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RESEARCH ARTICLE

Classification of Feature Engineering Techniques for Machine Learning under the Environment of Lattice Ordered T-Bipolar Soft Rings

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ABSTRACT The practice of adding new features or changing current features to enhance a machine-learning model's performance is known as feature engineering. It increases the prediction potential of machine learning and aids in revealing the data's underlying patterns. Different soft structures can be utilized to reach a decision where we have to decide the best alternative among the given choices. The structure of a ring is an algebraic structure that plays a vital role due to its characteristics. Moreover, a soft set is a valuable structure that can consider the parameterization tool. Also T-bipolar soft set is a parameterization tool that can consider the positive and negative aspects. Based on these observations we have developed the theory of lattice-ordered T-bipolar soft rings (LOTBSRs) and anti-lattice-ordered T-bipolar soft rings (ALOTBSRs). Moreover, we have defined the notions of OR product, extended union, and restricted union for LOTBSRs. Furthermore, the ideas of AND product, restricted intersection, and extended intersection are defined. To analyze the whole theory, we have proved some results related to these ideas. To construct the applications part of these developed notions, we have defined an algorithm and utilized these ideas in the decision-making scenarios for the classification of feature engineering techniques. In the end, we have some conclusion remarks.

INDEX TERMS Lattice (anti-lattice) ordered T-bipolar soft rings, feature engineering techniques.

I. INTRODUCTION

Feature engineering (FE) is an important phase in the machine learning pipeline that can have a big effect on the effectiveness, understandability, and generalization of models. It necessitates a thorough comprehension of the data and the problem domain, as well as innovation in the creation of features that effectively capture pertinent data. A mediocre model can be improved with carefully designed features to become a highly useful and accurate tool for making predictions and judgments. The importance of FE cannot be denied in machine learning procedure and we can list the importance of FE as follows

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1. The predictive ability of a model can be considerably increased by well-designed features. They can assist the model in identifying pertinent patterns and connections in the data, improving accuracy and generalization.
2. The associations between the characteristics and the target variable are frequently nonlinear in real-world problems. In order to capture these non-linear interactions, you can generate new features or modify old ones using feature engineering, which makes it simpler for your model to learn and make correct predictions.
3. Imputation is one feature engineering approach that can be used to address missing information. You can avoid losing important data and enhance model performance by filling in the values that are missing with relevant information.

To accommodate ambiguity or uncertainty in data, a soft set (SS) is a mathematical structure that expands on the idea of a classical (crisp) set. Molodtsov [1] created SSs in 1999 as a method for handling ambiguous data in modelling and decision-making. Molodtsov SS theory is a widely used mathematical framework for handling ambiguous and unclear data that other conventional methods are unable to handle. The construction of a SS proved to be a very interesting technique for handling the issue by taking into account the parameterization process. SSs theory has drawn increased interest recently as a method for resolving confusing data. SSs have found useful in a variety of fields, including data analysis [2] and multi-attribute decision-making [3] where handling erroneous or ambiguous information is essential. They offer an adaptable and simple method for manipulating and modelling ambiguous data inside a mathematical framework. Some operations on SS have been given in [4]. Ali et al. [5] proposed some new operations based on SS theory. Numerous hybrid structures have been produced to show the importance of SS like the notion of fuzzy SS (FSS) [6], intuitionistic fuzzy SS (IFSS) [7], Pythagorean fuzzy SS (PyFSS) [8], q-rung orthopair fuzzy SS (q-ROFSS) [9]. These structures have been utilized in different fields like Sut [10] established the application of FS to decision-making problems. Moreover, Celik et al. [11] provide the application of FSS in ring theory. Neog and Sut [12] delivered the application of FSS in medical diagnosis by using the idea of FS complement. Also, the application of IFSS matrices for disease diagnosis has been delivered in [13]. Akram and Dudek [14] presented the theory of IF hypergraphs with applications. Furthermore, Ali et al. [15] utilize the structure of interval-valued Pythagorean fuzzy set and proposed the idea of Einstein aggregation operators with their application to green supplier chain management. Shahzadi et al. [16] established the notion of PyFS graphs and established their applications.

A mathematical framework called bipolar soft sets (BSSs) is utilized in uncertainty modeling and decision-making. They were presented as an extension of traditional soft sets to deal with circumstances where decision-makers have varying degrees of belief in and unbelief about an element's membership in a given set. In circumstances where the data are ambiguous or uncertain, bipolar soft sets are very helpful. Two kinds of attempts have been made in this regard. First by the Shabir and Naz [17] and other attempts have been made by Karaaslan [18]. BSS has received attention and new ideas have been developed to discuss the importance of the introduced approach. We can see that Dalkilic [19] developed a decision-making approach to reduce the margin of error of decision-makers for BSS theory. Moreover, Ozturk [20] delivered the notion of bipolar soft points and established the relationship between bipolar soft points and BSSs. Musa and Asaad [21] work on the theory of bipolar hypersoft sets.

Both of these ideas given in [17] and [18] have some drawbacks observed by Mahmood [22]. These drawbacks are stated as follows

1. If we study the FS and SS, we can observe that both are defined by a single function, have the same set as their domain set, and in both cases, have a lattice codomain set.
2. Take notice that both the IFS and the double-framed SS [23] use two functions, each of which has a single set serving as its domain set, and both of which have a lattice codomain set for both functions.
3. However, we can see that this is not true for BSS and bipolar-valued fuzzy sets. We can see that none of the BSS efforts made in [17] and [18] succeeded in filling the available area. Mahmood [22] introduced the notion of a T-bipolar soft set (TBSS) to fill this gap. A lot of new advancements have been achieved in this area, and the concept of TBSS has been recognized as a noteworthy achievement and innovative tool.

A branch of abstract algebra called ring theory is concerned with the study of rings, which are algebraic structures. Numerous branches of mathematics, science, and engineering use rings extensively. TBSS can fill up all the above given gaps and it is a parameterization tool. So based on this observation and the importance of the theory of TBSS, we have delivered the idea of lattice-ordered TBSRs and anti-lattice-ordered TBSRs. We have defined the concepts of OR product, extended union, and restricted union for LOTBSRs. In addition, the concepts of AND product, restricted intersection, and extended intersection have been added. We have demonstrated various findings linked to these concepts to analyze the entire theory. We have constructed an algorithm and used these concepts in decision-making scenarios for the classification of feature engineering techniques to build the applications portion of these generated notions.

The rest of the article is organized as follows. In section II, we have overviewed the notion of SS, BSS, TBSS, and their fundamental operations. In section III, we have defined the concepts of OR product, extended union, and restricted union for LOTBSRs. In addition, the concepts of AND product, restricted intersection, and extended intersection have been added. Moreover, section IV deals with applications of the developed theory. Section V is related to the conclusion of the developed theory.

II. PRELIMINARIES

This part of the article is about the definition of SS, AND product, OR product, extended union, extended intersection for two SSs, BSS, TBSS and their basic properties.

Definition 1 [1]: Assume that U represents a universal set and E be the set of parameters. Now for any $\Theta \subseteq E$, a SS is a pair $\left(\hat{h}, \Theta\right)$, where $\hat{h} : \Theta \to P(U)$ is set-valued mapping.

Definition 2 [4]: Assume that U represents a universal set and E be the set of parameters and $\Theta_1, \ \Theta_2$ are the subsets of E. Also assume that $\left(\hat{h}_1, \Theta_1\right)$ and $\left(\hat{h}_2, \Theta_2\right)$ denote the SSs over U, then

1. AND product of these two SSs is denoted and defined by

$$\left(\hat{h}_1, \Theta_1\right) \bigwedge \left(\hat{h}_2, \Theta_2\right) = \left(\hat{h}_3, \Theta_1 \times \Theta_2\right)$$

where $\hat{h}_3(\breve{a}, Ъ) = \hat{h}_1(\breve{a}) \cap \hat{h}_2(Ъ)$ for all $(\breve{a}, Ъ) \in \Theta_1 \times \Theta_2$

2. OR product of these two SSs is denoted and defined by

$$\left(\hat{h}_1, \Theta_1\right) \vee \left(\hat{h}_2, \Theta_2\right) = \left(\hat{h}_3, \Theta_1 \times \Theta_2\right)$$

where $\hat{h}_3(\breve{a}, Ъ) = \hat{h}_1(\breve{a}) \cup \hat{h}_2(Ъ)$ for all $(\breve{a}, Ъ) \in \Theta_1 \times \Theta_2$.

Definition 3 [4]: Assume that U represents a universal set and E be the set of parameters and Θ_1, Θ_2 are the subset of E. Also assume that $\left(\hat{h}_1, \Theta_1\right)$ and $\left(\hat{h}_2, \Theta_2\right)$ denote the SSs over U, then extended union of these two SSs is denoted and defined by

$$\left(\hat{h}_1, \Theta_1\right) \cup_{extended} \left(\hat{h}_2, \Theta_2\right) = \left(\hat{h}_3, \Theta_1 \cup \Theta_2\right)$$

where

$$\hat{h}_3(\breve{a}) = \begin{cases} \hat{h}_1(\breve{a}) & ; \textit{if } \breve{a} \in \Theta_1 - \Theta_2 \\ \hat{h}_2(\breve{a}) & ; \textit{if } \breve{a} \in \Theta_2 - \Theta_1 \\ \hat{h}_1(\breve{a}) \cup \hat{h}_2(\breve{a}) & ; \textit{if } \breve{a} \in \Theta_1 \cap \Theta_2 \end{cases}$$

Definition 4 [5]: Assume that U represents a universal set and E be the set of parameters and Θ_1, Θ_2 are the subset of E. Also assume that $\left(\hat{h}_1, \Theta_1\right)$ and $\left(\hat{h}_2, \Theta_2\right)$ denote the SSs over U, then extended intersection of these two SSs is denoted and defined by

$$\left(\hat{h}_1, \Theta_1\right) \cap_{extended} \left(\hat{h}_2, \Theta_2\right) = \left(\hat{h}_3, \Theta_1 \cup \Theta_2\right)$$

where

$$\hat{h}_3(\breve{a}) = \begin{cases} \hat{h}_1(\breve{a}) & ; \textit{if } \breve{a} \in \Theta_1 - \Theta_2 \\ \hat{h}_2(\breve{a}) & ; \textit{if } \breve{a} \in \Theta_2 - \Theta_1 \\ \hat{h}_1(\breve{a}) \cap \hat{h}_2(\breve{a}) & ; \textit{if } \breve{a} \in \Theta_1 \cap \Theta_2 \end{cases}$$

Shabir and Naz [17] initiated the idea of BSS and they have used the same codomain set.

