,0)+\alpha_1\beta_1^{k_2}\right)+\alpha_1\beta_1^{k_2+1}\right\}\right.\\
&\quad\left.+\alpha_1\beta_1^{k_2+2}\right]P_{1,0}\\
&= \rho\left[\rho^2 A(k_2,0)+\rho^2\alpha_1\beta_1^{k_2}+\rho\alpha_1\beta_1^{k_2+1}\right.\\
&\quad\left.+\alpha_1\beta_1^{k_2+2}\right]P_{1,0}\\
&= \rho\left[\rho^2 A(k_2,0)+\alpha_1\beta_1^{k_2}\left\{\rho^2+\rho\beta_1+\beta_1^2\right\}\right]P_{1,0}\\
&= \left\{\rho^3 A(k_2,0)+\rho\alpha_1\beta_1^{k_2}\left[\frac{\beta_1^3-\rho^3}{\beta_1-\rho}\right]\right\}P_{1,0}
\end{aligned}
$$

In general,

$$
P_{k,0} = \left\{\rho^{k-k_2}A(k_2,0)+\alpha_1\beta_1^{k_2}\left[\frac{\beta_1^{k-k_2}-\rho^{k-k_2}}{\beta_1-\rho}\right]\right\}P_{1,0}\qquad k=k_2,k_2+1,k_2+2,\ldots
$$

Finally, from the law of total probability we have that

$$
\begin{aligned}
&\sum_{k=1}^{\infty}P_{k,0}+\sum_{k=0}^{\infty}P_{k,1}+\sum_{k=0}^{k_2-1}P_{k,2}\\
&\quad=\sum_{k=1}^{k_2-1}P_{k,0}+\sum_{k=k_2}^{\infty}P_{k,0}+\sum_{k=0}^{\infty}P_{k,1}+\sum_{k=0}^{k_2-1}P_{k,2}=1
\end{aligned}
$$

This gives

$$
\left\{
\begin{array}{l}
\sum_{k=1}^{k_2-1}\rho\left[\frac{\alpha_1\beta_1\left(\beta_1^{k-1}-\rho^{k-1}\right)}{\beta_1-\rho}+\frac{\alpha_2\beta_2\left(\beta_2^{k-1}-\rho^{k-1}\right)}{\beta_2-\rho}+\rho^{k-2}\right]\\
\qquad+\sum_{k=k_2}^{\infty}\rho^{k-k_2}A(k_2,0)\\
+\alpha_1\beta_1^{k_2}\rho\left[\frac{\beta_1^{k-k_2}-\rho^{k-k_2}}{\beta_1-\rho}\right]+\sum_{k=0}^{\infty}\alpha_1\beta_1^k+\sum_{k=0}^{k_2-1}\alpha_2\beta_2^k
\end{array}
\right\}P_{1,0}=1
$$

Let

$$
\begin{aligned}
A_1 &= \sum_{k=1}^{k_2-1}\rho\left[\frac{\alpha_1\beta_1\left(\beta_1^{k-1}-\rho^{k-1}\right)}{\beta_1-\rho}\right.\\
&\qquad\left.+\frac{\alpha_2\beta_2\left(\beta_2^{k-1}-\rho^{k-1}\right)}{\beta_2-\rho}+\rho^{k-2}\right]\\
&= \frac{\alpha_1\beta_1\rho}{\beta_1-\rho}\left[\frac{1-\beta_1^{k_2}}{1-\beta_1}-\frac{1-\rho^{k_2}}{1-\rho}\right]\\
&\quad+\frac{\alpha_2\beta_2\rho}{\beta_2-\rho}\left[\frac{1-\beta_2^{k_2}}{1-\beta_2}-\frac{1-\rho^{k_2}}{1-\rho}\right]+\left[\frac{1-\rho^{k_2}}{1-\rho}\right]\\
A_2 &= \sum_{k=k_2}^{\infty}\left\{\rho^{k-k_2}A(k_2,0)+\alpha_1\beta_1^{k_2}\rho\left[\frac{\beta_1^{k-k_2}-\rho^{k-k_2}}{\beta_1-\rho}\right]\right\}\\
&= \frac{A(k_2,0)}{1-\rho}+\frac{\alpha_1\beta_1^{k_2}\rho}{(1-\beta_1)(1-\rho)}\\
A_3 &= \sum_{k=0}^{\infty}\alpha_1\beta_1^k=\frac{\alpha_1}{1-\beta_1}\\
A_4 &= \sum_{k=0}^{k_2-1}P_{k,2}=\sum_{k=0}^{k_2-1}\alpha_2\beta_2^k=\frac{\alpha_2\left(1-\beta_2^{k_2}\right)}{1-\beta_2}
\end{aligned}
$$