Definition 5 [17]: Let U be a universal set and $\Theta \subseteq E$. Also, $\neg\Theta = \{\neg \mathscr{v}, \mathscr{v} \in \Theta\}$ denote the NOT set of Θ, then $\left(\hat{h}, Ị, \Theta\right)$ is said to be BSS where $\hat{h} : \Theta \to P(U)$ and $Ị : \neg\Theta \to P(U)$ and $\hat{h}(\mathscr{v}) \bigcap Ị(\neg \mathscr{v}) = \phi$ *(empty set)*.

Definition 6 [18]: Let E represent the set of parameter and $\Theta_1 \subseteq E$, $\Theta_2 \subseteq E$ such that $\Theta_1 \cup \Theta_2 = E$ and $\Theta_1 \cap \Theta_2 = \phi$ *(empty set)*. Then $\left(\hat{h}, Ị, \Theta\right)$ is said to be BSS where $\hat{h} : \Theta_1 \to P(U)$ and $Ị : \Theta_2 \to P(U)$ with $\hat{h}(\mathscr{v}) \cap Ị(\hbar(\mathscr{v})) = \phi$ where $\hbar : \Theta_1 \to \Theta_2$ is bijective mapping.

Definition 7 [22]: Assume that U represent the universal set and E be the set of parameters and $\Theta \subset E$. Also, let $\mathcal{L} \subset U$ and $\mathcal{G} = U - X$. then $\left(\hat{h}, Ị, \Theta\right)$ is said to be TBSS over U, where $\hat{h} : \Theta \to P(\mathcal{L})$ and $Ị : \Theta \to P(\mathcal{G})$. So TBSS is given by simply $\left(\hat{h}, Ị, \Theta\right) = \left\{\mathscr{v}, \hat{h}(\mathscr{v}), Ị(\mathscr{v}) : \hat{h}(\mathscr{v}) \in P(\mathcal{L}) \textit{ and } Ị(\mathscr{v}) \in P(\mathcal{G})\right\}$.

III. LATTICE (ANTI-LATTICE) ORDERED T-BIPOLAR SOFT RING

Here we will define the notion of lattice (anti-lattice) ordered T-bipolar soft rings. In this whole section, the notations $\mathcal{L}$ and $\mathcal{G}$ represent two distinct rings, ₱ represent the set of parameter and $U = \mathcal{L} \cup \mathcal{G}$.

Definition 8: A T-bipolar soft set $(ʄ, Ʀ, ₱)$ over two distinct rings $\mathcal{L}$ and $\mathcal{G}$ such that $U = \mathcal{L} \bigcup \mathcal{G}$ is called lattice (anti-lattice) ordered T-bipolar soft ring (LOTBSR) $(ʄ, Ʀ, ₱)$ if for all $\breve{a}_1, \breve{a}_2 \in ₱$, $ʄ(\breve{a}_1)$ and $ʄ(\breve{a}_2)$ are the subring of $\mathcal{L}$ and $Ʀ(\breve{a}_1)$, $Ʀ(\breve{a}_2)$ are the subring of $\mathcal{G}$ and if there exist order between elements of set of parameters that is if $\breve{a}_1 \leq \breve{a}_2$ then $ʄ(\breve{a}_1) \subseteq ʄ(\breve{a}_2)$ and $Ʀ(\breve{a}_1) \supseteq Ʀ(\breve{a}_2)$ $\left(ʄ(\breve{a}_1) \supseteq ʄ(\breve{a}_2) \text{ and } Ʀ(\breve{a}_1) \subseteq Ʀ(\breve{a}_2)\right)$ where $ʄ : ₱ \to P(\mathcal{L})$ and $Ʀ : ₱ \to P(\mathcal{G})$.

Example 1: Assume that $\mathcal{L} = Z_{12}$ and $\mathcal{G} = Z_{24}$ are two distinct rings such that $\mathbb{U} = \mathcal{L} \bigcup \mathcal{G}$ and $₱ = \{\breve{a}_1, \breve{a}_2, \breve{a}_3\}$ such that $\breve{a}_1 \leq \breve{a}_2 \leq \breve{a}_3$. Now if we define the mappings as

$$ʄ(\breve{a}_1) = \{\bar{0}, \bar{4}, \bar{8}\},$$
$$ʄ(\breve{a}_2) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$$
$$ʄ(\breve{a}_3) = Z_{12}$$

And

$$Ʀ(\breve{a}_1) = \{\bar{0}, \bar{4}, \bar{8}, \overline{12}, \overline{16}, \overline{20}\}, \; Ʀ(\breve{a}_2) = \{\bar{0}, \bar{8}, \overline{16}\},$$
$$Ʀ(\breve{a}_3) = \{\bar{0}, \overline{12}\}.$$

Then it is clear that $ʄ(\breve{a}_1) \subseteq ʄ(\breve{a}_2) \subseteq ʄ(\breve{a}_3)$ and $Ʀ(\breve{a}_1) \supseteq Ʀ(\breve{a}_2) \supseteq Ʀ(\breve{a}_3)$. Hence $(ʄ, Ʀ, ₱)$ is LOTBSR and given by

$$(ʄ, Ʀ, ₱) = \left\{ \begin{array}{c} \left(\breve{a}_1, \{\bar{0}, \bar{4}, \bar{8}\}, \{\bar{0}, \bar{4}, \bar{8}, \overline{12}, \overline{16}, \overline{20}\}\right), \\ \left(\breve{a}_2, \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}, \{\bar{0}, \bar{8}, \overline{16}\}\right), \\ \left(\breve{a}_3, Z_{12}, \{\bar{0}, \overline{12}\}\right) \end{array} \right\}.$$

Definition 9: Let $(ʄ_1, Ʀ_1, ₱_1)$ and $(ʄ_2, Ʀ_2, ₱_2)$ are two LOTBSRs, then their OR product is denoted and defined by

$$(ʄ_1, Ʀ_1, ₱_1) \bigvee (ʄ_2, Ʀ_2, ₱_2) = \left\{ \begin{array}{c} (x, y) \mid ʄ_1(x) \bigcup ʄ_2(y), \\ Ʀ_1(x) \bigcap Ʀ_2(y) \end{array} \right\}$$
$$\text{for all } (x, y) \in ₱_1 \times ₱_2.$$

Remark 1: In general the OR product of two LOTBSRs is not a LOTBSR.

Example 2: Let $\mathcal{L} = Z_{12}$ and $\mathcal{G} = Z_{24}$ be two distinct rings such that $\mathbb{U} = \mathcal{L} \bigcup \mathcal{G}$ and $₱_1 = \{\breve{a}_1, \breve{a}_2\}$ with $\breve{a}_1 \leq \breve{a}_2$ and $₱_2 = \{\breve{a}_1, \breve{a}_2, \breve{a}_3\}$ such that $\breve{a}_1 \leq \breve{a}_2 \leq \breve{a}_3$. Now if we define the mappings as

$$ʄ_1(\breve{a}_1) = \{\bar{0}, \bar{4}, \bar{8}\}, \; ʄ_1(\breve{a}_2) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$$

And

$$Ʀ_1(\breve{a}_1) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}, \overline{12}, \overline{14}, \overline{16}, \overline{18}, \overline{20}, \overline{22}\},$$
$$Ʀ_1(\breve{a}_2) = \{\bar{0}, \bar{4}, \bar{8}, \overline{12}, \overline{16}, \overline{20}\}$$

Then it is clear that $(ʄ_1, Ʀ_1, ₱_1)$ is LOTBSR.

Also for $\mathcal{L} = Z_{12}$ and $\mathcal{G} = Z_{24}$ with $\mathbb{U} = \mathcal{L} \bigcup \mathcal{G}$ and $Ᵽ_2 = \{ă_1, ă_2, ă_3\}$ such that $ă_1 \leq ă_2 \leq ă_3$ if we define

$$\mathfrak{f}_2(ă_1) = \{\bar{0}, \bar{6}\},$$
$$\mathfrak{f}_2(ă_2) = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\},$$
$$\mathfrak{f}_2(ă_3) = Z_{12}$$

And

$$Ʀ_2(ă_1) = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}\},$$
$$Ʀ_2(ă_2) = \{\bar{0}, \bar{6}, \overline{12}, \overline{18}\}, \; Ʀ_2(ă_3) = \{\bar{0}, \overline{12}\}$$

Then it is clear that $(\mathfrak{f}_2, Ʀ_2, Ᵽ_2)$ is LOTBSR.

Now as OR product is defined as

$$(\mathfrak{f}_1, Ʀ_1, Ᵽ_1) \bigvee (\mathfrak{f}_2, Ʀ_2, Ᵽ_2) = \left\{ \begin{array}{c} (x, y) \mid \mathfrak{f}_1(x) \bigcup \mathfrak{f}_2(y), \\ Ʀ_1(x) \bigcap Ʀ_2(y) \end{array} \right\}$$
$$\text{for all } (x, y) \in Ᵽ_1 \times Ᵽ_2.$$

As $Ᵽ_1 \times Ᵽ_2 = \left\{ \begin{array}{c} (ă_1, ă_1), (ă_1, ă_2), (ă_1, ă_3), \\ (ă_2, ă_1), (ă_2, ă_2), (ă_2, ă_3) \end{array} \right\}$

Now we can observe that

$$\mathfrak{f}_1(ă_1) \cup \mathfrak{f}_2(ă_1) = \{\bar{0}, \bar{4}, \bar{8}\} \cup \{\bar{0}, \bar{6}\} = \{\bar{0}, \bar{4}, \bar{6}, \bar{8}\}$$

Which is not a subring of $\mathcal{L} = Z_{12}$. Hence it is clear that OR product of two LOTBSRs need not to be a LOTBSR.

Theorem 1: Let $(\mathfrak{f}_1, Ʀ_1, Ᵽ_1)$ and $(\mathfrak{f}_2, Ʀ_2, Ᵽ_2)$ are two LOTBSRs, then OR product of these two LOTBSRs is a LOTBSR provided that $\mathfrak{f}_1(ă_1) \cup \mathfrak{f}_2(ă_2)$ is a subring of $\mathcal{L}$ for all $(ă_1, ă_2) \in Ᵽ_1 \times Ᵽ_2$.

Proof: According to definition of OR product for two LOTBSRs $(\mathfrak{f}_1, Ʀ_1, Ᵽ_1) \bigvee (\mathfrak{f}_2, Ʀ_2, Ᵽ_2) = (\mathfrak{f}_3, Ʀ_3, Ᵽ_3)$ where $\mathfrak{f}_3(ă_1, ă_2) = \mathfrak{f}_1(ă_1) \cup \mathfrak{f}_2(ă_2)$ and $Ʀ_3(ă_1, ă_2) = Ʀ_1(ă_1) \cap Ʀ_2(ă_2)$ for all $(ă_1, ă_2) \in Ᵽ_1 \times Ᵽ_2$. Now as $(\mathfrak{f}_1, Ʀ_1, Ᵽ_1)$ and $(\mathfrak{f}_2, Ʀ_2, Ᵽ_2)$ are two LOTBSRs, then $\mathfrak{f}_1(ă_1), \mathfrak{f}_2(ă_2)$ are subring of $\mathcal{L}$ and $Ʀ_1(ă_1), Ʀ_2(ă_2)$ are subring of $\mathcal{G}$. Now as intersection of any number of subring is a subring, so we can say that $Ʀ_3(ă_1, ă_2) = Ʀ_1(ă_1) \cap Ʀ_2(ă_2)$ is a suborn of $\mathcal{G}$ and it is given in the statement that $\mathfrak{f}_3(ă_1, ă_2) = \mathfrak{f}_1(ă_1) \cup \mathfrak{f}_2(ă_2)$ is a subring of $\mathcal{L}$. Hence, we can say that OR product of two LOTBSR is again LOTBSR.

As $(\mathfrak{f}_1, Ʀ_1, Ᵽ_1)$ and $(\mathfrak{f}_2, Ʀ_2, Ᵽ_2)$ are two LOTBSRs where both $Ᵽ_1$ and $Ᵽ_2$ are partially ordered sets. Now $ă_1 \preccurlyeq_{Ᵽ_1} ă_2$ for all $ă_1, ă_2 \in Ᵽ_1$ then $\mathfrak{f}_1(ă_1) \subseteq \mathfrak{f}_1(ă_2)$ and $Ʀ_1(ă_1) \supseteq Ʀ_1(ă_2)$ and $Ѣ_1 \preccurlyeq_{Ᵽ_2} Ѣ_2$ for all $Ѣ_1, Ѣ_2 \in Ᵽ_2$ then $\mathfrak{f}_2(Ѣ_1) \subseteq \mathfrak{f}_2(Ѣ_2)$ and $Ʀ_2(Ѣ_1) \supseteq Ʀ_2(Ѣ_2)$. Now as $\preccurlyeq_{Ᵽ_3}$ is a partially ordered relation among the element of $Ᵽ_1 \times Ᵽ_2$ such that $(ă_1, Ѣ_1) \preccurlyeq (ă_2, Ѣ_2)$ where $(ă_1, Ѣ_1), (ă_2, Ѣ_2) \in Ᵽ_1 \times Ᵽ_2 = Ᵽ_3$ and this ordered is induced by elements of $Ᵽ_1$ and $Ᵽ_2$. As $\mathfrak{f}_1(ă_1) \subseteq \mathfrak{f}_1(ă_2)$ and $Ʀ_1(ă_1) \supseteq Ʀ_1(ă_2)$ and $\mathfrak{f}_2(Ѣ_1) \subseteq \mathfrak{f}_2(Ѣ_2)$ and $Ʀ_2(Ѣ_1) \supseteq Ʀ_2(Ѣ_2)$ and $(ă_1, Ѣ_1) \preccurlyeq_{Ᵽ_3} (ă_2, Ѣ_2)$ then $\mathfrak{f}_1(ă_1) \cup \mathfrak{f}_2(Ѣ_1) \subseteq \mathfrak{f}_1(ă_2) \cup \mathfrak{f}_2(Ѣ_2)$ and $Ʀ_1(ă_1) \cap Ʀ_2(Ѣ_1) \supseteq Ʀ_1(ă_2) \cap Ʀ_2(Ѣ_2)$. It conclude that $\mathfrak{f}_3(ă_1, Ѣ_1) \subseteq \mathfrak{f}_3(ă_2, Ѣ_2)$ and $Ʀ_3(ă_1, Ѣ_1) \supseteq Ʀ_3(ă_2, Ѣ_2)$. Hence proved

Definition 10: Assume that $(\mathfrak{f}_1, Ʀ_1, Ᵽ_1)$ and $(\mathfrak{f}_2, Ʀ_2, Ᵽ_2)$ are two LOTBSRs then extended union of these two LOTBSRs is denoted and defined as

$$(\mathfrak{f}_1, Ʀ_1, Ᵽ_1) \cup_{extended} (\mathfrak{f}_2, Ʀ_2, Ᵽ_2)$$
$$= (\mathfrak{f}_3, Ʀ_3, Ᵽ_3); \; Ᵽ_3 = Ᵽ_1 \cup Ᵽ_2$$

$$\mathfrak{f}_3(ă) = \begin{cases} \mathfrak{f}_1(ă) & ; \textit{if } ă \in Ᵽ_1 - Ᵽ_2 \\ \mathfrak{f}_2(ă) & ; \textit{if } ă \in Ᵽ_2 - Ᵽ_1 \\ \mathfrak{f}_1(ă) \cup \mathfrak{f}_2(ă) & ; \textit{if } ă \in Ᵽ_1 \cap Ᵽ_2 \end{cases}$$

$$Ʀ_3(ă) = \begin{cases} Ʀ_1(ă) & ; \textit{if } ă \in Ᵽ_1 - Ᵽ_2 \\ Ʀ_2(ă) & ; \textit{if } ă \in Ᵽ_2 - Ᵽ_1 \\ Ʀ_1(ă) \cap Ʀ_2(ă) & ; \textit{if } ă \in Ᵽ_1 \cap Ᵽ_2 \end{cases}$$

Remark 2: The extended union of two LOTBSRs need not to be LOTBSRs

Example 3: For $\mathcal{L} = Z_{12}$ and $\mathcal{G} = Z_{18}$ with $\mathbb{U} = \mathcal{L} \bigcup \mathcal{G}$ and $Ᵽ_1 = \{ă_1, ă_2, ă_3\}$ such that $ă_1 \leq ă_2 \leq ă_3$ if we define

$$\mathfrak{f}_1(ă_1) = \{\bar{0}, \bar{6}\},$$
$$\mathfrak{f}_1(ă_2) = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\},$$
$$\mathfrak{f}_1(ă_3) = Z_{12}$$

And

$$Ʀ_1(ă_1), = Z_{18}, \; Ʀ_1(ă_2) = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}, \overline{12}, \overline{15}\},$$
$$Ʀ_1(ă_3) = \{\bar{0}, \bar{6}, \overline{12}\}$$

Then it is clear that $(\mathfrak{f}_1, Ʀ_1, Ᵽ_1)$ is LOTBSR.

Also assume that $\mathcal{L} = Z_{12}$ and $\mathcal{G} = Z_{18}$ with $\mathbb{U} = \mathcal{L} \bigcup \mathcal{G}$ and $Ᵽ_2 = \{ă_1, ă_2\}$ such that $ă_1 \leq ă_2$ if we define

$$\mathfrak{f}_2(ă_1) = \{\bar{0}, \bar{4}, \bar{8}\},$$
$$\mathfrak{f}_2(ă_2) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$$

And

$$Ʀ_2(ă_1) = Z_{18}, \; Ʀ_2(ă_2) = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}, \overline{12}, \overline{15}\}$$

Then it is clear that $(\mathfrak{f}_2, Ʀ_2, Ᵽ_2)$ is LOTBSR.