Thus,

$$
P_{1,0}=\frac{1}{A_1+A_2+A_3+A_4}
$$

Also, the expected number of customers in the system is given by

$$
\begin{aligned}
&E[N]\\
&=\sum_{k=1}^{k_2-1}kP_{k,0}+\sum_{k=k_2}^{\infty}kP_{k,0}+\sum_{k=0}^{\infty}kP_{k,1}+\sum_{k=0}^{k_2-1}kP_{k,2}\\
&=\left\{
\begin{array}{l}
\sum_{k=1}^{k_2-1}k\rho\left[\frac{\alpha_1\beta_1\left(\beta_1^{k-1}-\rho^{k-1}\right)}{\beta_1-\rho}+\frac{\alpha_2\beta_2\left(\beta_2^{k-1}-\rho^{k-1}\right)}{\beta_2-\rho}+\rho^{k-2}\right]\\
+\sum_{k=k_2}^{\infty}k\left[\rho^{k-k_2}A(k_2,0)+\frac{\alpha_1\beta_1^{k_2}\rho\left(\beta_1^{k-k_2}-\rho^{k-k_2}\right)}{\beta_1-\rho}\right]\\
+\sum_{k=0}^{\infty}k\alpha_1\beta_1^k+\sum_{k=0}^{k_2-1}k\alpha_2\beta_2^k
\end{array}
\right\}P_{1,0}
\end{aligned}
$$

Let

$$
\begin{aligned}
H_1 &= \sum_{k=1}^{k_2-1}k\rho\left[\frac{\alpha_1\beta_1\left(\beta_1^{k-1}-\rho^{k-1}\right)}{\beta_1-\rho}\right.\\
&\qquad\left.+\frac{\alpha_2\beta_2\left(\beta_2^{k-1}-\rho^{k-1}\right)}{\beta_2-\rho}+\rho^{k-2}\right]\\
&= \frac{\alpha_1\beta_1\rho}{\beta_1-\rho}\left[\frac{1+\beta_1^{k_2}\left(k_2\beta_1-k_2-1\right)}{(1-\beta_1)^2}\right.\\
&\qquad\left.-\frac{1+\rho^{k_2}\left(k_2\rho-k_2-1\right)}{(1-\rho)^2}\right]\\
&\quad+\frac{\alpha_2\beta_2\rho}{\beta_2-\rho}\left[\frac{1+\beta_2^{k_2}\left(k_2\beta_2-k_2-1\right)}{(1-\beta_2)^2}\right.\\
&\qquad\left.-\frac{1+\rho^{k_2}\left(k_2\rho-k_2-1\right)}{(1-\rho)^2}\right]\\
&\quad+\left[\frac{1+\rho^{k_2}\left(k_2\rho-k_2-1\right)}{(1-\rho)^2}\right]
\end{aligned}
$$

$$H_2 = \sum_{k=k_2}^{\infty} k\left[\rho^{k-k_2}A\left(k_2,0\right) + \frac{\alpha_1\beta_1^{k_2}\rho\left(\beta_1^{k-k_2}-\rho^{k-k_2}\right)}{\beta_1-\rho}\right]$$
$$= A\left(k_2,0\right)\left[\frac{\rho+k_2\left(1-\rho\right)}{\left(1-\rho\right)^2}\right] + \frac{\alpha_1\beta_1^{k_2}\rho}{\beta_1-\rho}\left[\frac{\beta_1+k_2\left(1-\beta_1\right)}{\left(1-\beta_1\right)^2} - \frac{\rho+k_2\left(1-\rho\right)}{\left(1-\rho\right)^2}\right]$$
$$H_3 = \sum_{k=0}^{\infty} k\alpha_1\beta_1^k = \frac{\alpha_1\beta_1}{\left(1-\beta_1\right)^2}$$
$$H_4 = \sum_{k=0}^{k_2-1} k\alpha_2\beta_2^k = \alpha_2\left[\frac{\beta_2-k_2\beta_2+\left(k_2-1\right)\beta_2^{k_2-1}}{\left(1-\beta_1\right)^2}\right]$$

Then we obtain

$$E[N] = [H_1+H_2+H_3+H_4]P_{1,0}$$

From Little's formula [15] we obtain the expected delay in the system as

$$E[T] = \frac{E[N]}{\lambda}$$

This completes the proof of Theorem 1.