Now as

$$(\mathfrak{f}_1, Ʀ_1, Ᵽ_1) \cup_{extended} (\mathfrak{f}_2, Ʀ_2, Ᵽ_2)$$
$$= (\mathfrak{f}_3, Ʀ_3, Ᵽ_3); \; Ᵽ_3 = Ᵽ_1 \cup Ᵽ_2$$

$$\mathfrak{f}_3(ă) = \begin{cases} \mathfrak{f}_1(ă) & ; \textit{if } ă \in Ᵽ_1 - Ᵽ_2 \\ \mathfrak{f}_2(ă) & ; \textit{if } ă \in Ᵽ_2 - Ᵽ_2 \\ \mathfrak{f}_1(ă) \cup \mathfrak{f}_2(ă) & ; \textit{if } ă \in Ᵽ_1 \cap Ᵽ_2 \end{cases}$$

$$Ʀ_3(ă) = \begin{cases} Ʀ_1(ă) & ; \textit{if } ă \in Ᵽ_1 - Ᵽ_2 \\ Ʀ_2(ă) & ; \textit{if } ă \in Ᵽ_2 - Ᵽ_1 \\ Ʀ_1(ă) \cap Ʀ_2(ă) & ; \textit{if } ă \in Ᵽ_1 \cap Ᵽ_2 \end{cases}$$

As $Ᵽ_3 = Ᵽ_1 \cup Ᵽ_2 = \{ă_1, ă_2, ă_3\}$ then

$$\mathfrak{f}_3(ă_1) = \mathfrak{f}_1(ă_1) \cup \mathfrak{f}_2(ă_1) = \{\bar{0}, \bar{6}\} \cup \{\bar{0}, \bar{4}, \bar{8}\}$$
$$= \{\bar{0}, \bar{4}, \bar{6}, \bar{8}\}$$

We can observe that $\mathfrak{f}_3(ă_1) = \mathfrak{f}_1(ă_1) \cup \mathfrak{f}_2(ă_1)$ is not a subring of $\mathcal{L}$. Hence we can say that extended union of two LOTBSRs need not to be LOTBSR.

Theorem 2: Let $(\mathfrak{f}_1, Ʀ_1, Ᵽ_1)$ and $(\mathfrak{f}_2, Ʀ_2, Ᵽ_2)$ are two LOTBSRs, then extended union is again a LOTBSR provided that $\mathfrak{f}_1(ă)$ is a subring of $\mathfrak{f}_2(ă)$ or $\mathfrak{f}_2(ă)$ is a subring of $\mathfrak{f}_1(ă)$.

Proof: Assume that $\mathfrak{f}_1(\breve{a})$ is a subring of $\mathfrak{f}_2(\breve{a})$ and $\mathfrak{f}_2(\breve{a})$ is a subring of $\mathfrak{f}_1(\breve{a})$ then in either case $\mathfrak{f}_1(\breve{a}) \cup \mathfrak{f}_2(\breve{a})$ is a subring of $\mathcal{L}$. Now consider $\mathfrak{f}_3(\breve{a})$. If $\breve{a} \in \mathbb{P}_1 \backslash \mathbb{P}_2$ then $\mathfrak{f}_3(\breve{a}) = \mathfrak{f}_1(\breve{a})$ or if $\breve{a} \in \mathbb{P}_2 \backslash \mathbb{P}_1$ then $\mathfrak{f}_3(\breve{a}) = \mathfrak{f}_2(\breve{a})$, so in either case $\mathfrak{f}_3(\breve{a})$ is the subring of $\mathcal{L}$. Now consider $\breve{a} \in \mathbb{P}_1 \cap \mathbb{P}_2$ then $\mathfrak{f}_3(\breve{a}) = \mathfrak{f}_1(\breve{a}) \cup \mathfrak{f}_2(\breve{a})$. Hence $\mathfrak{f}_3(\breve{a})$ is again subring of $\mathcal{L}$. Now consider $\mathfrak{R}_3(\breve{a})$. If $\breve{a} \in \mathbb{P}_1 \backslash \mathbb{P}_2$ then $\mathfrak{R}_3(\breve{a}) = \mathfrak{R}_1(\breve{a})$ or if $\breve{a} \in \mathbb{P}_2 \backslash \mathbb{P}_1$ then $\mathfrak{R}_3(\breve{a}) = \mathfrak{R}_2(\breve{a})$, so in either case $\mathfrak{R}_3(\breve{a})$ is the subring of $\mathcal{G}$. Now consider $\breve{a} \in \mathbb{P}_1 \cap \mathbb{P}_2$ then $\mathfrak{R}_3(\breve{a}) = \mathfrak{R}_1(\breve{a}) \cap \mathfrak{R}_2(\breve{a})$. Now as $\mathfrak{R}_1(\breve{a})$, $\mathfrak{R}_2(\breve{a})$ are subrings of $\mathcal{G}$ then their intersection $\mathfrak{R}_1(\breve{a}) \cap \mathfrak{R}_2(\breve{a})$ is also a subring of $\mathcal{G}$. Hence $\mathfrak{R}_3(\breve{a}) = \mathfrak{R}_1(\breve{a}) \cap \mathfrak{R}_2(\breve{a})$ is a subring of $\mathcal{G}$. Hence $(\mathfrak{f}_1, \mathfrak{R}_1, \mathbb{P}_1) \cup_{extended} (\mathfrak{f}_2, \mathfrak{R}_2, \mathbb{P}_2)$ is a TBSR.

As $(\mathfrak{f}_1, \mathfrak{R}_1, \mathbb{P}_1)$ and $(\mathfrak{f}_2, \mathfrak{R}_2, \mathbb{P}_2)$ are two LOTBSRs then for all $\breve{a}_1, \breve{a}_2 \in$ for all $\breve{a}_1, \breve{a}_2 \in \mathbb{P}_1$, $\mathfrak{f}_1(\breve{a}_1) \subseteq \mathfrak{f}_1(\breve{a}_2)$ and $\mathfrak{R}_1(\breve{a}_1) \supseteq \mathfrak{R}_1(\breve{a}_2)$. Similarly for all $\breve{a}_1, \breve{a}_2 \in \mathbb{P}_2$, $\mathfrak{f}_2(\breve{a}_1) \subseteq \mathfrak{f}_2(\breve{a}_2)$ and $\mathfrak{R}_2(\breve{a}_1) \supseteq \mathfrak{R}_2(\breve{a}_2)$. Then $\mathfrak{f}_1(\breve{a}_1) \cup \mathfrak{f}_2(\breve{a}_1) \subseteq \mathfrak{f}_1(\breve{a}_2) \cup \mathfrak{f}_2(\breve{a}_2)$ and $\mathfrak{R}_1(\breve{a}_1) \cap \mathfrak{R}_2(\breve{a}_1) \supseteq \mathfrak{R}_1(\breve{a}_2) \cap \mathfrak{R}_2(\breve{a}_2)$ as required.

Definition 11: Let $(\mathfrak{f}_1, \mathfrak{R}_1, \mathbb{P}_1)$ and $(\mathfrak{f}_2, \mathfrak{R}_2, \mathbb{P}_2)$ be two LOTBSRs, then restricted union of two LOTBSRs is denoted and defined by

$$(\mathfrak{f}_1, \mathfrak{R}_1, \mathbb{P}_1) \cup_{restr.} (\mathfrak{f}_2, \mathfrak{R}_2, \mathbb{P}_1) = \left\{ \begin{matrix} \breve{a}, \mathfrak{f}_1(\breve{a}) \cup \mathfrak{f}_2(\breve{a}), \\ \mathfrak{R}_1(\breve{a}) \cap \mathfrak{R}_1(\breve{a}) \text{ for all } \breve{a} \in \mathbb{P}_1 \cap \mathbb{P}_2 \end{matrix} \right\}$$

Remark 3: The restricted union of two LOTBSRs need not to be LOTBSR.

Example 4: Let $\mathcal{L} = Z_{24}$ and $\mathcal{G} = Z_{28}$ be two distinct rings such that $\mathbb{U} = \mathcal{L} \bigcup \mathcal{G}$ and $\mathbb{P}_1 = \{\breve{a}_1, \breve{a}_2\}$ with $\breve{a}_1 \leq \breve{a}_2$. Now if we define the mappings as

$$\mathfrak{f}_1(\breve{a}_1) = \{\overline{0}, \overline{12}\}, \mathfrak{f}_1(\breve{a}_2) = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}, \overline{16}, \overline{20}\}$$

And

$$\mathfrak{R}_1(\breve{a}_1) = \{\overline{0}, \overline{7}, \overline{14}, \overline{21}\}, \mathfrak{R}_1(\breve{a}_2) = \{\overline{0}, \overline{14}\}$$

Then it is clear that $(\mathfrak{f}_1, \mathfrak{R}_1, \mathbb{P}_1)$ is LOTBSR.

Also for $\mathcal{L} = Z_{24}$ and $\mathcal{G} = Z_{28}$ with $\mathbb{U} = \mathcal{L} \bigcup \mathcal{G}$ and $\mathbb{P}_2 = \{\breve{a}_1, \breve{a}_2, \breve{a}_3\}$ such that $\breve{a}_1 \leq \breve{a}_2 \leq \breve{a}_3$ if we define

$$\mathfrak{f}_2(\breve{a}_1) = \{\overline{0}, \overline{12}\}, \mathfrak{f}_2(\breve{a}_2) = \{\overline{0}, \overline{6}, \overline{12}, \overline{18}\}, \mathfrak{f}_2(\breve{a}_3) = Z_{24}$$

And

$$\mathfrak{R}_2(\breve{a}_1) = Z_{28}, \mathfrak{R}_2(\breve{a}_2) = \{\overline{0}, \overline{7}, \overline{14}, \overline{21}\}, \mathfrak{R}_2(\breve{a}_3) = \{\overline{0}, \overline{14}\}$$

Then it is clear that $(\mathfrak{f}_2, \mathfrak{R}_2, \mathbb{P}_2)$ is LOTBSR.

Now we can see that as $\breve{a}_2 \in \mathbb{P}_1 \cap \mathbb{P}_2$, $\mathfrak{f}_1(\breve{a}_2) \cup \mathfrak{f}_2(\breve{a}_2) = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}, \overline{16}, \overline{20}\} \cup \{\overline{0}, \overline{6}, \overline{12}, \overline{18}\} = \{\overline{0}, \overline{4}, \overline{6}, \overline{8}, \overline{12}, \overline{16}, \overline{18}, \overline{20}\}$ is not a subring of Z_{24}

Theorem 3: The restricted union of two LOTBSRs $(\mathfrak{f}_1, \mathfrak{R}_1, \mathbb{P}_1)$ and $(\mathfrak{f}_2, \mathfrak{R}_2, \mathbb{P}_2)$ is LOTBSR provided that $\mathfrak{f}_1(\breve{a}) \cup \mathfrak{f}_2(\breve{a})$ is a subring of $\mathcal{L}$ for all $\breve{a} \in \mathbb{P}_1 \cap \mathbb{P}_2$.

Proof: Similar to the proof of theorem (2).

Definition 12: Let $(\mathfrak{f}_1, \mathfrak{R}_1, \mathbb{P}_1)$ and $(\mathfrak{f}_2, \mathfrak{R}_2, \mathbb{P}_2)$ be two LOTBSRs. Then AND product is denoted and defined by

$$(\mathfrak{f}_1, \mathfrak{R}_1, \mathbb{P}_1) \bigwedge (\mathfrak{f}_2, \mathfrak{R}_2, \mathbb{P}_2) = \left\{ \left((\breve{a}_1, \breve{a}_2), \mathfrak{f}_1(\breve{a}_1) \bigcap \mathfrak{f}_2(\breve{a}_2), \mathfrak{R}_1(\breve{a}_1) \bigcup \mathfrak{R}_2(\breve{a}_2) \right) \text{ where } (\breve{a}_1, \breve{a}_2) \in \mathbb{P}_1 \times \mathbb{P}_2 \right\}.$$

Remark 4: AND product of two LOTBRS need not to be LOTBSR in general.

Example 5: Let $\mathcal{L} = Z_{30}$ and $\mathcal{G} = Z_{28}$ be two distinct rings such that $\mathbb{U} = \mathcal{L} \bigcup \mathcal{G}$ and $\mathbb{P}_1 = \{\breve{a}_1, \breve{a}_2, \breve{a}_3\}$ with $\breve{a}_1 \leq \breve{a}_2 \leq \breve{a}_3$ and $\mathbb{P}_2 = \{\breve{a}_1, \breve{a}_2\}$ such that $\breve{a}_1 \leq \breve{a}_2$. Now if we define the mappings as

$$\mathfrak{f}_1(\breve{a}_1) = \{\overline{0}, \overline{15}\}, \mathfrak{f}_1(\breve{a}_2) = \{\overline{0}, \overline{10}, \overline{20}\}, \mathfrak{f}_1(\breve{a}_3) = \{\overline{0}, \overline{5}, \overline{10}, \overline{15}, \overline{20}, \overline{25}\}$$

And

$$\mathfrak{R}_1(\breve{a}_1) = \{\overline{0}, \overline{7}, \overline{14}, \overline{21}\}, \mathfrak{R}_1(\breve{a}_2) = \{\overline{0}, \overline{14}\}, \mathfrak{R}_1(\breve{a}_3) = \{\overline{0}, \overline{14}\}$$

Then it is clear that $(\mathfrak{f}_1, \mathfrak{R}_1, \mathbb{P}_1)$ is LOTBSR.

Also for $\mathcal{L} = Z_{30}$ and $\mathcal{G} = Z_{28}$ with $\mathbb{U} = \mathcal{L} \bigcup \mathcal{G}$ and $\mathbb{P}_2 = \{\breve{a}_1, \breve{a}_2\}$ such that $\breve{a}_1 \leq \breve{a}_2$ if we define

$$\mathfrak{f}_2(\breve{a}_1) = \{\overline{0}, \overline{6}, \overline{12}, \overline{18}, \overline{24}\}, \mathfrak{f}_2(\breve{a}_2) = \{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}, \overline{24}, \overline{27}\}, \mathfrak{f}_2(\breve{a}_3) = Z_{30}$$

And

$$\mathfrak{R}_2(\breve{a}_1) = Z_{28}, \mathfrak{R}_2(\breve{a}_2) = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}, \overline{12}, \overline{14}, \overline{16}, \overline{18}, \overline{20}, \overline{22}, \overline{24}, \overline{26}\}, \mathfrak{R}_2(\breve{a}_3) = \{\overline{0}, \overline{4}, \overline{8}, \overline{12}, \overline{16}, \overline{20}, \overline{24}\}.$$

Then it is clear that $(\mathfrak{f}_2, \mathfrak{R}_2, \mathbb{P}_2)$ is LOTBSR.

Now as AND product is defined as

$$(\mathfrak{f}_1, \mathfrak{R}_1, \mathbb{P}_1) \bigwedge (\mathfrak{f}_2, \mathfrak{R}_2, \mathbb{P}_2) = \left\{ \left(\begin{matrix} (\breve{a}_1, \breve{a}_2), \mathfrak{f}_1(\breve{a}_1) \bigcap \mathfrak{f}_2(\breve{a}_2), \\ \mathfrak{R}_1(\breve{a}_1) \bigcup \mathfrak{R}_2(\breve{a}_2) \end{matrix} \right) \text{ where } (\breve{a}_1, \breve{a}_2) \in \mathbb{P}_1 \times \mathbb{P}_2 \right\}$$

As $\mathbb{P}_1 \times \mathbb{P}_2 = \left\{ \begin{matrix} (\breve{a}_1, \breve{a}_1), (\breve{a}_1, \breve{a}_2), (\breve{a}_2, \breve{a}_1), \\ (\breve{a}_2, \breve{a}_2), (\breve{a}_3, \breve{a}_1), (\breve{a}_3, \breve{a}_2) \end{matrix} \right\}$

Now we can observe that

$$\mathfrak{R}_1(\breve{a}_1) \cup \mathfrak{R}_2(\breve{a}_2) = \{\overline{0}, \overline{7}, \overline{14}, \overline{21}\} \cup \left\{ \begin{matrix} \overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}, \overline{12}, \overline{14}, \overline{16}, \\ \overline{18}, \overline{20}, \overline{22}, \overline{24}, \overline{26} \end{matrix} \right\} = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{7}, \overline{8}, \overline{10}, \overline{12}, \overline{14}, \overline{16}, \overline{18}, \overline{20}, \overline{21}, \overline{22}, \overline{24}, \overline{26}\}$$

As $Ꞧ_1(\breve{a}_1) \cup Ꞧ_2(\breve{a}_2)$ is not a subring of $\mathcal{G} = Z_{28}$. Hence it is clear that AND product of two LOTBSRs need not to be a LOTBSR.

Theorem 4: Let $(\mathfrak{f}_1, Ꞧ_1, Ᵽ_1)$ and $(\mathfrak{f}_2, Ꞧ_2, Ᵽ_2)$ are two LOTBSRs, then AND product of these two LOTBSRs is a LOTBSR provided that $Ꞧ_1(\breve{a}_1) \cup Ꞧ_2(\breve{a}_2)$ is a subring of $\mathcal{G}$ for all $(\breve{a}_1, \breve{a}_2) \in Ᵽ_1 \times Ᵽ_2$.

Proof: According to definition of AND product for two LOTBSRs

$$(\mathfrak{f}_1, Ꞧ_1, Ᵽ_1) \bigwedge (\mathfrak{f}_2, Ꞧ_2, Ᵽ_2) = \left\{ \begin{pmatrix} (\breve{a}_1, \breve{a}_2), \mathfrak{f}_1(\breve{a}_1) \bigcap \mathfrak{f}_2(\breve{a}_2), \\ Ꞧ_1(\breve{a}_1) \bigcup Ꞧ_2(\breve{a}_2) \end{pmatrix} \right. \textit{where } (\breve{a}_1, \breve{a}_2) \in Ᵽ_1 \times Ᵽ_2 \Big\} = (\mathfrak{f}_3, Ꞧ_3, Ᵽ_3)$$

where $\mathfrak{f}_3(\breve{a}_1, \breve{a}_2) = \mathfrak{f}_1(\breve{a}_1) \cap \mathfrak{f}_2(\breve{a}_2)$ and $Ꞧ_3(\breve{a}_1, \breve{a}_2) = Ꞧ_1(\breve{a}_1) \cup Ꞧ_2(\breve{a}_2)$ for all $(\breve{a}_1, \breve{a}_2) \in Ᵽ_1 \times Ᵽ_2$. Now as $(\mathfrak{f}_1, Ꞧ_1, Ᵽ_1)$ and $(\mathfrak{f}_2, Ꞧ_2, Ᵽ_2)$ are two LOTBSRs, then $\mathfrak{f}_1(\breve{a}_1)$, $\mathfrak{f}_2(\breve{a}_2)$ are subring of $\mathcal{L}$ and $Ꞧ_1(\breve{a}_1)$, $Ꞧ_2(\breve{a}_2)$ are subring of $\mathcal{G}$. Now as intersection of any number of subring is a subring, so we can say that $\mathfrak{f}_3(\breve{a}_1, \breve{a}_2) = \mathfrak{f}_1(\breve{a}_1) \cap \mathfrak{f}_2(\breve{a}_2)$ is a subring of $\mathcal{L}$ and it is given in the statement that $Ꞧ_3(\breve{a}_1, \breve{a}_2) = Ꞧ_1(\breve{a}_1) \cup Ꞧ_2(\breve{a}_2)$ is a subring of $\mathcal{G}$. Hence, we can say that AND product of two LOTBSR is again LOTBSR.