APPENDIX II

From Figure 3, modifications to equations for the partial vacation interruption policy yield the following results:

$$\lambda P_{k,0}+\lambda P_{k,1} = \mu P_{k+1,0} \quad k=k_2,k_2+1,\ldots,k_1-1 \quad (1)$$
$$\lambda P_{k,0} = \mu P_{k+1,0} \quad k=k_1,k_1+1,\ldots \quad (2)$$
$$(\lambda+\gamma_1)P_{0,1} = \mu P_{1,0} \quad (3)$$
$$(\lambda+\gamma_1)P_{k,1} = \lambda P_{k-1,1} \quad k=1,2,\ldots,k_1-1 \quad (4)$$

From equations (3) and (4) we obtain

$$P_{0,1} = \frac{\mu}{\lambda+\gamma_1}P_{1,0} = \alpha_1 P_{1,0}$$
$$P_{1,1} = \frac{\lambda}{\lambda+\gamma_1}P_{0,1} = \beta_1 P_{0,1} = \alpha_1\beta_1 P_{1,0}$$
$$P_{2,1} = \frac{\lambda}{\lambda+\gamma_1}P_{1,1} = \left(\frac{\lambda}{\lambda+\gamma_1}\right)^2 P_{1,1} = \alpha_1\beta_1^2 P_{1,0}$$
$$\vdots$$
$$P_{k,1} = \frac{\lambda}{\lambda+\gamma_1}P_{k-1,1} = \left(\frac{\lambda}{\lambda+\gamma_1}\right)^k P_{0,1} = \beta_1^k P_{0,1} = \alpha_1\beta_1^k P_{1,0} \quad k=0,1,\ldots,k_1-1$$

From (2) we obtain

$$P_{k,0} = \left(\frac{\lambda}{\mu}\right)^{k-k_1} P_{k_1,0} = \rho^{k-k_1}P_{k_1,0} k=k_1,k_1+1,\ldots$$

Thus,

$$\sum_{k=k_1}^{\infty} P_{k,0} = P_{k_1,0}\sum_{k=k_1}^{\infty}\rho^{k-k_1} = \frac{P_{k_1,0}}{1-\rho}$$

From (1) we have that

$$P_{k_2+1,0} = \frac{\lambda}{\mu}\left(P_{k_2,0}+P_{k_2,1}\right) = \rho\left(P_{k_2,0}+P_{k_2,1}\right)$$
$$P_{k_2+2,0} = \rho(P_{k_2+1,0}+P_{k_2+1,1}) = \rho\left\{\rho(P_{k_2,0}+P_{k_2,1})+P_{k_2+1,1}\right\}$$

As in the partial vacation interruption case, it can be shown that

$$P_{k,0} = \left\{\rho^{k-k_2}A(k_2,0)+\alpha_1\beta_1^{k_2}\left[\frac{\beta_1^{k-k_2}-\rho^{k-k_2}}{\beta_1-\rho}\right]\right\}P_{1,0} \quad k=k_2,k_2+1,k_2+2,\ldots,k_1-1$$

From (2) we obtain

$$P_{k,0} = \left(\frac{\lambda}{\mu}\right)^{k-k_1} P_{k_1,0} = \rho^{k-k_1}P_{k_1,0}k=k_1,k_1+1,\ldots$$

and

$$\mu P_{k_1,0} = \lambda P_{k_1-1,0}+\lambda P_{k_1-1,1} \Rightarrow P_{k_1,0} = \rho P_{k_1-1,0}+\rho P_{k_1-1,1}$$

which means that

$$P_{k_1,0} = \rho\left\{\rho^{k_1-k_2-1}A(k_2,0)+\alpha_1\beta_1^{k_2}\rho \times\left[\frac{\beta_1^{k_1-k_2-1}-\rho^{k_1-k_2-1}}{\beta_1-\rho}\right]+\alpha_1\beta_1^{k_1-1}\right\}P_{1,0} = A\left(k_1,0\right)P_{1,0}$$

where

$$A(k_1,0) = \rho\left\{\rho^{k_1-k_2-1}A(k_2,0)+\alpha_1\beta_1^{k_2}\rho \times\left[\frac{\beta_1^{k_1-k_2-1}-\rho^{k_1-k_2-1}}{\beta_1-\rho}\right]+\alpha_1\beta_1^{k_1-1}\right\}$$