As $(\mathfrak{f}_1, Ꞧ_1, Ᵽ_1)$ and $(\mathfrak{f}_2, Ꞧ_2, Ᵽ_2)$ are two LOTBSRs where both $Ᵽ_1$ and $Ᵽ_2$ are partially ordered sets. Now $\breve{a}_1 \preccurlyeq_{Ᵽ_1} \breve{a}_2$ for all $\breve{a}_1, \breve{a}_2 \in Ᵽ_1$ then $\mathfrak{f}_1(\breve{a}_1) \subseteq \mathfrak{f}_1(\breve{a}_2)$ and $Ꞧ_1(\breve{a}_1) \supseteq Ꞧ_1(\breve{a}_2)$ and $Ђ_1 \preccurlyeq_{Ᵽ_2} Ђ_2$ for all $Ђ_1, Ђ_2 \in Ᵽ_2$ then $\mathfrak{f}_2(Ђ_1) \subseteq \mathfrak{f}_2(Ђ_2)$ and $Ꞧ_2(Ђ_1) \supseteq Ꞧ_2(Ђ_2)$. Now as $\preccurlyeq_{Ᵽ_3}$ is a partially ordered relation among the element of $Ᵽ_1 \times Ᵽ_2$ such that $(\breve{a}_1, Ђ_1) \preccurlyeq (\breve{a}_2, Ђ_2)$ where $(\breve{a}_1, Ђ_1), (\breve{a}_2, Ђ_2) \in Ᵽ_1 \times Ᵽ_2 = Ᵽ_3$ and this ordered is induced by elements of $Ᵽ_1$ and $Ᵽ_2$. As $\mathfrak{f}_1(\breve{a}_1) \subseteq \mathfrak{f}_1(\breve{a}_2)$ and $Ꞧ_1(\breve{a}_1) \supseteq Ꞧ_1(\breve{a}_2)$ and $\mathfrak{f}_2(Ђ_1) \subseteq \mathfrak{f}_2(Ђ_2)$ and $Ꞧ_2(Ђ_1) \supseteq Ꞧ_2(Ђ_2)$ and $(\breve{a}_1, Ђ_1) \preccurlyeq_{Ᵽ_3} (\breve{a}_2, Ђ_2)$ then $\mathfrak{f}_1(\breve{a}_1) \cap \mathfrak{f}_2(Ђ_1) \subseteq \mathfrak{f}_1(\breve{a}_2) \cap \mathfrak{f}_2(Ђ_2)$ and $Ꞧ_1(\breve{a}_1) \cup Ꞧ_2(Ђ_1) \supseteq Ꞧ_1(\breve{a}_2) \cup Ꞧ_2(Ђ_2)$. It conclude that $\mathfrak{f}_3(\breve{a}_1, Ђ_1) \subseteq \mathfrak{f}_3(\breve{a}_2, Ђ_2)$ and $Ꞧ_3(\breve{a}_1, Ђ_1) \supseteq Ꞧ_3(\breve{a}_2, Ђ_2)$. Hence proved

Definition 13: Let $(\mathfrak{f}_1, Ꞧ_1, Ᵽ_1)$ and $(\mathfrak{f}_2, Ꞧ_2, Ᵽ_2)$ are two LOTBSRs, then restricted intersection of two LOTBSRs is denoted and defined by

$$(\mathfrak{f}_1, Ꞧ_1, Ᵽ_1) \cap_{restr.} (\mathfrak{f}_2, Ꞧ_2, Ᵽ_1) = \left\{\breve{a}, \mathfrak{f}_1(\breve{a}) \cap \mathfrak{f}_2(\breve{a}), Ꞧ_1(\breve{a}) \cup Ꞧ_1(\breve{a}) \textit{ for all } \breve{a} \in Ᵽ_1 \cap Ᵽ_2\right\}$$

Remark 5: The restricted intersection of two LOTBSRs need not to be LOTBSR

Example 6: Let $\mathcal{L} = Z_{30}$ and $\mathcal{G} = Z_{12}$ be two distinct rings such that $\mathbb{U} = \mathcal{L} \bigcup \mathcal{G}$ and $Ᵽ_1 = \{\breve{a}_1, \breve{a}_2\}$ with $\breve{a}_1 \leq \breve{a}_2$ and $Ᵽ_2 = \{\breve{a}_1, \breve{a}_2, \breve{a}_3\}$ such that $\breve{a}_1 \leq \breve{a}_2 \leq \breve{a}_3$. Now if we define the mappings as

$$\mathfrak{f}_1(\breve{a}_1) = \left\{\overline{0}, \overline{6}, \overline{12}, \overline{18}, \overline{24}\right\},$$
$$\mathfrak{f}_1(\breve{a}_2) = \left\{\overline{0}, \overline{3}, \overline{6}, \overline{9}, \overline{12}, \overline{15}, \overline{18}, \overline{21}, \overline{24}, \overline{27}\right\}$$

And

$$Ꞧ_1(\breve{a}_1) = Z_{12}, \; Ꞧ_1(\breve{a}_2) = \left\{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}\right\}$$

Then it is clear that $(\mathfrak{f}_1, Ꞧ_1, Ᵽ_1)$ is LOTBSR.

Also for $\mathcal{L} = Z_{30}$ and $\mathcal{G} = Z_{12}$ with $\mathbb{U} = \mathcal{L} \bigcup \mathcal{G}$ and $Ᵽ_2 = \{\breve{a}_1, \breve{a}_2, \breve{a}_3\}$ such that $\breve{a}_1 \leq \breve{a}_2 \leq \breve{a}_3$ if we define

$$\mathfrak{f}_2(\breve{a}_1) = \left\{\overline{0}, \overline{10}, \overline{20}\right\},$$
$$\mathfrak{f}_2(\breve{a}_2) = \left\{\overline{0}, \overline{5}, \overline{10}, \overline{15}, \overline{20}, \overline{25}\right\},$$
$$\mathfrak{f}_2(\breve{a}_3) = Z_{30}$$

And

$$Ꞧ_2(\breve{a}_1) = Z_{12}, \; Ꞧ_2(\breve{a}_2) = \left\{\overline{0}, \overline{3}, \overline{6}, \overline{9}\right\}, \; Ꞧ_2(\breve{a}_3) = \left\{\overline{0}\right\}$$

Then it is clear that $(\mathfrak{f}_2, Ꞧ_2, Ᵽ_2)$ is LOTBSR.

Now we can see that as $\breve{a}_2 \in Ᵽ_1 \cap Ᵽ_2$, $Ꞧ_1(\breve{a}_2) \cup Ꞧ_2(\breve{a}_2) = \left\{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}\right\} \cup \left\{\overline{0}, \overline{3}, \overline{6}, \overline{9}\right\} = \left\{\overline{0}, \overline{2}, \overline{3}, \overline{4}, \overline{6}, \overline{8}, \overline{9}, \overline{10}\right\}$ is not a subring of Z_{12}. Hence it is clear that restricted union is not a LOTBSR.

Theorem 5: The restricted union of two LOTBSRs $(\mathfrak{f}_1, Ꞧ_1, Ᵽ_1)$ and $(\mathfrak{f}_2, Ꞧ_2, Ᵽ_2)$ defined on two distinct rings $\mathcal{L}$ and $\mathcal{G}$ is LOTBSR provided that $Ꞧ_1(\breve{a}) \cup Ꞧ_2(\breve{a})$ is a subring of $\mathcal{G}$ for all $\breve{a} \in Ᵽ_1 \cap Ᵽ_2$.

Proof: Similar to the proof of theorem (4).

Definition 14: Let $(\mathfrak{f}_1, Ꞧ_1, Ᵽ_1)$ and $(\mathfrak{f}_2, Ꞧ_2, Ᵽ_2)$ be two TBSRs. Then the extended intersection of two LOTBSRs is denoted and defined by

$$(\mathfrak{f}_1, Ꞧ_1, Ᵽ_1) \cap_{extended} (\mathfrak{f}_2, Ꞧ_2, Ᵽ_2) = (\mathfrak{f}_3, Ꞧ_3, Ᵽ_3); \; Ᵽ_3 = Ᵽ_1 \cup Ᵽ_2$$

And

$$\mathfrak{f}_3(\breve{a}) = \begin{cases} \mathfrak{f}_1(\breve{a}) & ; \textit{ if } \breve{a} \in Ᵽ_1 - Ᵽ_2 \\ \mathfrak{f}_2(\breve{a}) & ; \textit{ if } \breve{a} \in Ᵽ_2 - Ᵽ_1 \\ \mathfrak{f}_1(\breve{a}) \cap \mathfrak{f}_2(\breve{a}) & ; \textit{ if } \breve{a} \in Ᵽ_1 \cap Ᵽ_2 \end{cases}$$

$$Ꞧ_3(\breve{a}) = \begin{cases} Ꞧ_1(\breve{a}) & ; \textit{ if } \breve{a} \in Ᵽ_1 - Ᵽ_2 \\ Ꞧ_2(\breve{a}) & ; \textit{ if } \breve{a} \in Ᵽ_2 - Ᵽ_1 \\ Ꞧ_1(\breve{a}) \cup Ꞧ_2(\breve{a}) & ; \textit{ if } \breve{a} \in Ᵽ_1 \cap Ᵽ_2 \end{cases}$$

Remark 6: The extended intersection for two LOTBSRs need not to be LOTBSR in general.