Thus

$$P_{k,0} = \rho^{k-k_1}A(k_1,0)P_{k_1,0} \quad k=k_1,k_1+1,\ldots$$

Finally, from the law of total probability we have that

$$\sum_{k=1}^{\infty}P_{k,0}+\sum_{k=0}^{k_1-1}P_{k,1}+\sum_{k=0}^{k_2-1}P_{k,2} = \sum_{k=1}^{k_2-1}P_{k,0}+\sum_{k=k_2}^{k_1-1}P_{k,0} + \sum_{k=k_1}^{\infty}P_{k,0}+\sum_{k=0}^{k_1-1}P_{k,1}+\sum_{k=0}^{k_2-1}P_{k,2} = 1$$

That is,

$$\left\{\begin{array}{l}\sum_{k=1}^{k_2-1}\rho\left[\frac{\alpha_1\beta_1\left(\beta_1^{k-1}-\rho^{k-1}\right)}{\beta_1-\rho}+\frac{\alpha_2\beta_2\left(\beta_2^{k-1}-\rho^{k-1}\right)}{\beta_2-\rho}+\rho^{k-2}\right] \\ +\sum_{k=k_2}^{k_1-1}\rho^{k-k_2}A\left(k_2,0\right)+\alpha_1\beta_1^{k_2}\rho\left[\frac{\beta_1^{k-k_2}-\rho^{k-k_2}}{\beta_1-\rho}\right] \\ +\sum_{k=k_1}^{\infty}\rho^{k-k_1}A(k_1,0)+\sum_{k=0}^{k_1-1}\alpha_1\beta_1^k+\sum_{k=0}^{k_2-1}\alpha_2\beta_2^k\end{array}\right\} P_{1,0} = 1$$

Let

$$B_1 = \sum_{k=1}^{k_2-1} \rho \left[\frac{\alpha_1\beta_1\left(\beta_1^{k-1} - \rho^{k-1}\right)}{\beta_1 - \rho} + \frac{\alpha_2\beta_2\left(\beta_2^{k-1} - \rho^{k-1}\right)}{\beta_2 - \rho} + \rho^{k-2} \right]$$
$$= \frac{\alpha_1\beta_1\rho}{\beta_1 - \rho}\left[\frac{1-\beta_1^{k_2}}{1-\beta_1} - \frac{1-\rho^{k_2}}{1-\rho}\right] + \frac{\alpha_2\beta_2\rho}{\beta_2 - \rho}\left[\frac{1-\beta_2^{k_2}}{1-\beta_2} - \frac{1-\rho^{k_2}}{1-\rho}\right] + \left[\frac{1-\rho^{k_2}}{1-\rho}\right]$$

$$B_2 = \sum_{k=k_2}^{k_1-1} \rho^{k-k_2}A(k_2,0) + \alpha_1\beta_1^{k_2}\rho\left[\frac{\beta_1^{k-k_2} - \rho^{k-k_2}}{\beta_1 - \rho}\right]$$
$$= A(k_2,0)\left[\frac{1-\rho^{k_1}}{1-\rho}\right] + \frac{\alpha_1\beta_1^{k_2}\rho}{(\beta_1-\rho)}\left[\frac{1-\beta_1^{k_2}}{1-\beta_1} - \frac{1-\rho^{k_2}}{1-\rho}\right]$$

$$B_3 = \sum_{k=k_1}^{\infty} \rho^{k-k_1}A(k_1,0) = \frac{A(k_1,0)}{1-\rho}$$

$$B_4 = \sum_{k=0}^{k_1-1} \alpha_1\beta_1^k = \alpha_1\left[\frac{1-\beta_1^{k_1}}{1-\beta_1}\right]$$

$$B_5 = \sum_{k=0}^{k_2-1} \alpha_2\beta_2^k = \alpha_2\left[\frac{1-\beta_2^{k_2}}{1-\beta_2}\right]$$

Thus, we obtain

$$P_{1,0} = \frac{1}{B_1+B_2+B_3+B_4+B_5}$$

Also, the mean number of customers in the system is given by:

$$E[N] = \sum_{k=1}^{k_2-1} kP_{k,0} + \sum_{k=k_2}^{k_1-1} kP_{k,0} + \sum_{k=k_1}^{\infty} kP_{k,0} + \sum_{k=0}^{k_1-1} kP_{k,1} + \sum_{k=0}^{k_2-1} kP_{k,2}$$
$$= \left\{ \begin{array}{l} \sum_{k=1}^{k_2-1} k\rho\left[\frac{\alpha_1\beta_1\left(\beta_1^{k-1}-\rho^{k-1}\right)}{\beta_1-\rho} + \frac{\alpha_2\beta_2\left(\beta_2^{k-1}-\rho^{k-1}\right)}{\beta_2-\rho} + \rho^{k-2}\right] \\ + \sum_{k=k_2}^{k_1-1} k\left[\rho^{k-k_2}A(k_2,0) + \frac{\alpha_1\beta_1^{k_2}\rho\left(\beta_1^{k-k_2}-\rho^{k-k_2}\right)}{\beta_1-\rho}\right] \\ + \sum_{k=k_1}^{\infty} k\rho^{k-k_1}A(k_1,0) + \sum_{k=0}^{k_1-1} k\alpha_1\beta_1^k + \sum_{k=0}^{k_2-1} k\alpha_2\beta_2^k \end{array} \right\} P_{1,0}$$

Let

$$J_1 = \sum_{k=1}^{k_2-1} k\rho \left[\frac{\alpha_1\beta_1\left(\beta_1^{k-1} - \rho^{k-1}\right)}{\beta_1 - \rho} + \frac{\alpha_2\beta_2\left(\beta_2^{k-1} - \rho^{k-1}\right)}{\beta_2 - \rho} + \rho^{k-2} \right]$$
$$= \frac{\alpha_1\beta_1\rho}{\beta_1-\rho}\left[\frac{1+\beta_1^{k_2}(k_2\beta_1-k_2-1)}{(1-\beta_1)^2} - \frac{1+\rho^{k_2}(k_2\rho-k_2-1)}{(1-\rho)^2}\right] + \frac{\alpha_2\beta_2\rho}{\beta_2-\rho}\left[\frac{1+\beta_2^{k_2}(k_2\beta_2-k_2-1)}{(1-\beta_2)^2} - \frac{1+\rho^{k_2}(k_2\rho-k_2-1)}{(1-\rho)^2}\right] + \left[\frac{1+\rho^{k_2}(k_2\rho-k_2-1)}{(1-\rho)^2}\right]$$

$$J_2 = \sum_{k=k_2}^{k_1-1} kP_{k,0}$$
$$= \sum_{k=k_2}^{k_1-1} k\left[\rho^{k-k_2}A(k_2,0) + \frac{\alpha_1\beta_1^{k_2}\rho\left(\beta_1^{k-k_2}-\rho^{k-k_2}\right)}{\beta_1-\rho}\right]$$
$$= A(k_2,0)\left[\frac{\rho-k_1\rho^{k_1}+(k_1-1)\rho^{k_1+1}}{(1-\rho)^2} + \frac{k_2\left(1-\rho^{k_2}\right)}{1-\rho}\right] + \frac{\alpha_1\beta_1^{k_2}\rho}{\beta_1-\rho}\left[\frac{\beta_1-k_1\beta_1^{k_1}+(k_1-1)\beta_1^{k_1+1}}{(1-\beta_1)^2} + \frac{k_2(1-\beta_1^{k_1})}{1-\beta_1} - \frac{\rho-k_1\rho^{k_1}+(k_1-1)\rho^{k_1+1}}{(1-\rho)^2} - \frac{k_2\left(1-\rho^{k_2}\right)}{1-\rho}\right]$$

$$J_3 = \sum_{k=k_1}^{\infty} k\rho^{k-k_1}A(k_1,0) = A(k_1,0)\frac{\rho+k_1(1-\rho)}{(1-\rho)^2}$$