Example 7: Let $\mathcal{L} = Z_{20}$ and $\mathcal{G} = Z_{18}$ be two distinct rings such that $\mathbb{U} = \mathcal{L} \bigcup \mathcal{G}$ and $Ᵽ_1 = \{\breve{a}_1, \breve{a}_2, \breve{a}_3\}$ with $\breve{a}_1 \leq \breve{a}_2 \leq \breve{a}_3$ and $Ᵽ_2 = \{\breve{a}_1, \breve{a}_2\}$ such that $\breve{a}_1 \leq \breve{a}_2$. Now if we define the mappings as

$$\mathfrak{f}_1(\breve{a}_1) = \left\{\overline{0}, \overline{10}\right\},$$
$$\mathfrak{f}_1(\breve{a}_2) = \left\{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}, \overline{12}, \overline{14}, \overline{16}, \overline{18}\right\},$$
$$\mathfrak{f}_1(\breve{a}_3) = Z_{20}$$

And

$$Ꞧ_1(\breve{a}_1) = Z_{18}, \; Ꞧ_1(\breve{a}_2) = \left\{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}, \overline{12}, \overline{14}, \overline{16}\right\},$$
$$Ꞧ_1(\breve{a}_3) = \left\{\overline{0}\right\}$$

Then it is clear that $(\mathfrak{f}_1, Ꞧ_1, Ᵽ_1)$ is LOTBSR.

Also for $\mathcal{L} = Z_{20}$ and $\mathcal{G} = Z_{18}$ with $\mathbb{U} = \mathcal{L} \bigcup \mathcal{G}$ and $Ᵽ_2 = \{\breve{a}_1, \breve{a}_2\}$ such that $\breve{a}_1 \leq \breve{a}_2$ if we define

$$\mathfrak{f}_2(\breve{a}_1) = \left\{\overline{0}, \overline{4}, \overline{8}, \overline{12}, \overline{16}\right\},$$

$$\mathfrak{f}_2(\breve{a}_2) = \{\overline{0},\ \overline{2},\ \overline{4},\ \overline{6},\ \overline{8},\ \overline{10},\ \overline{12}, \overline{14},\ \overline{16}, \overline{18}\}$$

And

$$\mathrm{R}_2(\breve{a}_1) = Z_{18},\ \mathrm{R}_2(\breve{a}_2) = \{\overline{0},\ \overline{3},\ \overline{6},\ \overline{9},\ \overline{12},\ \overline{15}\}$$

Then it is clear that $(\mathfrak{f}_2,\ \mathrm{R}_2,\ \mathrm{P}_2)$ is LOTBSR.

Now we can see that as $\breve{a}_2 \in \mathrm{P}_1 \cap \mathrm{P}_2$, $\mathrm{R}_1(\breve{a}_2) \cup \mathrm{R}_2(\breve{a}_2) = \{\overline{0},\ \overline{2},\ \overline{4},\ \overline{6},\ \overline{8},\ \overline{10},\ \overline{12},\ \overline{14},\ \overline{16}\} \cup \{\overline{0},\ \overline{3},\ \overline{6},\ \overline{9},\ \overline{12},\ \overline{15}\} = \{\overline{0},\ \overline{2},\ \overline{3},\ \overline{4},\ \overline{6},\ \overline{8},\ \overline{9},\ \overline{10},\ \overline{12},\ \overline{14},\ \overline{15},\ \overline{16}\}$ is not a suborn of Z_{18}. Hence it is clear that extended intersection need not to be LOTBSR in general.

Theorem 6: The extended intersection of two LOTBSRs is a LOTBSR if $\mathrm{R}_1(\breve{a})$ is a subring of $\mathrm{R}_2(\breve{a})$ or $\mathrm{R}_2(\breve{a})$ is a subring of $\mathrm{R}_1(\breve{a})$ for all $\breve{a} \in \mathrm{P}_1 \cap \mathrm{P}_2$.

Proof: The prove of this theorem trivial so it is omitted

IV. APPLICATION OF LATTICE ORDERED T-BIPOLAR SOFT RINGS

In this part of the article, our aim is to define an algorithm that can help us to utilize these developed notions for decision-making problems. For this purpose, we will define some basic definitions that can assist us to reach an appropriate result.

Definition 15: Let $\mathrm{P} = \{\breve{a}_1,\ \breve{a}_2,\ \breve{a}_3,\ \ldots,\ \breve{a}_m\} \subseteq E$ be the set of m alternatives and assume that $\mathcal{L}$ and $\mathcal{G}$ are two distinct rings such that $\mathbb{U} = \mathcal{L} \bigcup \mathcal{G}$ and $(\mathfrak{f},\ \mathrm{R},\ \mathrm{P})$ is said to be LOTBSR where $\mathfrak{f} : \mathrm{P} \to P(\mathcal{L})$ and $\mathrm{R} : \mathrm{P} \to P(\mathcal{G})$ are the set valued maps. Then we can represent $(\mathfrak{f},\ \mathrm{R},\ \mathrm{P})$ as

$$\mathbb{d}_{ijk} = (\mathfrak{e}_j,\ \mathfrak{f}_k) = \begin{cases} (0,\ 0)\ \ if\ \ \mathfrak{s}_j \notin \mathfrak{f}(\breve{a}_i)\ \ and\ \ \mathfrak{t}_k \notin \mathrm{R}(\breve{a}_i) \\ (1,\ 0)\ \ if\ \ \mathfrak{s}_j \in \mathfrak{f}(\breve{a}_1)\ \ and\ \ \mathfrak{t}_k \notin \mathrm{R}(\breve{a}_1) \\ (0,\ 1)\ \ if\ \ \mathfrak{s}_j \notin \mathfrak{f}(\breve{a}_1)\ \ and\ \ \mathfrak{t}_k \in \mathrm{R}(\breve{a}_1) \\ (1,\ 1)\ \ if\ \ \mathfrak{s}_j \in \mathfrak{f}(\breve{a}_1)\ \ and\ \ \mathfrak{t}_k \in \mathrm{R}(\breve{a}_1) \end{cases}$$

where $\mathfrak{M}^*_{ijk} = \mathfrak{e}_j$

$$\mathfrak{M}^{\boxdot}_{ijk} = \mathfrak{f}_k$$

Example 8: Let $\mathrm{P} = \{\breve{a}_1,\ \breve{a}_2,\ \breve{a}_3\} \subseteq E$ with $\breve{a}_1 \leq \breve{a}_2 \leq \breve{a}_3$ and $\mathcal{L} = Z_6 = \{\overline{0},\ \overline{1},\ \overline{2},\ \overline{3},\ \overline{4},\ \overline{5}\}$ and $\mathcal{G} = Z_9 = \{\overline{0},\ \overline{1},\ \overline{2},\ \overline{3},\ \overline{4},\ \overline{5}, \overline{6}\ ,\ \overline{7},\ \overline{8}\}$. Now if we define

$$\mathfrak{f}(\breve{a}_1) = \{\overline{0}\},\ \mathfrak{f}(\breve{a}_2) = \{\overline{0},\ \overline{2},\ \overline{4}\},\ \mathfrak{f}(\breve{a}_3) = \{\overline{0},\ \overline{1},\ \overline{2},\ \overline{3},\ \overline{4},\ \overline{5}\}$$

And

$$\mathrm{R}(\breve{a}_1) = Z_9,\ \mathrm{R}(\breve{a}_2) = \{\overline{0},\ \overline{3},\ \overline{6}\},\ \mathrm{R}(\breve{a}_3) = \{\overline{0}\}$$

Then $(\mathfrak{f},\ \mathrm{R},\ \mathrm{P})$ is LOTBSR and it is represented by

$$(\mathfrak{f},\ \mathrm{R},\ \mathrm{P}) = \left\{ \begin{array}{c} (\breve{a}_1,\ \{\overline{0}\},\ Z_9),\ (\breve{a}_2,\ \{\overline{0},\ \overline{2},\ \overline{4}\},\ \{\overline{0},\ \overline{3},\ \overline{6}\}), \\ (\breve{a}_3,\ \{\overline{0},\ \overline{1},\ \overline{2},\ \overline{3},\ \overline{4},\ \overline{5}\},\ \{\overline{0}\}) \end{array} \right\}$$

Now according to definition (15), the tabular representation of this LOTBSR is given in Table 1.

Definition 16: Let $\mathrm{P} = \{\breve{a}_1,\ \breve{a}_2,\ \breve{a}_3,\ \ldots,\ \breve{a}_{\mathfrak{r}}\}$ for $1 \leq i \leq \mathfrak{r}$, $\mathcal{L} = \{l_1,\ l_2,\ l_3,\ \ldots,\ m\}$ for $1 \leq j \leq m$, $\mathcal{G} = \{g_1,\ g_2,\ g_3,\ \ldots,\ g_n\}$ for $1 \leq k \leq n$ be two distinct rings with $\mathbb{U} = \mathcal{L} \bigcup \mathcal{G}$ and $(\mathfrak{f},\ \mathrm{R},\ \mathrm{P})$ be LOTBSRs.

$$Scor_i = \mathfrak{I}_i - \mathfrak{N}_i$$

$$\text{where } \mathfrak{I}_i = \sum\nolimits_{j,\ k} \mathfrak{M}^*_{ijk} \text{ and } \mathfrak{N}_i = \sum\nolimits_{j,\ k} \mathfrak{M}^{\boxdot}_{ijk}$$

Definition 17: Let $\mathrm{P} = \{\breve{a}_1,\ \breve{a}_2,\ \breve{a}_3,\ \ldots,\ \breve{a}_{\mathfrak{r}}\}$ for $1 \leq i \leq \mathfrak{r}$, $\mathcal{L} = \{l_1,\ l_2,\ l_3,\ \ldots,\ m\}$ for $1 \leq j \leq m$, $\mathcal{G} = \{g_1,\ g_2,\ g_3,\ \ldots,\ g_n\}$ for $1 \leq k \leq n$ be two distinct rings with $\mathbb{U} = \mathcal{L} \bigcup \mathcal{G}$ and $(\mathfrak{f},\ \mathrm{R},\ \mathrm{P})$ be LOTBSRs. Then $\breve{a}_i\ (1 \leq i \leq \mathfrak{r})$ is called best alternative if and only if $Scor_i > Scor_{i^\circ}$ for $(i \neq i^\circ)$.