$$J_4 = \sum_{k=0}^{k_1-1} kP_{k,1} = \sum_{k=0}^{k_1-1} k\alpha_1\beta_1^k = \alpha_1\left[\frac{\beta_1-k_1\beta_1^{k_1}+(k_1-1)\beta_1^{k_1+1}}{(1-\beta_1)^2}\right]$$

$$J_5 = \sum_{k=0}^{k_2-1} kP_{k,2} = \sum_{k=0}^{k_2-1} k\alpha_2\beta_2^k = \alpha_2\left[\frac{\beta_2-k_2\beta_2^{k_2}+(k_2-1)\beta_2^{k_2+1}}{(1-\beta_2)^2}\right]$$

Then we obtain

$$E[N] = [J_1+J_2+J_3+J_4+J_5]P_{1,0}$$

From Little's formula [15] we obtain the expected delay in the system as

$$E[T] = \frac{E[N]}{\lambda}$$

This completes the proof of Theorem 2.

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REFERENCES

[1] Y. Levy and U. Yechiali, "Utilization of idle time in an *M/G/1* queueing system," *Manage. Sci.*, vol. 22, no. 2, pp. 202–211, 1975.
[2] B. T. Doshi, "Queueing systems with vacations—A survey," *Queueing Syst.*, vol. 1, no. 1, pp. 29–66, 1986.
[3] B. T. Doshi, "Single-server queues with vacations," in *Stochastic Analysis of Computer and Communication Systems*, H. Takagi, Ed. Amsterdam, The Netherlands: Elsevier, 1990.
[4] H. Takagi, *Queueing Analysis: A Foundation of Performance Analysis, Volume 1: Vacation and Priority Systems, Part 1*. Amsterdam, The Netherlands: Elsevier, 1991.
[5] N. Tian and G. Zhang, *Vacation Queueing Models: Theory and Applications*. New York, NY, USA: Springer-Verlag, 2006.
[6] O. C. Ibe and O. A. Isijola, "*M/M/1* multiple vacation queueing systems with differentiated vacations," *Model. Simul. Eng.*, vol. 2014, Jul. 2014, Art. ID 158247.
[7] J. Li and N. Tian, "The M/M/1 queue with working vacations and vacation interruption," *J. Syst. Sci. Syst. Eng.*, vol. 16, no. 1, pp. 121–127, 2007.
[8] L. D. Servi and S. G. Finn, "*M/M/1* queues with working vacations (M/M/1/WV)," *Perform. Eval.*, vol. 50, no. 1, pp. 41–52, Oct. 2002.
[9] J.-H. Li and N.-S. Tian, "The discrete-time GI/Geo/1 queue with working vacations and vacation interruption," *Appl. Math. Comput.*, vol. 185, no. 1, pp. 1–10, Feb. 2007.
[10] J.-H. Li, N.-S. Tian, and Z.-Y. Ma, "Performance analysis of GI/M/1 queue with working vacations and vacation interruption," *Appl. Math. Model.*, vol. 32, no. 12, pp. 2715–2730, Dec. 2008.
[11] M. Zhang and Z. Hou, "Performance analysis of *M/G/1* queue with working vacations and vacation interruption," *J. Comput. Appl. Math.*, vol. 234, no. 10, pp. 2977–2985, Sep. 2010.
[12] G. Ayyappan, G. Sekar, and A. M. Ganapathi, "M/M/1 retrial queuing system with vacation interruptions under ERLANG-K service," *Int. J. Comput. Appl.*, vol. 2, no. 2, pp. 52–57, 2010.
[13] A. Krishnamoorthy and C. Sreenivasan, "An *M/M/2* Queueing System with heterogeneous servers including one with working vacation," *Int. J. Stochastic Anal.*, vol. 2012, May 2012, Art. ID 145867.
[14] C. Sreenivasan, S. R. Chakravarthy, and A. Krishnamoorthy, "*MAP/PH/1* queue with working vacations, vacation interruptions and *N* policy," *Appl. Math. Model.*, vol. 37, no. 6, pp. 3879–3893, Mar. 2013.
[15] J. D. C. Little, "A proof for the queuing formula: $L = \lambda W$," *Oper. Res.*, vol. 9, no. 3, pp. 383–387, Jun. 1961.

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