A. AlGORITHM

Let $\mathrm{P} = \{\breve{a}_1,\ \breve{a}_2,\ \breve{a}_3,\ \ldots,\ \breve{a}_{\mathfrak{r}}\}$ for $1 \leq i \leq \mathfrak{r}$, $\mathcal{L} = \{l_1,\ l_2,\ l_3,\ \ldots,\ m\}$ for $1 \leq j \leq m$, $\mathcal{G} = \{g_1,\ g_2,\ g_3,\ \ldots,\ g_n\}$ for $1 \leq k \leq n$ be two distinct rings with $\mathbb{U} = \mathcal{L} \bigcup \mathcal{G}$ and $(\mathfrak{f},\ \mathrm{R},\ \mathrm{P})$ be LOTBSRs. The overall algorithm for choosing the optimal result is given by

Step 1: Collect the data in tabular form for LOTBSRs.

Step 2: Find out the score values $\sigma_1,\ \sigma_2,\ \sigma_2,\ \ldots,\ \sigma_{\mathfrak{r}}$

Step 3: Find out the maximum score value as $\max_i \sigma_i = \sigma_{\mathfrak{T}}$

Step 4: $\sigma_{\mathfrak{T}}$ is the optimal value.

B. NUMERICAL EXAMPLE

Feature engineering:

A critical stage in preparing data for machine learning models is feature engineering. It entails developing new features (variables) or altering already existing ones to boost a machine learning algorithm's performance. In order to better comprehend and learn from the algorithm, it is necessary to extract pertinent information from the raw data and portray it in a suitable manner. Machine learning model performance can be dramatically impacted by effective feature engineering. In order to do this, you must have an extensive knowledge of the data, the issue domain, and the approaches to machine learning that you plan to utilize. Professional feature engineers can design features that collect pertinent data, decrease noise, and improve model generalization from the data. Through the use of a number of techniques, feature engineering enables us to combine or change existing features to produce new ones. These methods assist in drawing attention to the data's most significant patterns and connections, which in turn improves the machine learning model's capacity to learn from the data. Here are some common feature engineering techniques.

1) MISSING DATA IMPUTATION

When working with datasets that contain missing values, missing data imputation is an essential stage in the feature engineering process. The practice of adding new features or changing existing ones to enhance the functionality of machine learning models is known as feature engineering. Dealing with missing variables is crucial since many machine learning algorithms cannot handle them, and if they are not

TABLE 1. Tabular representation of LOTBSR $(ʃ, ℞, ₱) = \{(\breve{a}_1, \{\bar{0}\}, Z_9), (\breve{a}_2, \{\bar{0}, \bar{2}, \bar{4}\}, \{\bar{0}, \bar{3}, \bar{6}\}), (\breve{a}_3, \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}, \{\bar{0}\})\}$.

	$(\bar{0},\bar{0})$	$(\bar{0},\bar{1})$	$(\bar{0},\bar{2})$	$(\bar{0},\bar{3})$	$(\bar{0},\bar{4})$	$(\bar{0},\bar{5})$	$(\bar{0},\bar{6})$	$(\bar{0},\bar{7})$	$(\bar{0},\bar{8})$	$(\bar{1},\bar{0})$	$(\bar{1},\bar{1})$	$(\bar{1},\bar{2})$
$\breve{a}_1$	(1, 1)	(1, 1)	(1, 1)	(1, 1)	(1, 1)	(1, 1)	(1, 1)	(1, 1)	(1, 1)	(0, 1)	(0, 1)	(0, 1)
$\breve{a}_2$	(1, 1)	(1, 0)	(1, 0)	(1, 1)	(1, 0)	(1, 0)	(1, 1)	(1, 0)	(1, 0)	(0, 1)	(0, 0)	(0, 0)
$\breve{a}_3$	(1, 1)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 1)	(1, 0)	(1, 0)
	$(\bar{1},\bar{3})$	$(\bar{1},\bar{4})$	$(\bar{1},\bar{5})$	$(\bar{1},\bar{6})$	$(\bar{1},\bar{7})$	$(\bar{1},\bar{8})$	$(\bar{2},\bar{0})$	$(\bar{2},\bar{1})$	$(\bar{2},\bar{2})$	$(\bar{2},\bar{3})$	$(\bar{2},\bar{4})$	$(\bar{2},\bar{5})$
$\breve{a}_1$	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
$\breve{a}_2$	(0, 1)	(0, 0)	(0, 0)	(0, 1)	(0, 0)	(0, 0)	(1,1)	(1,0)	(1,0)	(1,1)	(1,0)	(1,0)
$\breve{a}_3$	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 1)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)
	$(\bar{2},\bar{6})$	$(\bar{2},\bar{7})$	$(\bar{2},\bar{8})$	$(\bar{3},\bar{0})$	$(\bar{3},\bar{1})$	$(\bar{3},\bar{2})$	$(\bar{3},\bar{3})$	$(\bar{3},\bar{4})$	$(\bar{3},\bar{5})$	$(\bar{3},\bar{6})$	$(\bar{3},\bar{7})$	$(\bar{3},\bar{8})$
$\breve{a}_1$	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
$\breve{a}_2$	(1,1)	(1,0)	(1,0)	(0,1)	(0,0)	(0,0)	(0,1)	(0,0)	(0,0)	(0,1)	(0,0)	(0,0)
$\breve{a}_3$	(1, 0)	(1, 0)	(1, 0)	(1, 1)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)
	$(\bar{4},\bar{0})$	$(\bar{4},\bar{1})$	$(\bar{4},\bar{2})$	$(\bar{4},\bar{3})$	$(\bar{4},\bar{4})$	$(\bar{4},\bar{5})$	$(\bar{4},\bar{6})$	$(\bar{4},\bar{7})$	$(\bar{4},\bar{8})$	$(\bar{5},\bar{0})$	$(\bar{5},\bar{1})$	$(\bar{5},\bar{2})$
$\breve{a}_1$	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
$\breve{a}_2$	(1,1)	(1,0)	(1,0)	(1,1)	(1,0)	(1,0)	(1,1)	(1,0)	(1,0)	(0,1)	(0,0)	(0,0)
$\breve{a}_3$	(1, 1)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 1)	(1, 0)	(1, 0)
	$(\bar{5},\bar{3})$	$(\bar{5},\bar{4})$	$(\bar{5},\bar{5})$	$(\bar{5},\bar{6})$	$(\bar{5},\bar{7})$	$(\bar{5},\bar{8})$						
$\breve{a}_1$	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0, 1)						
$\breve{a}_2$	(0,1)	(0,0)	(0,0)	(0,1)	(0, 0)	(0, 0)						
$\breve{a}_3$	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)	(1, 0)						

TABLE 2. Score values.

(ʃ, ℞, ₱)	$\mathfrak{I}_i$	$\mathfrak{N}_i$	$Scor_i = \mathfrak{I}_i - \mathfrak{N}_i$
$\mathfrak{a}_1$	9	54	−45
$\mathfrak{a}_2$	27	18	9
$\mathfrak{a}_3$	54	6	48

handled correctly, they might produce biased or erroneous results. Here are a few typical feature engineering methods for imputation of missing data (1) Mean, median and mod imputation (2) Constant value imputation (3) Linear regression imputation (4) Domain specific imputation, etc.

2) CATEGORICAL ENCODING

The act of converting categorical (nominal or ordinal) data into a numerical representation that machine learning algorithms can comprehend and utilize efficiently is known as categorical encoding in feature engineering. Encoding categorical features is a critical step in getting data ready for modeling because many machine learning models need numerical input. Here are a few typical feature engineering methods for categorical encoding like (1) One-hot encoding (2) Label encoding (3) Ordinal encoding, etc.

3) VARIABLE TRANSFORMATION

A key component of feature engineering, which is a crucial stage in the preparation of data for machine learning models is variable transformation. In order to make the features (variables) in your dataset better suited for modelling, you must edit or change them. By making the data more useful or by meeting specific modelling algorithm assumptions, this can aid in enhancing the performance of your machine learning models. Some common types of variable transformations used in variable transformations are (1) Normalizations (2) Log transformation (3) Box-Cox transformations (4) Square root and cube root transformation etc.

Assume that there are three kinds of feature engineering techniques that are (1) Missing data imputation (2) Categorical encoding (3) Variable transformation and our aim is to classify these techniques based on above defined algorithm. The overall discussion is given as follows

Step 1: Assume that data is given in the form of LOTBSRs as shown in Table 1.

Step 2: Now we find out the score values and their results are given in Table 2.

Step 3: Find out the maximum score value as $\max_i Scor_i = Scor_w = 48$

Step 4: $\breve{a}_3$ is the best option

V. CONCLUSION

In In this article, we have defined the concepts of OR product, extended union and restricted union for LOTBSRs. In addition, the concepts of AND product, restricted intersection and extended intersection have been added. We have proved some results based on developed notions. For the applicability of the introduced notions, we have defined an algorithm to support the introduced study. To cover the application part of these concepts, we have utilized these notion for the classification of feature engineering techniques for machine learning.

Conflict of Interest

About the publication of this manuscript, the authors declare that they have no conflict of interest.

Data Availability

The data will be available on reasonable request to the corresponding author.

Ethics Declaration Statement

The authors state that this is their original work and it is neither submitted nor under consideration in any other journal simultaneously.

Human and Animal Participants

This article does not contain any studies with human participants or animals performed by any of the authors.

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